Cosmological Moduli Problem and Thermal Inflation Models

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Abstract

In superstring theories, there exist various dilaton and modulus fields which masses are expected to be of the order of the gravitino mass $m_{3/2}$. These fields lead to serious cosmological difficulties, so called “cosmological moduli problem”, because a large number of moduli particles are produced as the coherent oscillations after the primordial inflation. We make a comprehensive study whether the thermal inflation can solve the cosmological moduli problem in the whole modulus mass region $m_\phi \sim 10 \text{ eV} - 10^4 \text{ GeV}$ predicted by both hidden sector supersymmetry (SUSY) breaking and gauge-mediated SUSY breaking models. In particular, we take into account the primordial inflation model whose reheating temperature is so low that its reheating process finishes after the thermal inflation ends. We find that the above mass region $m_\phi(\simeq m_{3/2}) \sim 10 \text{ eV} - 10^4 \text{ GeV}$ survives from various cosmological constraints in the presence of the thermal inflation.
I. INTRODUCTION

Supersymmetry (SUSY) is one of the most attractive extensions of the standard model. In virtue of the SUSY the electroweak scale can be stabilized against the radiative corrections. Furthermore, in the SUSY grand unified theories the unification of the standard gauge couplings can be realized.

However, overviewing the cosmology of the SUSY model, we are faced with various difficulties. One of the cosmological problems is the gravitino problem [1–3]. This problem still exists even if the universe experienced a primordial inflation, since the gravitino is reproduced by scatterings with particles in the thermal bath at the reheating epoch. In the hidden sector SUSY breaking (HSSB) models [4] the gravitino has a mass of the order of the electroweak scale: \( m_{3/2} \sim 10^2\text{–}10^3\text{ GeV} \), and it decays soon after the big bang nucleosynthesis (BBN). Since high energy photons produced by the gravitino decay might destroy the light elements (D, \(^3\text{He}, \(^4\text{He})\) synthesized by the BBN, the reheating temperature of the primordial inflation should be low enough not to conflict with the observations (e.g., see a recent analysis in Ref. [5]). On the other hand, the gauge mediated SUSY breaking (GMSB) models [6] predict a light gravitino of mass \( m_{3/2} \sim 10\text{ eV–}1\text{ GeV} \). If the gravitino mass is \( m_{3/2} \gtrsim 1\text{ keV} \) [1], a low enough reheating temperature of the inflation is also required to avoid the overclosure by the stable gravitino [7].

Furthermore, when one considers the SUSY models in the framework of the superstring theories [8], they suffer from a more serious problem, i.e., “cosmological moduli problem” [9–11]. As a general consequence of the superstring theories, various dilaton and modulus fields appear. (We call them as “moduli” throughout this paper.) These moduli fields \( \phi \) are expected to acquire masses of the order of the gravitino mass (\( m_{\phi} \approx m_{3/2} \)) from nonperturbative effects of the SUSY breaking [11]. Their lifetimes, since they have only the gravitationally suppressed interaction, are roughly estimated as

\[
\tau_{\phi} \sim N^{-1} \frac{M_{pl}^2}{m_{\phi}^4} \approx 10^{17} \text{ sec} N^{-1} \left( \frac{m_{\phi}}{100 \text{ MeV}} \right)^{-3},
\]

where \( N \) denotes the number of the decay channels and \( M_{pl} \) is the Planck scale \( M_{pl} = 1.2 \times 10^{19} \text{ GeV} \). The existence of moduli \( \phi \) with such long lifetimes leads to various cosmological difficulties, because a large number of moduli particles are produced as the coherent oscillations after the primordial inflation. In the HSSB models \( \phi \) decay soon after the BBN as the gravitino, and hence the light elements might also be destroyed [9–11]. On the other hand, for the lighter stable moduli predicted by the GMSB models, say \( m_{\phi} \lesssim 100 \text{ MeV} \), the energy of the oscillation lasting until the present overcloses the universe. Moreover, moduli with masses \( m_{\phi} \sim 0.1 \text{ MeV–}1 \text{ GeV} \) give too much contributions to the x(\( \gamma \))-ray background spectrum [12]. Therefore, moduli particles typically predicted by both HSSB and GMSB models bring a cosmological disaster.

One way to evade these difficulties due to the string moduli is to give heavy masses (\( \phi \gtrsim 100 \text{ TeV} \)) to all moduli, so that the decays of moduli take place before the BBN [13] and the lightest superparticles produced by the moduli decays do not overclose the universe [14]. Although a large entropy is produced by the moduli decay, the present observed baryon asymmetry can be naturally explained by the Affleck-Dine mechanism [15] as shown
in Ref. [16]. However, such heavy moduli masses (i.e., a heavy gravitino mass) could be only achieved by some specific models.

