Quantum synthesis of arbitrary unitary operators

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I. INTRODUCTION

Controlled manipulations with individual quantum systems such as trapped ions or cold atoms in atomic physics, molecules irradiated by laser fields, and Rydberg atoms interacting with quantized micromaser fields, provides us with a deeper understanding of fundamental principles of physics. Simultaneously, the possibility to control individual quantum systems opens new perspectives in application of quantum physics. Specifically, coherent control over dynamics of quantum systems is of vital importance for quantum computing and information processing [1,2].

Quantum information processing can be schematically devided into three stages. The first stage is the encoding of information into quantum systems, i.e. this corresponds to a preparation of states of quantum systems. The second stage is the information processing which in general is equivalent to a specific unitary evolution of the quantum system, i.e. this is an application of a given quantum algorithm. The third stage is the reading of output states of quantum registers (i.e. the “decoding” of information from quantum systems). Obviously this final stage is the measurement of quantum system and the reconstruction of relevant information.

There are several physical systems which are believed to be hot candidates for quantum processors. In particular, Cirac and Zoller [3] have shown that a system of trapped ions can be utilized as a prototype of a quantum computer. Therefore it is of great interest to understand how the three stages of the information processing as specified above can be implemented in this system.

(i) State preparation: Recently, several methods for deterministic synthesis (preparation) of vibrational states of trapped ions have been proposed. In particular, a scheme for preparation of quantum states of one and two-mode bosonic fields (e.g. 1 and 2-D quantum states of vibrational motion of trapped ions) have been proposed by Law and Eberly [4], and Kneer and Law [5] (see also [6] and for a more general discussion on the state preparation see [7]).

(ii) State measurement: There are various experimental techniques which allow to measure and reconstruct quantum states of trapped ions (for a review see [8]).

(iii) Arbitrary unitary evolution: One of the most important task in information processing is to design processors which take an arbitrary input and process it according to a specific prescription. Nature provides us with a restricted set of microscopic interactions. The question is whether we can synthesize out of these fundamental interactions an arbitrary unitary operator. In this paper we present a constructive algorithm for realization of any unitary operator which acts on a finite-dimensional Hilbert space. An algorithmic proof that any discrete finite-dimensional unitary matrix can be factorized into a sequence of two-dimensional beam splitter transformations was given by Reck et al. [9]. The problem of controlled dynamics of quantum systems has been addressed recently by Harel and Akulin [10] and by Lloyd and Braunstein [11]. Harel and Akulin [10] have proposed a method to attain any desired unitary evolution of quantum systems by switching on and off alternatively two distinct constant perturbations. The power of the method was shown in controlling the 1-D translational motion of a cold atom.

Our aim is to find a constructive algorithm to realize an arbitrary unitary operator \( \hat{V} \) which transforms any state \( |\psi\rangle \) of a single bosonic mode, e.g. an 1-D vibrational state of a trapped ion in \( x \) direction, to another state \( |\psi'\rangle = \hat{V}|\psi\rangle \). In particular, we consider a truncated \((N+1)\)-dimensional Hilbert space of the bosonic mode. Within this truncated Hilbert space the desired unitary operator \( \hat{V} \) in the number-state basis reads

\[
\hat{V} = \sum_{m,n=0}^{N,N} V_{n,m} |n\rangle \langle m|.
\]

Under action of the operator \( \hat{V} \) a given input state transforms as

\[
|\psi\rangle = \sum_{m=0}^{N} c_m |m\rangle \rightarrow \hat{V}|\psi\rangle = \sum_{n=0}^{N} c'_n |n\rangle
\]

where \( c'_n = \sum_{m=0}^{N} V_{n,m} c_m \). Our task is to represent by a
feasible physical process any operator $\hat{V}$ (1). The synthesis of operators enables thus to realize universal quantum gates for qubits which can be encoded into vibrational levels.

The paper is organized as follows. In Section II we briefly introduce physical tools which we use to realize arbitrary unitary operators for a trapped ion. The synthesis algorithm is described in Section III. The method is illustrated in Section IV where a realization of quantum gates which perform the Fourier transform is considered. We also discuss stability of the algorithm and possibilities to realize also non-unitary operators. We finish our paper with conclusions.

