Quantum creation of Universes with compact spacelike sections that have curvature $k$ either closed, flat or open, i.e. $k = \pm 1$, 0 are studied. In the flat and open cases, the superpotential of the Wheeler De Witt equation is significantly modified, and as a result the qualitative behaviour of a typical wavefunction differs from the traditional closed case. In the open case boundary conditions that include the Tunneling ones are allowed but not the no boundary choice. Restricting ourselves to the Tunneling boundary condition, and applying it in turn to each of these curvatures, it is shown that quantum cosmology actually suggests that the Universe be open, $k = -1$. In all cases sufficient inflation $\sim 60$ e-foldings is predicted: this is an improvement over classical measures that generally are ambiguous as to whether inflation is certain to occur.

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I. INTRODUCTION

Quantum cosmology is regarded as a possible way of obtaining the initial conditions required to start the evolution of a classical cosmological model, for a general introduction, see Ref. [1]. These initial conditions correspond to a number of arbitrary constants that determine entirely, in the absence of chaos, the future evolution of the model. In the simple Friedman-Lemaître-Robertson-Walker (FLRW) metric these constants determine the amount of matter present, and in the case of a scalar field source the initial balance of kinetic energy to potential energy of the scalar field. This determines how the initial expansion proceeds and whether the strong-energy condition is first violated to create an inflationary expansion. One could hope to overcome ambiguities in the prediction of whether inflation occurs that result from purely classical measures of probability [2].

The spatial curvature $k$ is also another constant to be supplied. Although with strong-energy satisfying matter e.g. radiation, it will only dominate at large scale factor to either force the universe to re-collapse or to keep eternally expanding. But when the strong-energy condition is violated the curvature instead dominates at small scale factors. Curvature is therefore especially important if the universe is assumed to be created in an inflationary state. But, because closed $k = 1$ models have finite size they have generally been thought to be most relevant for quantum cosmology as they will have finite action, see e.g. Ref. [3].

The archetypal model that has been considered is the closed DeSitter space that has at small scale factors a Euclidean or “forbidden region”. Starting at zero scale factor one can envision tunneling into the classically allowed region. Boundary conditions need to be supplied in order to make predictions: the two most widely used being the Hartle-Hawking (HH) [4] and Tunneling ones [5]. The Tunneling boundary condition can be formulated in a number of ways, see [6] for a recent discussion, but we will generally refer to [7] where the Tunneling condition is defined as “outgoing modes” only. Roughly speaking the HH condition makes use of the Euclidean nature of space time to smooth out singularities while the Tunneling one corresponds to allowing only outgoing modes, making it analogous to quantum $\alpha$ decay of an atom.

To overcome the limitation of only having closed models quantum creation of bubbles during any subsequent inflationary phase can enable locally open regions to form [8]. The use of these so-called Hawking-Turok instantons [9] are essentially making use of the fact that in DeSitter space all curvatures are equivalent. Different slicings of the 5-dimensional DeSitter hyperboloid correspond to different curvature $k$ when considered as a 4-dimensional model, see e.g. Ref. [10]. But if the final requirement is simply an open or flat universe this would seem, at best, a rather convoluted procedure for their production. Because open universes can also be compact [11] it seems possible to work directly with a universe of specific curvature, instead of by necessity starting with a closed universe that can later by quantum tunneling create locally open regions. This will be the subject of this paper, for a general reference to topology in cosmology see [11]. For some results this requirement of compactness seems unnecessary as one can let the volume become arbitrarily large. Or alternately quantize the Friedmann equation...
for the scale factor directly where the volume factor is a redundant multiplicative factor cf. [12].

Allowing $k = -1$ has a drastic effect on quantum creation scenarios as the forbidden region that is assumed to be tunneled through is no longer present. The classical singularity at zero scale factor is no longer isolated from the larger universe but instead classical evolution can start from arbitrarily small size.

The fact that curvature plays such an important role in quantum creation scenarios is worrying since the standard notions of curved space, and their associated metrics, are likely to require extensive modifications as the quantum gravity epoch is approached. As a first approximation to including the effect of quantization of the curvature, we will also consider that the curvature is initially a quantum ($\hbar$) variable that is allowed to take values within an ensemble. With this assumption we will argue that creation is in some sense “more likely” if this initial tunneling is not required which leads one to conclude that open universes are favoured. A somewhat related work [13] has concluded that quantum tunneling is favoured if open universes are favoured. A somewhat related work [13] has concluded that quantum tunneling is favoured if the strong-energy condition is only just being violated. This can be understood since this corresponds to making the barrier to be tunneled through is no longer present. The classical singularity at zero scale factor is no longer isolated.

In order to pass to the Hamiltonian formalism, conjugate momenta must be calculated. They are given by:

$$\pi_a = -\frac{c^2 v_k}{16\pi G} \frac{12a\dot{a}}{N}, \quad \pi_{\phi} = \frac{a^3 v_k \dot{\phi}}{cN}. \quad (5)$$

From the last expressions, the canonical Hamiltonian can be deduced. It reads:

$$H_c = N\left(-16\pi G \frac{\pi_a^2}{c^2 v_k} 24a + \frac{c^2 \pi_{\phi}^2}{v_k 2a^3} - \frac{c^4 v_k}{16\pi G} 6ka + \frac{v_k}{c} a^3 V(\phi) \right). \quad (6)$$

We are now in a position where the quantization à la Dirac can be carried out cf. [1,15]. It consists in replacing the two momenta according to the rule:

$$\pi_a^2 \rightarrow -h^2 a^{-p} \frac{\partial}{\partial a} \left(a p \frac{\partial}{\partial a} \right), \quad \pi_{\phi}^2 \rightarrow -h^2 \frac{\partial^2}{\partial \phi^2}, \quad (7)$$

where $p$ takes into account the factor ordering ambiguity. The second rule is that the action of the operator $H_c$ on the wave function $\Psi(a,\phi)$ gives zero. This leads to the WDW equation:

$$\frac{\partial^2}{\partial a^2} \Psi(a,\phi) + \frac{p}{a} \frac{\partial}{\partial a} \Psi(a,\phi) - \frac{6}{\kappa a^2} \frac{\partial^2}{\partial \phi^2} \Psi(a,\phi) - \frac{36 \nu_0^2}{\kappa^2 h c^2} a_0^2 \left(\frac{a}{a_0}\right)^2 |k - \left(\frac{a}{a_0}\right)^2| \Psi(a,\phi) = 0, \quad (8)$$

where $\kappa \equiv 8\pi G/c^4$ and $a_0 \equiv [v_0 V(\phi)/3]^{-1/2}$. This equation is not exactly soluble in its present form so we will

\[ \text{II. DESCRIPTION OF THE MODEL} \]

We consider the quantization of the following FLRW metric:

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right). \quad (1)$$

It is well-known that the metric does not fix the global topology [11]. In this article, we will generally assume that the spacelike sections are compact (i.e. they have a finite volume) with constant curvature characterized by $k = 0, \pm 1$. Different topologies correspond to different ranges of variation for the coordinates $(r, \theta, \varphi)$. The volume of the spacelike hypersurfaces is formally given by:

$$v_k \equiv \int d^3 x \sqrt{h}, \quad (2)$$

where we have written the metric as: $ds^2 = -N^2(t)dt^2 + a^2(t)h_{ij}dx^idx^j$. Then, the Einstein-Hilbert action plus the boundary term for this minisuperspace is given by the following expression:

$$S_{E-H} = \frac{c^3}{16\pi G} v_k \int dt Na^3 \left( \frac{6k}{a^2} - \frac{6}{N^2} \left(\frac{\dot{a}}{a}\right)^2 \right). \quad (3)$$

