REGULARITY CONDITIONS FOR VECTORIAL STATIONARY PROCESSES

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1. Vectorial stationary processes

In this paper we study regularity conditions of vectorial stationary processes. We simplify the approach given in [P3] and improve the results of [P3].

By a *vectorial stationary process* (in the wide sense) with discrete time we mean a sequence

$$\{x(t)\}_{t \in \mathbb{Z}}, \quad x(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}^t \quad (1.1)$$

of columns of real random variables such that $\mathbb{E}|x_j(t)|^2 < \infty$ and the *covariance matrix* $Q_{s,t}$,

$$Q_{s,t} = \{\mathbb{E}x_{j_1}(s)x_{j_2}(t)\}_{1 \leq j_1, j_2 \leq n} \quad (1.2)$$

depends only on $s - t$ ($\mathbb{E}$ stands for mathematical expectation). The most important class of stationary processes consists of *Gaussian processes* since the covariance matrix of Gaussian variables determines their mutual distribution. We refer the reader to [R], [IR], [DM], and [PK] for general information about stationary processes.

With the process (1.1) we associate the subspaces $\mathcal{X}_0$ (past) and $\mathcal{X}^m$ (future starting at moment m) of the space L^2 with respect to the probability measure defined by

$$\begin{aligned} \mathcal{X}_0 &\stackrel{\text{def}}{=} \text{span}\{x_j(t) : t < 0, 1 \leq j \leq n\}, \\ \mathcal{X}^m &\stackrel{\text{def}}{=} \text{span}\{x_j(t) : t \geq m, 1 \leq j \leq n\}, \end{aligned}$$

where “span” means closed linear span.

It is well known that for a stationary process with covariance matrix given by (1.2) there exists a matrix-valued positive measure M on the unit circle $\mathbb{T}$ whose Fourier coefficients satisfy

$$Q_{s,t} = \hat{M}(s - t), \quad s, t \in \mathbb{Z}.$$

The measure M is called the *spectral measure* of the process (1.1).

The process (1.1) is called *regular* if $\bigcap_{m \geq 0} \mathcal{X}^m = \{\mathbb{O}\}$. In this case M is absolutely continuous with respect to Lebesgue measure (see [R]). The density W of

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M with respect to Lebesgue measure is called the *spectral density* of the process. It is also well known (see [R]) that $\text{rank } W(\zeta)$ is constant for almost all $\zeta \in \mathbb{T}$, it is called the *rank of the process*. If the rank is equal to n , the process is said to be of *full rank*. The process is regular and has full rank if and only if $\log \det W \in L^1$ (see [R]).

In this paper we consider only processes of full rank. We refer the reader to [I2] where the study of certain regularity conditions for general processes is reduced to the case of processes of full rank.

Let W be the density of the process (1.1). We identify the space $\text{span}\{x_j(t) : t \in \mathbb{Z}, 1 \leq j \leq n\}$ with the space $L^2(W)$ of $\mathbb{C}^n$ -valued functions in the following way:

$$x_j(t) \mapsto \begin{pmatrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \\ z^t \\ \mathbb{O} \\ \vdots \\ \mathbb{O} \end{pmatrix}$$

(z^t stands in the j th position). It is easy to see that this is a unitary map. It transforms $\mathcal{X}_0$ onto $H_-^2(W)$ and $\mathcal{X}^m$ onto $z^m H^2(W)$, where

$$H^2(W) \stackrel{\text{def}}{=} \text{span}\{z^t c : t \geq 0, c \in \mathbb{C}^n\},$$

$$H_-^2(W) \stackrel{\text{def}}{=} \text{span}\{z^t c : t < 0, c \in \mathbb{C}^n\}.$$

We denote by $\mathcal{P}_0$ and $\mathcal{P}^m$ the orthogonal projections from $L^2(W)$ onto $H_-^2(W)$ and $z^m H^2(W)$.

It is well known (see [WM], [R]) that if W is the density of a regular process of full rank, then there exist outer matrix functions Ψ and $\Psi_\#$ in the Hardy class H^2 such that

$$W = \Psi^* \Psi = \Psi_\# \Psi_\#^*. \quad (1.3)$$

The matrix function Ψ is unique modulo a left unitary constant factor while the function $\Psi_\#$ is unique modulo a right unitary constant factor.

Recall that an $n \times n$ matrix function Ψ of class H^2 is called *outer* if

$$\text{span}\{z^m \Psi d : m \in \mathbb{Z}_+, d \in \mathbb{C}^n\} = H^2(\mathbb{C}^n).$$

A matrix function Θ is called *inner* if $\Theta^*(\zeta)\Theta(\zeta)$ is the identity matrix almost everywhere on $\mathbb{T}$.

Throughout this paper we adopt the following convention. If X is a space of scalar functions, then for a vector (or matrix) function F we say $F \in X$ if all entries of F are in X .