II. TOOLS FOR SYNTHESIS: LASER STIMULATED PROCESSES

Our realization of the unitary transformation $\hat{V}$ given by Eq.(1) is based on an enlargement of the Hilbert subspace of the given system. Namely, the transformed bosonic mode corresponds to one vibrational mode (in $x$ direction, for concreteness) of a quantized center-of-mass motion of an ion confined in the 2-D trapping potential. Within our synthesis procedure the vibrational $x$-mode becomes entangled with the auxiliary degrees of freedom (ancilla) which are represented by the second vibrational mode (e.g., quantized vibrational motion in $y$ direction) and three internal electronic levels $|a\rangle, |b\rangle, |c\rangle$ of the ion. The particular choice of the physical system is motivated by a feasibility of highly coherent control over motional degrees of freedom as demonstrated in recent experiments [8] which makes trapped ions to be a hot candidates for quantum processors.

A physical realization of the desired operator $\hat{V}$ for an ion confined in a 2-D trapping potential consists of a sequential switching (on/off) of laser fields which irradiate the ion. Namely, we utilize four types of laser stimulated interactions which are associated with the following (effective) interaction Hamiltonians:

$$\hat{H}^{(1,m)} = (\Delta_y + \hat{s}_x)(|a\rangle\langle a| - |b\rangle\langle b|) + g_1|a\rangle\langle b| + g_1^*|b\rangle\langle a|,$$

$$\hat{H}^{(2)} = g_2|b\rangle\langle c|F(\hat{a}_y^\dagger \hat{a}_y)|b\rangle + g_2^*|c\rangle\langle b|,$$

$$\hat{H}^{(3)} = g_3|b\rangle\langle c| + g_3^*|c\rangle\langle b|,$$

$$\hat{H}^{(4)} = g_4|b\rangle\langle c|F(\hat{a}_x^\dagger \hat{a}_x)|c\rangle + g_4^*|c\rangle\langle b|,$$

where $\hat{s}_x = \sum_m s_m |m\rangle_x \langle m|$ with $s_m = \chi \left[ 1 + e^{-2\eta^2}L_m^0 \right]$ and $F(\hat{a}_q \hat{a}_q) = e^{-\eta^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k\eta^{2k}}{k!} \hat{a}_q^k \hat{a}_q^k$. The Lamb-Dicke parameters are defined as $\eta_q = \omega_q/(c\sqrt{2m_\omega \nu_q})$ (assuming units such that $\hbar = 1$) where $\omega_q, \nu_q$ are frequencies of the laser and vibrational mode in direction $q$ ($q = x, y$), respectively. Further, $L_m^0$ denote the Laguerre polynomial; $m_\omega$ is the mass of the ion.

The dynamical Stark shift operator $\hat{s}_x = \sum_m s_m |m\rangle_x \langle m|$ is induced by a detuned standing-wave laser field applied in $x$ direction [5]. For $\chi >> g_1$ the effective Hamiltonian $\hat{H}^{(1,m)}$ addresses the states with fixed number $m$ of excitations (indicated by the superscript) in the mode $x$ while the detuning $\Delta_y$ of the laser field, applied in the orthogonal $y$ direction, equal to the dynamical Stark shift $-s_m$. In other words, when the interaction is governed by the Hamiltonian $\hat{H}^{(1,m)}$ with $\Delta_y = -s_m$ there is an exchange of the population only between the states $|m, n\rangle \otimes |a\rangle \leftrightarrow |m, n\rangle \otimes |b\rangle$ with the given number of quanta $m$ in the vibrational mode $x$ and any number of phonons $n$ in the mode $y$. The populations of the other number states (with number of excitations in the mode $x$ different from $m$) effectively do not change due to a large detuning. However, there are significant phase shifts of the amplitudes of the off-resonant states. It should be stressed that the addressing of states with a phase shifts of the amplitudes of the off-resonant states.

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The considered interactions (3) are quite typical for a trapped ion. The effective interaction Hamiltonians represent a classical driving of the ion [see $H^{(2)}$] and a non-linear Jaynes-Cummings model [12] [see $H^{(3)}$] when the lasers applied in given directions ($x, y$) are tuned to appropriate vibrational sidebands. These Hamiltonians are thoroughly discussed in papers [5,13].

