String Propagation in Bianchi Type I models: Dynamical Anisotropy Damping and Consequences

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Abstract

A generic ansatz is introduced which provides families of exact solutions to the equations of motion and constraints for Null strings in Bianchi type I cosmological models. This is achieved irrespective of the form of the metric. Within classes of dilaton cosmologies a backreaction mapping relation is established where the null string leads to more or less anisotropic members of the family. The equations of motion and constraints for the generic model are casted in first order form and integrated both analytically and numerically.

I. Introduction

In recent years string theory has emerged as the most promising candidate for the consistent quantization of gravity and a unified description of all the fundamental interactions. A consistent quantum theory of gravity is necessary part of a unified theory of all interactions, because pure gravity (a model containing only gravitons) cannot be a physical and realistic theory. This is the strongest motivation for the study of strings in curved spacetimes [1-3].

In this context strings moving in curved spacetimes have gained considerable attention, since they could provide clues to a proper generalization of the theory [2-3].

Among other notable features, string theory has become a theory that gives interesting answers towards other fields, such as the physics of Black Holes, Cosmology, Galaxy formation etc [2-4].

In the cosmological context inflationary models have become natural outcomes of the theory. Yet a treatment of anisotropy which is a basic feature of Cosmological spacetimes has not be considered in the framework of string theory.
Anisotropic Cosmological spacetimes are candidates for a description of the early Universe, because it is known that inhomogeneous and anisotropic features may have prevailed in the primary stages of the evolution of the Universe.

Since present day observations suggest that the Universe is highly homogeneous and isotropic, it would be interesting to examine whether string theory could account for a possible altering of anisotropic features that may have prevailed in the primordial stages of the Universe.

Generic Bianchi I type cosmological solutions can occur in the context of more general dynamical theories [5]. As it is known dilaton fields appear naturally at the low energy limit of string theory, coupled with Einstein-Maxwell fields [6-8]. Also, dilaton fields appear as a result of dimensional reduction, of the Kaluza-Klein type Lagrangian [9,10]. The presence of a homogeneous primordial magnetic fields implies that the cosmological model is necessary anisotropic [5]. Much work has been done on magnetic Bianchi cosmologies in order to establish their main properties [11-14]. The introduction of the dilaton fiels renders the system in a completely integrable form [5]. It would be interesting to consider the general effects of the introduction of null strings in the above context, particularly in view of the possible interaction of null strings with the dilaton and/or EM fields and the fact that dilaton fields appear in the low energy-limit of string theory. The purpose of this paper is to study and examine these classes of models.

The null string is a collection of massless point particles moving along the geodesics of curved spacetime [15]. Schild [16] was the first to consider null or tensionless strings (see also [17]). Null strings can be considered as the zeroth order term, of the perturbation expansion with respect to the string tension [18-20]. The evolution of null strings in different curved spacetime manifolds has been considered extensively [21-23].

This paper is organized as follows: In section II we present the dynamical equations of bosonic and null string theory in general Bianchi type I models and we specify the classes of Dilaton-Bianchi cosmological spacetimes under consideration. The generic procedure for obtaining classes of null-string solutions is explained in section III and applied to the specified cosmological solutions. In section IV the energy-momentum tensor of string configurations is studied and the effects of string tension on the anisotropic features of these cosmological spacetimes is examined through the backreaction process. The thermodynamical properties of the general class of solutions, through the ansatz introduced, are studied in section V. In section VI, without further specification, the equations of motion and constraints for Bianchi type I models in their first order form, are integrated both analytically and numerically.
II. Strings in Bianchi type I Models

The action for a bosonic string in a curved-spacetime background is given (for a D-dimensional spacetime) by [1]

\[
S = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \alpha^\beta(\tau, \sigma) \partial_\alpha X^M \partial_\beta X^N G_{MN}(X)
\]  

(1)

where \((M, N = 0, 1, \ldots, (D - 1)\) are spacetime indices, \((\alpha, \beta = 0, 1)\) are worldsheet indices and \(T = (2\pi \alpha')^{-1}\) is the string tension.

Variation of the action with respect to the "fields" which are the string coordinates \(X^M(\tau, \sigma)\), gives the following equations of motion and the constraints in the conformal gauge \((h_{\alpha\beta}(\tau, \sigma) = \exp(\phi(\tau, \sigma))\eta_{\alpha\beta})\) where \(\phi(\tau, \sigma)\) is an arbitrary function and \(\eta_{\alpha\beta} = \text{diag}(-1, +1)\) [2]

\[
\ddot{X}^M - (X^M)'' + \Gamma^M_{AB}[\dot{X}^A \dot{X}^B - X^A X^B] = 0 \tag{2}
\]

\[
G_{AB}(X)[\dot{X}^A \dot{X}^B + X^A X^B] = 0 \tag{3}
\]

\[
G_{AB}(X)\dot{X}^A X^B = 0 \tag{4}
\]

where the dot stands for \(\partial_\tau\) and the prime stands for \(\partial_\sigma\).

The equations of motion and constraints for the null-strings are given by

\[
\ddot{X}^M + \Gamma^M_{AB} \dot{X}^A \dot{X}^B = 0 \tag{5}
\]

\[
G_{AB}(X)\dot{X}^A \dot{X}^B = 0 \tag{6}
\]

\[
G_{AB}(X)\dot{X}^A X^B = 0 \tag{7}
\]

The action (1) is invariant under general reparametrizations \(\delta^\alpha = \delta^{\alpha}(\sigma^\beta)\) whereas the null string equations of motion and constraints only under \(\tau_1 = f(\tau, \sigma), \sigma_1 = g(\sigma)\) [15].

We consider the four-dimensional Bianchi type I model as the background spacetime, where the string coordinates are denoted by \(t = t(\tau, \sigma), X = X(\tau, \sigma), Y = Y(\tau, \sigma), Z = Z(\tau, \sigma)\).

The metric tensor is given by

\[
ds^2 = -dt^2 + \alpha^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2 \tag{8}
\]

where \(\alpha, b, c\), are differentiable functions of the cosmic time. The Bianchi I spacetime represents an expanding universe since the volume element of any spacelike hypersurface of simultaneity is constantly increasing [24,26]

\[
\sqrt{-G} = \sqrt{-(3)} G = a(t)b(t)c(t) \tag{9}
\]

The equations of motion for \(X^\mu = X^\mu(\tau, \sigma)\) \((\mu = 0, 1, 2, 3)\) are given

\[
\ddot{t} - t'' = a\dot{a}[-(\dot{X})^2 + (X')^2] + b\dot{b}[-(\dot{Y})^2 + (Y')^2] + c\dot{c}[-(\dot{Z})^2 + (Z')^2] \tag{10}
\]

\[
\partial_\tau[a^2(t)\dot{X}] = \partial_\sigma[a^2(t)X'] \tag{11}
\]

\[
\partial_\tau[b^2(t)\dot{Y}] = \partial_\sigma[b^2(t)Y'] \tag{12}
\]

\[
\partial_\tau[c^2(t)\dot{Z}] = \partial_\sigma[c^2(t)Z'] \tag{13}
\]
The constraints become
\[ - \left[ (\dot{t})^2 + (\dot{t}')^2 \right] + a^2(t) \left[ (\dot{X})^2 + (X')^2 \right] + 
+ b^2(t) \left[ (\dot{Y})^2 + (Y')^2 \right] + c^2(t) \left[ (\dot{Z})^2 + (Z')^2 \right] = 0 \]  
(14)
\[ -i\dot{t} + a^2(t) \dot{X}X' + b^2(t) \dot{Y}Y' + c^2(t) \dot{Z}Z' = 0 \]  
(15)
We will be concerned with the case \((t) \geq 0\) and normalize our solutions so that \((\tau) \geq 0\).

