Supersymmetry and Singular Potentials

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Abstract

The breaking of supersymmetry due to singular potentials in supersymmetric quantum mechanics is critically analyzed. It is shown that, when properly regularized, these potentials respect supersymmetry, even when the regularization parameter is removed.
1 Introduction

Supersymmetry is a beautiful and, simultaneously, a tantalizing symmetry [1-7]. On the one hand, supersymmetry leads to field theories and string theories with exceptional properties [8-9]. The improved ultraviolet behavior, the natural solution of the hierarchy problem are just a few of the nice features of supersymmetric theories. On the other hand, supersymmetry also predicts degenerate superpartner states (superpartner states with degenerate mass) corresponding to every physical particle state of the theory. Unfortunately, such superpartners with degenerate masses are not observed experimentally and, consequently, one expects that supersymmetry must be spontaneously (dynamically) broken much like the spontaneous breaking of ordinary symmetries in physical theories. However, unlike ordinary symmetries, spontaneous breaking of supersymmetry has so far proved extremely difficult in the conventional framework. Consequently, in the context of supersymmetry, one constantly looks for alternate, unconventional methods of breaking of this symmetry [6-7]. There is, of course, the breaking of supersymmetry due to instanton effects which is well understood. However, several authors, in recent years have suggested that supersymmetry may be broken in the presence of singular potentials or boundaries in a nonstandard manner [10-12]. We will explain the details of the proposed mechanism later, but the gist of the argument is that in such systems, the superpartner states may not belong to the physical Hilbert space thereby leading to a breaking of supersymmetry. This would, of course, explain why the superpartner states would not be observable. Even more interesting is the possibility that since such a breaking is nonstandard, the usual theorems of supersymmetry breaking may not apply and the ground state may continue to have zero energy thereby leading to a solution of the cosmological constant problem also.

The examples, where such a breaking has been discussed, are simple quantum mechanical models which nonetheless arise from the non-relativistic limit of some field theories. It is for this reason that, in an earlier paper, we had examined [13] a candidate relativistic 2 + 1 dimensional field theory to see if the manifestation of such a mechanism was possible in a field theory. However, a careful examination of the theory revealed that supersymmetry prevails at the end although it might appear naively, in the beginning, that supersymmetry would be broken in the nonstandard manner. This prompted us to re-analyze the quantum mechanical models, where this mechanism was demonstrated, more carefully in order to understand if supersymmetry is truly broken in these models and if so what may be the distinguishing features in these models from a relativistic field theory. A systematic and critical examination, once again, reveals that when carefully done, supersymmetry is manifest even in such singular quantum mechanical models which is the main result of this paper.

Since our discussion would be entirely within the context of one dimensional supersymmetric quantum mechanics, let us establish the essential notations here. Given a superpotential, $W(x)$, we can define a pair of supersymmetric potentials as

$$V_+ = \frac{1}{2} \left( W^2(x) + W'(x) \right), \quad V_- = \frac{1}{2} \left( W^2(x) - W'(x) \right)$$

(1)

where “prime” denotes differentiation with respect to $x$. With $\hbar = 1$ and $m = 1$, we can, then,
define a pair of Hamiltonians which describe a supersymmetric system as

\[
\begin{align*}
H_+ &= -\frac{1}{2} \frac{d^2}{dx^2} + V_+ \\
H_- &= -\frac{1}{2} \frac{d^2}{dx^2} + V_-
\end{align*}
\]  

In fact, defining the supercharges as

\[
Q = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + W(x) \right), \quad Q^\dagger = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + W(x) \right)
\]  

we recognize that we can write the pair of Hamiltonians in eq. (2) also as

\[
H_+ = Q^\dagger Q, \quad H_- = QQ^\dagger
\]  

It is clear now that if \( |\psi_-\rangle \) represents an eigenstate of \( H_- \) with a nonzero energy \( E \), then, \( Q^\dagger |\psi_-\rangle \) would be an eigenstate of \( H_+ \) with the same energy, namely,

\[
\begin{align*}
H_- |\psi_-\rangle &= QQ^\dagger |\psi_-\rangle = E |\psi_-\rangle \\
H_+ (Q^\dagger |\psi_-\rangle) &= Q^\dagger QQ^\dagger |\psi_-\rangle = Q^\dagger (QQ^\dagger |\psi_-\rangle) = E (Q^\dagger |\psi_-\rangle)
\end{align*}
\]  

In other words, the states \( |\psi_-\rangle \) and \( Q^\dagger |\psi_-\rangle \) (alternately, we can also denote them as \( Q |\psi_+\rangle \) and \( |\psi_+\rangle \) respectively) would correspond to the degenerate superpartner states. All the eigenstates of the two Hamiltonians \( H_+ \) and \( H_- \) would be degenerate except for the ground state with vanishing energy which would correspond to the state satisfying

\[
Q |\psi_+\rangle = 0, \quad \text{or}, \quad Q^\dagger |\psi_-\rangle = 0
\]  

For a given superpotential, at most one of the two conditions in eq. (6) can be satisfied (that is, at most, only one of the two conditions in (6) would give a normalizable state). Namely, the ground state with vanishing energy is unpaired and can belong to the spectrum of either \( H_+ \) or \( H_- \) depending on which of the conditions leads to a normalizable state. This corresponds to the case of unbroken supersymmetry. For example, with \( W(x) = -\omega x \), the pair of Hamiltonians

\[
\begin{align*}
H_+ &= \frac{1}{2} \left( -\frac{d^2}{dx^2} + \omega^2 x^2 - \omega \right) \\
H_- &= \frac{1}{2} \left( -\frac{d^2}{dx^2} + \omega^2 x^2 + \omega \right)
\end{align*}
\]  

describe the supersymmetric harmonic oscillator and it can be easily checked that the normalizable ground state belongs to the spectrum of \( H_+ \) (In fact, we will choose \( H_+ \) to have the ground state throughout this paper.).

If, on the other hand, the superpotential is such that neither of the states in eq. (6) is normalizable, then, supersymmetry is known to be broken by instanton effects [6]. For example, this can
happen when the superpotential is an even polynomial. In contrast, the new mechanism described, in the presence of a boundary or a singular potential [10-12], corresponds to the case where the action of the supercharges takes a state out of the physical Hilbert space (although \( QQ^\dagger \) and \( Q^\dagger Q \) belong to the Hilbert space) so that, say, if \( |\psi_+\rangle \) is in the Hilbert space, \( Q|\psi_+\rangle \) would not belong to the Hilbert space leading to the breaking of the degeneracy of states and, therefore, supersymmetry.

In this paper, we carefully analyze the models [10-12] where this mechanism is thought to be operative. We note that a quantum mechanical potential with a singular structure is best studied with a regularization because this brings out the correct boundary conditions naturally. Furthermore, the regularization must be chosen carefully preserving supersymmetry when one is dealing with a supersymmetric potential with a singular structure. Keeping this in mind, we examine a supersymmetric model with a boundary, namely, the supersymmetric oscillator on the half line [10] in section 2. We briefly recapitulate the results of a harmonic oscillator on the half line and then show through a careful analysis that supersymmetry is, in fact, manifest in such a model in spite of the boundary. In section 3, we analyze a model with a singular potential, namely, the oscillator with a \( \frac{1}{x^2} \) potential. This potential is quite interesting and, in spite of several studies in the literature [14-16], the analysis is not quite complete and we present a systematic and complete analysis of this problem. In section 4, we take up the study of the supersymmetric oscillator with a \( \frac{1}{x^2} \) potential [11] and show that supersymmetry is, in fact, manifest in this system as well, in spite of the singular nature of the potential. There are several interesting aspects of this model which emerge from a careful analysis which we bring out. In section 5, we solve this problem algebraically as well which supports the analysis of section 4. In section 6, we describe the solution to a puzzle raised in the literature [11] in the context of the supersymmetric oscillator with a \( \frac{1}{x^2} \) potential and present our conclusions in section 7.

## 2 Super “Half” Oscillator

To understand the super “half” oscillator, it is useful to recapitulate briefly the results of the “half” oscillator. Let us consider a particle moving in the potential

\[
V(x) = \begin{cases} 
\frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\
\infty & \text{for } x < 0
\end{cases}
\]

The spectrum of this potential is quite clear intuitively. Namely, because of the infinite barrier, we expect the wave function to vanish at the origin leading to the conclusion that, of all the solutions of the oscillator on the full line, only the odd solutions (of course, on the “half” line there is no notion of even and odd) would survive in this case. While this is quite obvious, let us analyze the problem systematically for later purpose.