With the factorizations (1.3) we associate the *phase function* $U \stackrel{\text{def}}{=} \Psi_{\#}^* \Psi^{-1}$. It follows easily from (1.3) that U is a unitary-valued function. The phase function U is uniquely determined by W modulo left and right unitary constant factors.

2. Regularity conditions

We are going to study different regularity conditions for stationary processes, i.e., conditions of weak dependence of the future on the past.

For $m \in \mathbb{Z}_+$ we define the numbers

$$\rho_m(W) \stackrel{\text{def}}{=} \|\mathcal{P}_0 \mathcal{P}^m\|.$$

Clearly, $\rho_m(W)$ is the cosine of the angle between $H_-^2(W)$ and $z^m H^2(W)$.

Definition. The process is called *completely regular* if

$$\lim_{m \rightarrow \infty} \rho_m(W) = 0. \quad (2.1)$$

For Gaussian processes condition (2.1) is equivalent to the condition

$$\lim_{m \rightarrow \infty} \sup\{|P(A \cap B) - P(A)P(B)| = 0 : A \in \Sigma_0, B \in \Sigma^m\} = 0,$$

where Σ_0 is the σ -algebra of events measurable with respect to $\mathcal{X}_0$, Σ^m is the σ -algebra of events measurable with respect to $\mathcal{X}^m$, and P is the probability measure [KR] (see also [PK]).

If the process satisfies the following stronger regularity condition

$$\rho_m(W) \leq \text{const}(1 + m)^{-\alpha}, \quad \alpha > 0,$$

we call it *completely regular* of order α .

In [Y] an important notion of canonical correlations of two subspaces of a Gaussian space was introduced. If G_1 and G_2 are such subspaces, the sequence of *canonical correlations* $\lambda_k(G_1, G_2)$ can be defined as the sequence of the singular values $s_k(\mathcal{P}_{G_1} \mathcal{P}_{G_2})$ of the product of the orthogonal projections onto G_1 and G_2 .

Definition. Let $0 < p < \infty$. A Gaussian process is called *p-regular* if the sequence $\lambda_k(\mathcal{X}_0, \mathcal{X}^0)$ belongs to the sequence space ℓ^p .

In other words, a process is *p-regular* if $\mathcal{P}_0 \mathcal{P}^0$ belongs to the Schatten-von Neumann class $\mathcal{S}_p$.

Recall that a Gaussian process is 2-regular if and only if it is *absolutely regular*, i.e.,

$$\lim_{m \rightarrow \infty} \mathbb{E} \sup_{A \in \Sigma^m} \|\mathbb{E}_{\Sigma_0}(\chi_A - P(A))\|^2 = 0,$$

where $\mathbb{E}_{\Sigma_0}$ is conditional expectation with respect to Σ_0 [IR] (see also [PK]).

Note also that a Gaussian process is 2-regular if and only if it is informationally regular (see [GY] for the definition and the proof).

Finally, we say that a process is *p-regular of order α* if

$$\|\mathcal{P}_0\mathcal{P}^m\|_{\mathcal{S}_p} \leq \text{const}(1+m)^{-\alpha}.$$

In the scalar case the completely regular processes were described in [HS] and [Sa]: the process is completely regular if and only if its spectral density w admits a representation $w = |q|^2 e^\varphi$, where q is a polynomial whose zeros are on $\mathbb{T}$ and φ is a real function in the space VMO of functions of vanishing mean oscillation.

The process is α -regular if and only if $w = |q|^2 e^\varphi$, where q is a polynomial whose zeros are on $\mathbb{T}$ and φ is a real function in the Hölder–Zygmund class Λ_α . This result was obtained in [I1].

The p -regular processes were described in [IR] for $p = 2$, in [P1] for $1 \leq p < \infty$ and in [P2] for $0 < p < 1$. Namely, the process is p -regular if and only if $w = |q|^2 e^\varphi$, where q is a polynomial whose zeros are on $\mathbb{T}$ and φ is a real function in the Besov space $B_p^{1/p}$ (see also [PK]).

Note that for different regularity conditions the proofs were different and some of them very complicated. In [PK] a unified approach was found which allowed the authors to find a single method to prove the above results. That approach is based on Hankel operators. In addition to that it was shown in [PK] that the process is p -regular of order α if and only if $w = |q|^2 e^\varphi$, where q is a polynomial whose zeros are on $\mathbb{T}$ and φ is a real function in the Besov space $B_{p\infty}^{1/p+\alpha}$.

Let us return now to the case of vectorial processes.