The open string boundary conditions demand
\[ (X^M)'(\tau, \sigma = 0) = (X^M)'(\tau, \sigma = \pi) = 0 \]  
(16)
while closed strings
\[ X^M(\tau, \sigma = 0) = X^M(\tau, \sigma = 2\pi) \]  
(17)
Introducing light-cone coordinates, \(\chi^\pm \equiv (\tau \pm \sigma)\), the equations of motion become
\[ \partial_+ \partial_- t + \alpha \dot{\alpha} (\partial_+ X)(\partial_- X) + b \dot{b} (\partial_+ Y)(\partial_- Y) + 
+ c \dot{c} (\partial_+ Z)(\partial_- Z) = 0 \]  
(18)
\[ \alpha \partial_+ \partial_- X + \dot{\alpha} [\partial_+ X \partial_- t + \partial_+ t \partial_- X] = 0 \]  
(19)
\[ b \partial_+ \partial_- Y + \dot{b} [\partial_+ Y \partial_- t + \partial_+ t \partial_- Y] = 0 \]  
(20)
\[ c \partial_+ \partial_- Z + \dot{c} [\partial_+ Z \partial_- t + \partial_+ t \partial_- Z] = 0 \]  
(21)
and the constraints read
\[ - \left[ (\partial_+ t)^2 \pm (\partial_- t)^2 \right] + a^2(t) \left[ (\partial_+ X)^2 \pm (\partial_- X)^2 \right] + 
+ b^2(t) \left[ (\partial_+ Y)^2 \pm (\partial_- Y)^2 \right] + c^2(t) \left[ (\partial_+ Z)^2 \pm (\partial_- Z)^2 \right] = 0 \]  
(22)
Recently, cosmological solutions of the Bianchi type I, possessing dilaton and electromagnetic fields have been considered [5]. The solutions consist of two families of particular solutions for the scale factors and the dilaton field. The action is given by (note however several misprints in [5])
\[ S = \int \sqrt{-g} (R + \lambda g^\mu{}^\nu \omega_{\mu} \omega_{\nu} + \frac{1}{2} \exp(-2\omega) F_{\alpha \beta} F^{\alpha \beta}) d^4 x \]  
(23)
and the equations of motion read
\[ \Box \omega = \frac{e^{-2\omega}}{2\lambda} F_{\mu \nu} F^{\mu \nu} \]  
(24)
\[ (e^{-2\omega} F^{\mu \nu})_{; \nu} = 0 \]  
(25)
\[ G_{\mu \nu} = T^{(MATT)}_{\mu \nu} \]  
(26)
\[ \Box \equiv g^{\mu \nu} \nabla_\mu \nabla_\nu \]  
(27)
\[ T^{(MATT)}_{\mu \nu} \equiv -\lambda \left( \omega_{\mu} \omega_{\nu} - \frac{1}{2} g_{\mu \nu} \omega_{\alpha} \omega^{\alpha} \right) - 
- \exp(-2\omega) \left( F_{\mu \alpha} F^{\alpha \nu} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right) \]  
(28)
with \( \omega \) the dilaton field. The first family of particular solutions is given by

\[
\alpha(\tau) = A_0 e^{\omega_0 \tau}, \quad b(\tau) = B_0 e^{\omega_1 \tau}, \quad c(\tau) = C_0 e^{\omega_2 \tau}, \quad \omega(\tau) = W_1 \tau + W_0 \tag{29}
\]

where the constants satisfy \( \lambda W_1 + 2(x_0 y_0 + y_0 z_0 + z_0 x_0) = 0 \) with \( \lambda \) the dilaton coupling constant and \( \tau \) defined with respect to the cosmic time \( t \) as \( dt = \alpha(\tau)b(\tau)c(\tau)d\tau \). The second family is given by

\[
\alpha(\tau) = A_0 \tau^{-p}, \quad b(\tau) = B_0 \tau^p, \quad c(\tau) = C_0 \tau^p, \quad \omega(\tau) = W_0 \ln \tau \tag{30}
\]

with \( p = \lambda / (\lambda - 2) \), \( W_0 = 2 / (\lambda - 2) \). We want the kinetic term of the dilaton field to be positive, i.e. \( \lambda < 0 \) which requires \( 0 < p < 1 \).

There exist also three families of general solutions. The first one which we shall deal with, is given in terms of the initial expansion rates along the three directions \((\omega_1, \omega_2, \omega_3)\). Here \( \Omega_0 \equiv \omega_1 \omega_2 + \omega_2 \omega_3 + \omega_1 \omega_3 \), \( r^2 \equiv A^2(1 + A^2) \), \( \lambda = -2A^2 \) and \( s \equiv A^2 \omega_1 \).

\[
\omega(\tau) = -\frac{s}{r^2} \tau + \omega_0 - \frac{A^2}{r^2} \ln \left[ \sinh \left( \frac{\beta r}{A^2} \right) \right] \tag{31}
\]

\[
\alpha(\tau) = A_0 e^{-\left(s A^2 / r^2\right) \tau} \left[ \sinh \left( \frac{\beta r}{A^2} \right) \right]^{-A^4 / r^2} e^{\omega_1 \tau} \tag{32}
\]

\[
b(\tau) = B_0 e^{\left(s A^2 / r^2\right) \tau} \left[ \sinh \left( \frac{\beta r}{A^2} \right) \right]^{A^4 / r^2} e^{\omega_2 \tau} \tag{33}
\]

\[
c(\tau) = C_0 e^{\left(s A^2 / r^2\right) \tau} \left[ \sinh \left( \frac{\beta r}{A^2} \right) \right]^{A^4 / r^2} e^{\omega_3 \tau} \tag{34}
\]

\[
\beta^2 \equiv \Omega_0 + (s / r)^2 \tag{35}
\]

III. The Generic Ansatz and Exact solutions

We denote the target space coordinates of the string by \( t = t(\tau, \sigma) \), \( X = X(\tau, \sigma) \), \( Y = Y(\tau, \sigma) \), \( Z = Z(\tau, \sigma) \), and introduce the ansatz, \( t = t[\phi] \), \( X = X[\phi] \), \( Y = Y[\phi] \), \( Z = Z[\phi] \), for the string coordinates, with \( \phi = \phi(\tau, \sigma) \) a function to be determined. Hence we postulate an explicit functional-form dependence for the string coordinates.

The metric tensor functions of Eq (8) assume the form

\[
\alpha^2(t) = \alpha^2[t(\phi)] = \tilde{\alpha}^2(\phi) \tag{36}
\]

\[
b^2(t) = b^2[t(\phi)] = \tilde{b}^2(\phi) \tag{37}
\]

\[
c^2(t) = c^2[t(\phi)] = \tilde{c}^2(\phi) \tag{38}
\]
Now we can define the functions \( \xi = \xi[\phi(\tau, \sigma)] \), \( \eta = \eta[\phi(\tau, \sigma)] \), \( \zeta = \zeta[\phi(\tau, \sigma)] \), through the differential relations

\[
\begin{align*}
\text{d}\xi &\equiv \tilde{a}(\phi)dX = \tilde{a}(\phi)X_\phi d\phi \\
\text{d}\eta &\equiv \tilde{b}(\phi)dY = \tilde{b}(\phi)Y_\phi d\phi \\
\text{d}\zeta &\equiv \tilde{c}(\phi)dZ = \tilde{c}(\phi)Z_\phi d\phi
\end{align*}
\]

(39) (40) (41)

The comma’s with respect to \( \phi \) denote total differentiation of the indicated functions, these being considered as single-variable functions of it. The constraints (22), become

\[
\xi^2 + \eta^2 + \zeta^2 - t^2 = 0
\]

(42)

Eqs (19)-(21) become

\[
\begin{align*}
\xi_{,\phi}(\partial_+ \partial_- \phi) + \xi_{,\phi\phi}(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\alpha I}{\alpha} \right] t_\phi \xi_{,\phi}(\partial_+ \phi)(\partial_- \phi) = 0 \\
\eta_{,\phi}(\partial_+ \partial_- \phi) + \eta_{,\phi\phi}(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\beta I}{\beta} \right] t_\phi \eta_{,\phi}(\partial_+ \phi)(\partial_- \phi) = 0 \\
\zeta_{,\phi}(\partial_+ \partial_- \phi) + \zeta_{,\phi\phi}(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\gamma I}{\gamma} \right] t_\phi \zeta_{,\phi}(\partial_+ \phi)(\partial_- \phi) = 0
\end{align*}
\]

(43) (44) (45)

Eq (18) yields

\[
\begin{align*}
t_\phi(\partial_+ \partial_- \phi) + t_{,\phi\phi}(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\alpha I}{\alpha} \right] (\xi_{,\phi})^2(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\beta I}{\beta} \right] (\eta_{,\phi})^2(\partial_+ \phi)(\partial_- \phi) + \\
&+ \left[ \frac{\gamma I}{\gamma} \right] (\zeta_{,\phi})^2(\partial_+ \phi)(\partial_- \phi) = 0
\end{align*}
\]

(46)

We consider here three classes of solutions, corresponding to the choices

\begin{align*}
\text{I.} & \quad (\partial_+ \partial_- \phi) = 0 \\
\text{II.} & \quad (\partial_+ \partial_- \phi) = \kappa(\partial_+ \phi)(\partial_- \phi) \\
\text{III.} & \quad (\partial_+ \partial_- \phi) = \mu t_\phi(\partial_+ \phi)(\partial_- \phi)
\end{align*}

(47) (48) (49)

for the function \( \phi = \phi(\chi^\pm) \). Here \( \kappa, \mu \neq 0 \) are non-zero constants. The general solution of the first equation (wave equation) is given by

\[
\phi = \phi(\chi^\pm) = H(\chi^+) + G(\chi^-)
\]

(50)
while for Eq(48),
\[
\phi = \phi(\chi^\pm) = \frac{-1}{\kappa} \ln \left[ H(\chi^+) + G(\chi^-) \right]
\]

(51)
where \( H, G \) are arbitrary integration functions of the indicated arguments.