First, let us note that singular potentials are best studied in a regularized manner because this is the only way that appropriate boundary conditions can be determined correctly. Therefore, let
us consider the particle moving in the regularized potential

\[ V(x) = \begin{cases} \frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\ \frac{c^2}{2} & \text{for } x < 0 \end{cases} \]  

(9)

with the understanding that the limit \( |c| \to \infty \) is to be taken at the end. The Schrödinger equation

\[ \left( -\frac{1}{2} \frac{d^2}{dx^2} + V(x) \right) \psi(x) = \epsilon \psi(x) \]

can now be solved in the two regions. Since \( |c| \to \infty \) at the end, for any finite energy solution, we have the asymptotically damped solution, for \( x < 0 \),

\[ \psi^{(II)}(x) = A e^{(c^2 - 2\epsilon)^{1/2} x} \]  

(10)

Since the system no longer has reflection symmetry, the solutions, in the region \( x > 0 \), cannot be classified into even and odd solutions. Rather, the normalizable (physical) solution would correspond to one which vanishes asymptotically. The solutions of the Schrödinger equation, in the region \( x > 0 \), are known as the parabolic cylinder functions \([17]\) and the asymptotically damped physical solution is given by

\[ \psi^{(I)}(x) = B U(-\left(\frac{\epsilon}{\omega} + \frac{1}{2}\right), \sqrt{2\omega} x) \]  

(11)

The parabolic cylinder function, \( U(a, x) \), of course, vanishes for large values of \( x \). For small values of \( x \), it satisfies

\[ U(a, x) \xrightarrow{x \to 0} \frac{\sqrt{\pi}}{2^{\frac{1}{2}(2a+1)} \Gamma\left(\frac{3}{4} + \frac{a}{2}\right)} \]

\[ U'(a, x) \xrightarrow{x \to 0} -\frac{\sqrt{\pi}}{2^{\frac{1}{2}(2a-1)} \Gamma\left(\frac{1}{4} + \frac{a}{2}\right)} \]  

(12)

It is now straightforward to match the solutions in eqs. (10, 11) and their first derivatives across the boundary at \( x = 0 \) and their ratio gives

\[ \frac{1}{\sqrt{c^2 - 2\epsilon}} = -\frac{1}{2} \frac{\Gamma\left(-\frac{\epsilon}{\omega}\right)}{\Gamma\left(-\frac{\epsilon}{\omega} + \frac{1}{2}\right)} \]  

(13)

It is clear, then, that as \( |c| \to \infty \), this can be satisfied only if

\[ \frac{-\epsilon}{2\omega} + \frac{1}{2} \xrightarrow{|c| \to \infty} -n, \quad n = 0, 1, 2, \cdots \]  

(14)

In other words, when the regularization is removed, the energy levels that survive are the odd ones, namely, (remember that the zero point energy is already subtracted out in (8) or (9))

\[ \epsilon_n = \omega(2n + 1) \]  

(15)
The corresponding physical wave functions are nontrivial only on the half line \( x > 0 \) and have the form

\[
\psi_n(x) = B_n U(-\frac{2n + \frac{3}{2}}{2}, \sqrt{2\omega} x) = \tilde{B}_n e^{-\frac{1}{2} \omega x^2} H_{2n+1}(\sqrt{\omega} x)
\]  

(16)

Namely, only the odd Hermite polynomials survive leading to the fact that the wave function vanishes at \( x = 0 \). Thus, we see that the correct boundary condition naturally arises from regularizing the singular potential and studying the problem systematically.

We now turn to the analysis of the supersymmetric oscillator on the half line. One can define a superpotential [10]

\[
W(x) = \begin{cases} 
-\omega x & \text{for } x > 0 \\
\infty & \text{for } x < 0 
\end{cases}
\]  

(17)

which would, naively, lead to the pair of potentials

\[
V_{\pm}(x) = \begin{cases} 
\frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\
\infty & \text{for } x < 0 
\end{cases}
\]  

(18)

Since, this involves singular potentials, we can study it, as before, by regularizing the singular potentials as

\[
V_+(x) = \begin{cases} 
\frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\
\frac{c_+^2}{2} + 2 & \text{for } x < 0 
\end{cases}
\]

\[
V_-(x) = \begin{cases} 
\frac{1}{2}(\omega^2 x^2 + \omega) & \text{for } x > 0 \\
\frac{c_-^2}{2} - 2 & \text{for } x < 0 
\end{cases}
\]  

(19)

with the understanding that \( |c_\pm| \to \infty \) at the end.

The earlier analysis can now be repeated for the pair of potentials in eq. (19). It is straightforward and without going into details, let us simply note the results, namely, that, in this case, we obtain

\[
\epsilon_{+,n} = \omega(2n + 1) \quad \psi_{+,n}(x) = B_{+,n} e^{-\frac{1}{2} \omega x^2} H_{2n+1}(\sqrt{\omega} x) \\
\epsilon_{-,n} = 2\omega(n + 1) \quad \psi_{-,n}(x) = B_{-,n} e^{-\frac{1}{2} \omega x^2} H_{2n+1}(\sqrt{\omega} x)
\]  

(20)

Here \( n = 0, 1, 2, \cdots \). There are several things to note from this analysis. First, only the odd Hermite polynomials survive as physical solutions since the wave function has to vanish at the origin. This boundary condition arises from a systematic study involving a regularized potential. Second, the energy levels for the supersymmetric pair of Hamiltonians are no longer degenerate. Furthermore, the state with \( \epsilon = 0 \) no longer belongs to the Hilbert space (since it corresponds to an even Hermite polynomial solution). This leads to the conventional conclusion that supersymmetry is broken in such a case and let us note, in particular, that in such a case, it would appear that the superpartner states do not belong to the physical Hilbert space (Namely, in this case, the supercharge is an odd
operator and hence connects even and odd Hermite polynomials. However, the boundary condition selects out only odd Hermite polynomials as belonging to the physical Hilbert space.

There is absolutely no doubt that supersymmetry is broken in this case. The question that needs to be addressed is whether it is a dynamical property of the system or an artifact of the regularization (and, hence the boundary condition) used. The answer is quite obvious, namely, that supersymmetry is broken mainly because the regularization (and, therefore, the boundary condition) breaks supersymmetry. In other words, for any value of the regularizing parameters, \(|c_\pm| = |c_-|\), the pair of potentials in eq. (19) do not define a supersymmetric system and hence the regularization itself breaks supersymmetry. Consequently, the breaking of supersymmetry that results when the regularization is removed cannot be trusted as a dynamical effect.

**Regularized Superpotential**

Another way to understand this is to note that for a supersymmetric system, it is not the potential that is fundamental. Rather, it is the superpotential which gives the pair of supersymmetric potentials through Riccati type relations. It is natural, therefore, to regularize the superpotential which would automatically lead to a pair of regularized potentials which would be supersymmetric for any value of the regularization parameter. Namely, such a regularization will respect supersymmetry and, with such a regularization, it is, then, meaningful to ask if supersymmetry is broken when the regularization parameter is removed at the end. With this in mind, let us look at the regularized superpotential

\[ W(x) = -\omega x \theta(x) + c \theta(-x) \]  

(21)

Here \(c\) is the regularization parameter and we are supposed to take \(|c| \to \infty\) at the end. Note that although, at this level, both the signs of the regularization parameter are allowed, existence of a normalizable ground state (see eq. (6)) selects out \(c > 0\) (otherwise, the regularization would have broken supersymmetry through instanton effects as we have mentioned earlier).