The matter is described by a scalar field whose action can be written as:

$$S_\phi = -\frac{v_k}{c} \int dt Na^3 \left( -\frac{1}{2N^2} \dot{\phi}^2 + V(\phi) \right). \quad (4)$$

In section II, we describe the minisuperspace model for the case where the spacelike sections are compact (i.e. they have a finite volume) with constant curvature characterized by $k = 0, \pm 1$. Different topologies correspond to different ranges of variation for the coordinates $(r, \theta, \varphi)$. The volume of the spacelike hypersurfaces is formally given by:

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where $\kappa \equiv 8\pi G/c^4$ and $a_0 \equiv [v_0 V(\phi)/3]^{-1/2}$. This equation is not exactly soluble in its present form so we will
make a further assumption. Although we still let the wave function keep a scalar field dependence we will ignore the 2nd derivative term w.r.t. \( \phi \) in the WDW equation. This should be valid during any “slow roll” regime where the scalar potential plays the role of an effective cosmological constant, i.e. \( a_0 = (3/\Lambda)^{1/2} \). As a consequence we find that \( a_0^2/(\kappa^2 L^2 c^2) = 3\rho p / (512\pi^2\hbar^2\rho L) \) where \( L \) is the Planck length and \( \rho p \) the Planck energy density, \( \rho \equiv c^2/(bG^2) \). \( \rho_\Lambda \) is the energy density of the effective cosmological constant, \( \rho_\Lambda \equiv \Lambda/\kappa \). Finally, it is convenient to work with a dimensionless scale factor expressed in units of the Planck length. Therefore we redefine the scale factor according to: \( a \rightarrow L/a \). Equation (8) can then be rewritten as:

\[
\frac{d^2 \Psi(a)}{da^2} + \frac{p}{a} \frac{d \Psi(a)}{da} - \frac{27\pi^2}{128\pi^3} \frac{\rho p}{\rho_\Lambda} (\frac{a}{a_0})^2 [k - (\frac{a}{a_0})^2] \Psi(a) = 0. \tag{9}
\]

The last equation determines the superpotential:

\[
U(a;k) = \frac{27\pi^2}{128\pi^3} \frac{\rho p}{\rho_\Lambda} (\frac{a}{a_0})^2 [k - (\frac{a}{a_0})^2]. \tag{10}
\]

A related WDW equation has been obtained by using a normalization of the scale factor to remove any volume factor divergences [12]. In the following figure, the superpotentials for the three cases \( k = 0, \pm 1 \) are displayed. Note that for \( k = 0, -1 \), the superpotentials are always negative. This is a crucial difference in comparison with the \( k = 1 \) case. This means that when \( k = 0, -1 \) there is no possibility of tunneling anymore since a zero energy system is always above the superpotential. This will have important consequences which are now investigated.

![Superpotentials for different values of k. The full line represents the case k = 1, the dotted line the case k = 0 and the dashed line the case k = -1.](image)

In order to obtain exact solutions of the WDW equation we choose to work with the factor ordering given by \( p = -1 \). This permits to work with analytical exact solutions. Introducing the variable \( z(a;k) \) defined by cf. [7]:

\[
z(a;k) \equiv (\frac{3}{8\pi})^{1/3} \frac{v_k^{2/3}}{\rho_\Lambda} [k - (\frac{a}{a_0})^2], \tag{11}
\]

the general solution can be expressed in terms of Airy functions of first and second kind [16]:

\[
\frac{\Psi(a;k)}{\alpha \Psi(0;k)} = \frac{\alpha A_l[z(a;k)] + \beta B_l[z(a;k)]}{\alpha A_l[z(0;k)] + \beta B_l[z(0;k)]} \equiv \frac{a}{D(k)}, \tag{12}
\]

where the coefficients \( \alpha \) and \( \beta \) are arbitrary complex numbers determined by the choice of a state for the wave function of the Universe.

We can make some comments at this point. The presence of the denominator in the previous equation comes from the requirement that the wave function be regular everywhere in the minisuperspace. Recall that the wave function depends on the scalar field and the approximation made previously was only adopted for computational convenience. Let us therefore reconsider the full WDW equation given by formula (8). It is convenient to define the quantity \( \alpha \equiv \ln(a/a_0) \). Then the WDW equation becomes:

\[
\frac{\partial^2 \Psi(\alpha,\phi)}{\partial \alpha^2} + (p-1) \frac{\partial \Psi(\alpha,\phi)}{\partial \alpha} - \frac{6}{\kappa} \frac{\partial^2 \Psi(\alpha,\phi)}{\partial \phi^2} - \frac{27\pi^2}{128\pi^3} \frac{\rho p}{\rho_\Lambda} \alpha [k - e^{2\alpha}] \Psi(\alpha,\phi) = 0. \tag{13}
\]

This equation is separable. If we define \( \Psi(\alpha,\phi) \equiv e^{\frac{i\lambda}{4} \alpha} f(\alpha) g(\phi) \) and if \( \lambda \) is the separation constant then the solution can be written as:

\[
\Psi(\alpha,\phi) = \int c(\lambda)e^{\frac{i\lambda}{4} \alpha} f(\alpha) e^{\sqrt{\lambda} \phi} d\lambda, \tag{14}
\]

where \( c(\lambda) \) are \textit{a priori} arbitrary coefficients and the function \( f_\lambda(\alpha) \) satisfies the equation:

\[
\frac{d^2 f_\lambda(\alpha)}{d\alpha^2} + \left( \lambda^2 - \frac{(1-p)^2}{4} \right) f_\lambda(\alpha) - \frac{27\pi^2}{128\pi^3} \frac{\rho p}{\rho_\Lambda} \alpha [k - e^{2\alpha}] f_\lambda(\alpha) = 0. \tag{15}
\]

Note that when considering quantum wormhole solutions with massless scalar fields the oscillatory divergence can be regulated by a proper choice of the coefficients \( c(\lambda) \) [17–19]. But we will rather consider single wavefunction like solutions for which such a scheme is absent [20]. When the scale factor becomes small, \( a \ll a_0 \), Eq. (15) can be readily solved. The solution is \( f_\lambda(\alpha) \approx e^{i\omega_{\lambda,p} \alpha} \) where \( \omega_{\lambda,p} \) can be expressed as:

\[
\omega_{\lambda,p} = \sqrt{\lambda^2 - \frac{(1-p)^2}{4}}. \tag{16}
\]

In particular when \( p = -1 \), i.e. the factor ordering adopted in this article, one has \( \omega_{\lambda,p} = -1 = \sqrt{\lambda^2 - 1} \). We
are now in a position where the regularity of the wavefunction in the vicinity of $a = 0$ can be studied. It crucially depends on the value of the separation constant, see also Ref. [20] where a similar treatment has been performed. If $\lambda \in [-1,1]$ then the wavefunction is regular as the scale factor goes to zero whereas otherwise the wavefunction exhibits rapid oscillations. For these values of $\lambda$ the wavefunction remains finite. Nevertheless, the “wiggliness” of the wavefunction is also a kind of singularity and it seems reasonable not to consider this possibility. In this paper, we restrict ourselves to the case $\lambda = 0$ for which the wavefunction is regular at $a = 0$. This means that $\Psi(a = 0, \phi)$ does not depend on $\phi$ and it justifies the presence of the denominator $D(k)$ in Eq. (12), see also Refs. [7]. It is clear from the previous discussion that this is not the most general case since there exists non-vanishing values of $\lambda$ such that the wavefunction is regular at the origin. However, to our knowledge, there does not exist a general study of the boundary conditions for an arbitrary value of the separation constant. Moreover, it turns out that the two most widely discussed choices of boundary conditions, i.e. Hartle-Hawking and Vilenkin states, picked out the value $\lambda = 0$ as will be demonstrated below, see also Ref. [20] (in these two cases, $D(k)$ is responsible for the appearance of the factors $e^{\pm 1/V(\phi)}$ in the probability density functions). So as a first approach it seems reasonable to consider the case $\lambda = 0$ only. Let us emphasize that Eq. (12) allows us to study all the boundary conditions such that $\Psi(a = 0, \phi)$ is independent of the scalar field and not only the Hartle-Hawking and Vilenkin states.