I. The equations of motion (33)-(36) and the constraint (32) become a set of four second-order ordinary differential equations and a first order differential constraint, namely

\[
t_{,\phi\phi} + \left[ \frac{\alpha_t}{\alpha} \right] (\xi_{,\phi})^2 + \left[ \frac{b_t}{b} \right] (\eta_{,\phi})^2 + \left[ \frac{c_t}{c} \right] (\zeta_{,\phi})^2 = 0
\]

(52)

\[
\xi_{,\phi\phi} + \left[ \frac{\alpha_t}{\alpha} \right] t_{,\phi} \xi_{,\phi} = 0
\]

(53)

\[
\eta_{,\phi\phi} + \left( \frac{b_t}{b} \right) t_{,\phi} \eta_{,\phi} = 0
\]

(54)

\[
\zeta_{,\phi\phi} + \left( \frac{c_t}{c} \right) t_{,\phi} \zeta_{,\phi} = 0
\]

(55)

and

\[
\xi_{,\phi}^2 + \eta_{,\phi}^2 + \zeta_{,\phi}^2 - t_{,\phi}^2 = 0
\]

(56)

Eqs (53)-(55) integrate to

\[
\xi_{,\phi} = \frac{c_x}{\alpha}, \quad \eta_{,\phi} = \frac{c_y}{b}, \quad \zeta_{,\phi} = \frac{c_z}{c}
\]

(57)

where \( c_x, c_y, c_z \) are arbitrary constants, while substituting Eqs (57) into (52), and integrating once, results in the same equation to be satisfied with the case when one substitutes Eqs (57) into (56), which is

\[
[t_{,\phi}]^2 = \frac{c_x^2}{\alpha^2(t)} + \frac{c_y^2}{b^2(t)} + \frac{c_z^2}{c^2(t)}
\]

(58)

Therefore for every given set of scale factors \( \alpha(t), b(t), c(t) \) one solves the ordinary differential equation (58) for \( t = t(\phi) \). Then through Eqs (57) one determines \( \xi = \xi(\phi), \eta = \eta(\phi), \zeta = \zeta(\phi) \), with \( \phi = \phi(\chi^\pm) \) satisfying Eq (47). Finally from Eqs (39)-(41) one obtains the explicit expressions for the string coordinates.

II. Following the same line of reasoning, we obtain

\[
\xi_{,\phi} = \frac{c_x}{\alpha} e^{-\kappa \phi}, \quad \eta_{,\phi} = \frac{c_y}{b} e^{-\kappa \phi}, \quad \zeta_{,\phi} = \frac{c_z}{c} e^{-\kappa \phi}
\]

(59)

where \( c_x, c_y, c_z \) are arbitrary constants and

\[
[t_{,\phi}]^2 = \left[ \frac{c_x^2}{\alpha^2(t)} + \frac{c_y^2}{b^2(t)} + \frac{c_z^2}{c^2(t)} \right] e^{-2\kappa \phi}
\]

(60)
Now $\phi = \phi(\chi^\pm)$ satisfies eq (II) or equivalently Eq (51).

III. In the same manner, we obtain

$$\xi_\phi = \frac{c_x}{t^\mu \alpha}, \quad \eta_\phi = \frac{c_y}{t^\mu b}, \quad \zeta_\phi = \frac{c_z}{t^\mu c}$$

(61)

where $c_x, c_y, c_z$ are arbitrary constants and

$$[t^\mu t_{\phi}]^2 = \left[ \frac{c_x^2}{\alpha^2(t)} + \frac{c_y^2}{b^2(t)} + \frac{c_z^2}{c^2(t)} \right]$$

(62)

Now $\phi = \phi(\chi^\pm)$ satisfies Eq (49) after $t = t(\phi)$ as the solution of Eq (62) is substituted. The interesting fact is that given any set of Bianchi-type Scale-factors, irrespective of the general dynamical theory that they come one proceeds directly to a string solution depending on two arbitrary integration functions and seven constants.

The invariant string size is defined as the induced metric on the worldsheet [3,15]

$$ds^2 = G_{AB}(X) dX^M dX^N = G_{AB}(X) X^M_{\phi} X^N_{\phi} (-d\tau^2 + d\sigma^2)$$

(63)

which vanishes due to the constraints (22). Therefore the worldsheet hypersurface is null and we have classes of string solutions that are null-string solutions. It is straightforward to verify that Eqs (5)-(7) are satisfied for the choices of the $\phi = \phi(\sigma^\alpha)$, if we insert the small dimensionless parameter of the null-string perturbation expansion $c^2 = 2\lambda T$ ($0 < c < 1$) [19] in the terms that contain the ($\sigma$) derivatives in Eqs (57)-(59) and let $(c \to 0)$. For example Eq (59) will become $(\ddot{\phi} - c^2 \phi^\prime) = \mu(t_{\phi}/t)((\dot{\phi})^2 - c^2(\phi')^2)$.

However, if we let the parameter $(c)$ to be small $(c \ll 1)$ we can always reconcile the fact that the solutions are null-string solutions (i.e. have null worldsheet manifold) with the fact that they also satisfy the full bosonic string equations of motion (2)-(4). Indeed, for this case while the solutions satisfy the full string equations (2)-(4), when expanded with respect to the parameter $(c)$ they also satisfy in a very good approximation the null equations (5)-(7). This will permit us to use the string energy-momentum tensor derived from the action (1) and study the effects of the string tension on the anisotropic features of spacetime, at least in the first order case with respect to the perturbation parameter $(c)$. Thus these classes of solutions are more general than solutions that arise from Eqs (5)-(7).

For the second family of particular solutions, Eqs (30), in the case of $p = + (1/2)$ and $p = (1/6)$ the equations are integrable in closed form. One obtains the following, performing the integration for (I) (and in a completely similar way for the cases II, III, Eqs (48)-(49)),

1. For $p = + (1/2)$

$$\alpha(t) = A_0 \left[ \frac{3t}{2A_0 B_0 C_0} \right]^{-1/3} \equiv a_0 t^{-1/3}$$

(64)

$$b(t) = B_0 \left[ \frac{3t}{2A_0 B_0 C_0} \right]^{1/3} \equiv b_0 t^{1/3}$$

(65)
\[ c(t) = C_0 \left[ \frac{3t}{2A_0B_0C_0} \right]^{1/3} \equiv c_0 t^{1/3} \] (66)

\[ \omega(t) = \frac{2W_0}{3} \ln \left[ \frac{3t}{2A_0B_0C_0} \right] \] (67)

and for the solution

\[ \left( \frac{c_0^2 t^{4/3}}{\alpha_0^2} + c_{yz}^2 \right)^{1/2} = \left( \frac{2c_0^2}{3\alpha_0^2} \phi + \phi_0 \right) \] (68)

\[ X_{\phi} = [c_x/\alpha^2(t)] \] (69)

\[ Y_{\phi} = [c_y/b^2(t)] \] (70)

\[ Z_{\phi} = [c_z/c^2(t)] \] (71)

\[ c_{yz}^2 \equiv (c_y^2/b_0^2) + (c_z^2/c_0^2) \] (72)

\[ t = t(\phi) \] (73)

2. For \( p=+(1/6) \)

\[ \alpha(t) = A_0 \left[ \frac{7t}{6A_0B_0C_0} \right]^{-1/7} \] (74)

\[ b(t) = B_0 \left[ \frac{7t}{6A_0B_0C_0} \right]^{1/7} \] (75)

\[ c(t) = C_0 \left[ \frac{7t}{6A_0B_0C_0} \right]^{1/7} \] (76)

\[ \omega(t) = \frac{6W_0}{7} \ln \left[ \frac{7t}{6A_0B_0C_0} \right] \] (77)

and for the solution

\[ \left( \frac{c_x^2 t^{4/7}}{\alpha_0^2} + c_{yz}^2 \right)^{1/2} \left( \frac{c_x^2 t^{4/7}}{\alpha_0^2} - 2c_{yz}^2 \right) = \frac{3c_x^4}{4\alpha_0^4} (\phi + \phi_0) \] (78)

\[ X_{\phi} = [c_x/\alpha^2(t)] \] (79)

\[ Y_{\phi} = [c_y/b^2(t)] \] (80)

\[ Z_{\phi} = [c_z/c^2(t)] \] (81)

\[ t = t(\phi) \] (82)

As an explicit demonstration of the integration procedure for the case \( p=1/2 \) we have the complete solution for the choice \( \phi_0 = c_{yz} \), as follows. We insert Eq (68) into Eqs (69)-(71) implicitly, through Eqs (64)-(66) and obtain

\[ X(\phi) = \left( \frac{c_x^2}{\alpha_0^2} \right) \left( \frac{4c_x}{3} \right)^{1/2} \left[ \left( \frac{3\alpha_0^2}{4c_x} \right) \left[ 2 \left( \frac{c_x}{3\alpha_0^2} \right) \phi + 1 \right] \sqrt{\frac{c_x}{3\alpha_0^2} \phi^2 + \phi} \right] - \]
where $\phi$ satisfies the wave equation (47).