The regularized superpotential now leads to the pair of regularized supersymmetric potentials

\[ V_+(x) = \frac{1}{2} \left[ (\omega^2 x^2 - \omega) \theta(x) + c^2 \theta(-x) - c \delta(x) \right] \]

\[ V_-(x) = \frac{1}{2} \left[ (\omega^2 x^2 + \omega) \theta(x) + c^2 \theta(-x) + c \delta(x) \right] \]

(22)

which are supersymmetric for any \(c > 0\). Let us note that the difference here from the earlier case where the potentials were directly regularized (see eq. (19)) lies only in the presence of the \(\delta(x)\) terms in the potentials. Consequently, the earlier solutions in the regions \(x > 0\) and \(x < 0\) continue to hold. However, the matching conditions are now different because of the delta function terms. Carefully matching the wave function and the discontinuity of the first derivative across \(x = 0\) for each of the wavefunctions and taking their ratio, we obtain the two conditions

\[ \frac{1}{(c^2 - 2\epsilon_+)^{1/2} - c} = -\frac{1}{2\sqrt{\omega}} \frac{\Gamma(-\frac{\epsilon_+}{2\omega})}{\Gamma(-\frac{\epsilon_+}{2\omega} + \frac{1}{2})} \]  

(23)
\[ \frac{1}{(c^2 - 2c_\omega)^{1/2} + c} = \frac{1}{2\sqrt{\omega}} \frac{\Gamma\left(-\frac{c_\omega}{\omega} + \frac{1}{2}\right)}{\Gamma\left(-\frac{c_\omega}{\omega} + 1\right)} \]  

(24)

It is now clear that, as \( c \to \infty \), (23) and (24) give respectively

\[ \begin{align*}
\epsilon_{+,n} &= 2\omega n \\
\epsilon_{-,n} &= 2\omega (n + 1)
\end{align*} \]  

(25)

The corresponding wave functions, in this case, have the forms

\[ \begin{align*}
\psi_{+,n}(x) &= B_{+,n} e^{-\frac{1}{2}\omega x^2} H_{2n}(\sqrt{\omega} x) \\
\psi_{-,n}(x) &= B_{-,n} e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x)
\end{align*} \]  

(26)

This is indeed quite interesting for it shows that the spectrum of \( H_+ \) contains the ground state with vanishing energy. Furthermore, all the other states of \( H_+ \) and \( H_- \) are degenerate in energy corresponding to even and odd Hermite polynomials as one would expect from superpartner states. Consequently, it is quite clear that if the supersymmetric “half” oscillator is defined carefully by regularizing the superpotential, then, supersymmetry is manifest in the limit of removing the regularization. This should be contrasted with the general belief that supersymmetry is broken in this system (which is a consequence of using boundary conditions or, equivalently, of regularizing the potentials in a manner which violates supersymmetry).

### Alternate Regularization

Of course, we should worry at this point as to how regularization independent our conclusion really is. Namely, our results appear to follow from the matching conditions in the presence of singular delta potential terms and, consequently, it is worth investigating whether our conclusions would continue to hold with an alternate regularization of the superpotential which would not introduce such singular terms to the potentials. With this in mind, let us choose a regularized superpotential of the form

\[ W(x) = -\omega x \theta(x) - \lambda x \theta(-x) \]  

(27)

Here \( \lambda \) is the regularization parameter and we are to take the limit \( |\lambda| \to \infty \) at the end. Once again, we note that, although both signs of \( \lambda \) appear to be allowed, existence of a normalizable ground state (see eq. (6)) would select \( \lambda > 0 \).

This regularized superpotential would now lead to the pair of supersymmetric potentials of the form

\[ \begin{align*}
V_+(x) &= \frac{1}{2} \left[ (\omega^2 x^2 - \omega) \theta(x) + (\lambda^2 x^2 - \lambda) \theta(-x) \right] \\
V_-(x) &= \frac{1}{2} \left[ (\omega^2 x^2 + \omega) \theta(x) + (\lambda^2 x^2 + \lambda) \theta(-x) \right]
\end{align*} \]  

(28)
There are no singular delta potential terms with this regularization. In fact, the regularization merely introduces a supersymmetric pair of oscillators for \( x < 0 \) whose frequency is to be taken to infinity at the end.

Since there is a harmonic oscillator potential for both \( x > 0 \) and \( x < 0 \), the solutions are straightforward. They are the parabolic cylinder functions which we have mentioned earlier. Now matching the wave function and its first derivative at \( x = 0 \) for each of the Hamiltonians and taking the ratio, we obtain

\[
\frac{1}{\sqrt{\lambda}} \frac{\Gamma\left(\frac{-\epsilon + 2}{2\lambda}\right)}{\Gamma\left(\frac{-\epsilon + 2}{2\lambda} + \frac{1}{2}\right)} = \frac{1}{\sqrt{\omega}} \frac{\Gamma\left(\frac{-\epsilon}{2\omega} + \frac{1}{2}\right)}{\Gamma\left(\frac{-\epsilon}{2\omega} + \frac{1}{2} + \frac{1}{2}\right)} \quad (29)
\]

\[
\frac{1}{\sqrt{\lambda}} \frac{\Gamma\left(\frac{-\epsilon + 2}{2\lambda} + \frac{1}{2}\right)}{\Gamma\left(\frac{-\epsilon + 2}{2\lambda} + 1\right)} = \frac{1}{\sqrt{\omega}} \frac{\Gamma\left(\frac{-\epsilon}{2\omega} + \frac{1}{2}\right)}{\Gamma\left(\frac{-\epsilon}{2\omega} + 1\right)} \quad (30)
\]

It is clear now that, as \( \lambda \to \infty \), eqs. (29) and (30) give respectively

\[
\epsilon_+, n = 2\omega n \\
\epsilon_-, n = 2\omega(n + 1)
\quad (31)
\]

The corresponding wave functions are given by

\[
\psi_+, n(x) = B_+, n e^{-\frac{1}{2}\omega x^2} H_{2n}(\sqrt{\omega} x) \\
\psi_-, n(x) = B_-, n e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x)
\quad (32)
\]

These are, of course, the same energy levels and wave functions as obtained in eqs. (25) and (26) respectively showing again that supersymmetry is manifest. Furthermore, this shows that this conclusion is independent of the regularization used as long as the regularization preserves supersymmetry which can be achieved by properly regularizing the superpotential.

### 3 Oscillator with \( \frac{1}{x^2} \) Potential

In the last section, we showed that, in the presence of one kind of singularity, namely, a boundary, supersymmetry is unbroken. In what follows, we will study another class of supersymmetric models, namely, the supersymmetric oscillator with a \( \frac{1}{x^2} \) potential, where there is a genuine singularity in the potential not necessarily arising from a boundary. A naive analysis of this model [11] also shows that supersymmetry is broken by such a singular potential (for certain parameter ranges). However, this conclusion can be understood, again, as a consequence of regularizing the potential which, as we have seen before, does not respect supersymmetry. In stead, we will show through a careful analysis that, when the superpotential is regularized, supersymmetry is manifest in this model as well (with a lot of interesting features). In this section, however, we will systematically
analyze only the quantum mechanical system corresponding to an oscillator in the presence of a $\frac{1}{x^2}$ potential (postponing the discussion of the supersymmetric case to the next section). This system has been analyzed by several people [14-16] and the most complete analysis appears to be in ref. [16]. However, we feel that, while the energy levels derived in [16] are correct, the wave functions are not (namely, the extensions of the solutions from the positive to the negative axis are incomplete and the wave functions, of course, become quite crucial when one wants to extend the analysis to a supersymmetric system) and, consequently, we present a careful analysis of this system regularizing the singular potential in a systematic manner. With the supersymmetric system in mind (to follow in the next section), we write the potential for the system as (with $\hbar = m = \omega = 1$)

$$V(x) = \frac{1}{2} \left[ \frac{g(g + 1)}{x^2} + x^2 - 2g + 1 \right]$$

(33)

Consequently, the Schrödinger equation that we want to study has the form

$$\left[ \frac{d^2}{dx^2} - \frac{g(g + 1)}{x^2} - x^2 + (2\epsilon + 2g - 1) \right] \psi(x) = 0$$

(34)

The singular potential is repulsive for $g > 0$ or $g < -1$ while it is attractive for $-1 < g < 0$. Furthermore, for ease of comparison, let us make the identifications with the notations of ref. [16] (note that our energies are shifted since we have in mind the supersymmetric system to study later)

$$\lambda = \frac{1}{2} (\alpha^2 - \frac{1}{4}) = \frac{g(g + 1)}{2}$$

$$E = \epsilon + g - \frac{1}{2}$$

(35)

It is also worth noting here that the Schrödinger equation in (34) is invariant under

$$g \leftrightarrow -(g + 1)$$

$$\epsilon \leftrightarrow \epsilon + 2g + 1$$

(36)

This symmetry, of course, would also be reflected in the solutions. Furthermore, the fixed point of this symmetry, namely, $g = -\frac{1}{2}$ separates the two branches (namely, for every value of $\lambda$ there exist two distinct values of $g$ corresponding to two distinct branches separated at the branch point) in the parameter space.