The dynamics governed by the interaction Hamiltonians (3) can be separated into independent 2-D subspaces. Switching on a particular interaction “channel” associated with one of the interaction Hamiltonians $H^{(p)}$ (3) for a time $\tau$ is described as the action of the corresponding unitary time-evolution operator $\hat{U}(\rho) = \exp[-iH^{(p)}\tau]$ on the state vector of the system under consideration.

III. SYNTHESIS OF TRANSFORMATIONS

To implement the desired transformation $\hat{V}$ (1) for an ion confined in the 2-D trapping potential we realize a mapping $|^0\psi\rangle \to |0\psi\rangle = |0\rangle \otimes |\psi\rangle$ of two–mode bosonic states (here subscripts indicate particular vibrational modes; in what follows the subscripts will be omitted for given ordering of modes). To be more explicit, the realization of the transformation $\hat{V}$ can be expressed as the mapping in the extended Hilbert space of two bosonic modes and internal electronic levels in the following form:

$$|0, \psi\rangle \otimes |b\rangle = \hat{N}\hat{P}_{|b\rangle} \hat{B} \hat{A} \psi, 0 \rangle \otimes |a\rangle$$

where the operators $\hat{A}$ and $\hat{B}$ represent a sequence of four types of unitary operations. The final projection $\hat{P}_{|b\rangle}$ on the state $|b\rangle$ selects conditionally the right outcome ($\hat{N}$ is a proper normalization constant). In the subsequent step one could use the two–mode linear coupler based on laser stimulated Raman transitions [14] to swap the states of the vibrational modes, i.e., $|0, \psi\rangle \to |\psi, 0\rangle$. The additional $\pi$-pulse can be used to flip the electronic state from the level $|b\rangle$ into the initial level $|a\rangle$. 
The operators $\hat{A}$, $\hat{B}$, $P_{|b\rangle}$ appearing in the Eq.(4) indicate four essential steps which lead us subsequently to the desired transformations.

Step A. The operator $\hat{A}$ "spreads" the amplitudes $c_m$'s of the component number states $|n_x,0\rangle \otimes |a\rangle$ over the whole Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_m$ so that the entangled state of the composed system becomes

$$|\psi^A\rangle = \sum_{m,n=0}^{N,N} V_{n,m} c_m \exp(\tau \sqrt{i\phi_n}) |m,n\rangle \otimes |a\rangle. \quad (5)$$

This task can be done by the method of 1-D quantum state synthesis associated with the action of the operator $\hat{A}(m)$ to "prepare" the superposition $\sum_{n=0}^{N} V_{n,m} |m,n\rangle \otimes |b\rangle$. After that the electronic level $|b\rangle$ is flipped back to $|a\rangle$ via the action of $\hat{U}_{-\pi/2}^{(1,m)}$, where $\phi_m$'s represent proper phase factors which will be discussed later, see Eq.(11).

The appropriate interaction parameters for our 1-D quantum state synthesis can be found when we solve the inverse task which is given by the inverse transformation: $|m,0\rangle \otimes |b\rangle = \hat{A}^\dagger(m) \sum_{n} V_{n,m} |m,n\rangle \otimes |b\rangle$. The inverse task is based on "sweeping" down the probability of the component states of the given superposition in the $y$ mode into the vacuum. Therefore the subscripts of the unitary operators $\hat{U}_{(p)}$ ($p = 2,3$) in (7) indicate that interaction parameters $g_{p\tau} = \exp(i\phi) |g_p\rangle$ have to be chosen in such way that after the action of the $\hat{U}_{(p)}$ the amplitude (population) of the component state $|\Phi\rangle$ becomes equal to zero. The action of the operator $\hat{A}^\dagger(m)$ is shown schematically in Fig. 1. We have applied the procedure introduced by Law and Eberly for synthesis of 1-D bosonic states in the straightforward way. Therefore we refer readers to the original paper [4] for other details (generalized Hamiltonians to operate beyond the Lamb–Dicke limit can be found in [13]). Note that in our case we apply the state–synthesis procedure only in $y$ direction and the resulting state (5) remains unknown.

To resume, the action of the operator $\hat{A}$ encodes the matrix elements $V_{n,m}$ (multiplied with the amplitudes $c_m$'s of unknown state) into rows of the 2D vibrational “lattice” of number states $|m,n\rangle$. In the next step an appropriate superposing of columns within the 2D vibrational-number-state “lattice” is required.