However we will be concerned with the most general case Eq (39) and keep the parameter $p$ unspecified. We have

$$\alpha(t) = A_0 \left( \frac{(1 + p)t}{A_0 B_0 C_0} \right)^{-p/(1+p)}$$  \hspace{1cm} (86)

$$b(t) = B_0 \left[ (1 + p)t \right]^{p/(1+p)}$$  \hspace{1cm} (87)

$$c(t) = C_0 \left[ (1 + p)t \right]^{p/(1+p)}$$  \hspace{1cm} (88)

$$\omega(t) = \frac{W_0}{1 + p} \ln \left[ \frac{(1 + p)t}{A_0 B_0 C_0} \right]$$  \hspace{1cm} (89)

and

$$t_{\phi} = t^{-(p+\mu+pq)/(1+p)} \left[ \left( \frac{c_x}{\alpha_0^2} \right) t^{4p/(1+p)} + c_y^2 \right]^{1/2}$$  \hspace{1cm} (90)

$$X_{\phi} = [c_x/t^{\mu} \alpha^2(t)]$$  \hspace{1cm} (91)

$$Y_{\phi} = [c_y/t^{\mu} b(t)]$$  \hspace{1cm} (92)

$$Z_{\phi} = [c_z/t^{\mu} c(t)]$$  \hspace{1cm} (93)

$$t = t(\phi)$$  \hspace{1cm} (94)

The cases that are integrable in closed form corresponding to the above two cases are precisely $\mu = (-1 + 2p)/(1 + p)$, and $\mu = (-1 + 6p)/(1 + p)$, but we will retain the general form throughout.

For the case of the solution (30), from Eq (28) we get (following the standard procedure of [5])

$$8\pi T_{00}^{(MATT)} = \frac{\lambda}{2} \frac{W_0^2}{(1 + p)^2 t^2} - \frac{F_0^2}{2B_0^2 C_0^2} \left[ A_0 B_0 C_0 \right]^{2/(1+p)} \left[ (W_0 + 2p) \right]$$  \hspace{1cm} (95)

$$8\pi T_{xx}^{(MATT)} = \left[ \frac{\lambda}{2} \frac{W_0^2}{(1 + p)^2 t^2} + \frac{F_0^2}{2B_0^2 C_0^2} \left[ A_0 B_0 C_0 \right]^{2/(1+p)} \left[ (W_0 + 2p) \right] \right] \alpha^2(t)$$  \hspace{1cm} (96)
\[ 8\pi T^{(MATT)}_{yy} = \left[ -\frac{\lambda}{2} \frac{W_0^2}{(1+p)^2t^2} - \frac{F_0^2}{2B_0^2C_0^2} \left\{ \frac{A_0B_0C_0}{(1+p)t} \right\}^{\frac{2}{(1+p)}(W_0+2p)} \right] b^2(t) \] (97)

\[ 8\pi T^{(MATT)}_{zz} = \left[ -\frac{\lambda}{2} \frac{W_0^2}{(1+p)^2t^2} - \frac{F_0^2}{2B_0^2C_0^2} \left\{ \frac{A_0B_0C_0}{(1+p)t} \right\}^{\frac{2}{(1+p)}(W_0+2p)} \right] c^2(t) \] (98)

**IV. The Energy-Momentum Tensor**

The energy-momentum tensor for the string is obtained from the functional derivative of the action (1) at the spacetime point \(X\). We obtain [4]

\[ \sqrt{-G} T^{AB}(X) = -\frac{T}{2} \int d\tau d\sigma (\dot{X}^A \dot{X}^B - X_A' X_B') \delta^{(4)}(X - X(\tau, \sigma)) \] (99)

where

\[ \delta^{(4)}(X - X(\tau, \sigma)) = \prod_{M=0}^{3} \delta(X^M - X^M(\tau, \sigma)) \] (100)

It is useful to integrate the energy-momentum tensor in a three-dimensional volume that completely encompasses the string [3]

\[ I^{MN}(X) = \int \sqrt{-G} T^{MN}(X) d^3X \] (101)

For the ansatz introduced, \( t = t[\phi(\tau, \sigma)] \), \( X = X[\phi] \), \( Y = Y[\phi] \), \( Z = Z[\phi] \) we obtain

\[ I^{MN}(X) = -\frac{T}{2} \int \frac{d\sigma dt (\tau, \sigma)}{t} X^M_{\phi} X^N_{\phi} (\dot{\phi})^2 - (\phi')^2 |_{t=t[\phi]} \delta[t - t(\tau, \sigma)] \] (102)

Finally using Eq (9), an effective energy-momentum tensor for the string is derived

\[ T^{MN}_{(STRING)}(X) = -\frac{1}{4\pi \alpha} \frac{1}{\alpha(t)b(t)c(t)} \left[ \frac{X^M_{\phi} X^N_{\phi}}{t_{\phi}} \right]_{t=t[\phi]} \int d\sigma d\phi \frac{(\phi)^2 - (\phi')^2}{\phi} \delta[\phi - \phi(\tau, \sigma)] \] (103)

We demonstrate the evaluation of this integral for the case that \( \phi = \phi(\chi^\pm) \) satisfies the wave equation (47), for a closed string. We have

\[ \hat{I} = \int d\sigma d\phi \frac{(\phi)^2 - (\phi')^2}{\phi} \delta[\phi - \phi(\tau, \sigma)] = \int_{0}^{2\pi} d\sigma \left[ \frac{(\phi)^2 - (\phi')^2}{\phi} \right]_{\tau=\tau(\phi, \sigma)} \] (104)
where the dependence of \( \tau \) is now in the variable \( \phi \) defined by the cosmic time \( t = t(\phi) \). For closed strings satisfying the wave equation we have

\[
\phi = \tilde{\phi} + 2q\alpha' \tau + i\sqrt{\alpha} \sum_{n \neq 0} \frac{1}{n} [\phi_n e^{-in(\tau - \sigma)} + \tilde{\phi}_n e^{-in(\tau + \sigma)}]
\]

(105)

with the reality condition \( \phi^* = \phi_{-n} \ (n \in \mathbb{Z}) \) and a similar relation satisfied by the tilded operators. We can invert approximately this relation to obtain \( \tau \approx (\phi/2q\alpha') \) and finally

\[
\hat{I} = 4\pi q\alpha' + \frac{8\pi}{q} \sum_{n > 0} Re[\phi_n \tilde{\phi}_n e^{2i\sigma}] \]

(106)

In the general case, one must evaluate

\[
\hat{I} \equiv \int d\tau d\phi \left[ \frac{(\dot{\phi})^2 - (\phi')^2}{\dot{\phi}} \delta[\phi - \phi(\tau, \sigma)] \right] = \int d\sigma dt(\tau, \sigma) \left[ \frac{(\dot{\phi})^2 - (\phi')^2}{\dot{\phi}} \delta[t - t(\tau, \sigma)] \right]
\]

(107)

where \( \phi = \phi(\chi^\pm) \) satisfies Eq (39). This task is difficult to be achieved in its full generality. However we can estimate the dependence on the cosmic-time \( t \). From Eq (49), neglecting the primed factors we obtain approximately \( [\ln(\dot{\phi})^2, \phi] = [\ln t^\mu], \phi \) and since the fraction in Eq (107) is roughly proportional to \( (\dot{\phi}) \) we have \( \hat{I} \propto t^{\mu/2} \).