Here we would like to emphasize that this cosmological moduli problem could not be solved by the primordial inflation, even if one assumed an extremely low reheating temperature about 10 MeV which is limited from below by the BBN observation. This is a crucial difference from the gravitino problem. Therefore, we require some extra mechanism other than the primordial inflation to dilute the moduli mass density sufficiently.

One of such dilution mechanisms is the thermal inflation model proposed by Lyth and Stewart [18]. The thermal inflation occurs before the electroweak phase transition and produces tremendous entropy (just) before the BBN epoch, which leads to sufficient dilution of the moduli density. In Ref. [18] it was shown that the moduli problem in the HSSB models can be solved by the thermal inflation.

On the other hand, the attempts to solve the problem in the GMSB models by the thermal inflation were done in Refs. [19,12,22,23]. It was shown by Ref. [19] that it can solve the overclosure problem for the stable moduli whose masses are $m_\phi \sim 10 \text{ eV} - 10^{-1} \text{ GeV}$. However, in Refs. [12,22,23], the stringent x(γ)-ray background constraint was found to exclude the modulus mass region $m_\phi \sim 10^{-4} - 1 \text{ GeV}$. Thus the thermal inflation can cure a part of the cosmological difficulties of the string moduli predicted by both HSSB and GMSB models.

However, most of previous works (except Ref. [18]) assumed that the reheating process of the primordial inflation be completed before the moduli oscillations start, i.e., only the inflation models with relatively high reheating temperatures were considered. Even in Ref. [18], the reheating process is assumed to end before the thermal inflation starts. Without this restriction one might obtain wider allowed region for the modulus mass. In this paper, therefore, we make a comprehensive study whether the thermal inflation can solve the cosmological problem of the moduli particles with masses $m_\phi \sim 10 \text{ eV} - 10^4 \text{ GeV}$ (i.e., the whole modulus mass region predicted by both HSSB and GMSB models), especially taking into account the inflation model whose reheating temperature is low enough so that its reheating process finishes after the thermal inflation ends. We find that the above mass region $m_\phi (\approx m_{3/2}) \sim 10 \text{ eV} - 10^4 \text{ GeV}$ survives from various cosmological constraints if we con-

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1 Some other approach to the solution has been proposed [17], but it is difficult to construct a realistic model.

2 The other possible mechanism is the oscillating inflation by Moroi [20] which works in the GMSB models. A mini-inflation takes place while a scalar field corresponding to the flat direction oscillates along the logarithmic potential induced by the GMSB mechanism and it significantly dilutes the relic density of the moduli. See also Ref. [21].

3 More precisely, the thermal inflation can solve the moduli problem if one assumes the primordial inflation model which reheating process completes after the moduli begin the coherent oscillations.

4 Here we do not include a constraint from the present baryon asymmetry which excludes some modulus mass regions. See the discussion below.
consider the primordial inflation with sufficiently low reheating temperature in addition to the thermal inflation.

The organization of this paper is as follows. In Sec. II we first review about the cosmological difficulties of the string moduli whose masses are \( m_\phi \sim 10 \, \text{eV} - 10^{4} \, \text{GeV} \). The (original) thermal inflation model proposed by Lyth and Stewart is explained in Sec. III. Then, in Sec. IV, we study whether this original thermal inflation could solve the moduli problem, considering various models of the primordial inflation. In the original thermal inflation model, however, there appears an \( R \)-axion which is the NG boson from the spontaneous breaking of the \( R \)-symmetry. In the GMSB models the \( R \)-axion is light enough so that the flaton, which causes the thermal inflation, almost decays into them. Then, as shown in Ref. [23], the original model could not produce a sufficient entropy to dilute the light moduli predicted by the GMSB. To avoid this difficulty we introduce in Sec. V the “modified” thermal inflation by Ref. [23], which forbids the flaton decay into \( R \)-axions. Then, we investigate the solution of the moduli problem by this modified model in Sec. VI. Finally, in Sec. VII, we discuss two extra problems when one assumes the tremendous entropy production enough to dilute the string moduli, i.e., problems of baryogenesis and dark matter. We find that the problem of the baryon asymmetry is so serious that we have only two possible solution so far; (i) sufficient baryons can be generated through the Affleck-Dine mechanism [15] for the modulus mass region \( m_\phi (\simeq m_{3/2}) \lesssim 1 \, \text{MeV} \) and (ii) for \( m_\phi (\simeq m_{3/2}) \sim 1 - 10^{4} \, \text{GeV} \) the electroweak baryogenesis can produce the present observed baryon asymmetry.

II. COSMOLOGICAL DIFFICULTIES OF MODULI PARTICLES

First of all, we explain the cosmological difficulties due to the string moduli particles. In the following analysis we assume only one modulus field, \( \phi \), with the mass \( m_\phi \simeq m_{3/2} \) to derive the conservative constraints, although we can easily extend the analysis for more general cases.