Step B. In the second step the operator $\hat{B}$ creates the state in which the amplitudes of the states $|0,n\rangle \otimes |b\rangle$ ($n = 0, ...N$) are proportional to $c_n = \sum_{m=0}^N V_{n,m} c_m$. The state of the system $|\psi^B\rangle = \hat{B}|\psi^A\rangle$ after this synthesis step reads

$$|\psi^B\rangle = \frac{1}{\sqrt{N+1}} |0,\psi'\rangle \otimes |b\rangle + \sum_{m=0}^N \sum_{n=0}^N z_{m,n} |m,n\rangle \otimes |a\rangle. \quad (8)$$

The operator $\hat{B}$ can be written in the form

$$\hat{B} = \prod_{m=0}^{N-1} \hat{B}(m) \hat{U}_{-\pi/2}^{(1,N)} \hat{U}_{-\pi/2}^{(1,m)} \hat{U}_{-\pi/2}^{(1,0)} \hat{U}_{-\pi/2}^{(1,m)}, \quad (9)$$

where

$$\hat{B}(m) = \hat{U}_{-\pi/2}^{(1,m)} \hat{U}_{-\pi/2}^{(1,m)} \hat{U}_{-\pi/2}^{(1,m)} \hat{U}_{-\pi/2}^{(1,m)}, \quad (10)$$

The subscripts of the unitary operators $U_{(p)}^{(p)}$ indicate

*In our synthesis algorithm we neglect off-resonant transitions between internal levels in $H_{(1,m)}$. Strictly speaking, the equality sign applies only in the limit $\chi/|g_1| \rightarrow \infty$. 

Fig. 1. The action of the operator $\hat{A}^\dagger(m)$ on the row with a fixed number $m$ of phonons in the mode $x$. The population of the component number states $|m\rangle \otimes |a\rangle$ is "spread" down into the vacuum $|0\rangle_y$ in the mode $y$.
The action of the operator $\hat{B}$ on a particular column with a given number $n$ of phonons in the $y$ mode is shown schematically in Fig. 2. Simultaneously, the operator $\hat{B}$ acts in the same way on “parallel” columns with different $n$. As illustrated in the upper part of Fig. 2, the operator $\hat{U}^{(1,N)}_{1^{-i\tau}}$ (for $N = 5$) transfers the population from the internal level $|a\rangle$ to $|b\rangle$ only in the row with the fixed number of quanta $m = 5$ in the mode $x$. Further, this population is transferred to the internal level $|b\rangle$ in the neighboring row with the number of quanta $m = 4$ performing transformations $\hat{U}^{(3)}_{1^{-i\tau/2}}$, $\hat{U}^{(4)}_{1^{-i\tau/2}}$, $\hat{U}^{(5)}_{1^{-i\tau/2}}$.

FIG. 2. The action of the operator $\hat{B}$ on the column with $n$ vibrational quanta in the mode $y$. The same changes occur simultaneously on “parallel” columns with different $n$. The upper figure shows the action of $\hat{U}^{(1,N)}_{1^{-i\tau/2}}$ (for $N = 5$) followed by the basic sequence of operations $\hat{B}(m)$ (for $m = 4$). Decreasing the number of quanta $m$ in $x$ direction the basic sequence $\hat{B}(m)$ is recursively repeated as shown in the lower figure. After the action of the operator $\hat{B}$ the value of the amplitude $s_n = (\sum_{m=0}^{N} V_{n,m} c_m) / \sqrt{N+1}$.

The equal superposition of the amplitudes $V_{n,4} c_4$ and $V_{n,5} c_5$ in the row with $m = 4$ is obtained after action of $\hat{U}^{(1,N)}_{1^{-i\arctan(1/\sqrt{N+1})}}$, i.e. undergoing one half of the Rabi flipping. Decreasing $m$ (the number of quanta in the mode $x$) the basic sequence $\hat{B}(m)$ is recursively repeated as indicated in the lower part of Fig. 2. Note that $\hat{U}^{(1,m)}_{1^{-i\arctan(1/\sqrt{N+1})}}$ is responsible for adding of the amplitude $V_{n,m} c_m$ (with a proper weight) to previously superposed amplitudes $\sum_{m=0}^{N} V_{n,m} c_m$ performing an adequate part of the Rabi flipping. After the action of the whole operator $\hat{B}$ the value of the amplitude $s_n = \frac{\sum_{m=0}^{N} V_{n,m} c_m}{\sqrt{N+1}}$. The transformed state $|\psi^B\rangle$ is given in Eq.(8).