Therefore we obtain

\[
T_{(STRING)}^{MN}(t) \propto -\frac{1}{4\pi\alpha'} \frac{t^{\mu/2}}{\alpha(t)c(t)} \left[ \frac{X_M^N X^N_M}{\dot{t}^\phi} \right]_{\phi = \phi(t)}
\]

(108)

With the aid of Eq (108) we write the functional dependence of the energy-momentum tensor components of the string and the "matter" content of the model that corresponds to Eqs (86)-(94).

\[
T_{(STRING)}^{00} \propto \sqrt{\left( \frac{c^2}{\alpha_0^2} \right) t^4 p^4/(1+p) + c_{yz}^2 t^{\mu/2}}
\]

(109)

\[
T_{(STRING)}^{0x} \propto \frac{t^p/(1+p)}{t^{\mu/2}}
\]

(110)

\[
T_{(STRING)}^{0y} = T_{(STRING)}^{0z} \propto \frac{1}{t^b p/(1+p) t^{\mu/2}}
\]

(111)

\[
T_{(STRING)}^{xx} \propto \frac{t^p/(1+p) t^3 p/(1+p)}{t^3 \sqrt{\left( \frac{c^2}{\alpha_0^2} \right) t^4 p^4/(1+p) + c_{yz}^2}}
\]

(112)

\[
T_{(STRING)}^{yy} = T_{(STRING)}^{zz} = \frac{12}{12}
\]
\[ T^{yz}_{\text{STRING}} = T^{zy}_{\text{STRING}} \propto \frac{t^{(p+\mu+p\mu)/(1+p)}}{t^{2p/(1+p)}t^{3\mu/2} \sqrt{(c_0^2/\alpha_0^2)}t^{4p/(1+p)} + c_{yz}^2} \] (113)

\[ T^{xz}_{\text{STRING}} = T^{zx}_{\text{STRING}} \propto \frac{t^{(p+\mu+p\mu)/(1+p)}}{t^{p/(1+p)}t^{3\mu/2} \sqrt{(c_0^2/\alpha_0^2)}t^{4p/(1+p)} + c_{yz}^2} \] (114)

Correspondingly, for the "matter" content of the model, they are given by Eqs (95)-(98) with \( T^{xx} = \alpha^{-4}(t)T_{xx} \), \( T^{yy} = b^{-4}(t)T_{yy} \), \( T^{zz} = c^{-4}(t)T_{zz} \), using the fact that \( 0 < p < 1 \) with \( \lambda = (2p/(p-1)) < 0 \)

\[ 8\pi T^{00}_{\text{MATT}} = \frac{pw_0^2}{(1-p)(1+p)^2t^2} - \frac{F_0^2}{2B_0^2C_0^2} \left[ A_0B_0C_0 \right] \frac{t^{2p}}{t}\frac{(W_0+2p)}{(1+p)t} \] (115)

\[ 8\pi T^{xx}_{\text{MATT}} = \left[ \frac{1}{A_0^2} \right] \frac{pw_0^2}{(1-p)(1+p)^2t^2} + \frac{F_0^2}{2B_0^2C_0^2} \left[ A_0B_0C_0 \right] \frac{t^{2p}}{t}\frac{(W_0+2p)}{(1+p)t} \] (116)

\[ 8\pi T^{yy}_{\text{MATT}} = \left[ \frac{1}{B_0^2} \right] \frac{pw_0^2}{(1-p)(1+p)^2t^2} - \frac{F_0^2}{2B_0^2C_0^2} \left[ A_0B_0C_0 \right] \frac{t^{2p}}{t}\frac{(W_0+2p)}{(1+p)t} \] (117)

\[ 8\pi T^{zz}_{\text{MATT}} = \left[ \frac{1}{C_0^2} \right] \frac{pw_0^2}{(1-p)(1+p)^2t^2} - \frac{F_0^2}{2B_0^2C_0^2} \left[ A_0B_0C_0 \right] \frac{t^{2p}}{t}\frac{(W_0+2p)}{(1+p)t} \] (118)

For small values of the cosmic time \( (t \to 0) \) or for large values, \( (t \to +\infty) \), demanding functional identification, between the string energy-momentum density \((00)\) component, Eq (109), and the EM part of the "matter" content, Eq (115), results in

\[ \mu = \frac{4(W_0+p)}{(1+p)} \quad (t \to 0) \] (119)

\[ \mu = \frac{4(W_0+2p)}{(1+p)} \quad (t \to +\infty) \] (120)

The limit of \( t \to 0 \) is taken by neglecting the first term in the square root of Eq (109), while the limit of \( t \to +\infty \) by neglecting the second term. The same argument, regarding the pair of dilaton and EM parts of the "matter" content in Eq (115), yields \( W_0 = (1-p) \), therefore,

\[ \mu = \frac{4}{(1+p)} \quad (t \to 0) \] (121)

\[ \mu = 4 \quad (t \to +\infty) \] (122)
For the second family of particular solutions Eqs (30), when the Universe is not highly anisotropic, we have \( p \approx 0 \) and so the components of the string energy-momentum tensor Eqs (109)-(114) do not exceed the \((00)\) component Eq (109), as \( t \) approaches zero. We can now form a sort of backreaction process, regarding the EM field \( F_0 \) as a small perturbing factor that can be omitted. The coefficients of the dual Universes are related by

\[
p(1-p) \pm \left( \frac{T}{2} \right) c_{yz} = p'(1-p') \frac{(1+p')}{(1+p)^2}
\]

(123)

In Eq (123) \( T \equiv (1/2\pi\alpha') \) is the string tension. Using Eq (109), we assume that the presence of the null-string acts as an additional new source, so that the primed term corresponds to another member of the family, Eq (115). Eq (95) has also been used and \( W_0 = (1-p) \) is substituted. This relation is valid in the immediate vicinity of \( 0 \approx p \leq 1/3 \). We obtain now the range of the cosmic-time where this coupling occurs. The fraction at the r.h.s. of Eq (123) is dimensionless and has a maximum equal to \((1/8)\), at \( p' = (1/3) \). So the last term on the l.h.s. of Eq (123) has to be dimensionless and of the same order. Using Eq (90) we can therefore write \( T t^{(4+p)/(1+p)} t_{\phi} \approx 0.1 \).

It is well known that the string tension can be taken to be of the order of the Planck energy scale \([3]\) \( T_{Pl} = (1/2\pi\alpha') \approx 10^{38}(GeV)^2 \) where the Planck mass is given by \( M_{Planck} = (hc/G)^{1/2} \approx 10^{19}(GeV/c^2) \) and correspondingly the characteristic time scale of quantum gravity is \( t_{QG} \approx 10^{-43}sec \) while for the GUT’s we have \( t_{GUT} \approx 10^{-30}sec \). The gravitational constant is \( G \approx 10^{-30}(GeV)^{-2} \). Since we have null strings we use in the action (1) the tension \( T \) to be much smaller than the one for the tensionful strings thus \( (T/T_{Pl}) \approx 10^{-\kappa} \) with \( \kappa \) a large integer.

The quantity \( t_{\phi} \) has the form of inverse angular frequency, and for closed strings it is proportional to the modes of oscillation that are excited. We can therefore set equal to an integer \( n \), \((n=1,2,...)\). So we obtain

\[
\left( \frac{T}{T_{Pl}} \right) \left( \frac{t}{t_{QG}} \right)^{(4+p)/(1+p)} \approx 0.1n
\]

(124)

Taking logarithms we obtain

\[
\left( \frac{4+p}{1+p} \right) \log_{10} \left( \frac{t}{t_{QG}} \right) = \kappa - 1 + log_{10}n
\]

(125)

This formula provides the range of cosmic-time, with respect to the used value of the string tension, where the coupling occurs. As the value of the string tension is increased \((\kappa \to 0)\) or we have excitation of higher modes of the string \((\text{increasing } n)\), the range approaches the Planckian regime. We conclude that when the string action enters with a negative sign, Eq (1) we get a transition to less anisotropic member of the family but also because of the presence of the dilaton field in this family of solutions it exhibits also a sort of chaotic behavior, in which case one can get to a more anisotropic member of the family (see Fig 1.).
V. The General Case

We consider now the first family of general solutions Eq (31)-(35). These correspond to a Universe which is singular at \( t \to 0 \), where the energy density of the magnetic field and the scalar fields diverges and the spatial volume tends to zero [5]. Using Eq (12) of [5], it is not possible to obtain in closed form the dependance of the scale-factors on the cosmic-time \( t \). (note that it should be written as \( dt = \alpha(t) b(\tau)c(\tau)d\tau \). However we can examine the cases of the strong and weak coupling limits of the Dilaton field. These correspond to the limits \( A^2 \to +\infty \) and \( A^2 \to 0 \) respectively.