**Regularized Potential**

The Schrödinger equation in (34) can be solved quite easily for $x > 0$ as was also done in [16]. However, to determine correctly how this wavefunction should be extended to the negative axis, it is more suitable to regularize the potential near the origin and study the problem carefully. Let us
consider a potential of the form
\[
V(x) = \begin{cases} 
\frac{1}{2} \left[ \frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right] & \text{for } |x| > R \\
\frac{1}{2} \left[ \frac{g(g+1)}{R^2} + R^2 - 2g + 1 \right] & \text{for } |x| < R
\end{cases}
\]  
(37)

Namely, we have regularized the potential in a continuous manner preserving the symmetry in eq. (36) with the understanding that the regularization parameter \( R \to 0 \) at the end. With this regularization, the Schrödinger equation has to be analyzed in three distinct regions. However, since the potential has reflection symmetry, we need to analyze the solutions only in the regions \(-R < x < R\) and \(x > R\).

The potential is a constant in the region \(-R < x < R\) and hence the Schrödinger equation is quite simple here. The solutions can be classified into even and odd ones and take the forms
\[
\psi^{(II)\text{even}}(x) = A(R) \cosh \kappa x \\
\psi^{(II)\text{odd}}(x) = B(R) \sinh \kappa x
\]  
(38)

where we have defined
\[
\kappa = \sqrt{\frac{g(g+1)}{R^2} + R^2 - (2\epsilon + 2g - 1)} \approx \frac{\sqrt{g(g+1)}}{R} 
\]  
(39)

Since \( R \) is small (and we are to take the vanishing limit at the end), the last equality holds only if \( g \neq 0 \) or \(-1\) which we will assume. The special values of \( g \) corresponding to the absence of a singular potential have to be treated separately and we will come back to this at the end of this section. We note here that the normalization constants, \( A \) and \( B \), can, in principle depend on the regularization parameter which we have allowed for in writing down the form of the solutions in eq. (38).

The potential is much more complicated in the region \( x > R \). However, if we make the definitions
\[
\xi = x^2, \quad \psi^{(I)}(x) = e^{-\frac{1}{2} \xi} \xi^{\frac{1}{2}} u(\xi)
\]  
(40)

where
\[
s = (g + 1) \quad \text{or} \quad -g
\]  
(41)

the Schrödinger equation of (34) takes the form
\[
\xi \frac{d^2 u}{d\xi^2} + (s + \frac{1}{2} - \xi) \frac{du}{d\xi} + \frac{1}{2}(\epsilon + g - s - 1)u = 0
\]  
(42)

An equation of the form
\[
\xi \frac{d^2 u}{d\xi^2} + (b - \xi) \frac{du}{d\xi} - au = 0
\]  
(43)
is known as the confluent hypergeometric equation [17] and the only solution of this equation which is damped for large values of the coordinate has the form

\[ U(a, b, \xi) = \frac{\Gamma(1 - b)}{\Gamma(1 + a - b)} M(a, b, \xi) + \frac{\Gamma(b - 1)}{\Gamma(a)} \xi^{1-b} M(1 + a - b, 2 - b, \xi) \] (44)

Here \( M(a, b, \xi) \) are known as the confluent hypergeometric functions which have the series expansion

\[ M(a, b, \xi) = 1 + \frac{a}{b} \xi + \frac{a(a + 1)}{b(b + 1)} \frac{\xi^2}{2!} + \cdots \]

and satisfy

\[ M(a, b, \xi) \xrightarrow{\xi \to 0} 1 \] (45)

It is clear now that eq. (42) simply is the confluent hypergeometric equation and the asymptotically damped physical solutions are nothing other than the \( U(a, b, \xi) \) functions with appropriate parameters. Since \( s \) takes two possible values (see eq. (41)), it would seem that there would be two independent solutions of eq. (42). However, it can be easily checked that the two solutions corresponding to the two values of \( s \) are really proportional to each other and not independent. Thus, we can write the general solution of the Schrödinger equation, for \( x > 0 \), as

\[ \psi^{(I)}(x) = C(R) e^{-\frac{1}{2}x^2} \times \left[ \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(\frac{1}{2} - g - \frac{1}{2})} x^{g+1} M(1 - \frac{\epsilon}{2}, g + \frac{3}{2}, x^2) + \frac{\Gamma(g + \frac{1}{2})}{\Gamma(1 - \frac{g}{2})} x^{-g} M(\frac{1}{2} - g - \frac{\epsilon}{2}, -g + \frac{1}{2}, x^2) \right] \] (46)

Once again, we have allowed for a dependence of the normalization constant, \( C \), on the regularization parameter, \( R \). However, for a nontrivial solution to exist, we require that

\[ C(R) \xrightarrow{R \to 0} C \neq 0 \]

So far, we have the general solutions, in the two regions, where energy is not quantized and which should arise from the matching conditions. Furthermore, we have not bothered to evaluate the solution in the region \( x < -R \) which clearly would be the same as in the region \( x > R \). However, the matching conditions would determine how we should extend the solutions in the region \( x > R \) to the region \( x < -R \). Therefore, let us now examine the matching conditions systematically since there are two possible cases.

(i) Even Solution

We can match the even solution of the region \(-R < x < R\) and its derivative with those of the region \( x > R \) at \( x = R \). Taking the ratio and remembering that \( R \) is small (which is to be taken to zero at the end), we obtain to the leading order in \( R \)

\[ \sqrt{g(g + 1)} \tanh \sqrt{g(g + 1)} = \frac{(g + 1) \Gamma(-g - \frac{1}{2})}{\Gamma(\frac{1}{2} - g - \frac{1}{2})} R^{g+1} - g \frac{\Gamma(g + \frac{1}{2})}{\Gamma(1 - \frac{g}{2})} R^{-g} \] (47)
Since the left hand side is independent of $R$, for consistency, the right hand side must also be and this can happen in two different ways.

First, for $g > -\frac{1}{2}$, it is clear that relation (47) can be satisfied if (we assume from now on that $n = 0, 1, 2, \ldots$)

$$1 - \frac{\epsilon}{2} = -n + f_1(g)R^{2g+1}$$

or,

$$\epsilon_n = 2(n + 1) - 2f_1(g)R^{2g+1}$$

(48)

with a suitable choice of $f_1(g)$.

On the other hand, for $g < -\frac{1}{2}$, if

$$\frac{1}{2} - g - \frac{\epsilon}{2} = -n + f_2(g)R^{-2g-1}$$

or,

$$\epsilon_n = (2n - 2g + 1) - 2f_2(g)R^{-2g-1}$$

(49)

relation (47) can be satisfied with a suitable choice of $f_2(g)$. It is clear that the two possible branches of the solution simply reflect the symmetry in eq. (36).

This analysis shows that when the regularization is removed (namely, $R \to 0$), we have an even extension of the solution of the forms

$$g > -\frac{1}{2} : \quad \epsilon_n = 2(n + 1)$$

(50)

with

$$\psi_n(x) = C_n \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g + \frac{3}{2}, x^2) \left\{ \begin{array}{ll} x^{g+1} & \text{for } x > 0 \\ |x|^{g+1} & \text{for } x < 0 \end{array} \right.$$ (51)

$$g < -\frac{1}{2} : \quad \epsilon_n = 2n - 2g + 1$$

(52)

with

$$\psi_n(x) = C_n \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + \frac{1}{2}, x^2) \left\{ \begin{array}{ll} x^{-g} & \text{for } x > 0 \\ |x|^{-g} & \text{for } x < 0 \end{array} \right.$$ (53)

(ii) Odd Solution

We can also match the odd solution of the region $-R < x < R$ and its derivative with those of the region $x > R$ at $x = R$ and taking the ratio, we obtain to leading order

$$\sqrt{g(g + 1)} \coth \sqrt{g(g + 1)} = \frac{(g + 1) \Gamma(-g - \frac{1}{2}) R^{g+1} - g \Gamma(g + \frac{1}{2}) R^{-g}}{\Gamma(-g - \frac{1}{2}) R^{g+1} + \Gamma(g + \frac{1}{2}) R^{-g}}$$