At this step we should notice that each action of the “elementary” unitary operator $\hat{U}^{(1,m)}_{1^{-i\tau/2}}$, associated with the interaction Hamiltonian $\hat{H}^{(1,m)}$ with $\Delta_g = -s_m$, causes significant phase shifts on off-resonant rows with the number of quanta in the mode $x$ different from $m$. These phase shifts have to be compensated in advance in order “to superpose” the amplitudes via the action of $\hat{B}(m)$ as described above [see the rôle of $\hat{U}^{(1,m)}_{1^{-i\arctan(1/\sqrt{N+1})}}$]. This compensation can be done when we include appropriate phase shifts $\phi_m$’s directly in the operator $\hat{A}$ [see Eq.(7)]. The explicit expression reads

$$\phi_m = -\frac{\pi}{2} \sum_{k=0}^{m-1} f_m^{(k)}(\pi) - \sum_{k=m+1}^{N} f_m^{(k)}(\arctan \frac{1}{\sqrt{N-k+1}})$$

(11)

where $f_m^{(k)}(\tau) = \arg[\cos(\Omega_m^{(k)} \tau) + i \frac{\Omega_m^{(k)}}{\Omega_m^{(k)}} \sin(\Omega_m^{(k)} \tau)]$ with $\Omega_m^{(k)} = \sqrt{(s_k - s_m)^2 + |g|^2}$. The origin of the expression (11) can be traced back to the operators $\hat{U}^{(1,m)}_{1^{-i\tau}}$ in the steps $A$ and $\hat{B}$. The aim is to cancel the phase shifts of the amplitudes in order to “superpose” them on the $m$-th row by means of $\hat{B}(m)$. Therefore the first sum in (11) compensates (in advance) the subsequent shifts in $A$ due to $\hat{U}^{(1,k)}_{1^{-i\tau}}$ for $k = 0, \ldots, m - 1$. The second sum compensates the shifts which will take place during “superposing” operations $\hat{B}(k)$ for $k = m + 1, \ldots, N$ which precede $\hat{B}(m)$.

**Step C.** Comparing the state $|\psi^B\rangle$ [Eq.(8)] with the desired one [Eq.(4)] we see that the target state is entangled to the internal level $|b\rangle$. On the other hand, all undesired component states $|m \neq 0, n \rangle \otimes |a\rangle$ are now contributing into Eq.(8) with nonvanishing amplitudes. However (fortunately), all the undesired components are entangled with the internal level $|a\rangle$. Therefore we can perform a conditional measurement to project the state vector (8) on the internal level $|b\rangle$. To be more specific, the internal state of the ion can be determined by driving transition from the level $|a\rangle$ to an auxiliary level $|r\rangle$.
and observing the fluorescence signal. No signal (no interaction with probing field) means that the undisturbed ion is occupying the level $|b\rangle$ being in the motional state $|0,\psi\rangle$. It means that after the conditional measurement [indicated in (4) by the projector $\hat{P}_{|b\rangle}$] the state vector (8) is reduced to the desired state vector (4). The probability to find the right outcome for the unitary transformations is equal to $\frac{1}{N+1}$.

In spite of the involved conditional selection of the right outcomes, our algorithm is universal as the sequence of the “elementary” operations (with appropriate interaction parameters) which represents the desired transformation is always independent of input states. Moreover, in contrary to conditional measurement schemes known from quantum state preparation, in our case the probability of the right outcome is constant being also independent of input states.

IV. DISCUSSION

One of important applications of the operator synthesis is a realization of universal quantum gates for qubits which are encoded in vibrational levels. The number states of the vibrational mode can represent a quantum register.

To illustrate our synthesis procedure we considered a realization of the operator which “rotates” the population between $N$ vibrational Fock states under consideration.

$$\hat{V}_R = \frac{1}{\sqrt{N+1}} \sum_{j=0}^{N-1} (j+1) |j+1\rangle \langle j | + |0\rangle \langle N|.$$  

(12)

It corresponds to a cyclic “rotation” of the quantum register. The operator $\hat{V}_R$ represents the unitary exponential phase operator of the Pegg–Barnett formalism [15].