We should distinguish between the sort of singularity discussed above, present for example with a massless scalar field case [17], which would tend to be displayed by a rapid oscillation in the wavefunction as $a \to 0$ (recall kinetic energy $\sim$ “wigginess” of wavefunction, see eg. [21]) if not regulated and those considered by Ref. [22] which are related to the choice of the factor ordering. There, for the closed case, it was claimed that only the no-boundary state is regular in the limit $a \to 0$ for any choice of $p$. It was also shown that the Tunneling solution is regular in this limit only if $p < 1$. This behaviour was caused by the wavefunction having growing and decaying, actually Modified Bessel functions, solutions in the forbidden region. Typically the required Tunneling solution diverges like $\Psi \sim a^{(1-p)/2}$ in the limit $a \to 0$. However, this sort of divergence is present universally. Solving Eq. (9) for $k = \Lambda = 0$ gives the solution

$$\Psi(a) = c_1 + c_2 a^{1-p},$$

with $c_1$ and $c_2$ arbitrary constants. So the factor ordering divergence remains even in flat empty space. These divergences appear intrinsic to these models and one can conceive that they should be removed by a renormalization scheme. It was further suggested, in the context of quantum wormhole that such factor ordering divergences are of no great concern [23]. In practice, as $a \to 0$, one should anyway include a more realistic strong-energy satisfying matter component. As emphasized by Gott and Li [24] such a matter source should be expected if only of a size due to quantum “zero point” fluctuations. One can think of this as being because the scalar potential has a “fuzziness” due to quantum uncertainty in the limit $a \to 0$. For the case of radiation, given here by a parameter $A$ the WDW equation is then given by, see e.g. [25]

$$\frac{d^2 \Psi(a)}{da^2} + \frac{p}{a} \frac{d \Psi(a)}{da} + A \Psi(a) = 0$$

and now with solution

$$\Psi(a) = c_1 a^{\frac{1-p}{2}} J_{\frac{-1}{2}}(\sqrt{Aa}) + c_2 a^{\frac{1-p}{2}} Y_{\frac{-1}{2}}(\sqrt{Aa}),$$

where $J_{(p-1)/2}$ and $Y_{(p-1)/2}$ are Bessel functions of order $(p-1)/2$ [16]. But as pointed out by [26] the Y Bessel function still diverges for $p \geq 1$ and this term is part of the Tunneling boundary condition’s solution. However, the presence now of oscillatory behaviour complicates the adoption of the HH boundary condition’s solution. It sometimes is given by the prescription “outgoing” plus “ingoing”, if so it would also suffer this same divergence, although if it was simply the J Bessel function term it would be normalizable [26]. But regardless of such considerations for the chosen $p = -1$ case the no-boundary state as well as the Tunneling state are immune from factor ordering divergences and are a priori both allowed.

Let us now turn to the study of the behaviour of the wave functions given in Eq. (12). The case $k = 1$ is standard. We choose the spacelike sections to be spheres although of course other possibilities are allowed. This means that $v_1 = 2\pi^2$. The evolution of the Universe can be viewed as the motion of a fictitious particle with zero energy in the potential given by Eq. (10). The particle starts to the left of the potential and can proceed to tunnel through the barrier. While in the region $a < a_0$ the wavefunction exponentially decays. Then, when the scale factor is beyond the barrier $a > a_0$, the wavefunction becomes oscillatory. This behaviour is illustrated in the following figure* for the boundary conditions $\alpha = 1$ and $\beta = i$, which correspond to the Tunneling wavefunction [7]. The value of $\rho_\Lambda$ is chosen such that $\rho_{D1}/\rho_\Lambda = 1000$. Since we have $a_0^2 = (3/8\pi)\rho_{D1}/\rho_\Lambda$, this corresponds to a dimensional $a_0$ equal to $\approx 10.9\rho_{D1}$. This value can be thought of as being the size at which the Universe is first created.

*In this article, the figures have been obtained using the Mathematica (version 3.0) and IDL software packages
It can also be noticed that the oscillations in the region $a < a_0$ are more pronounced for the $k = -1$ case than for when $k = 0$. This is due to the different shapes of the superpotential. The particle can roll faster down the steeper $k = -1$ superpotential compared to the $k = 0$ case.

The fact that the wavefunctions behave very differently for different values of $k$ also has an impact on the choice of the quantum state. Studying this question is the purpose of the next section.

### III. MEASURE AND THE INITIAL STATE

To be able to make predictions and to calculate probabilities, we need a suitable measure. If one chooses a surface in the minisuperspace perpendicular to the “a” direction, then the component of the current associated with the WDW equation through this surface is given by [15]:

$$ j = \frac{i}{2}a^p(\Psi^* \partial_a \Psi - \Psi \partial_a \Psi^*), $$

where, in our case, $p = -1$. As is well-known this current is not positive definite since the signature of the (mini)-superspace is Lorentzian. However, in the WKB approximation, where the wave function can be written as $\Psi \sim Ce^{iS}$, this current becomes positive definite and permits the calculation of conditional probabilities [29].

Let us first calculate the behaviour of the numerator of Eq. (12) when the scale factor becomes large. We have $\lim_{a \to +\infty} z(a; k) = -\infty$ for any value of $k$. Using the asymptotic behaviour of the Airy functions, we find:

$$ \lim_{a \to +\infty} N(a; k) = \frac{[z(a; k)]^{-\frac{1}{2}}}{2\sqrt{\pi}} \left( \left( \frac{\alpha}{k} + \beta \right) e^{\frac{2\alpha}{\sqrt{\pi}}[z(a; k)]^\frac{3}{2} + \frac{\alpha}{4}} - \left( \frac{\alpha}{k} - \beta \right) e^{-\frac{2\alpha}{\sqrt{\pi}}[z(a; k)]^\frac{3}{2} - \frac{\alpha}{4}} \right). $$

(21)
Let us now turn to consider the form of the denominator. This time the three cases must be treated separately. Let us start with the usual case, i.e. \( k = +1 \). The value of the function \( z(\alpha;+1) \) when the scale factor vanishes is given by:

\[
z(0;+1) = \left( \frac{3}{8\pi} \right)^{4/3} (v_1)^{2/3} \left( \frac{\rho_{PL}}{\rho_\Lambda} \right)^{2/3}.
\] (22)

Semi-classical considerations are supposed to be valid only if we are in a region where \( \rho_{PL}/\rho_\Lambda \gg 1 \). We can therefore work with the approximation that \( z(0;+1) \gg 1 \). In that case, we obtain:

\[
D(k = +1) \approx \frac{\beta}{2\sqrt{\pi}} \left[ z(0;+1) \right]^{-1/4} e^{2j\pi z(0;+1)^{3/2}}.
\] (23)

The case \( k = 0 \) is rather simple since we have \( z(0;0) = 0 \).