After the primordial inflation ends, the modulus field \( \phi \) is considered to be displaced from the true minimum and the displacement is of the order of \( M_G (M_G = 2.4 \times 10^{18} \, \text{GeV}) \). This is because of the additional SUSY breaking effect due to large vacuum energy of the inflaton. Then, when the expansion rate of the universe (i.e., the Hubble parameter \( H \)) becomes comparable to the modulus mass, the modulus starts to oscillate around its true minimum with the initial amplitude \( \phi_0 \sim M_G \). At this time, the energy density of this oscillation is given by \( \rho_\phi = \frac{m_\phi^2 \phi_0^2}{2} \) and the cosmic temperature of the universe is estimated as

\[
T_\phi \simeq \left( \frac{90}{\pi^2 g_\ast} \right)^{1/4} \sqrt{m_\phi M_G} \simeq 7.2 \times 10^8 \, \text{GeV} \left( \frac{m_\phi}{1 \, \text{GeV}} \right)^{1/2},
\]

where \( g_\ast (\simeq 200) \) counts the effective degrees of freedom of the radiation. Thus the ratio between \( \rho_\phi \) to the entropy density \( s \) is given by

\[
\frac{\rho_\phi}{s} \simeq \frac{1}{2} \frac{m_\phi^2 \phi_0^2}{\frac{8 \pi}{45} g_\ast T_\phi^3} \simeq 0.89 \times 10^8 \, \text{GeV} \left( \frac{m_\phi}{1 \, \text{GeV}} \right)^{1/2} \left( \frac{\phi_0}{M_G} \right)^2.
\]
until the modulus decays) if “no” extra entropy is produced. Since, at \( T = T_\phi \), the energy density of the radiation, \( \rho_R \), is comparable to that of the modulus (\( \rho_R \sim T^4_\phi \sim \rho_\phi \)) and is diluted faster as \( R^{-4} \), the modulus oscillation soon dominates the whole energy of the universe.

In deriving Eq. (3) we have assumed that the reheating process of the primordial inflation be completed before the beginning of the modulus oscillation, i.e., the decay rate \( \Gamma_{\varphi I} \) of the inflaton be larger than \( m_\phi \).\(^5\) On the other hand, when \( \Gamma_{\varphi I} \ll m_\phi \), the energy of the modulus oscillation is expected to be diluted by the entropy production of the primordial inflation.\(^6\) In this case, when the reheating process completes at \( H \approx \Gamma_{\varphi I} \), the ratio \( \rho_\phi/s \) is estimated as

\[
\frac{\rho_\phi}{s} \approx \frac{1}{8} T_{RI} \left( \frac{\phi_0}{M_G} \right)^2.
\]

Here \( T_{RI} \) denotes the reheating temperature of the primordial inflation which is given by

\[
T_{RI} \approx \left( \frac{90}{\pi^2 g_*} \right)^{1/4} \sqrt{\Gamma_{\varphi I} M_G}.
\]

In order to keep the success of the BBN it should be higher than about 10 MeV and we obtain

\[
\frac{\rho_\phi}{s} > 1.25 \times 10^{-3} \text{ GeV} \left( \frac{\phi_0}{M_G} \right)^2.
\]

Comparing with Eq.(3), it is seen that the primordial inflation with a low \( T_{RI} \) does dilute the modulus energy significantly. However, even if we assume the primordial inflation with extremely low reheating temperature, the energy of the modulus oscillation is still large. It leads to disastrous effect on the thermal history of the universe after the BBN if the modulus lifetime is long enough.

The modulus, in fact, has a very long lifetime since it has only gravitationally suppressed interaction. From the naive dimensional analysis the modulus lifetime is estimated as shown in Eq. (1). Then we can roughly say that the modulus with a mass \( m_\phi \ll 100 \text{ MeV} \) has a longer lifetime than the present age of the universe \( \sim 10^{17} \text{ sec} \).

We can say more precisely about the modulus lifetime for the dilaton particle which is the most plausible candidate among various moduli. The dilaton has the following couplings to the kinetic terms of the gauge fields:

\[
\mathcal{L} = \frac{b}{4 M_G} F_{\mu\nu}^2,
\]

\(^5\)In other words the reheating temperature of the primordial inflation is higher than \( T_\phi \) given by Eq. (2).