(54)
Clearly, the analysis following from eq. (47) goes through identically so that we conclude that in the limit $R \to 0$, we have an odd extension of the solution of the forms

$$g > -\frac{1}{2} : \quad \epsilon_n = 2(n + 1)$$

with

$$\psi_n(x) = C_n \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g + 3/2, x^2) \begin{cases} x^{g+1} & \text{for } x > 0 \\ -|x|^{g+1} & \text{for } x < 0 \end{cases}$$

and

$$g < -\frac{1}{2} : \quad \epsilon_n = 2n - 2g + 1$$

with

$$\psi_n(x) = C_n \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + 1/2, x^2) \begin{cases} x^{-g} & \text{for } x > 0 \\ -|x|^{-g} & \text{for } x < 0 \end{cases}$$

Understanding of the Result

The conclusion following from this analysis, therefore, is that every energy level of this system is doubly degenerate. Both even and odd extensions of the solution are possible for every value of the energy level. The energy levels, as given in eqs. (50) and (52) (or, alternately, (55) and (57)) are, of course, identical to those obtained in [16]. The crucial difference is in the structure of the wave functions, namely, that both even and odd extensions of the solution are possible for every value of the energy (Incidentally, the solutions we have obtained in terms of confluent hypergeometric functions also coincide with generalized Laguerre polynomials as was obtained in ref. [16].). It is crucial, therefore, to ask if such a conclusion is physically plausible. To understand this question, let us recapitulate the results from a simple quantum mechanical model which is well studied. Namely, let us look at a particle moving in a potential of the form

$$V(x) = \begin{cases} \gamma \delta(x) & \text{for } |x| < a \\ \infty & \text{for } |x| > a \end{cases}$$

It is well known that the solutions of this system can be classified into even and odd ones with energy levels ($\hbar = m = 1$)

$$E_n^{\text{even}} = \frac{n^2\pi^2}{2(a + \frac{1}{\gamma})^2}$$

$$E_n^{\text{odd}} = \frac{n^2\pi^2}{2a^2}$$

The even and the odd solutions, of course, have distinct energy values for any finite strength of the delta potential. However, when $\gamma \to \infty$, both the even and the odd solutions become degenerate in energy. Namely, a delta potential with an infinite strength leads to a double degeneracy of every
energy level corresponding to both even and odd solutions. The connection of this example with the problem we are studying is intuitively clear. Namely, we can think of

\[
\frac{g(g+1)}{x^2} = \lim_{\eta \to 0} \frac{g(g+1)}{x^2 + \eta^2} = \lim_{\eta \to 0} \left( \frac{\pi g(g+1)}{\eta} \right) \left( \frac{1}{\pi x^2 + \eta^2} \right)
\]

It is clear that for \( g \neq 0 \) or \(-1\), the singular \( \frac{1}{x^2} \) potential behaves like a delta potential with an infinite strength and it is quite natural, therefore, that this system has both even and odd solutions degenerate in energy.

It is also clear from this analysis that it is meaningless to take the \( g = 0 \) or \(-1\) limit from the results obtained so far simply because the characters of the two problems are quite different. As we have argued, for any finite value of \( g \) not coinciding with those special values, the potential behaves, at the origin, like a delta potential of infinite strength while for the special values, there is no such potential. The two cases are related in a drastically discontinuous manner. As a result, one cannot treat the \( \frac{g(g+1)}{x^2} \) as a perturbation and obtain the full, correct solution simply because there is nothing perturbative (small) about this potential for any “nontrivial” value of \( g \). Another way of saying this is to re-emphasize what we have already observed following eq. (39), namely, the character of \( \kappa \) and, therefore, the matching conditions change depending on whether or not \( g \) differs from the special values 0, -1.

To see how the standard results of the harmonic oscillator would emerge from this analysis, let us work out the case only for \( g = 0 \). In this case, the matching conditions for the even solution would lead to the relation for the ratios

\[
-(2\epsilon - 1)R = \frac{\Gamma\left(\frac{1}{2}\right) - (2\epsilon - 1)R \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - \frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2} - \frac{\epsilon}{2}\right) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - \frac{1}{2}\right)}}
\]

which can be satisfied only if

\[
\frac{1}{2} - \frac{\epsilon}{2} = -n, \quad \text{or} \quad \epsilon_n = (2n + 1)
\]

We recognize these to be the even levels of the oscillator (remember the shifted zero point energy of the system in eq. (33) for \( g = 0 \)). Similarly, matching the odd solution and its derivative leads to

\[
R = \frac{\Gamma\left(-\frac{1}{2}\right) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - \frac{1}{2}\right)}}{\Gamma\left(\frac{1}{2} - \frac{\epsilon}{2}\right) - (2\epsilon - 1)R \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - \frac{1}{2}\right)}}
\]

which can be satisfied only if

\[
\frac{1}{2} - \frac{\epsilon}{2} = -n, \quad \text{or} \quad \epsilon_n = 2(n + 1)
\]

These are, of course, the odd energy levels of the harmonic oscillator. (The corresponding wavefunctions are the even and odd Hermite polynomials respectively.) There is no longer any degeneracy
of these levels. In other words, the matching conditions change drastically and so do the solutions of the problem depending on whether or not \( g \) equals one of the special values 0, -1 and the results for the special values cannot be (and, in fact, should not be) obtained from the general result in a limiting manner.

4 Supersymmetric Oscillator with \( \frac{1}{x^2} \) Potential

In this section, we will analyze the supersymmetric version of the case studied in the last section. Let us consider a superpotential of the form [11]

\[
W(x) = \frac{g}{x} - x
\]  

so that the pair of supersymmetric potentials would have the form

\[
V_+ (x) = \frac{1}{2} \left[ \frac{g(g-1)}{x^2} + x^2 - 2g - 1 \right]
\]

\[
V_- (x) = \frac{1}{2} \left[ \frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right]
\]

To analyze this problem, we should, of course, regularize the superpotential. However, even before introducing the regularization, let us observe some general features associated with this system, namely, that this supersymmetric system would have a ground state satisfying

\[
Q \psi_0(x) = 0 \\
\text{or}, \psi_0(x) = \left( \frac{x}{a} \right)^g e^{-\frac{1}{2}(x^2-a^2)} \psi_0(a)
\]  

which would be damped for asymptotically large values of the coordinate. On the other hand, from the behavior of this wave function for small values of \( x \), it is clear that normalizability of the wave function would require that \( g > -\frac{1}{2} \). We also note that since the supersymmetric system involves two Hamiltonians with different \( g \) dependence, the symmetry observed in the previous section, namely, \( g \leftrightarrow -(g+1) \) (or, alternately, \( g \leftrightarrow -(g-1) \)) cannot be a symmetry of the whole system (Another way of saying this is to note that the superpotential has no such symmetry. We will come back to this question later in this section.). Furthermore, we cannot naively take over the results from the previous section since, as we have seen earlier, regularizing the potential directly may not respect supersymmetry.

Therefore, to study this problem systematically, we regularize the superpotential as

\[
W(x) = \theta(x-|R|) \left( \frac{g}{x} - x \right) + \theta(|R|-|x|) \left( \frac{g}{R} - R \right) \frac{x}{R}
\]  

Here, as before, \( R \) is the regularization parameter which should be taken to zero at the end and we have regularized the superpotential such that it is continuous across the boundary. This has
the nice feature that there are no delta potential terms in the potentials. In fact, the regularized superpotential leads to the pair of supersymmetric potentials of the forms

\begin{align*}
V_+(x) &= \frac{1}{2} \left[ \theta(x - |R|) \left( \frac{g(g - 1)}{x^2} + x^2 - 2g - 1 \right) + \theta(R - |x|) \left( \left( \frac{g}{R^2} - 1 \right)^2 x^2 + \left( \frac{g}{R^2} - 1 \right) \right) \right] \\
V_-(x) &= \frac{1}{2} \left[ \theta(x - |R|) \left( \frac{g(g + 1)}{x^2} + x^2 - 2g + 1 \right) + \theta(R - |x|) \left( \left( \frac{g}{R^2} - 1 \right)^2 x^2 - \left( \frac{g}{R^2} - 1 \right) \right) \right] \quad (67)
\end{align*}

It is worth noting here that for \( g = 0 \), the singular potential at the origin is not present in both the Hamiltonians (namely, it is truly not there). On the other hand, for \( g = 1 \) or \(-1\), although the singular potential disappears from only one of the Hamiltonians, the complete system remembers about it through supersymmetry as is clear from the structure of the regularized potentials.