We obtain

\[
\lim_{A \to +\infty} \left[ \frac{\sinh \left( \frac{\beta r \tau}{A^2} \right)}{A^2} \right]^{-(A^4/r^2)} = \frac{1}{\sinh(\beta \tau)} \quad (126)
\]

\[
\lim_{A \to 0} \left[ \frac{\sinh \left( \frac{\beta r \tau}{A^2} \right)}{A^2} \right]^{-(A^4/r^2)} = 1 \quad (127)
\]

For the strong coupling limit we get

\[
t = \frac{e^{\nu \tau}}{\beta^2 - \nu^2} [\beta \cosh(\beta \tau) - \nu \sinh(\beta \tau)] + \text{constant} \quad (128)
\]

\[
\nu \equiv (2\omega_1 + \omega_2 + \omega_3) \quad (129)
\]

where the integration constant is chosen so that the cosmic and worldsheet times to assume zero value simultaneously. Now we can study the asymptotic behaviour of the solutions in two particular cases:

i) \( \tau \to 0 \), \( t \simeq \frac{\beta}{\beta^2 - \nu^2} [e^{\nu \tau} - 1] \quad (130) \)

Here we get \( \beta = (\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1)^2 + \omega_1^2 \)

\[
\tau = \frac{1}{\nu} \ln \left[ 1 + \frac{(\beta^2 - \nu^2)}{\beta} t \right] \\
\alpha(t) = \frac{A_0}{\sinh(\beta \tau)} \\
b(t) = B_0 e^{(\omega_1 + \omega_2)\tau} \sinh(\beta \tau) \\
c(t) = C_0 e^{(\omega_1 + \omega_3)\tau} \sinh(\beta \tau) \quad (131)
\]

The energy-momentum tensor for the string is given by Eq (108) together with Eqs (62) and (91)-(93). We now give a physical interpretation of the parameter \( \mu \), in Eq (49), which is related to the thermodynamical interaction of the string with the geometry. From Eq (130), close to \( \tau \simeq 0 \) we have that \( \tau \simeq [(\beta^2 - \nu^2)/\nu \beta] t \) and the
scale factors become

\[ \alpha(t) = \frac{A_0 \nu}{(\beta^2 - \nu^2)t} \]

\[ b(t) = B_0 \frac{\beta^2 - \nu^2}{\nu} t \]

\[ c(t) = C_0 \frac{\beta^2 - \nu^2}{\nu} t \] \hspace{1cm} (132)

From Eq (108), inside the three dimensional volume that completely encompasses the string, \( V = \alpha(t)b(t)c(t) = A_0 B_0 C_0 (\beta^2 - \nu^2)t/\nu \), using Eqs (101)-(103) we obtain the total energy \( E^{(STR)}(t) \) of the string and the pressures as follows

\[
E^{(STR)}(t) = -\frac{1}{4\pi \alpha'} \frac{1}{t^{n/2}} \left[ \frac{c_x^2}{\alpha^2(t)} + \frac{c_y^2}{b^2(t)} + \frac{c_z^2}{c^2(t)} \right]^{1/2}
\]

\[ p_x = -\frac{1}{4\pi \alpha'} \frac{c_x^2}{t^{n/2}V \alpha^2(t)} \]

\[ p_y = -\frac{1}{4\pi \alpha'} \frac{c_y^2}{t^{n/2}V b^2(t)} \]

\[ p_z = -\frac{1}{4\pi \alpha'} \frac{c_z^2}{t^{n/2}V c^2(t)} \] \hspace{1cm} (133)

It is well known [25] (see also [26], p.449) that if \( F^{(STR)}(t) \) is the free energy of the string and \( \bar{\beta} \) the string temperature, we have

\[ E^{(STR)} = F^{(STR)} + \left( \frac{1}{\bar{\beta}} \right) S^{(STR)} \]

\[ p_k = -\left( \partial F^{(STR)}/\partial V \right)_{\bar{\beta}} \] \hspace{1cm} (134)

where \( S^{(STR)} = \bar{\beta}^2 \left( \partial F^{(STR)}/\partial \bar{\beta} \right) \) is the string entropy, in the three-dimensional volume. So if we consider isothermal motion of the string, then its temperature does not depend on time. Using Eq (133), and the fact that the volume \( V \) is proportional to the cosmic time \( t \), we have

\[ \frac{dS^{(STR)}}{dt} \propto -\frac{\bar{\beta} \mu}{2} \frac{E^{(STR)}}{V} \] \hspace{1cm} (135)

so it is a measure of the entropy exchange between the string and the spacetime geometry.

If we consider adiabatic motions of the three-dimensional volume that contains the string i.e. one allows for time dependance of the temperature \( \bar{\beta} = \bar{\beta}(t) \) in order that there exists no entropy exchange then one concludes that \( \mu \) is a measure for the adjustment of the string temperature that is required for no entropy exchange. Using again Eq (134) we demand that there exists no entropy exchange \( dS^{(STR)}/dt = 0 \) resulting in

\[ \frac{d\bar{\beta}(t)}{dt} \propto -\frac{\mu}{4\pi \alpha'} \frac{1}{t^{n/2+2}} \] \hspace{1cm} (136)
VI. Dynamical Isotropization and Flatness by Null Strings

In this Section, we integrate numerically the Einstein field equations which determine the evolution of a Bianchi Type I model, with matter content consisting solely of null-strings. A null-string may be considered as representing a collection of points moving independently along null geodesics [16]. The corresponding equations of motion and constraints are now given by Eqs. (5) - (7). In this case, Eq. (7) ensures that each of these points propagates in a direction perpendicular to the string.

In general, it is expected that the introduction of a null-string structure in a cosmological model will react back on the curved background, modifying its dynamical characteristics. In what follows, we are interested in determining the way that this backreaction procedure affects on the evolution of the Universe. Accordingly, we need to calculate the components of the energy-momentum tensor attributed to a spacetime region due to the propagation of a null-string, in an otherwise vacuum cosmological model, with metric in the form of Eq. (8). These quantities are subsequently inserted into the r.h.s. of the corresponding field equations. The resulting dynamical system is highly non-linear and therefore, solutions can be obtained only through certain numerical techniques, where the concept of attractor plays an important role: If some special spacetime represents an attractor for a wide range of initial conditions, such a spacetime is naturally realized asymptotically.

In a Bianchi Type I spacetime, the equations of motion for a null-string in the zeroth order approximation to \( c^2 = 2\lambda T \), are

\[
\ddot{x} + a \frac{da}{dt} \dot{x}^2 + b \frac{db}{dt} \dot{y}^2 + c \frac{dc}{dt} \dot{z}^2 = 0
\]

(137)

\[
\dot{x} + 2b \frac{db}{dt} \dot{y} = 0
\]

(138)

where a dot denotes derivative with respect to the world-sheet coordinate \( \tau \). The corresponding constraints [Eqs. (6) and (7)] are written in the form

\[
\ddot{i} = a^2 \dot{x}^2 + b^2 \dot{y}^2 + c^2 \dot{z}^2
\]

\[
\dddot{i} = 2a^2 \dot{x} \dot{x}' + b^2 \dot{y} \dot{y}' + c^2 \dot{z} \dot{z}'
\]

(139)

where the prime denotes differentiation with respect to \( \sigma \). Eqs. (138) are evaluated as follows

\[
\dot{x} = \frac{x_0(\sigma)}{a^2}
\]
\[ \dot{y} = \frac{y_0(\sigma)}{b^2} \]
\[ \dot{z} = \frac{z_0(\sigma)}{c^2} \]  (140)

In this case, provided that \( \dot{t} \neq 0 \), from Eq. (137) we deduce
\[ t^2 = t_0^2(\sigma) + \frac{x_0^2(\sigma)}{a^2} + \frac{y_0^2(\sigma)}{b^2} + \frac{z_0^2(\sigma)}{c^2} \]  (141)

from which, by virtue of Eq. (139), we obtain \( t_0(\sigma) = 0 \). Now, with the aid of Eqs. (140) and (141), we may determine the components of the total energy-momentum tensor [Eq. (101)] attributed to a spacetime region due to the propagation of a null-string in it. As regards the corresponding (00) component, we have
\[ T^{00} = -\frac{T}{2} \int d\sigma \int dt \dot{t}^2 \delta(t - t') \]  (142)

In Eq. (142), we perform the substitution \((\tau, \sigma) \rightarrow [t(\tau, \sigma), \sigma] \) in a way such that
\[ dt \ d\sigma = \dot{t} \ d\tau \ d\sigma \]  (143)

thus obtaining
\[ T^{00} = -\frac{T}{2} \int d\sigma \int dt \dot{t} \delta(t - t') \]  (144)

Now, by virtue of Eq. (141) and the identity
\[ \int d\mu(k) f(k') \delta(k - k') = f(k) \]  (145)

Eq. (144) is finally written in the form
\[ T^{00} = \mp \frac{T}{2} \int_0^{2\pi} d\sigma \frac{1}{abc} \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2} \]  (146)