Therefore, the denominator can be written as:

\[
D(k = 0) = Ai(0)(\alpha + \sqrt{3}\beta),
\] (24)

where \( Ai(0) = 3^{-2/3}/\Gamma(2/3) \approx 0.35502 \). Finally, we turn to the case \( k = -1 \). Now we have:

\[
z(0;-1) = -\left( \frac{3}{8\pi} \right)^{4/3} (v_{-1})^{2/3} \left( \frac{\rho_{PL}}{\rho_\Lambda} \right)^{2/3}.
\] (25)

Therefore, in the semi-classical regime, we have \( |z(0;-1)| \gg 1 \) and \( z(0;-1) < 0 \). The presence of the minus sign turns out to be crucial. This time the asymptotic expansion of the Airy functions for large negative \( z \) must be used contrary to the case \( k = 1 \) where the asymptotic expansion for large positive \( z \) has been utilized. Thus, in this limit, the product \( D(k = -1)D^*(k = -1) \) is given by:

\[
D(k = -1)D^*(k = -1) \approx \frac{|\alpha|^2}{2\pi(1+t^2)\sqrt{-z(0;-1)}} \left[ (1+\rho^2-2\rho\cos\psi)t^2 + 2(1-\rho^2)t 
+ 1 + \rho^2 + 2\rho\cos\psi \right],
\] (26)

where \( t \equiv \tan((2/3)|-z(0;-1)|^{3/2}) \) and \( \beta/\alpha \equiv \rho e^{i\psi} \).

There is now a danger of obtaining a divergence in the current (see the following section) when the product \( D(k = -1)D^*(k = -1) \) vanishes. The discriminant of the second order polynomial in \( t \) in Eq. (26) is given by:

\[
\Delta = -16\rho^2 \sin^2\psi.
\] (27)

Therefore, the polynomial has no real roots except when \( \Delta = 0 \). This corresponds to the following cases:

\[
\rho = 0, \quad \text{and/or} \quad \psi = n\pi,
\] (28)

where \( n \) is an integer. Each wavefunction in the minisuperspace is characterized by the numbers \( (\rho, \psi) \) and thus belongs to a two-dimensional space. It has been shown by Gibbons and Grishchuk [30] that this space of the wavefunctions is in fact a sphere of unit radius. The polar coordinates \((\theta, \varphi)\) of a wave function are calculated according to the formulas:

\[
\theta = 2\tan^{-1} \left( \frac{1}{\rho} \right), \quad \varphi = \psi.
\] (29)

Therefore, the subspace of singular wavefunctions defined by Eqs. (28) is just the great circle going through the north and south poles such that \( \varphi = 0, \pi \). This shows that almost all the wavefunctions are regular except those belonging to this subspace. In the case \( k = 1 \), the situation is different since all the wavefunctions are regular. The Tunneling (Vilenkin’s) wave function is such that:

\[
\alpha = 1, \quad \beta = i,
\] (30)

which means that its coordinates are \((\pi/2, \pi/2)\); it is regular. On the other hand, the Hartle-Hawking state is given by:

\[
\alpha = 1, \quad \beta = 0,
\] (31)

i.e. it is represented by the south pole of the sphere and thus belongs to the subspace of the singular wavefunctions. The divergence occurs when \( t = -1 \) (there is only one solution since \( \Delta = 0 \)), that is to say when \( \rho_\Lambda \) satisfies the following expression:

\[
\frac{\rho_{PL}}{\rho_\Lambda} = \frac{8\pi^3}{v_{-1}} + \frac{32\pi^3}{3v_{-1}},
\] (32)

where \( m \) is an integer. This problem is more serious that it might first appear because one should not think of \( \rho_\Lambda \) as being strictly constant (as we have approximated during the previous calculations). It rather slowly varies in time during the “slow roll” inflationary period. This means that \( \rho_\Lambda \) could easily pass through one of these dangerous values and so cause a divergence in the wave function. Note that this would be across any chosen \( a = const \) surface, where the semi-classical analysis should be valid, and so more serious that the previously mentioned factor-ordering type divergences that can occur as \( a \to 0 \). It is also in addition to the problems of simply having real wavefunctions, that do not allow the current \( j \) to be interpreted directly [12]. This divergent behaviour is like that found in the context of the quantization of spacetimes which admit a classical change of signature [14].

To know if a divergent value is actually passed through requires a knowledge of the volume \( v_{-1} \). For \( m = 0 \), we have \( \rho_{PL}/\rho_\Lambda \approx 248/v_{-1} \). Depending on the value of \( v_{-1} \) it could turn out that we are no longer in a regime where the semi-classical approximation is valid and so the danger could be ignored as being outside the range of validity. But for bigger values of \( m \), we certainly will be within this region of semi-classical validity. For example, if \( v_{-1} \approx 0.94 \) and \( m = 3 \), one has \( \rho_{PL}/\rho_\Lambda \approx 1316 \).
Therefore, it seems that the Tunneling wave function can easily be generalized to describe the quantum creation of hyperbolic Universes with compact spacelike section whereas the Hartle Hawking wave function leads to important difficulties. It is interesting to see that considering the quantum creation of compact Universes can lead to some progress in the debate Tunneling vs no boundary. In the following section, we will consider that the wave function is placed in the Tunneling or Vilenkin’s quantum state.

But before ending this section, we would like to discuss the important case of the Hartle-Hawking wave function in more detail. We will concentrate on the case $k = 1$ versus $k = -1$. The first step of the derivation consists in establishing the Euclidean equations of motion. With the help of Eqs. (3) and (4), it is easy to see that in the gauge $\bar{N} = 0$ they read:

\[
\begin{align*}
\frac{a''}{a} &= -\frac{\kappa}{3} \phi'^2 - \frac{N^2}{3} \kappa V(\phi), \\
\phi'' + 3 \frac{a'}{a} \phi' - N^2 \frac{\partial V(\phi)}{\partial \phi} &= 0, \\
\frac{a'^2}{a^2} - \frac{\kappa}{6} \phi'^2 - k \frac{N^2}{a^2} + \kappa N^2 \frac{V(\phi)}{3} &= 0,
\end{align*}
\]  

where a prime denotes derivative with respect to $\tau \equiv -it$. The hypersurface on which we evaluate the wave function is the hypersurface such that $\tau = 1$ for which we have:

\[
a(\tau = 1) \equiv \bar{a} \text{ and } \phi(\tau = 1) \equiv \phi.
\]

The no boundary conditions are [15]:

\[
a(\tau = 0) = 0, \quad \bar{a} \frac{d\phi}{d\tau}(\tau = 0) = 0.
\]  

In the slow roll regime, the solutions satisfying all the boundary conditions are found by integrating the Euclidean equations of motion. They are given by:

\[
a(\tau) = \frac{i\bar{a} \sin(N \sqrt{\kappa V / 3} \tau)}{\sin\left(\frac{N \sqrt{\kappa V}}{3} \tau\right)}, \quad \phi(\tau) = \phi.
\]  

The value of $k$ does not appear explicitly in these equations but is in fact hidden in the algebraic equation satisfied by the lapse function:

\[
\sin^2\left(N \sqrt{\frac{\kappa V}{3}} \right) = \frac{\kappa \bar{a}^2 V}{3 k}.
\]  

At this point, one should consider the two cases separately. The case $k = 1$ is standard. The solution of the previous equation can be expressed as:

\[
N_{n,k=1}^\pm = \sqrt{\frac{3}{\kappa V}} \left[ (n + \frac{1}{2}) \pi \pm \cos^{-1}\left( \bar{a} \sqrt{\frac{\kappa V}{3}} \right) \right].
\]  

It is common to consider only the case $n = 0$ together with the minus sign, for a fuller discussion see Ref. [15]. Everything is now present and the Euclidean action can be calculated along this solution. Then the Hartle-Hawking wave function is given by $\Psi_{HH}(a, \phi) \approx \exp(-S_E)$ (we have dropped the bars in order to avoid cumbersome notation) and can be expressed as:

\[
\Psi_{HH}(a, \phi) \approx \exp\left\{ \frac{6v_1}{\kappa^2 c V} \left[ 1 - \left( 1 - a^2 \frac{\kappa V}{3} \right)^{3/2} \right] \right\}.
\]  

In the region where $a \ll a_0$ the previous wave function can be written as $\Psi_{HH}(a, \phi) \approx \exp\left[ (3iv_1 a^2)/\kappa c \right]$ and we see that this is indeed independent of the scalar field and therefore that we have picked out the value $\lambda = 0$.