It was shown in [P3] that

$$\|\mathcal{P}_0\mathcal{P}^m\| = \|H_{z^m U}\| \quad \text{and} \quad \|\mathcal{P}_0\mathcal{P}^m\|_{\mathcal{S}_p} = \|H_{z^m U}\|_{\mathcal{S}_p}, \quad 0 < p < \infty.$$

Recall that for a function $\Phi \in L^\infty(\mathbb{M}_{n,n})$ taking values in the space $\mathbb{M}_{n,n}$ of $n \times n$ matrices the *Hankel operator* $H_\Phi : H^2(\mathbb{C}^n) \rightarrow H_-^2(\mathbb{C}^n) = L^2(\mathbb{C}^n) \ominus H^2(\mathbb{C}^n)$ is defined by

$$H_\Phi F = \mathbb{P}_- \Phi F,$$

where $\mathbb{P}_-$ is the orthogonal projection of $L^2(\mathbb{C}^n)$ onto $H_-^2(\mathbb{C}^n)$.

The *Toeplitz operator* $T_\Phi : H^2(\mathbb{C}^n) \rightarrow H^2(\mathbb{C}^n)$ is defined by

$$T_\Phi F = \mathbb{P}_+ \Phi F,$$

where $\mathbb{P}_+$ is the orthogonal projection of $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$.

As in the scalar case (see [PK]) the above regularity conditions can be expressed in terms of $\mathbb{P}_- U$, where U is a phase function of the process. Namely, the process is completely regular if and only if $\mathbb{P}_- U \in VMO$; the process is completely regular of order α if and only if $\mathbb{P}_- U \in \Lambda_\alpha$; the process is p -regular if and only if $\mathbb{P}_- U \in B_p^{1/p}$ (this follows from the fact that $H_\Phi \in \mathcal{S}_p$ if and only if $\mathbb{P}_- \Phi \in B_p^{1/p}$ which was

established in [P1] for $1 \leq p < \infty$, and in [P2] and [Se] for $0 < p < 1$); and the process is p -regular of order α if and only if $\mathbb{P}_-U \in B_{p,\infty}^{1/p+\alpha}$.

Let us introduce a more general notion. For a space X of functions on $\mathbb{T}$ we say that a process is X -regular if $\mathbb{P}_-U \in X$.

Completely regular vectorial process of order α were described in [I2]: these are processes whose weights can be represented as $W = Q^*W_1Q$, where Q is a polynomial such that the zeros of $\det Q$ are on $\mathbb{T}$ and W_1 is a matrix function invertible in Λ_α . Note that the proof given in [I2] is very complicated.

In [P3] Hankel operators were used to study regularity conditions for vectorial processes. In particular, a simple proof was found in [P3] to characterize completely regular processes of order α . The study of completely regular processes was reduced in [P3] to the case of so-called minimal processes.

Recall that the process (1.1) is called *minimal* if for each $t \in \mathbb{Z}$ and each j , $1 \leq j \leq n$, the function $x_j(t)$ does not belong to the closed linear span of the remaining functions, i.e.,

$$x_j(t) \notin \text{span}\{x_k(s) : s \neq t \text{ or } k \neq j\}.$$

In the scalar case the minimal processes were characterized by Kolmogorov [K] who proved that the process is minimal if and only if the spectral density w satisfies the condition $w^{-1} \in L^1$. It is easy to prove the same result for vectorial processes.

Theorem 2.1. *Let W be the spectral density of a vectorial stationary process. Then the process is minimal if and only if $W^{-1} \in L^1$.*

This fact must be familiar to experts. Its proof is similar to the proof in the scalar case and we omit it.

Suppose that W is the density of a completely regular stationary process with phase function U determined by factorizations (1.3). Consider the Toeplitz operator T_U on $H^2(\mathbb{C}^n)$. It was shown in [P3] that T_U has dense range in $H^2(\mathbb{C}^n)$. It was also proved in [P3] that the process with the density W is minimal if and only if $\text{Ker } T_U = \{0\}$ (see [P3]).

In the next section we essentially improve and simplify results of [P3] and reduce the study of X -regular processes to the case of minimal processes for a large class of function spaces X .

We consider in this paper two classes of function spaces X . The first class consists of so-called $\mathcal{R}$ -spaces. We refer the reader to [PK] where a formal definition is given. Roughly speaking X is an $\mathcal{R}$ -space if one can determine whether a function φ belongs to X in terms of the singular values of the Hankel operators H_φ and $H_{\bar{\varphi}}$. For us the most important examples of $\mathcal{R}$ -spaces are VMO and $B_p^{1/p}$ ($\varphi \in VMO$ if and only if H_φ and $H_{\bar{\varphi}}$ are compact; $\varphi \in B_p^{1/p}$ if and only if H_φ and $H_{\bar{\varphi}}$ belong to the Schatten-von Neumann class S_p).

The second class consists of spaces $X \subset C(\mathbb{T})$ satisfying the following axioms:

- (A1) If $f \in X$, then $\bar{f} \in X$ and $\mathbb{P}_+ f \in X$;
- (A2) X is a Banach algebra with respect to pointwise multiplication;
- (A3) for every $\varphi \in X$ the Hankel operator H_φ is a compact operator from X_+ to X_- ;
- (A4) if $f \in X$ and f does not vanish on $\mathbb{T}$, then $1/f \in X$.