**Spectrum of \( H_+ \)**

Let us now analyze the spectrum of the two different Hamiltonians systematically. First, let us note that, for the Hamiltonian \( H_+ \), the potential is that of a harmonic oscillator in the region \(-R < x < R\). We can write down the even and the odd solutions in this region as [17] (We will assume throughout that \( g \neq 0 \))

\begin{align*}
\psi_+^{(II)}(x) = A_+(R) e^{-\frac{1}{2} \left( \frac{g}{R^2} - 1 \right) x^2} M(- \frac{\epsilon_+ R^2}{g - R^2} + 1, \frac{1}{2}, \frac{g}{R^2} - 1) x^2 \\
\psi_+^{(II)}(x) = B_+(R) x e^{-\frac{1}{2} \left( \frac{g}{R^2} - 1 \right) x^2} M(- \frac{\epsilon_+ R^2}{g - R^2} + 1, \frac{3}{2}, \frac{g}{R^2} - 1) x^2 \quad (68)
\end{align*}

The solution in the region \( x > R \) can also be obtained from an analysis as given in the earlier section and leads to

\begin{align*}
\psi_+^{(I)}(x) = C_+(R) e^{-\frac{1}{2} x^2} \\
\times \left[ \frac{\Gamma(-g + \frac{1}{2})}{\Gamma\left( \frac{1}{2} - g - \frac{\epsilon_+}{2} \right)} x^g M\left( - \frac{\epsilon_+}{2} + \frac{1}{2}, g + \frac{1}{2}, x^2 \right) + \frac{\Gamma\left( g - \frac{1}{2} \right)}{\Gamma\left( -\frac{\epsilon_+}{2} - g + \frac{3}{2} \right)} x^{-g+1} M\left( \frac{1}{2} - g - \frac{\epsilon_+}{2}, -g + \frac{3}{2}, x^2 \right) \right] \quad (69)
\end{align*}

Once again, we can match the solutions in the two regions and their derivatives across \( x = R \) to obtain the relevant quantization conditions. As before, there are two possibilities.

(i) **Even Solution**

If we match the even solution in eq. (68) and its derivative with those of eq. (69) at \( x = R \) and take the ratio, we obtain to leading order in \( R \) (remember \( R \) is small and is to be taken to zero at
Clearly, this can be satisfied only if \((n = 0, 1, 2, \cdots)\)

\[- \frac{\epsilon_+}{2} = -n, \quad \text{or,} \quad \epsilon_+, n = 2n\]  

(71)

Thus, we obtain that there exists an even extension of the solution of the form (when the regularization is removed)

\[
\psi_{+, n}(x) = C_{+, n} \frac{\Gamma(-g + \frac{1}{2})}{\Gamma(-g + \frac{1}{2} - n)} \left\{ \begin{array}{ll}
    x^g & \text{for } x > 0 \\
    -|x|^g & \text{for } x < 0 
\end{array} \right.
\]  

(72)

with the energy levels given by

\[
\epsilon_+, n = 2n
\]  

(73)

Furthermore, we note from equation (72) that normalizability of the solution restricts that this solution is physical only for \(g > -\frac{1}{2}\).

(ii) Odd Solution

We can also match the odd solution in eq. (68) and its derivative with those of eq. (69) and the ratio leads to the appropriate quantization conditions. Without going into technical details, let us simply note here that there are two possibilities in this case. For \(g > \frac{1}{2}\), we have in the limit \(R \rightarrow 0\)

\[
\epsilon_+, n = 2n
\]  

(74)

with

\[
\psi_{+, n}(x) = C_{+, n} \frac{\Gamma(-g + \frac{1}{2})}{\Gamma(-g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g + \frac{1}{2}, x^2) \left\{ \begin{array}{ll}
    x^g & \text{for } x > 0 \\
    -|x|^g & \text{for } x < 0 
\end{array} \right.
\]  

(75)

On the other hand, for \(g < \frac{1}{2}\),

\[
\epsilon_+, n = 2n - 2g + 1
\]  

(76)

with

\[
\psi_{+, n}(x) = C_{+, n} \frac{\Gamma(g - \frac{1}{2})}{\Gamma(g - \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g - \frac{3}{2}, x^2) \left\{ \begin{array}{ll}
    x^{g-1} & \text{for } x > 0 \\
    -|x|^{g-1} & \text{for } x < 0 
\end{array} \right.
\]  

(77)

This completes the analysis of the spectrum for \(H_+\). We see that it consists of a set of even solutions with vanishing energy as we would expect from supersymmetry, but it also contains additional physical solutions.
Spectrum of $H_-$

We can similarly analyze the spectrum for the supersymmetric partner Hamiltonian $H_-$. The even and the odd solutions in the region $-R < x < R$ have the forms

$$
\psi_-(x) = A_-(R) e^{-\frac{1}{2}(\frac{g}{R^2} - 1)x^2} M(-\frac{\epsilon_- R^2}{g - R^2}, \frac{1}{2}, (\frac{g}{R^2} - 1)x^2)
$$

$$
\psi_-(x) = B_-(R) x e^{-\frac{1}{2}(\frac{g}{R^2} - 1)x^2} M(-\frac{\epsilon_- R^2}{g - R^2}, \frac{3}{2}, (\frac{g}{R^2} - 1)x^2)
$$

while the solution in the region $x > R$ now takes the form

$$
\psi_+(x) = C_-(R) e^{-\frac{1}{2}x^2} \times \left[ \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{3}{2} - n)} x^{g+1} M(1 - \frac{\epsilon_-}{2}, g + \frac{3}{2}, x^2) + \frac{\Gamma(g + \frac{1}{2})}{\Gamma(1 - \frac{\epsilon_-}{2})} x^{-g} M(\frac{1}{2} - g - \frac{\epsilon_-}{2}, g + 1, x^2) \right]
$$

The matching of the solutions and their derivatives can now be done in the standard manner and we simply give the results in the limit $R \to 0$ here. There exists a normalizable even solution only when $g < \frac{1}{2}$ of the form

$$
\psi_{-n}(x) = C_{-n} \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + \frac{1}{2}, x^2) \left\{ \begin{array}{ll} x^{-g} & \text{for } x > 0 \\ |x|^{-g} & \text{for } x < 0 \end{array} \right.
$$

with

$$
\epsilon_{-n} = 2n - 2g + 1
$$

On the other hand, there are two possible odd solutions. For $g > -\frac{1}{2}$, we have

$$
\psi_{-n}(x) = C_{-n} \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{3}{2} - n)} x^{g+1} M(-n, g + \frac{3}{2}, x^2) \left\{ \begin{array}{ll} x^{-g} & \text{for } x > 0 \\ -|x|^{g+1} & \text{for } x < 0 \end{array} \right.
$$

with

$$
\epsilon_{-n} = 2(n + 1)
$$

while, for $g < -\frac{1}{2}$, we have

$$
\psi_{-n}(x) = C_{-n} \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + \frac{1}{2}, x^2) \left\{ \begin{array}{ll} x^{-g} & \text{for } x > 0 \\ -|x|^{-g} & \text{for } x < 0 \end{array} \right.
$$

with

$$
\epsilon_{-n} = 2n - 2g + 1
$$

This completes the determination of the spectrum of the supersymmetric pair of Hamiltonians, $H_+$ and $H_-$. We note that we have many more states than would be needed from the point of view of supersymmetry and, therefore, it is crucial to understand the solutions in a systematic manner which we do next.
Analysis of the Result

The large number of solutions obtained is really very interesting and to appreciate their presence, let us analyze their behavior in some detail. First, let us note that (as we had pointed out in eq. (65) and in the subsequent discussion) a supersymmetric ground state would exist for our system only if \( g > -\frac{1}{2} \). Indeed, there exists such a branch of the solutions in our model, namely, the even solutions of \( \hat{H}_+ \) and one of the two sets of odd solutions of \( \hat{H}_- \). They are, in fact, degenerate in energy except for the ground state which is an even state with vanishing energy belonging to the spectrum of \( \hat{H}_+ \). The solutions have the relative odd parity appropriate to be superpartner states and, in fact, it is not hard to check using the properties of the confluent hypergeometric functions [17], that, for \( g > -\frac{1}{2} \),

\[
Q^\dagger(g)\psi^-_{-,n}(x) = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \frac{g}{x} - x \right) \psi^-_{-,n}(x) = -\sqrt{2} \frac{C_{-,n}}{C_{+,n+1}} \psi^\even_{+,n+1}(x)
\]

as we would expect from states which are superpartners of each other. This shows that the system of solutions, at least, contains the supersymmetric set that we are interested in.