In the same fashion, we obtain
\[ T^{01} = \mp \frac{T}{2} \int_0^{2\pi} x_0(\sigma) \frac{c}{a^2(t)} \ d\sigma = I^{10} \]
\[ T^{02} = \mp \frac{T}{2} \int_0^{2\pi} y_0(\sigma) \frac{1}{b^2(t)} \ d\sigma = I^{20} \]
\[ T^{03} = \mp \frac{T}{2} \int_0^{2\pi} z_0(\sigma) \frac{1}{c^2(t)} \ d\sigma = I^{30} \]  (147)

\[ I^{12} = \mp \frac{T}{2} \int_0^{2\pi} x_0 y_0 \frac{c}{ab} \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2} \ d\sigma = I^{21} \]
\[ I^{13} = \mp \frac{T}{2} \int_0^{2\pi} x_0 z_0 \frac{b}{ac} \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2} \ d\sigma = I^{31} \]
\[ I^{23} = \mp \frac{T}{2} \int_0^{2\pi} y_0 z_0 \frac{a}{bc} \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2} \ d\sigma = I^{32} \]  (148)
and

\[ I^{11} = \pm \frac{T}{2} \int_0^{2\pi} \frac{x_0^2 b c}{a^3 \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2}} d\sigma \]

\[ I^{22} = \pm \frac{T}{2} \int_0^{2\pi} \frac{y_0^2 c a}{b^3 \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2}} d\sigma \]

\[ I^{33} = \pm \frac{T}{2} \int_0^{2\pi} \frac{a b}{c^3 \sqrt{x_0^2 b^2 c^2 + y_0^2 c^2 a^2 + z_0^2 a^2 b^2}} d\sigma \]  

(149)

Taking into account a circular (planar) string, namely

\[ x_0(\sigma) = \cos \sigma \quad y_0(\sigma) = \sin \sigma \quad z_0 = \text{const.} \]  

(150)

which propagates along the anisotropic \( z \)-direction of an axisymmetric Bianchi Type I model, i.e.

\[ a(t) = b(t) \]  

(151)

we obtain \( t \neq t(\sigma) \). Then, the only non-zero components of the corresponding total energy-momentum tensor are

\[ I^{00} = \mp \frac{\pi T}{a c} \sqrt{c^2 + z_0^2 a^2} \]

\[ I^{11} = \mp \frac{\pi T}{2a^3} \sqrt{c^2 + z_0^2 a^2} = I^{22} \]

\[ I^{33} = \mp \frac{\pi T}{c^3 z_0^2} \sqrt{c^2 + z_0^2 a^2} \]  

(152)

and

\[ I^{03} = \mp \frac{\pi T}{c^2} z_0 = I^{30} \]  

(153)

Due to the homogeneity of the spacetime under consideration, we may impose that \( z_0 = 0 \) [29]. In this case, the resulting total energy-momentum tensor is diagonal

\[ I^{00} = \mp \frac{\pi T}{a(t)} \]

\[ I^{11} = \mp \frac{\pi T}{2a^3(t)} = I^{22} \]

\[ I^{33} = 0 \]  

(154)

Now, the corresponding physical quantity to be inserted into the r.h.s. of the field equations, may be defined as: \([T^{\mu \nu}] = [I^{\mu \nu} \text{ per unit of proper–comoving volume}]\), thus resulting to

\[ T^{00} = \mp \frac{\pi T}{c a^3} \]

\[ T^{11} = \mp \frac{\pi T}{2ca^3} = T^{22} \]

\[ T^{33} = 0 \]  

(155)

19
In this respect, the Einstein field equations which determine the evolution of a Bianchi Type I model with matter in the form of null-strings, are written in the form

\[
\begin{align*}
\frac{1}{a^2} \left( \frac{da}{dt} \right)^2 + 2 \frac{1}{a} \frac{da}{dt} \frac{1}{c} \frac{dc}{dt} &= -4\pi \left( \frac{G}{\alpha'} \right) \frac{1}{a^3} c \\
\frac{1}{a} \frac{d^2 a}{dt^2} + \frac{1}{c} \frac{d^2 c}{dt^2} + \frac{1}{a} \frac{da}{dt} \frac{1}{c} \frac{dc}{dt} &= 2\pi \left( \frac{G}{\alpha'} \right) \frac{1}{a^3} c \\
2 \frac{1}{a} \frac{d^2 a}{dt^2} + \frac{1}{a} \frac{da}{dt} \frac{1}{c} \frac{dc}{dt} &= 0
\end{align*}
\] (156) - (158)

For the purpose of numerical analysis, in what follows we shall treat the dimensionless constant \( \frac{G}{\alpha'} \) (where \( \alpha' \) is Regge slope) as a free parameter, to be varied at will. In fact, this parameter may be used to denote the regime at which gravitational phenomena dominate over the string effects (\( \frac{G}{\alpha'} > 1 \)) and vice versa (\( \frac{G}{\alpha'} < 1 \)).

In principle, we may integrate the system of Eqs. (156) - (158) to obtain the exact form of the unknown scale functions, \( a(t) \) and \( c(t) \), thus determining the evolution of the cosmological model under consideration. Since this is not an easy task, we may get a good estimation of their dynamic behaviour through numerical integration. From Eqs. (156) - (158) only two are truly independent. The third one corresponds to an additional constraint to be satisfied by the solutions of this system. As such, we choose Eq. (156). The remaining independent field equations (157) and (159) may be recast in the form of a first order system, as follows

\[
\begin{align*}
\frac{dH_1}{dt} &= -H_1^2 - \frac{1}{2} H_1 H_3 \\
\frac{dH_3}{dt} &= -4\pi \left( \frac{G}{\alpha'} \right) a^2 c - H_3^2 - \frac{1}{2} H_1 H_3
\end{align*}
\] (159) - (160)

and

\[
\begin{align*}
\frac{da}{dt} &= a H_1, \quad \frac{dc}{dt} = c H_3
\end{align*}
\] (161)

where \( H_1 \) and \( H_3 \) are the anisotropic Hubble parameters. Accordingly, in what follows, we integrate numerically the system of Eqs. (159) - (161).

To avoid possible implications of small variations on the dynamical parameters involved, the constraint (156) is checked to be satisfied with an accuracy of \( 10^{-10} \) along numerical integration. Both Hubble parameters are measured in units of \( 10^{-4} \) sec\(^{-1} \), being normalized with respect to \( \sqrt{G} \). The initial conditions imposed on the dynamical variables of the problem (\( H_1^0, H_3^0, a_0, c_0 \)), are chosen so that: (a) \( a_0 = c_0 \), i.e. at the origin, the two factor spaces (the isotropic \( xy \)-plane and the anisotropic \( z \)-direction) are separated, but of the same linear dimension. (b) \( H_1^0 > 0 \), i.e. initially the isotropic space expands, in accordance to what we observe at the present epoch. As regards the anisotropic direction it may be either expanding or contracting, a thing that depends on the exact value of \( \frac{G}{\alpha'} \), i.e. on the regime at which we are working at (classical or quantum).

The time coordinate is measured in dimensionless units, being normalized with respect to the Planck time, \( t \rightarrow t/t_{\text{Pl}} \) (\( t_{\text{Pl}} = \sqrt{G} \sim 10^{-43} \) sec). The limits of numerical
integration range from $t_0 = 10$ to $t_f = 10^5$. The upper limit coincides with the origin of the GUTs epoch, $t_{GUT} = 10^5 t_{Pl}$, corresponding to the end of the string era. On the other hand, the lower limit ($t_0 = 10 t_{Pl}$) is chosen as being safely away from the Planck epoch, since, in the absence of a self-consistent quantum theory of gravity, there is always a region of ambiguity around $t = 0$, of the order of the Planck time.

The solutions to the system of differential equations (159) - (161) may be represented as curves in a $H_1 - H_3$ plane. Any point located on them, always satisfies the constraint condition (156), as well. Thus, these curves actually represent orbits of the dynamical system under study. Each curve, corresponding to a different set of initial conditions, is bounded by fixed points (or infinities) and represents a different type of evolution for the Universe. In what follows, we focus attention on the existence and the evolution of attracting points in the $H_1 - H_3$ plane. The reason rests in the physical meaning of the attractor: No matter what the behaviour of a cosmological model at the origin might be, it will always end up to evolve as indicated by the location of the attracting point in the $H_1 - H_3$ plane.

In order to study a possible dissipation of the existing anisotropy, we furthermore define the so called anisotropy measure [30], as

$$\frac{\Delta H}{H} = \frac{3}{2} \frac{H_1 - H_3}{2H_1 + H_3}$$

(162)

This dimensionless quantity is a measure of the variance in the expansion rates at different directions and hence it represents the degree of anisotropy. Accordingly, we examine its dynamical behaviour during numerical integration. We expect that, at the end of the backreaction process, $\Delta H/H$ must settle down to a constant value that approaches zero, corresponding to a dynamical isotropization of the anisotropic background.