Here for a function space $X \subset L^2$ we use the notation

$$X_+ = X \cap H^2; \quad X_- = X \cap H_-^2.$$

If (X) satisfies (A1), (A2), (A4) and the trigonometric polynomials are dense in X , then X also satisfies (A3). However, (A3) is satisfied for many nonseparable Banach algebras X (see [P4] for details where a general sufficient condition for a space X to satisfy (A3) is given). In particular, the spaces Λ_α , $0 < \alpha < \infty$, $B_{p,\infty}^{1/p+\alpha}$, $1 \leq p < \infty$, $0 < \alpha < \infty$, satisfy (A1)–(A4).

In [P3] it was shown that if U is a unitary-valued matrix function such that the Toeplitz operator $T_U : H^2(\mathbb{C}^n) \rightarrow H^2(\mathbb{C}^n)$ has dense range and $\mathbb{P}_- U \in X$ then $U \in X$ for any $\mathcal{R}$ -space X .

In [P4] the same was proved for any function space X satisfying (A1)–(A4). Strictly speaking in [P4] this was proved for spaces X in which the trigonometric polynomials are dense and for nonseparable Banach algebras satisfying certain natural axioms. However, recently R. Alexeev observed that the reasoning in [P4] works for spaces satisfying the above axioms (A1)–(A4).

Those results allow us to conclude the following.

Theorem 2.2. *Let X be an $\mathcal{R}$ -space or X be a space satisfying (A1)–(A4). Suppose that U is a phase function of a vectorial stationary process. Then the process is X -regular if and only if $U \in X$.*

In §4 we see that for spaces X satisfying (A1)–(A4) the X -regular processes can be characterized in the following way which is similar to the scalar case. Namely, the process is X -regular if and only if its density W can be represented as $W = Q^* W_1 Q$, where Q is a polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and $\log W_1 \in X$.

However, it turns out that for completely regular processes the situation is different. In [TV] it was shown that there exists a function matrix Φ in VMO which takes self-adjoint values and such that the process with density $\exp \Phi$ is not completely regular.

In [TV] the following characterization of vectorial completely regular processes was obtained. Let W be the density of a minimal vectorial process. The process is completely regular if and only if

$$\lim_{|J| \rightarrow 0} \left\| \left(\frac{1}{|J|} \int_J W d\mathbf{m} \right)^{1/2} \left(\frac{1}{|J|} \int_J W^{-1} d\mathbf{m} \right)^{1/2} \right\| = 1,$$

where the limit is taken over subarcs of $\mathbb{T}$ and $|J| \stackrel{\text{def}}{=} \mathbf{m}(J)$ is the Lebesgue measure of J .

Note, however, that this characterization does not generalize to the case of p -regular processes. In §5 we obtain necessary conditions for a process to be completely regular or p -regular in terms of matrix entries of the density W .

3. Reduction to the case of minimal processes

In this section we reduce the study of stationary processes satisfying different regularity conditions to the case of minimal processes.

The following theorem reduces the study of an arbitrary completely regular processes to the case of minimal processes. It improves Theorem 3.5 of [P3] while the approach given here simpler than that of [P3].

Theorem 3.1. *Let W be the density of a completely regular stationary process with phase function U determined by (1.3). Let $N = \text{Ker } T_U$. Then there exists an analytic polynomial matrix function Q such that $\deg \det Q = N$, all zeros of $\det Q$ are on $\mathbb{T}$, and*

$$W = Q^* W_1 Q, \quad (3.1)$$

where W_1 is the density of a minimal completely regular process which has a phase function U_1 satisfying $U_1 = \Pi U$, where Π is a polynomial inner function of size $n \times n$.

Proof. Consider the factorizations (1.3). Let $U = \Psi_{\#}^* \Psi^{-1}$ be the phase function. Since $U \in VMO$ (see §2), it follows that the Toeplitz operator T_U on $H^2(\mathbb{C}^n)$ is Fredholm. Indeed, this follows from the identities

$$I - T_U^* T_U = H_U^* H_U, \quad I - T_U T_U^* = H_{U^*}^* H_{U^*}$$

and the fact that the Hankel operators H_U and H_{U^*} are compact (see [P3] and [PK] for details).

By Simonenko's theorem [Si] the function U admits a factorization

$$U = \Phi_2^* \Lambda \Phi_1^{-1},$$

where Φ_1 and Φ_2 are outer matrix functions in H^2 such that $\Phi_1^{-1}, \Phi_2^{-1} \in H^2$ and Λ is a diagonal matrix function of the form

$$\Lambda = \begin{pmatrix} z^{d_1} & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & z^{d_2} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & z^{d_n} \end{pmatrix}, \quad d_1, \dots, d_n \in \mathbb{Z}.$$

Moreover, $\text{ind } T_U = -\sum_{j=1}^n d_j = N$. It is easy to see that $d_j \leq 0$, $1 \leq j \leq n$. Indeed, since T_U has dense range in $H^2(\mathbb{C}^n)$ (see §2), it follows that $\text{Ker } T_U^* = \{\mathbb{O}\}$, and if $d_j > 0$ for some j , then $\Phi_2^{-1}e_j \in \text{Ker } T_U^*$ (here e_j is the j th standard basis vector of $\mathbb{C}^n$).