Let us now turn to the case $k = -1$. The solution to the algebraic equation giving the lapse function can be written as:

\[
N_{k=-1} = i \sqrt{\frac{3}{\kappa V}} \sinh^{-1}\left( \bar{a} \sqrt{\frac{\kappa V}{3}} \right).
\]  

Interestingly enough the solution is now unique. The explicit expression of the scale factor can be easily deduced. It reads:

\[
a(\tau) = \sqrt{\frac{3}{\kappa V}} \sinh\left[ \sinh^{-1}\left( \bar{a} \sqrt{\frac{\kappa V}{3}} \right) \right].
\]  

In the same manner, we can evaluate the Euclidean action and therefore find the corresponding no boundary state. One obtains:

\[
\Psi_{HH}(a, \phi) \approx \exp\left\{ i \frac{6v_{-1}}{\kappa^2 c V} \left[ 1 - \left( 1 + a^2 \frac{\kappa V}{3} \right)^{3/2} \right] \right\},
\]  

where we have again suppressed the bars. In the limit where the scale factor goes to zero, this wave function can be written as $\Psi_{HH}(a, \phi) \approx \exp\left[ - (3iv_{-1} a^2)/\kappa c \right]$. It confirms that we have picked out a vanishing separation constant. This also shows that in the Euclidean region, the wave function is oscillatory. One could easily find the no boundary wavefunction in the classical region with the WKB matching method. Of course, we would recover the fact that the denominator is now a trigonometric function which was the main reason for arguing that the Vilenkin state is preferable. In fact we can go a step further and say that the very concept of a no boundary wavefunction is in danger by this kind of analysis. Indeed, from Eq. (35) we see that the no boundary conditions (36) imply that:

\[
\frac{1}{N^2} \left( \frac{da}{d\tau} \right)^2 = k.
\]  

In order for the metric $ds^2 = N^2 d\tau^2 + a^2(\tau) d\Omega_3^2$ to be Euclidean and regular at $\tau = 0$ we need $a(\tau) \approx N \tau$ which implies $k = +1$! In the case $k = -1$, it is easy to see that we have in fact $a(\tau) \approx -iN \tau$. This means that $ds^2 \approx N^2 (d\tau^2 - \tau^2 d\Omega_3^2)$. The corresponding manifold is no longer Euclidean. In addition, the metric found above
is very reminiscent of that used by Hawking and Turok [9] in their analysis. Therefore it seems that there exists an interesting link between the approach advocated here and the one of Ref. [9]. This requires further studies which are beyond the scope of the present article.

To end this section, let us come back to the question of the regularity of the wave function. We have just shown that the requirement $|\Psi(a,k)| < \infty$ everywhere in the minisuperspace and for any $k$ favours the Tunneling wavefunction over the Hartle Hawking state. On the other hand requiring that the wavefunction be regular as $a \to 0$ for every factor ordering leads to a Hartle Hawking state for the $k = 1$ case [22]. Therefore, the two requirements are strictly not compatible if the analysis was extended to $p > 1$. However, we have previously mentioned that in more realistic models the actual presence of non-inflationary matter could also complicate the adoption of the HH wavefunction.

Regardless of these complications, and because factor ordering problems, which occur in the unrealistic limit $a \to 0$, appear less serious than the divergences found in this section for $k = -1$ models, we will next consider the predictions with these Tunneling wavefunctions.

IV. PREDICTIONS

In this section, we address the problem of computing physical predictions from the previously obtained wavefunctions. It is well known that, due to the fact of obtaining non-normalizable wavefunctions, this is a difficult task. In Ref. [15], a method to overcome this problem has been proposed. The idea is to use the current defined by Eq. (20) in the WKB regime. This leads to well-defined probabilities. Using the asymptotic form of the wavefunction given by Eq. (21), the following expression for the current, valid for any $k$, can be found:

$$j(k) = \frac{2}{\pi} \left(\frac{3}{8\pi}\right)^{1/3} \frac{\rho_{k}^{2/3}}{D(k)D^{*}(k)\rho_{\Lambda}} \left(\frac{\rho_{Pl}}{\rho_{\Lambda}}\right)^{-1/3}. \quad (45)$$

As expected this expression no longer depends on the scale factor. Going further requires a knowledge of $D(k)$ so we must now treat the three cases separately. Let us first consider the case $k = 1$. The expression of $D(k = 1)$ can be deduced from Eq. (23). This leads to:

$$j(k = 1) = \frac{3\rho_{Pl}}{\pi} e^{-\frac{3\pi}{8\pi} \rho_{Pl} / \rho_{\Lambda}}. \quad (46)$$

We recover the well-known expression for the closed case. The coefficient of proportionality is not of great interest since the probability distribution is not normalizable and instead must be used to calculate conditional probabilities, see below. The ratio $\rho_{Pl}/\rho_{\Lambda}$ only appears in the argument of the exponential function. This is because the factor $[z(0, +1)]^{-1/2}$ in the $D(k = 1)D^{*}(k = 1)$ term cancels exactly the term $(\rho_{Pl}/\rho_{\Lambda})^{-1/3}$ in Eq. (45). The expression of the current in the case $k = 0$ can be established from Eq. (24). It reads:

$$j(k = 0) = \frac{1}{4\pi A^{2}(0)} \left(\frac{3}{\pi}\right)^{1/3} \frac{\rho_{0}^{2/3}}{\left(\frac{\rho_{Pl}}{\rho_{\Lambda}}\right)^{1/3}}. \quad (47)$$

Since $D(k = 0)$ is just a constant, this does change the dependence in $(\rho_{Pl}/\rho_{\Lambda})^{-1/3}$ of Eq. (46). Finally, we turn to the case $k = -1$. From Eq. (26), we have that $D(k = -1)D^{*}(k = -1) = \left(1/\pi\right) [-z(0, -1)]^{-1/2}$. This time no exponential function appears as was the case for $k = 1$. This is due to the fact that we have chosen the boundary condition that $D(k = -1)$ simply be a phase. However, now for the case $k = -1$, the factor $\rho_{Pl}/\rho_{\Lambda}$ simply cancels out. As a consequence, one finds:

$$j(k = -1) = \frac{3\rho_{Pl}}{4\pi}. \quad (48)$$

The current turns out to be independent of $\rho_{\Lambda}$.