Depending on the exact value of the free parameter $G/\alpha'$, we consider three different cases:

(i) Gravitational phenomena are comparable to string effects

In this case, we have $G/\alpha' = 1$. The time evolution of the Hubble parameters for a particular set of initial conditions, $(H_1^0, H_3^0) = (1, -1.5)$, is presented in Fig. 2a. The cosmic-time coordinate is measured in units of $10^5 t_{Pl}$. We see that, although at the beginning both Hubble parameters evolve anisotropically, after some time which is approximately of the order $\Delta t \approx 8 \times 10^3 t_{Pl}$, their evolution becomes identical. In the later stages (for $t \geq 3 \times 10^4 t_{Pl}$), both evolution rates are reduced to a static value approximately equal to zero. In this respect, a complete isotropization of the model is achieved (i.e. $\Delta H/H = 0$) quite rapidly (see Fig. 2b). Notice that the evolution rate of the anisotropic direction ($H_3$) never exceeds in value the corresponding parameter of the isotropic space ($H_1$).

The above results indicate that, during the cosmological evolution there exists an attracting point in the $H_1 - H_3$ plane. The explicit location of this point is presented in Fig. 3. We observe that, for a wide range of initial conditions (including
both expansion and contraction of the anisotropic direction), every orbit of the dynamical system under consideration ends up to a fixed point very close to $(0,0)$, i.e. the Minkowski spacetime. This point is stable and according to the definition of the attractor, the cosmological model under study is not only isotropic, but also asymptotically (as $t \to \infty$) flat.

(ii) Gravity dominates over string effects

In this case, we take $\frac{G}{\alpha^4} = 10$. The time evolution of the Hubble parameters corresponding to the initial conditions of the previous case, is presented in Fig. 4a. The cosmic-time coordinate is also measured in units of $10^3 t_{Pl}$. We see that at the end of the backreaction process, an initially large degree of anisotropy is subsequently minimized, so that, asymptotically, the evolution rates of both factor spaces are reduced to the static value $(0,0)$. However, in this case, the anisotropy dissipation rate is quite lower and the isotropization of the model is completed only after a period of $\Delta t > 3 \times 10^4 t_{Pl}$ (see Fig. 4b). We furthermore observe that, although the dynamical behaviour of $H_1$ is more or less the same as in the previous case, the anisotropic direction undergoes a period ($\approx 10^2 t_{Pl}$) of accelerating (inflationary) expansion, during which the value of $H_3$ exceeds $H_1$.

In this case also, we verify the existence of an attractor in the $H_1 - H_3$ plane. The explicit location of this point is presented in Fig. 5. Notice that the dynamical behaviour of the various orbits is almost identical (there is no spreading of the orbits in the $H_1 - H_3$ plane), although they correspond to quite different sets of initial conditions. Again, the attractor coincides to the stable point $(0,0)$, i.e. the Minkowski spacetime. Therefore, the cosmological model under consideration is also asymptotically flat.

(iii) String effects dominate over gravity

In this case, we take $\frac{G}{\alpha^4} = 0.1$. The corresponding results are more or less similar to the previous ones, with only one major difference: We do not have orbits corresponding to negative initial conditions of $H_3$. For $H_3^0 < 0$, the constraint equation (156) is not satisfied any more. Accordingly, a dominant null-string structure does not permit contraction of the anisotropic dimension and therefore, the cosmological model under consideration does not admit a Kasner-type solution. Again, we verify the existence of an attractor in the $H_1 - H_3$ plane, which coincides to the stable point $(0,0)$ corresponding to Minkowski spacetime, while anisotropy dissipation occurs rather rapidly (see Figs. 6 and 7).

In concluding, we may state that the introduction of a null-string structure in a Bianchi Type I spacetime, may act in favour of both isotropization and flatness of the anisotropic background. Moreover, in the case where the string effects are either comparable or dominant with respect to the gravitational ones, the dynamical behaviour of $H_3$ indicates that the string actually decelerates the evolution of the
anisotropic direction (e.g. see Figs. 2a and 4a), thus serving as a factor for counter-inflation, as well (in connection see [26]).

VII. Discussion and Conclusions

In the first part of this paper the introduction of a generic ansatz has provided classes of exact, null-string solutions in a Bianchi Type I spacetime. This can be carried out irrespectively of the dynamical theory under which the corresponding cosmological solutions may originate and in fact, one has to use only the explicit functional form of the scale factors. This procedure is applied explicitly to the case of dilaton-Bianchi Type I cosmological models.

We may realize the possibility of backreaction of the strings on the curved background, by assuming that the dilaton and electromagnetic fields have the same cosmic-time dependence with the string. This specifies the values of the parameter $\mu$, in such a way, that allows transition to a less or more anisotropic member of the same family of the cosmological solutions. The range of values of the cosmic-time parameter in which this process is valid, depends on the exact value of the string tension and the number of the string modes that are excited, in the first order approximation. If the three dimensional volume that encompasses the string is considered as a whole, its interaction with the spacetime geometry has a rate which is proportional to the parameter that characterizes the general class of solutions.

Finally, if one assumes that in the primordial stages of the evolution of the Universe, high spacetime curvature resulted in particle production [27], then one is led to examine if string-like properties of particles can also lead to anisotropy damping. In this respect, the strings, being considered as objects carrying stress and energy, seem to provide the means for anisotropy damping in epochs where the spacetime could have different expansion rates along different directions. This is examined in the last part of the paper, where the cosmological field equations regarding a null-string distribution propagating in a Bianchi type I spacetime, are integrated without any further assumptions. This integration indicates that, for a wide range of initial conditions, the null-string structure acts in favour of both the isotropization and the flatness of an, initially, curved anisotropic background.

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References


Figure Captions

- **Figure 1**: Plot of the function \( f(p) = p(1-p)/(1+p)^2 \)

- **Figure 2a**: The time evolution of the anisotropic Hubble parameters \( H_1 \) (squares) and \( H_3 \) (solid line), for a particular set of initial conditions \([(H^0_1, H^0_3) = (1, -1.5)]\, in the case where \( \frac{G}{\alpha^2} = 1 \). The time coordinate is measured in units of \( 10^3 \, t_{Pl} \). Notice that, the cosmological model becomes both isotropic (for \( t \geq 8 \times 10^3 \, t_{Pl} \)) and flat (as \( t \to \infty \)).

- **Figure 2b**: Plot of the anisotropy measure \( \frac{\Delta H}{H} \) versus time (in units of \( 10^3 \, t_{Pl} \)), as regards the evolution of the Hubble parameters corresponding to the set of initial conditions \((H^0_1, H^0_3) = (1, -1.5)\), in the case where \( \frac{G}{\alpha^2} = 1 \). The isotropization of the model is completed after \( 8 \times 10^3 \, t_{Pl} \).

- **Figure 3**: The orbits of the dynamical system determined by the cosmological field equations, for five different sets of initial conditions, in the \( H_1 - H_3 \) plane. All orbits approach asymptotically at the attracting point \((0,0)\).

- **Figure 4a**: The time-evolution of \( H_1 \) (squares) and \( H_3 \) (solid line), for the same set of initial conditions as in Fig. 2a, in the case where \( \frac{G}{\alpha^2} = 10 \). The time coordinate is measured in units of \( 10^3 \, t_{Pl} \). Notice that in this case, the anisotropic direction undergoes a period of accelerating (inflationary) expansion before the model becomes both isotropic (for \( t > 3 \times 10^4 \, t_{Pl} \)) and flat (as \( t \to \infty \)).

- **Figure 4b**: Similar to Figure 2b, except that \( \frac{G}{\alpha^2} = 10 \). Notice that the isotropization of the model is quite slower in this case.

- **Figure 5**: Similar to Figure 3, except that \( \frac{G}{\alpha^2} = 10 \). Again, all orbits end at the attracting point \((0,0)\).

- **Figure 6a**: The time evolution of \( H_1 \) (squares) and \( H_3 \) (solid line) for a particular set of initial conditions \([(H^0_1, H^0_3) = (1, 2)]\, in the case where \( \frac{G}{\alpha^2} = 0.1 \). The time coordinate is measured in units of \( 10^3 \, t_{Pl} \). Notice that the dominant null-string acts in favour of a counter-inflation of the anisotropic dimension, thus resulting to a rapid isotropization of the cosmological model.

- **Figure 6b**: Plot of the anisotropy measure \( \frac{\Delta H}{H} \) versus time (in units of \( 10^3 \, t_{Pl} \)), as regards the evolution of the Hubble parameters corresponding to the set of initial conditions \((H^0_1, H^0_3) = (1, 2)\), in the case where \( \frac{G}{\alpha^2} = 0.1 \).

- **Figure 7a**: Similar to Figure 6a, except that \((H^0_1, H^0_3) = (1, 0)\).

- **Figure 7b**: Similar to Figure 6b, except that \((H^0_1, H^0_3) = (1, 0)\).