Consider the unitary-valued function $V = \bar{z}U$. Clearly, T_V is Fredholm and $\text{ind } T_V = \dim \text{Ker } T_V = N + n$. It is easy to see that $\text{Ker } T_V$ coincides with the subspace

$$L = \left\{ \Phi_1 \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} : q_j \text{ is a polynomial in } H^2, \deg q_j \leq -d_j \right\}.$$

Indeed, L is obviously a subset of $\text{Ker } T_V$ and $\dim L = N + n = \dim \text{Ker } T_V$.

Since $V = \bar{z}\Psi_{\#}^*\Psi^{-1}$, it follows that $\Psi e_j \in \text{Ker } T_V$ for $1 \leq j \leq m$. Hence,

$$\Psi e_j = \Phi_1 \begin{pmatrix} q_{1j} \\ q_{2j} \\ \vdots \\ q_{nj} \end{pmatrix},$$

where the q_{rj} are polynomials in H^2 such that $\deg q_{rj} \leq -d_j$. Clearly, $\Psi = \Phi_1 Q$, where $Q = \{q_{rj}\}_{1 \leq r, j \leq n}$. We have $W = \Psi^*\Psi = Q^*\Phi_1^*\Phi_1 Q$.

Let us show that $\deg \det Q = N$ and the zeros of $\det Q$ lie on $\mathbb{T}$. Since $\deg q_{rj} \leq -d_j$, it follows that $\deg \det Q \leq -\sum_{j=1}^n d_j = N$.

We have

$$U = \Psi_{\#}^*\Psi^{-1} = \Psi_{\#}^*Q^{-1}\Phi_1^{-1}.$$

On the other hand,

$$U = \Phi_2^*\Lambda\Phi_1^{-1}.$$

Consequently, $\Phi_2^*\Lambda = \Psi_{\#}^*Q^{-1}$, and so $\Psi_{\#}^* = \Phi_2^*\Lambda Q$. Put

$$Q_{\#} \stackrel{\text{def}}{=} Q^*\Lambda^* = \Psi_{\#}\Phi_2^{-1}.$$

Clearly, $Q_{\#}$ is an outer function in H^1 . On the other hand, $Q_{\#}$ is a trigonometric polynomial. Since, $\Lambda Q = Q_{\#}^*$, it follows that

$$\bar{z}^N \det Q = \overline{\det Q_{\#}}. \quad (3.2)$$

Moreover, both $\det Q$ and $\det Q_{\#}$ are polynomial outer functions. If $\deg \det Q < N$, it follows from (3.2) that $\det Q_{\#}(0) = 0$ which contradicts the fact that $\det Q_{\#}$ is outer. Hence, $\deg \det Q = N$.

Since $\deg Q$ is an outer function, it cannot have zeros in $\mathbb{D}$. On the other hand, it is easy to see from (3.2) that if $\det Q$ has a zero outside $\text{clos } \mathbb{D}$, then $\det Q_{\#}$ has

a zero in $\mathbb{D}$ which is impossible since $\det Q_\#$ is an outer function. Therefore all zeros of $\det Q$ lie on $\mathbb{T}$.

Consider the weight $W_1 \stackrel{\text{def}}{=} \Phi_1^* \Phi_1$. Then $W = Q^* W_1 Q$ and $W_1^{-1} \in L^1$. Since $U = \Phi_2^* \Lambda \Phi_1^{-1}$ and U is unitary-valued, we have

$$U^* U = (\Phi_1^*)^{-1} \Lambda^* \Phi_2 \Phi_2^* \Lambda \Phi_1^{-1},$$

and so

$$\Lambda^* \Phi_2 \Phi_2^* \Lambda = \Phi_1^* \Phi_1 = W_1.$$

Hence,

$$(\Lambda^* \Phi_2)^* (\Lambda^* \Phi_2) = W_1.$$

The function $\Lambda^* \Phi_2$ belongs to H^2 . However, it is not outer unless Λ is the constant identity matrix. Let us find an outer function with the same property. Consider the function $(\Lambda^* \Phi_2)^t = \Phi_2^t \Lambda^*$. Since a square matrix function is outer if and only if its determinant is outer (see [N], Lect. I, Sect. 6), it follows that the function Φ_2^t is also outer. Consequently, the subspace