In the second example the synthesis procedure is applied for the unitary operator of the quantum Fourier transform defined as [16]

$$\hat{V}_{QFT} = \frac{1}{\sqrt{N+1}} \sum_{m,n=0}^{N-1} \exp \left( 2\pi i \frac{mn}{N+1} \right) |m\rangle \langle n|.$$  

(13)

The operator of the quantum Fourier transform represents an important tool in quantum computing [16].

As a check one could drive also transition from the level $|c\rangle$ to another auxiliary level $|\psi\rangle$. Errors in the synthesis procedure are thus indicated by presence of the fluorescence signal.

![Fig. 3](image-url)

**FIG. 3.** The fidelity of outputs to the ideally transformed states as function of the ratio $\chi/|g_1|$ for the quantum Fourier transform $\hat{V}_{QFT}$ (thick solid line) and the exponential phase operator $\hat{V}_R$ (thin solid line). Lamb–Dicke parameters are $\eta_x = \eta_y = 0.4$.

In the presented synthesis procedure we have neglected transitions between internal levels on off-resonant “rows” when the interaction Hamiltonian $\hat{H}^{(1,m)}$ is applied. Strictly speaking, the off-resonant transitions in $\hat{H}^{(1,m)}$ can be neglected only in the limit $\chi/|g_1| \to \infty$. Our estimation of the error due to the finite values of the ratio $\chi/|g_1|$ (feasible in practice) is based on the fidelity of the outputs to the ideally transformed states (4). The fidelity of two states $|\Phi\rangle$, $|\Phi'\rangle$ is defined as their squared scalar product $|\langle\Phi'|\Phi\rangle|^2$. As a testing input state of the $x$-mode we can take a uniform superposition of involved number states, i.e., $|\psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=0}^{N} |m\rangle_x$ (the initial internal state of the ion is $|a\rangle$ and the vibrational $y$-mode is in the vacuum). The figure 3 shows that the fidelity is close to one for large ratios $\chi/|g_1| \gg 1$ for both operators $\hat{V}_R$ and $\hat{V}_{QFT}$. In such case both the validity of the approximate Hamiltonian $\hat{H}^{(1,m)}$ is justified [5] and the level flipping ($|a\rangle \leftrightarrow |b\rangle$) on off-resonant “rows” (as the source of non-ideal fidelity) can be neglected.

Let us note that for nonunitary transformations the probability depends on initial states and can be from the interval $(0,1)$. However, the realization of the transformation (given by the sequence of the “elementary” operations with appropriate interaction parameters) is always independent of input states.

In Fig. 3 we consider parameters outside of the Lamb–Dicke regime. To operate in the Lamb–Dicke regime requires a further increase of the ratio $\chi/|g_1| \gg 1$ to reach a fidelity close to one.
V. CONCLUSIONS

In this paper we have proposed a constructive algorithm for synthesis of operators, a superior task to synthesis of quantum states. Our method allows us to find analytical expressions for switching times and interaction parameters of the utilized laser stimulated processes via which an arbitrary unitary dynamics can be realized. One of important applications of the operator synthesis is a realization of universal quantum gates for qubits which are encoded in vibrational levels. As an example we consider realization of the discrete Fourier transform.

The solution of the problem we present in our paper is neither unique nor optimal (in a sense of number of elementary operations used for a construction of the given unitary operator). The optimization of the procedure is the problem which has to be solved. The other problem which deserves attention is the stability of the algorithm with respect to noise inherent in the system. In fact, one can consider two types of uncertainties which might play an important rôle. Firstly, it is the noise induced by the environment, i.e. the elementary gates are not unitary. The second source of noise (kind of technical noise) is due to the fact that it is not possible to keep the interaction times and parameters to be fixed as given by the theory. Fluctuations in these parameters might reduce the fidelity of the realization of the desired unitary evolution. We will address these questions elsewhere.

ACKNOWLEDGMENTS

We thank Gil Harel and Vladimir Akulin for correspondence, and Jason Twamley for discussions. This work was supported in part by the Slovak Academy of Sciences (project VEGA), by the GACR (201/98/0369), and by the Royal Society.