This, therefore, raises the question about the roles of the other solutions that we have found. Both \( \hat{H}_+ \) and \( \hat{H}_- \) have solutions on another common branch, namely, for \( g < \frac{1}{2} \) and it is worth investigating whether, they, too, define a supersymmetric set of solutions. They are degenerate in energy (see eqs. (76) and (81)) and they have the correct relative odd parity structure (solutions of \( \hat{H}_+ \) are odd while those for \( \hat{H}_- \) are even). However, it is easily verified that

\[
Q^\dagger(g)\psi^\even_{-,n}(x) \neq \psi^\odd_{+,n+1}(x)
\]

even up to normalization constants. On the other hand, it can also be checked equally easily that

\[
Q^\dagger(-g)\psi^\odd_{+,n}(x) = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} - \frac{g}{x} - x \right) \psi^\odd_{+,n}(x) = -\sqrt{2} \frac{C_{+,n}}{C_{-,n+1}} \psi^\even_{-,n+1}(x)
\]

Thus, there appears to be some sort of supersymmetry for these states, but there is no apparent state of vanishing energy. Furthermore, there is, of course, the third set of solutions of \( \hat{H}_+ \) and \( \hat{H}_- \) which do not even share the same branch.

The meaning of all these solutions becomes quite clear once we realize that the supersymmetrization of a given Hamiltonian, in this case, is not unique. There are, in fact, various possibilities. For example,

(i) we can choose as the superpotential

\[
W(x) = \frac{g}{x} - x
\]
as we have done so that the supersymmetric pair potentials would be

\[ V_+(x) = \frac{1}{2} \left[ \frac{g(g-1)}{x^2} + x^2 - 2g - 1 \right] \]

\[ V_-(x) = \frac{1}{2} \left[ \frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right] \]  (90)

Here, as we have already seen, a normalizable ground state would exist only for \( g > -\frac{1}{2} \) (The ground state belongs to \( H_+ \)).

(ii) If, on the other hand, we choose

\[ W(x) = \frac{g-1}{x} - x \]  (91)

the supersymmetric pair of potentials become

\[ V_+(x) = \frac{1}{2} \left[ \frac{(g-1)(g-2)}{x^2} + x^2 - 2g + 1 \right] \]

\[ V_-(x) = \frac{1}{2} \left[ \frac{g(g-1)}{x^2} + x^2 - 2g + 3 \right] \]  (92)

where a normalizable ground state would exist only if \( g > \frac{1}{2} \).

(iii) Similarly, we could have chosen

\[ W(x) = -\frac{g+1}{x} - x \]  (93)

which would lead to a pair of supersymmetric potentials

\[ V_+(x) = \frac{1}{2} \left[ \frac{(g+1)(g+2)}{x^2} + x^2 + 2g + 1 \right] \]

\[ V_-(x) = \frac{1}{2} \left[ \frac{g(g+1)}{x^2} + x^2 + 2g + 3 \right] \]  (94)

where a normalizable ground state would exist only for \( g < -\frac{1}{2} \).

(iv) However, there is also the possibility of choosing

\[ W(x) = -\frac{g}{x} - x \]  (95)

In some sense, this is the generalization of the idea that a spin \( \frac{1}{2} \) particle, say for example, can belong to distinct multiplets of the form \((0, \frac{1}{2})\) or \((\frac{1}{2}, 1)\). (As is clear now, the transformation \( g \leftrightarrow -(g+1) \) or \( g \leftrightarrow -(g-1) \) alluded to earlier would simply take one supersymmetric system to another and cannot be a symmetry of one supersymmetric system.)
giving the supersymmetric potentials

\begin{align*}
V_+(x) &= \frac{1}{2} \left[ \frac{g(g+1)}{x^2} + x^2 + 2g - 1 \right] \\
V_-(x) &= \frac{1}{2} \left[ \frac{g(g-1)}{x^2} + x^2 + 2g + 1 \right]
\end{align*}

(96)

Namely, \( H_+ \) and \( H_- \) could also have reversed their roles (This basically determines which potential contains the ground state). In this case, a normalizable ground state would exist only for \( g < \frac{1}{2} \).

The meaning of all the solutions is clear now. First, the solutions corresponding to the branch \( g > -\frac{1}{2} \) define the supersymmetric solutions corresponding to the supersymmetrization in eq. (89). The solutions corresponding to the branch \( g < \frac{1}{2} \) also define a supersymmetric set of solutions corresponding to the supersymmetrization in eq. (95). (It is clear now why the solutions were related to each other as in eq. (88).) The fact that the spectrum contains a zero energy state, as we would expect in a supersymmetric system, follows from the fact that the potentials in (96) are shifted with respect to those in (90). Finally, the other solutions that do not even share the same branch correspond to the supersymmetrizations in eqs. (91) and (93). Supersymmetry is manifest in spite of the singular behavior of the potentials. It is also clear from this analysis that distinct solutions really correspond to distinct supersymmetrizations and in deriving conclusions regarding supersymmetry, one should be very careful in identifying the correct solutions from among the whole set.

5 Algebraic Solution

The supersymmetric system studied in the previous section is a very special system which can also be solved algebraically. Therefore, to further understand the properties of this system, we present here a short derivation of the algebraic solution of this system.

Let us consider the supersymmetric system of potentials in eq. (64). Defining the supercharges as in (3), namely,

\begin{align*}
Q(g) &= \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + \frac{g}{x} - x \right) \\
Q^\dagger(g) &= \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \frac{g}{x} - x \right)
\end{align*}

(97)

we can write the supersymmetric pair of Hamiltonians also as

\begin{align*}
H_+(g) &= Q^\dagger(g)Q(g) \\
H_-(g) &= Q(g)Q^\dagger(g)
\end{align*}

(98)
However, from the structure of the potentials in eq. (64), it is clear that we can write

\[ H_g = Q(g)Q(g) = Q(g + 1)Q(g + 1) + 2 = H_{g+1} + 2 \]  

(99)

Such a Hamiltonian, \( H_{g+1} \) (equivalently, the potential) is known as shape invariant [18,12]. The ground state of \( H_{g+1} \) satisfies

\[ Q(g)\psi_{g,0}(g,x) = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \frac{g}{x} - x \right) \psi_{g,0}(x) = 0 \]

or,

\[ \psi_{g,0}(g,x) \approx x^g e^{-\frac{1}{2}x^2} \]  

(100)

This solution is normalizable and physical only if

\[ g > -\frac{1}{2} \]  

(101)

Furthermore, the ground state energy of \( H_{g+1} \), in such a case, is zero,

\[ \epsilon_{g,0} = 0 \]  

(102)

Since \( H_{g+1} \) and \( H_g \) are supersymmetric partners and share all the energy levels except for the ground state of \( H_{g+1} \), it is clear that the ground state energy of \( H_g \) would correspond to the first excited state energy of \( H_{g+1} \) and from the relations in eqs. (99) and (102), it follows, therefore, that the energy of the first excited state of \( H_{g+1} \) is

\[ \epsilon_{g+1} = 2 \]  

(103)

Let us note here that the ground state wavefunction of \( H_g \) (from the structure in eq. (99)) would follow to have the form

\[ \psi_{g,0}(g + 1, x) \approx x^{g+1} e^{-\frac{1}{2}x^2} \]  

(104)

and would be normalizable only if

\[ g > -\frac{3}{2} \]

and this would automatically hold if eq. (101) is true.

We can, in fact, construct a sequence of Hamiltonians, in such a case, of the form

\[ H_0 = H_g = Q(g)Q(g) \]

\[ H_1 = H_{g+1} = Q(g)Q(g) = Q(g+1)Q(g+1) + 2 \]

\[ H_2 = Q(g+1)Q(g+1) + 2 = Q(g+2)Q(g+2) + 4 \]

\[ \vdots \]

\[ H_s = Q(g+s)Q(g+s) + 2s \]  

(105)

From the structure of the Hamiltonians, it is clear that the ground state of any Hamiltonian in the sequence would be normalizable if eq. (101) holds. Furthermore, every adjacent pair of Hamiltonians, \( H_s \) and \( H_{s-1} \) would define a supersymmetric pair of Hamiltonians and share all
the energy levels except for the ground state of $H^{(s-1)}$ which would have an energy eigenvalue $2(s-1)$. The energy levels of our original pair of Hamiltonians, $H_+(g)$ and $H_-(g)$ now follow from this to be

$$\epsilon_{+,n} = 2n, \quad \epsilon_{-,n} = 2(n+1)$$

(106)

These are of course, the energy levels obtained in our earlier analysis (see eqs. (71) and (83)).