$$\{z^j \Phi_2^t \Lambda^* x : j \in \mathbb{Z}_+, x \in \mathbb{C}^n\}$$

has finite codimension in $H^2(\mathbb{C}^n)$. Since this subspace is obviously invariant under multiplication by z , it follows that the function $\Phi_2^t \Lambda^*$ admits a representation of the form $\Phi_2^t \Lambda^* = B \Xi$, where B is a finite Blaschke–Potapov product of finite degree (see [N]). Since $\det(\Phi_2^t \Lambda^*) = \det(B \Xi)$ and both $\det \Phi_2$ and $\det \Xi$ are outer functions, it follows that $\det B = \text{const} \det \Lambda^* = \text{const} z^N$. It is easy to see that B is a polynomial inner $n \times n$ function. Therefore the function $\Phi_\# \stackrel{\text{def}}{=} \Lambda^* \Phi_2 \Pi^* = \Xi^t$ is outer, where $\Pi \stackrel{\text{def}}{=} B^t$. Clearly, Π is a polynomial inner function of size $n \times n$. It is also easy to see that

$$\Phi_\# \Phi_\#^* = W_1 = \Phi_1^* \Phi_1.$$

These factorizations determine the phase function

$$U_1 = \Phi_\#^* \Phi_1^{-1} = \Pi \Phi_2^* \Lambda \Phi_1^{-1} = \Pi U$$

which completes the proof. ■

Remark. Let us note the following interesting identities which can be verified very easily:

$$Q^* Q = Q_\# Q_\#^*; \quad W = Q_\# \Phi_2 \Phi_2^* Q_\#^*.$$

Theorem 3.2. *Suppose that X is a space of functions on $\mathbb{T}$ satisfying (A1)–(A4) or X is a linear $\mathcal{R}$ -space such that $X \subset VMO$. Let W be the density of a completely vector process which is represented in the form of (3.1), where W_1 and Q are as in the statement of Theorem 3.1. Then the initial process is X -regular if and only if the process with density W_1 is X -regular.*

Proof. By Theorem 2.2, the process is X -regular if and only if the phase function belongs to X . The result follows now immediately from Theorem 3.1. ■

4. Spectral characterizations for X satisfying (A1)–(A4)

In this section we give characterize X -regular vectorial stationary processes in terms of spectral densities for spaces X satisfying (A1)–(A4) and consider several important cases.

Theorem 4.1. *Let X be a space of functions on $\mathbb{T}$ which satisfies (A1)–(A4). Suppose that W is the density of a completely regular vectorial process. The following statements are equivalent:*

- (i) *the process is X -regular;*
- (ii) *W admits a representation Q^*W_1Q , where Q is an analytic polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and W_1 is invertible in X ;*
- (iii) *W admits a representation $Q^*\exp(\Upsilon)Q$, where Q is an analytic polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and Υ is a matrix function in X which takes self-adjoint values.*

Proof. By Theorem 3.1 it is sufficient to assume that the process is minimal, i.e., $W^{-1} \in L^1$. Consider the factorizations (1.3) and let $U = \Psi_{\#}^*\Psi^{-1}$ be the corresponding phase function.

Suppose first that the process is X -regular, i.e., $\mathbb{P}_-U \in X$. In [P3] it was shown that $\Psi \in X$ and $\Psi^{-1} \in X$ which implies that $W = \Psi^*\Psi \in X$ and W is invertible in X (in [P3] this was shown for spaces X in which the trigonometric polynomials are dense or spaces satisfying the axioms (B1)–(B3) (see [P3])); however, the same proof works for arbitrary spaces satisfying (A1)–(A4).

Suppose that W is invertible in X . Then it is well known (see [CG], [P4]) that $\Psi, \Psi_{\#}$ are invertible in X . Hence, $U \in X$.

It is obvious that (ii) is equivalent to (iii). ■

In particular the classes Λ_{α} , $0 < \alpha < \infty$, B_1^1 , and $B_{p,\infty}^s$, $1 \leq p < \infty$, $1/p < s < \infty$, satisfy (A1)–(A4) which implies the following facts.

Corollary 4.2. *Let W be the density of a completely regular stationary process. Then the process is completely regular of order α if and only if $W = Q^*W_1Q$, where Q is an analytic polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and W_1 is invertible in Λ_{α} .*

Recalled that this result was first obtained in [I2] by a different method.

Corollary 4.3. *Let W be the density of a completely regular stationary process. Then the process is 1-regular if and only if $W = Q^*W_1Q$, where Q is an analytic polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and W_1 is invertible in B_1^1 .*

Corollary 4.4. *Let W be the density of a completely regular stationary process and let $1 \leq p < \infty$. Then the process is p -regular of order α if and only if $W = Q^*W_1Q$, where Q is an analytic polynomial matrix function such that the zeros of $\det Q$ are on $\mathbb{T}$ and W_1 is invertible in $B_{p\infty}^{1/p+\alpha}$.*

5. The case of completely regular and p -regular processes

In this section we find necessary conditions for vectorial processes to be completely regular or p -regular. Then we consider a class of processes for which these conditions are also sufficient. Recall that Theorem 3.1 reduces the general case to the case of minimal processes. As we have already mentioned in §2, unlike the scalar case complete regularity for minimal processes is not equivalent to the condition $\log W \in VMO$ (see [TV]).