As already mentioned, the current can now be used to calculate conditional probabilities. For example, the probability of having an initial value $\phi_{i}$ of the scalar field, greater than the value needed to solve the problems of standard cosmology, $\phi_{suf}$, knowing that $0 < \phi < \phi_{suf}$ can be allowed. In the last inequality $\phi_{suf}$ is the value at which semi-classical considerations cease to be valid, i.e. when the potential reaches the Planck scale, $\rho_{Pl} = \rho_{\Lambda} = V(\phi) = m_{Pl}^{4}$ in the Planck system of units. Also, the fact that the minimal value for $\phi$ is zero is not a problem here because the wave function is the Tunneling one. This would no longer be true if the state were the HH one. We will evaluate the conditional probabilities for the prototype chaotic inflationary scenario, with scalar potential of the form $V(\phi) = (\lambda/4!\phi)^{4}$. In this context, the initial value of the field necessary to get $N$ e-folds is $\phi_{i} = [N + 1]/\pi m_{Pl}$. We consider that sufficient inflation is obtained when $N = 60$ then we find that the scalar field has to start at $\phi_{i} \equiv \phi_{suf} = 4.4 m_{Pl}$. The value of $\phi_{suf}$ is given by $\phi_{suf} = 24^{1/4}\lambda^{-1/4} m_{Pl}$. For chaotic inflation, one has $\lambda \sim 10^{-16}$ in order to reproduce the value of the quadrupole of the Cosmic Microwave Background Radiation anisotropy, $Q_{rms-PS}$, measured by the COBE satellite, i.e. $Q_{rms-PS} \approx 18 \times 10^{-6} K$ [31]. This implies $\phi_{suf} \approx 1.2 \times 10^{4} m_{Pl}$. The probability to have sufficient inflation in these three cases is computed according to the formula [15]:

$$P(k; \phi_{i} > \phi_{suf}; 0 < \phi_{i} < \phi_{suf}) \equiv \frac{\int_{0}^{\phi_{suf}} j(k) d\phi}{\int_{0}^{\phi_{suf}} j(k) d\phi}.$$  

(49)

It will be more convenient in the following to rewrite the previous definition of the conditional probabilities as $P(k) = 1 - R(k; N, \lambda)$ where $R(k; N, \lambda)$ is given by:

$$R(k; N, \lambda) \equiv \frac{\int_{0}^{\phi_{suf}} j(k) d\phi}{\int_{0}^{\phi_{suf}} j(k) d\phi}.$$  

(50)
We are now going to calculate $R$ in the three cases. Let us start with the case $k = 1$. The current $j(1)$ is given by Eq. (46). Using the change of variable $u \equiv 9m_{
u}^{2}/(\lambda \phi^{2})$ and the formula (3.381.6) of Ref. [32] one can easily show that the denominator is given by $(24/9)^{5/8}e^{-3/16}W_{-\frac{3}{2}}, \lambda, \frac{1}{2}(9/24)$ where $W_{\mu, \nu}$ is a Whittaker function. Thus, $R_{-\frac{3}{2}}, \lambda, \frac{1}{2}(9/24) \approx 0.52$ we find that the denominator is approximately equal to 0.80. In the same manner, the numerator is equal to:

$$
\left[ \frac{9\pi^{2}}{\lambda(N+1)^{2}} \right]^{-5/8} e^{-\frac{9\pi^{2}}{2\pi(N+1)^{1/2}}} W_{-\frac{3}{2}, \lambda, \frac{1}{2}} \left( \frac{9\pi^{2}}{\lambda(N+1)^{2}} \right).
$$

(51)

Since the parameter $\lambda$ appears at the denominator of the argument of the Whittaker function, this has a very large value. Therefore, we can use the asymptotic expansion of the Whittaker function for large values of its argument given by [32]:

$$
\lim_{|z| \to \infty} W_{\mu, \nu}(z) = e^{-\frac{z}{2}} z^{\mu} \left( 1 + O \left( \frac{1}{z} \right) \right).
$$

(52)

Thus, the value of the function $R(1; N, \lambda)$ can be expressed as:

$$
R(1; N, \lambda) \approx 0.0045 \lambda^{3/2} (N+1)^{2} e^{-\frac{9\pi^{2}}{2\pi(N+1)^{1/2}}}.
$$

(53)

This expression is valid for any value of $N$ and $\lambda$ provided that $\lambda$ is a small number. Putting $N = 60$ and $\lambda = 10^{-15}$, we find that $R(1; 60, 10^{-15}) \approx 10^{-13}$, an extremely small number which has its origin in the presence of the parameter $\lambda$ in the exponential factor. We conclude that $P(1) \approx 1$. The calculation of $P(0)$ is easier. Using Eq. (47), we find that:

$$
R(0; N, \lambda) = 24^{-\frac{3}{2}} \left( \frac{N+1}{\pi} \right)^{3/2} \lambda^{3/2}.
$$

(54)

This gives $R(0; 60, 10^{-15}) \approx 1.5 \times 10^{-9}$ leading to $P(0) \approx 1$ again. Finally, the case $k = -1$ is straightforward. Using Eq. (48), we can easily establish that:

$$
R(-1; N, \lambda) = 24^{-\frac{1}{2}} \left( \frac{N+1}{\pi} \right)^{1/2} \lambda^{1/2}.
$$

(55)

This results in $R(-1; 60, 10^{-15}) \approx 1.67 \times 10^{-4}$. Therefore, one can say that $P(-1) \approx 1$. The conclusion is that, in the three cases, the probability turns out to be close to one. This means that sufficient inflation is a prediction of the Tunneling wavefunction whatever the value of $k$ is. In this respect, the three cases are equally compatible with there being a near definite prediction of inflation occurring. This is actually a significant improvement over classical notions of whether inflation will occur. Which, due to an infinite divergence over an arbitrary scale factor, gives an ambiguous prediction even for apparently inflationary potentials [2]. We note however that the probability is closer to one in the case $k = 1$ than in the cases $k = 0, -1$.

We can pursue this reasoning a step further by also allowing $k$ to be quantum ($q$) variable. Such a possibility could be a first step in modeling quantum fluctuations in the geometry. Unlike the classical case where the curvature is simply a fixed ($c$) number constant we are now assuming that the curvature also is in a quantum ensemble. The conditional probability for having a given $k$ knowing that $k = 0, \pm 1$ can be calculated for a fixed value of the scale factor. This probability is formally defined according to the equation:

$$
P(a; k) \equiv \frac{|\Psi(a; k)|^{2}}{\sum_{l=0,\pm 1} |\Psi(a; l)|^{2}}.
$$

(56)

The use of this definition requires some comments. Let us first recall how the definition of Eq. (49) can be justified. Let $M$ be the minisuperspace and $W_{\text{WKB}} \subset M$ the region of the minisuperspace where the wavefunction can be well approximated by the WKB wavefunction. In non relativistic quantum mechanics, the time component of the current, $|\Psi|^{2}$, gives the probability density function. Since the wavefunction is normalizable, the probability of finding a particle in the interval $[a, b]$ is given by $\int_{a}^{b} dx|\Psi|^{2} / \int_{-\infty}^{\infty} dx|\Psi|^{2}$. In quantum cosmology, the current is positive definite only in the WKB regime. In this regime, $\Psi(a; k)$ is not normalizable, i.e. $\int_{M_{\text{WKB}}} dq|\Psi|^{2}$, where $dq$ is the volume in $M$, is not finite. This does not mean that $\int_{M} \mu(q^{*})dq^{*}|\Psi|^{2}$ is infinite since the measure $\mu(q^{*})$ is a priori not known. In the WKB regime, the wavefunction is by definition peaked over $M_{\text{WKB}}$. Therefore, one has $\int_{M} \mu(q^{*})dq^{*}|\Psi|^{2} \approx \int_{M_{\text{WKB}}} dq^{*}|\Psi|^{2}$. Thus, the same rule as in ordinary quantum mechanics says that the probability of finding the system in the region $R \subset M_{\text{WKB}}$ is given by $\int_{R} dq^{*}|\Psi|^{2} / \int_{M_{\text{WKB}}} dq^{*}|\Psi|^{2}$. Eq. (49) is a special case of this more general formula.