The wave functions for the original Hamiltonians can also be obtained from an analysis of the spectrum of this sequence of Hamiltonians. It is easily seen that [19]

$$\psi_{+,n}(g,x) \propto Q^\dagger(g)\psi_{+,n-1}(g+1,x)$$

(107)

which can be iterated to give

$$\psi_{+,n}(g,x) \propto Q^\dagger(g)Q^\dagger(g+1)\cdots Q^\dagger(g+n-1)\psi_{+,0}(g+n,x)$$

(108)

And since, we know the ground state wave function, $\psi_{+,0}$ (see eq. (100)), all the wave functions for the higher states can be explicitly constructed. Furthermore, once we know $\psi_{+,n}(g,x)$, the superpartner states can also be obtained from the relation that (see (5) and the discussion following)

$$\psi_{-,n}(g,x) \propto Q(g)\psi_{+,n}(g,x)$$

(109)

It is worth asking if these algebraically constructed states also coincide with the states we have constructed in eqs. (72) and (82) respectively. It is easy to verify the relation that

$$\left(\frac{d}{dx} + \frac{g}{x} - x\right) e^{-\frac{1}{2}x^2} x^{g+1} M(-n+1,g+\frac{3}{2},x^2) = (2g+1) e^{-\frac{1}{2}x^2} x^g M(-n,g+\frac{1}{2},x^2)$$

(110)

which is the appropriate recursion relation in eq. (107) for our system. Furthermore, noting the form of the ground state wave function in eq. (100)

$$e^{-\frac{1}{2}x^2} x^g = e^{-\frac{1}{2}x^2} x^g M(0,g+\frac{1}{2},x^2)$$

(111)

and choosing an even function for the ground state, as we normally do, (as far as we can see, the algebraic method cannot determine \textit{a priori} whether the function should be even/odd), it is clear that all the states associated with $H_+(g)$ would be even states and would coincide with those in eq. (72). Once these are identified, the superpartner states also follow to coincide with those in eq. (82) and are odd. This concludes the algebraic construction of the supersymmetric spectrum of the system and shows again that supersymmetry is manifest.

\section{The Puzzle}

An interesting puzzle has been raised in the literature [11] in connection with the supersymmetric system of eq. (64) and in this section, we describe the resolution of the puzzle from a systematic
analysis of the problem. Namely, let us look at the pair of supersymmetric potentials in eq. (64)

\[
V_+(x) = \frac{1}{2} \left[ \frac{g(g - 1)}{x^2} + x^2 - 2g - 1 \right]
\]

\[
V_-(x) = \frac{1}{2} \left[ \frac{g(g + 1)}{x^2} + x^2 - 2g + 1 \right]
\]

and note that, for \( g = 1 \), the potentials take the form

\[
V_+(x) = \frac{1}{2} \left( x^2 - 3 \right)
\]

\[
V_-(x) = \frac{1}{2} \left( \frac{2}{x^2} + x^2 - 1 \right)
\]

(112)

As we have noted earlier, \( g = 1 \) is not particularly a special point in the parameter space (only \( g = 0 \) is) and, consequently, our analysis would continue to hold for this value of the coupling as well. Furthermore, this value of the coupling is consistent with \( g > -\frac{1}{2} \) and, therefore, we would expect the spectrum to be supersymmetric with the ground state energy of \( H_+(g = 1) \) vanishing as the analysis of the last two sections have shown. On the other hand, if we look at the potential \( V_+ \) in eq. (112), it is clear that it is a harmonic oscillator potential with the zero point energy shifted by \( -\frac{3}{2} \). It would appear, therefore, that the ground state energy of \( H_+(g = 1) \) would be negative contrary to what our analysis has shown and in violation of the theorems of supersymmetry (The supersymmetry algebra tells us that the ground state energy in a supersymmetric theory has to be positive semi-definite.). This, therefore, raises an interesting puzzle and it has been discussed in the literature from various points of view. However, the resolution of this puzzle is really quite simple as we show next.

In concluding that the potential \( V_+ \) in eq. (112) would lead to a negative energy, we have, of course, tacitly assumed the ground state wave function for the system to be the standard oscillator wavefunction, namely,

\[
\psi_0(x) \propto e^{-\frac{1}{2}x^2}
\]  

(113)

which, as it should be, is an even function. On the other hand, from our analysis of the last two sections (see eq. (72) or (100)), we would conclude that the ground state wave function for our supersymmetric system would have the form

\[
\psi_+(x) \propto e^{-\frac{1}{2}x^2} |x|
\]  

(114)

It can also be explicitly checked that this indeed has a vanishing energy consistent with our general analysis. Therefore, understanding the question of the negative energy reduces to understanding the role of the wave function in eq. (113). It is, of course, clear from the explicit spectrum of the system we have constructed that it is not part of our supersymmetric system. It must, therefore, belong to a different supersymmetric system since as we have seen earlier, distinct supersymmetrizations lead to distinct forms of the wave functions.

In fact, let us choose

\[
\tilde{W}(x) = -\frac{g - 1}{x} - x
\]  

(115)
which would lead to the pair of supersymmetric potentials of the forms

\[
\tilde{V}_+(x) = \frac{1}{2} \left[ \frac{g(g - 1)}{x^2} + x^2 + 2g - 3 \right] \\
\tilde{V}_-(x) = \frac{1}{2} \left[ \frac{(g-1)(g-2)}{x^2} + x^2 + 2g - 1 \right]
\] (116)

It is clear that this supersymmetric system would have a normalizable, even ground state of the form

\[
\tilde{\psi}_0(x) \propto |x|^{1-g} e^{-\frac{1}{2}x^2}
\] (117)

which is normalizable for \( g < \frac{3}{2} \). It is also clear that this wave function coincides with (113) when \( g = 1 \) (which is in its range of validity). In other words, the wave function in eq. (113) does not belong to the supersymmetric spectrum of (64), rather, it is a legitimate ground state of the supersymmetrization in eq. (115). For \( g = 1 \) the potentials of (116), of course, have the form

\[
\tilde{V}_+(x) = \frac{1}{2} \left[ x^2 - 1 \right] \\
\tilde{V}_-(x) = \frac{1}{2} \left[ x^2 + 1 \right]
\] (118)

which is nothing other than the supersymmetric oscillator which, as we know, has a zero energy ground state.

Thus, there is really no puzzle and the theorems of supersymmetry are, indeed, on firm footing. The crucial thing to learn from this analysis is what we have emphasized earlier, namely, in dealing with supersymmetry in such a system it is essential to identify the appropriate solution relevant for a particular supersymmetric system because every distinct solution corresponds to a distinct supersymmetrization of the system.

### 7 Conclusion

In this paper, we have studied systematically two classes of supersymmetric quantum mechanical models - one consisting of a singular boundary and the other with a singular potential. We have shown that, contrary to the conventional understanding [10-12], supersymmetry is manifest in these systems. In particular, for a system with a singular potential such as \( \frac{1}{x^2} \), we have shown that the solution of the Schrödinger equation leads to several distinct solutions corresponding to distinct supersymmetrizations of the system. Consequently, it becomes quite important to identify the appropriate wavefunctions when supersymmetric properties are being investigated. We have also solved the system with a singular potential algebraically using the ideas of shape invariance which further corroborates the results of our direct analysis of the Schrödinger equation, namely, that supersymmetry is manifest in this system. Furthermore, we have shown how a puzzle raised in the literature finds a natural resolution when one does a careful analysis restricting oneself to the appropriate branch of the supersymmetric solutions. Finally, we would like to conclude by noting
that supersymmetry is known to be robust at short distances (high energies). The singularities discussed in the quantum mechanical models occur at short distances and, therefore, it is intuitively quite clear that they are unlikely to break supersymmetry. Our detailed, systematic analysis only reinforces this.

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