Theorem 5.1. *Let $W = \{w_{jk}\}_{1 \leq j,k \leq n}$ be the density of a minimal vectorial process. If the process is completely regular, then*

$$\log w_{jj} \in VMO, \quad 1 \leq j \leq n, \quad \text{and} \quad \frac{w_{jk}}{w_{jj}^{1/2} w_{kk}^{1/2}} \in VMO, \quad 1 \leq j, k \leq n.$$

Theorem 5.2. *Let $W = \{w_{jk}\}_{1 \leq j,k \leq n}$ be the density of a minimal vectorial process and let $0 < p < \infty$. If the process is p -regular, then*

$$\log w_{jj} \in B_p^{1/p}, \quad 1 \leq j \leq n, \quad \text{and} \quad \frac{w_{jk}}{w_{jj}^{1/2} w_{kk}^{1/2}} \in B_p^{1/p}, \quad 1 \leq j, k \leq n.$$

Recall that S_1 -regular processes are described in Corollary 4.4.

We need the following fact.

Theorem 5.3. *Let $X = VMO$ or $X = B_p^{1/p}$, $0 < p < \infty$. Suppose that u is a unimodular function in X and φ is a bounded function such that $\mathbb{P}_-u\varphi \in X$. Then $\mathbb{P}_-\varphi \in X$.*

Proof. Suppose that $X = VMO$. It is well known (see e.g., [PK]) that the Toeplitz operator T_u is Fredholm. Multiplying u by z^m if necessary, we can reduce

the problem to the case when T_u is invertible. Let us define the operator $\tilde{T}_u$ on H_-^2 by

$$\tilde{T}_u f = \mathbb{P}_- u f, \quad f \in H_-^2.$$

Exactly the same argument as for the Toeplitz operator T_u proves that $\tilde{T}_u$ is invertible on H_-^2 . The condition $\mathbb{P}_- u \varphi \in X$ means that the Hankel operator $H_{u\varphi}$ is compact. We have to prove that H_φ is compact. We have

$$H_{u\varphi} f = \mathbb{P}_- u \varphi f = \mathbb{P}_- u \mathbb{P}_- \varphi f + \mathbb{P}_- u \mathbb{P}_+ \varphi f = \tilde{T}_u H_\varphi f + H_u T_\varphi f.$$

By the hypotheses H_u is compact, T_φ is bounded. Hence, $H_u T_\varphi$ is compact, and so $\tilde{T}_u H_\varphi$ is compact. Since $\tilde{T}_u$ is invertible, it follows that H_φ is compact.

In the case $X = B_p^{1/p}$ the proof is the same. The only difference is that instead of the compactness condition we have to consider the membership of S_p . ■

Note that the lemma easily generalizes to the case of an arbitrary linear $\mathcal{R}$ -space. For $X = VMO$ another proof has been found by S. Semmes.

Proof of Theorem 5.1. Suppose that

$$x(t) = (x_1(t) \ x_2(t) \ \cdots \ x_n(t))^t, \quad k \in \mathbb{Z},$$

is minimal and completely regular. Then for each j , $1 \leq j \leq n$, the scalar process

$$x_j(t), \quad t \in \mathbb{Z},$$

is also completely regular and minimal. It is easy to see that the density of this process is w_{jj} . Hence, $\log w_{jj} \in VMO$.

Suppose now that $j \neq k$. Consider functions $F, G \in H^2(W)$ of the following form.

$$F = \{f_l\}_{1 \leq l \leq n}, \quad f_l = 0 \text{ if } l \neq k, \quad f_k = f \in H^2(w_{kk}),$$

$$G = \{g_l\}_{1 \leq l \leq n}, \quad g_l = 0 \text{ if } l \neq j, \quad g_j = g \in H_-^2(w_{jj}).$$

Then $\|F\|_{H^2(W)} = \|f\|_{H^2(w_{kk})}$ and $\|G\|_{H^2(W)} = \|g\|_{H_-^2(w_{jj})}$.