The interpretation of Eq. (56) is roughly the same. Although it is, of course, more contentious to apply this reasoning to obtain the geometry, here represented by $k$, compared to how we previously obtained the initial matter component $\phi$. The matter calculation is a fairly straightforward adaptation, as stated above, of usual quantum mechanics reasoning whereas the quantization of the geometry might require more extensive alterations to quantum mechanics. We will proceed with the notion that $k$ is not initially fixed but is rather undefined in a quantum state with “equipartition” among all possible $k$. In the context of the histories approach of quantum mechanics, $P(a; k)$ represents the probability that the Universe “choose” one of these possible histories. It will not come as a surprise that the Universe goes down the lowest potential case.

We would like to emphasize how $P(a; k)$ differs from the notion of the probability of a change of topology once the Universe has been created, i.e. the probability of having, for example, $k = 1$ for some value of the scale factor and then $k = 0$ for another (bigger) value. If topology
changes are required from one classical model to another	hen an explicit time dependent curvature should be in-
troduced. Such a construction with $dk/dt \neq 0$ has been
explicitly made in Ref. [33] and it has been shown that,
in this case, a passage to a midi-superspace description
is mandatory. However in the present example we are
assuming that the curvature is first a quantum variable
and not fixed in a particular classical state. What clas-
sical curvature state the universe first evolves to is our
present concern, not whether topology changes still occur
once this classical state is first achieved. As the universe
becomes increasingly classical, by for example gaining en-
ergy from falling down the potential, then the curvature
will no longer be in a quantum superposition but will be-
come increasingly in a specific classical state. How this
“measurement” takes place will be a rather complex pro-
cess and likely to depend on quantum mechanical inter-
pretational questions. But for our present purposes, this
rough notion that the curvature will eventually “crystal-
lize out” into a classical state should suffice.

On the following figures, the evolution of the three
probabilities as a function of the scale factor are dis-
played. The first figure is for the case $\rho_{Pl}/\rho_\Lambda = 10$.
This means that the dimensional quantity $a_0$ is equal to
$a_0 \approx l_{Pl}$. Let us also recall that we have taken $v_1 = 2\pi^2$,
v_0 = 1 and $v_{-1} \approx 0.94$. The choice of the volume can in-
fluence the behaviour of the probabilities for small scale
factors but in the limit of big scale factors, they are
mainly determined by the ratio $\rho_{Pl}/\rho_\Lambda$.

The second figure represents the case where $\rho_{Pl}/\rho_\Lambda = 100$.
This corresponds to $a_0 \approx 3.5l_{Pl}$.

The third and last figure represents the case $\rho_{Pl}/\rho_\Lambda = 1000$.
This corresponds to $a_0 \approx 10l_{Pl}$ as already mentioned. This case could be considered as the
most realistic one since inflation takes place at an energy
comparable to the GUT scale, i.e. $\rho_\Lambda \approx 10^{16}$GeV.

Finally, the third and last figure represents the case
$\rho_{Pl}/\rho_\Lambda = 1000$ which corresponds to $a_0 \approx 10.9l_{Pl}$ as al-
dready mentioned. This case could be considered as the
most realistic one since inflation takes place at an energy
comparable to the GUT scale, i.e. $\rho_\Lambda \approx 10^{16}$GeV.

Let us now comment on these figures in more detail.
In each case, we can point out the following features. At
vanishing scale factor, the probabilities are all assumed
equal to $1/3$ for any value of $k$. Provided the probabili-
ities are roughly equal the actual values will not affect the
predictions significantly. In the region where the scale
factor is still of order $a_0$, the behaviour of the probabili-
ties are rapidly evolving with $a$. Finally, when the scale
factor becomes large in comparison with $a_0$, the proba-
bilities tend to a constant value which depends on the
ratio $\rho_{Pl}/\rho_\Lambda$. This suggests some interesting ideas. For
example, the fact that the probabilities remain similar in
the region $a < a_0$ could mean that here the topology can
easily fluctuate due to quantum effects. On the other

![FIG. 5](image) Probabilities $P(k)$ for $\rho_{Pl}/\rho_\Lambda = 10$. The solid line
represents the case $k = 1$, the dotted line is the $k = 0$ case
and dashed line is the $k = -1$ case.

![FIG. 6](image) Probabilities $P(k)$ for $\rho_{Pl}/\rho_\Lambda = 100$. The solid line
represents the case $k = 1$, the dotted line is the $k = 0$ case
and dashed line is the $k = -1$ case.

![FIG. 7](image) Probabilities $P(k)$ for $\rho_{Pl}/\rho_\Lambda = 1000$. The solid line
represents the case $k = 1$, the dotted line is the $k = 0$ case
and dashed line is the $k = -1$ case.
hand, when the scale factor is such that \( a \gg a_0 \), the
probabilities differ significantly and are almost constant
as a function of \( a \). This can be easily understood if one
looks at the asymptotic behaviour of \( |\Psi(a; k)|^2 \) when \( a \)
becomes large. Using Eq. (12), one finds that:

\[
|\Psi(a; k = +1)|^2 \approx \frac{4}{a} \left( \frac{3}{8\pi} \right)^{1/2} \left( \frac{\rho_{ni}}{\rho_\Lambda} \right)^{1/2} e^{-\frac{2\pi a}{\eta_c/\rho_\Lambda}}, \quad (57)
\]

\[
|\Psi(a; k = 0)|^2 \approx \frac{1}{4\pi A_l^2(0)} \left( \frac{8\pi}{3} \right)^{1/6} \frac{1}{\alpha v_0^{1/3}} \left( \frac{\rho_{ni}}{\rho_\Lambda} \right)^{1/6}, \quad (58)
\]

\[
|\Psi(a; k = -1)|^2 \approx \frac{1}{4\pi A_l^2(0)} \left( \frac{3}{8\pi} \right)^{1/2} \left( \frac{\rho_{ni}}{\rho_\Lambda} \right)^{1/2}. \quad (59)
\]

In each case \( |\Psi(a; k)|^2 \) behaves as \( 1/a \) and therefore \( P(a; k) \)
becomes independent of the scale factor when \( a \gg a_0 \). Fluctuations in topology are more likely, regardless of the scale factor, provided the energy density
is near Planck values. But as the value of \( \rho_\Lambda \) reduces this
rapidly becomes less likely. When \( \rho_{ni}/\rho_\Lambda \gg 1 \), it is clear
from the figures that a definite prediction can be made
since one of the probabilities becomes equal to one:

\[
P(-1) \approx 1, \quad P(0) \approx 0, \quad P(+1) \approx 0. \quad (60)
\]

The fact that the greater the ratio \( \rho_{ni}/\rho_\Lambda \) is, the better
the prediction is [i.e. the closer to one \( P(-1) \) is] means
that when inflation takes place at energy well below the
Planck scale (typically the GUT scale seems to be the
most physical case), topology changes become strictly
forbidden as the scale factor becomes large. In practice
as the scalar field rolls down the potential the effective
cosmological constant \( \rho_\Lambda \) reduces and topology changes
become increasingly unlikely.