Let h_j , $1 \leq j \leq n$, be scalar outer functions such that $|h_j|^2 = w_{jj}$. We have

$$\begin{aligned} \rho_m(W) &\geq \sup\{|(z^m F, G)_{L^2(W)}| : \|f\|_{H^2(w_{kk})} \leq 1, \|g\|_{H_-^2(w_{jj})} \leq 1\} \\ &= \sup\{|\int_{\mathbb{T}} z^m w_{jk} f g d\mathbf{m}| : \|f\|_{H^2(w_{kk})} \leq 1, \|g\|_{H_-^2(w_{jj})} \leq 1\} \\ &= \sup\{|\int_{\mathbb{T}} z^m w_{jk} h_k^{-1/2} \xi \overline{h_j^{-1/2}} \eta d\mathbf{m}| : \|\xi\|_{H^2} \leq 1, \|\eta\|_{H_-^2} \leq 1\} \\ &= \left\| H_{z^m \frac{w_{jk}}{h_k^{1/2} h_j^{1/2}}} \right\| = \left\| H_{z^m u \frac{w_{jk}}{w_{kk}^{1/2} w_{jj}^{1/2}}} \right\|, \end{aligned}$$

where

$$u \stackrel{\text{def}}{=} \frac{w_{kk}^{1/2} w_{jj}^{1/2}}{h_k^{1/2} h_j^{1/2}}.$$

Since $\log w_{jj} \in VMO$ and $\log w_{kk} \in VMO$, it follows that u is a unimodular function in VMO . Put

$$\varphi \stackrel{\text{def}}{=} \frac{w_{jk}}{w_{kk}^{1/2} w_{jj}^{1/2}}.$$

Since the matrix function

$$\begin{pmatrix} w_{jj} & w_{jk} \\ w_{jk} & w_{kk} \end{pmatrix}$$

is positive definite, it follows that $\varphi \in L^\infty$.

The condition $\rho_n(W) \rightarrow 0$ implies now that

$$\lim_{n \rightarrow \infty} \|H_{z^m u \varphi}\| = 0$$

which means that $\mathbb{P}_-(u\varphi) \in VMO$. By Theorem 5.3, $\varphi \in VMO$ which completes the proof. ■

Proof of Theorem 5.2. The proof is similar to that of Theorem 5.1. First, for each j the process

$$x(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}^t, \quad k \in \mathbb{Z},$$

is minimal and p -regular. This implies that $\log w_{jj} \in B_p^{1/p}$ for $1 \leq j \leq n$,

Let now $j \neq k$. Take the operator $\mathcal{P}_0 \mathcal{P}^0$ on $L^2(W)$ and consider the operator given by the following Hermitian form:

$$(\mathcal{P}_0 \mathcal{P}^0 F, G),$$

where as in the proof of Theorem 5.1

$$F = \{f_l\}_{1 \leq l \leq n}, \quad f_l = \mathbb{O} \text{ if } l \neq k, \quad f_k = f \in H^2(w_{kk}),$$

$$G = \{g_l\}_{1 \leq l \leq n}, \quad g_l = \mathbb{O} \text{ if } l \neq j, \quad g_j = g \in H_-^2(w_{jj}).$$

This operator must be in $\mathbf{S}_p$ and the reasoning similar to that given in the proof of Theorem 5.1 shows that the Hankel operator $H_{u\varphi}$ must be in $\mathbf{S}_p$, where u and φ are defined in the same way as in the proof of Theorem 5.1. This means that $\mathbb{P}_-(u\varphi) \in B_p^{1/p}$ (see [P1], [P2]), and by Theorem 5.3, this implies that $u \in B_p^{1/p}$. ■

Consider now the class of weights W for which the coordinate projections

$$F \mapsto f_j, \quad f_j(\zeta) = ((F(\zeta), e_j)_{\mathbb{C}^n}) e_j,$$

are bounded in $L^2(W)$. It is easy to see that this condition is equivalent to the condition that

$$\sup_{\zeta \in \mathbb{T}} \|W^{1/2}(\zeta) e_j\|_{\mathbb{C}^n} \|W^{-1/2}(\zeta) e_j\|_{\mathbb{C}^n} < \infty, \quad 1 \leq j \leq n. \quad (5.1)$$

Theorem 5.4. Suppose that W is the density of a minimal vectorial process which satisfies (5.1). If

$$\log w_{jj} \in VMO, \quad 1 \leq j \leq n, \quad \text{and} \quad \frac{w_{jk}}{w_{jj}^{1/2} w_{kk}^{1/2}} \in VMO, \quad 1 \leq j, k \leq n,$$

then the process is completely regular.

Theorem 5.5. Suppose that W is the density of a minimal vectorial process which satisfies (5.1) and let $0 < p < \infty$. If

$$\log w_{jj} \in B_p^{1/p}, \quad 1 \leq j \leq n, \quad \text{and} \quad \frac{w_{jk}}{w_{jj}^{1/2} w_{kk}^{1/2}} \in B_p^{1/p}, \quad 1 \leq j, k \leq n.$$

then the process is p -regular.

It is easy to see that condition (5.1) allows one to reduce the estimation of $\|(\mathcal{P}_0 \mathcal{P}^m F, G)\|_{L^2(W)}$ to the case when F and G have only one nonzero coordinate. Similarly, one can estimate $\|\mathcal{P}_0 \mathcal{P}^m\|_{S_p}$.

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