Finally, let us comment on the prediction itself. It
says, since an inflationary phase is expected to give a fi-
nal \( a \approx 10^{30} a_0 \), that our Universe is most likely to be
open, or at worst flat, a quite interesting statement in-
dering this, as many studies erroneously assume. Likewise
possible problem is avoided. But one must bear in mind
wavefunction be independent of the matter as

\[
\Omega_{\text{ini}} < 1 \quad \text{to a value very close to one even}
\]

at the present age of the universe. The actual form of
the scalar potential is crucial for determining such prop-
erties, a quantum description alone does not mandate
such values for \( \Omega_0 \). Once the inflationary phase finishes
the matter behaves effectively like that of dust [2] so now
obeying the strong-energy condition. Eventually the cur-
vature term will start again to dominate the dynamics,
with the open model expanding infinitely into the fu-
ture. Curvature can play an important role at both
the beginning and end of the universe’s evolution because of

the drastically different change in the behaviour of matter:
from being a cosmological constant to becoming like
dust. Any small residual cosmological constant would
also come to dominate at later times. Recent develop-
ment, although still contentious, have suggested such a \( \Lambda 
\)
term is necessary to explain the Supernova data [34], but
such a value is still extremely small \( \rho_\Lambda \sim 10^{-120} \). Under-
standing such fine detail while quantum cosmology takes
a rather “broad brush” approach to calculating the vari-
ous quantities remains a difficulty. We would just remind
readers that the wormhole, and related mechanisms, that
were suggested should predict \( \Lambda = 0 \) exactly [35], could
instead give other values which are only approximately
zero -see e.g. [36].

V. CONCLUSIONS

In general relativity the global topology has to be im-
posed as an initial condition. In quantum cosmology
most studies have concentrated on elliptic space which for
the FLRW model is simply a closed universe. However,
hyperbolic and flat spaces are mathematically speaking
more numerous and many are also compact. These give
rise to geometries that are locally described by the FLRW
metric with \( k = -1, 0 \). For the simple DeSitter model the
\( k = -1, 0 \) cases are no longer distinguished by the pres-
ence of a forbidden or Euclidean region at small scale
factors.

Because the Euclidean nature of the model is now ab-
sent it might seem that the smooth geometric picture
of the Hartle-Hawking no-boundary proposal is a serious
loss. But a major weakness of “quantum creation of the
universe” ideas is that the forbidden region is anyway
eroded by the presence of strong-energy satisfying mat-
ter. Such matter will generally prevent the Euclidean na-
ture of the model for small scale factors. But on general
grounds such matter should be present, if only because of
“zero-point” fluctuations. A Lorentzian region will tend
to form anyway as \( a \to 0 \).

With a scalar field source a possible problem still re-
 mains since a singularity caused by the kinetic energy of
the scalar field blowing up could occur. The kinetic
energy of the scalar field behaves as a ‘stiff’ equation of
state. This causes an effective \( -a^{-2} \) term in the WDW
potential that would tend to create an infinite ‘wiggli-
ness’ in the wavefunction as \( a \to 0 \). By ensuring that the
wavefunction be independent of the matter as \( a \to 0 \) this
possible problem is avoided. But one must bear in mind
that one is removing the singularity by fiat, one should
not claim that the process of quantization alone is achiev-
ing this, as many studies erroneously assume. Likewise
in the usual closed models the Hartle Hawking boundary
condition only gives a Euclidean region by imposing that
the matter fields are not allowed to dominate as \( a \to 0 \).

Once it is accepted that a Lorentzian region is any-
way present for the smallest scale factors it is no longer
a major fundamental difference whether one then has a forbidden region away from the origin. In the flat and open cases the forbidden region indeed goes away and we have argued that these cases are in a sense more favourable. Further, in the open case the requirement that the wavefunction should remain finite at arbitrary $a$ allows boundary conditions that include Vilenkin’s “out-going only” but not the no boundary choice.

For the specific Tunneling boundary condition, comparisons between models that only differ by their value of $k$ can be made. If the initial curvature is given by a quantum ensemble, so allowing any possible $k$, then one can conclude that open universes are more likely. Although the models considered have only a cosmological constant as their matter source. This seem to contradict the notion that for DeSitter space “all curvatures are equivalent”. But because we consider the evolution from $a = 0$ and regularized the wavefunction around this point we have derived “propagators” to go to arbitrarily large scale factors. The universe is given a choice from its conception which path to follow. In the closed model one can think of the universe being “held up” waiting to tunnel through the barrier. While the open models gain a “push” in falling down a steeper WDW potential compared to the flat case. Once the scale factor becomes sufficiently large the possibility of topology change depends entirely on the energy density of the scalar field driving inflation. As this reduces below Planck values the prediction rapidly gives that an open $k = -1$ universe is favoured. Interestingly at small scale factors and/or Planck energy densities the universe might “pin ball” between various possibilities before settling into one final curvature. The “no hair” property of DeSitter space, of having finite causal horizon, might also allow various regions to develop different curvatures. Although the open case would still dominate pockets of closed curvature could also be created. This could depend on quantum interpretations, whether “many worlds” or “single-history” quantum theories are possible cf. [37].

Regardless of the actual value of curvature inflation is predicted in all cases. This resolves an ambiguity in classical measures of inflation. If quantum cosmology did nothing else but gave a definite predictions for inflation to occur it would be very significant. The only drawback is that the scalar potential must be chosen to be of the correct shape, just as the potential of the Hydrogen atom has to be provided before one does any quantum mechanics. The potential is further constrained by the need to create sufficiently small fluctuations and gravitational waves: roughly speaking inflation should occur at $\text{GUT} \sim 10^{14}\text{GeV}$ energy scales [38,39]. Ultimately inflation is a classical phenomena that can not be driven by quantum notions alone. We have modeled curvature as being described by a quantum variable and subject to notion of probability analogous to the usual initial matter distribution calculation. There is the worry that in trying to determine the curvature, which is also a part of the potential, we are going beyond the scope of what quantum mechanics should be used to predict. We have assumed that the universe is free to take the path of least resistance i.e. follow the open case, but maybe there is actually no freedom in this choice and it is pre-ordained what curvature the universe should take before the universe comes into existence. This is a rather deep problem that affects quantum cosmology in general, what variables are free to roam and which are fixed externally imposed constants? Is there a classical scaffolding surrounding the initial universe or is every variable initially a quantum variable? Quantum cosmology seems ambiguous why certain variables (e.g. $\phi$) are given by distribution functions, although constrained by the chosen boundary conditions, while others (e.g. $k$) are imposed with no longer apparent quantum uncertainty. Recall that all classical notions seem suspect as the Planck epoch is approached cf. [40]. But in the meantime it seems that, in admittedly simplistic models, different curvatures can be considered and an argument made that the open case is favoured. If the curvature is initially fixed and so not subject to quantum uncertainty, then one can still argue that open universes are as possible as closed ones with Tunneling like boundary conditions. Understanding better the measure of possible topologies would seem the next helpful step to see if arguments can be made to favour a specific choice cf. [41].

In summary, we have considered quantum cosmological models with arbitrary curvature. Although the models are all compact they are all possible candidates for quantum creation of the universe models. Unlike always starting from closed models which have a forbidden region at small scale factors, one can work directly with the curvature of one’s choice. Compared to recent instanton methods of creating an open universe from an initial closed one, one cuts out the unnecessary closed stage. One has the further prediction that open universes are favoured followed by flat ones provided in some sense the universe has the choice of deciding its curvature. The closed universe case is strongly suppressed in comparison. One can still obtain $\Omega_0 \approx 1$ in all cases since a long period of inflation is strongly predicted. Unfortunately this large inflationary period would appear to wipe out any interesting “multiple images” due to topological effects that are presently being searched for. Only for lesser amount of inflation would such effects be apparent in the patterns of Cosmic Background radiation or multiple galaxy images -see [11]. But interestingly with an inflationary matter source, the curvature still dominates the dynamics both at the beginning and the end of the universe, this seems reasonable that the end of the universe should reflect its origins. This is unlike the conventional big bang model where the curvature only dominates in the far future and “only matter matters” during its initial phase.
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[40] C. Schiller, Does matter differ from vacuum, preprint gr-qc/9610066.