\(^6\)Note that the modulus oscillation is considered to start at least after the primordial inflation ends, since \( m_\phi \ll H_I \) where \( H_I \) denotes the Hubble parameter during the inflation and \( H_I = V_I/(\sqrt{3} M_G) \) with the inflaton’s vacuum energy \( V_I \).
where we introduce an order one parameter $b$ which is determined by the string model and its compactification. For example, the dilaton has a coupling $b = \sqrt{2}$ [24] in some compactification of the M-theory [25]. If the dilaton mass is light enough, say $m_\phi \lesssim 1$ GeV, it dominately decays into two photons [12] and the lifetime of the dilaton is estimated as

$$\tau_\phi = 7.6 \times 10^{17} \text{ sec} \left( \frac{m_\phi}{100 \text{ MeV}} \right)^{-3}.$$ (8)

Moreover, the dilaton could decay into two gluons similarly if kinematically allowed. Then the dilaton lifetime becomes shorter by a factor of $1/9$ counting only the number of the final states. In addition, if $m_\phi$ is heavier than the electroweak scale, various decay modes may arise through the gravitational interaction with the SUSY standard model particles, and its decay lifetime becomes shorter. Comparing with the naive estimation Eq. (1), the both results are not different very much. Thus, in the present analysis we assume that the modulus lifetime be the same as the dilaton one and it decay into only two photons and two gluons if possible through the interaction (7) with $b = 1$ for simplicity.

We turn to see constraints on the modulus energy from various cosmological observations. First of all, the stable modulus of mass $m_\phi \lesssim 100$ MeV is constrained from the overclosure limit. The present energy density of the modulus oscillation should be smaller than the critical density of the universe $\rho_{cr}$. This requirement leads to to

$$\frac{\rho_\phi}{s} \lesssim \frac{\rho_{cr}}{s_0} = 3.6 \times 10^{-9} h^2 \text{ GeV},$$ (9)

where $s_0$ is the present entropy density and $h$ denotes the present Hubble parameter in units of 100 km/sec-Mpc$^{-1}$. Note that even if the modulus is unstable, we obtain similar upper bound on the modulus abundance requiring that the decay products (i.e., photons or gluons) should not overclose the universe. Since the decay products are relativistic and its energy density decreases faster than the modulus one, the constraint on the abundance becomes weaker than Eq. (9).

If the lifetime of the modulus is longer than the time of the recombination $\tau_\phi \gtrsim 10^{12}$ sec, i.e., $m_\phi \gtrsim 1$ GeV, a stringent constraint comes from the the observed cosmic x($\gamma$)-ray background spectrum [12]. The photon flux produced by the modulus decay directly contributes to the x($\gamma$)-ray backgrounds and we obtain the upper bound on the modulus abundance by requiring that they should not exceed the present observed spectrum. (One may find the detail analysis in Refs. [12,23].) In fact this gives us a more stringent constraint than the above overclosure limit (9) for $1 \text{ GeV} \gtrsim m_\phi \gtrsim 100$ keV. Therefore, this is very stringent constraint on the models of the GMSB where $m_\phi \simeq m_{3/2} \lesssim 1$ GeV as first pointed out by Ref. [12].

The modulus abundance is also constrained from the spectrum of the cosmic microwave background radiation (CMBR). If the modulus lifetime is about $10^6$ sec $\lesssim \tau_\phi \lesssim 10^{12}$ sec, extra radiation energy produced by the modulus decay may cause the spectral distortion of the cosmic microwave background from the blackbody one. The observation by the COBE satellite [26] gives the following upper bounds on the modulus abundances:

$$\frac{\rho_\phi}{s} \lesssim 2.8 \times 10^{-13} \text{ GeV} \left( \frac{10^{10} \text{ sec}}{\tau_\phi} \right)^{1/2}$$ (10)
for $10^6 \left( \frac{\Omega_B h^2}{0.0125} \right) \text{sec} \lesssim \tau_\phi \lesssim 1.4 \times 10^9 \left( \frac{\Omega_B h^2}{0.0125} \right) \text{sec.}$

$$\frac{\rho_\phi}{s} \lesssim 2.6 \times 10^{-13} \text{ GeV} \left( \frac{10^{10} \text{ sec}}{\tau_\phi} \right)^{1/2}$$

for $1.4 \times 10^9 \left( \frac{\Omega_B h^2}{0.0125} \right) \text{sec} \lesssim \tau_\phi \lesssim 10^{12} \text{ sec.}$

Here $\Omega_B$ is the density parameter for baryons.

Furthermore, the BBN plays a significant role to set limits to the moduli abundance. First, if the modulus oscillation exits at the cosmic temperature about 1 MeV, its extra energy accelerates the expansion of the universe and the weak interaction freezes out earlier. This leads to higher neutron-proton number ratio and overproduction of $^4$He. To keep the success of the BBN, the modulus energy density should be smaller than that of one neutrino species [e.g., see Ref. [5]], i.e.,

$$\frac{\rho_\phi}{s} \lesssim 1.2 \times 10^{-4} \text{ GeV},$$

for $\tau_\phi > 1 \text{ sec.}$

Furthermore, when the modulus mass becomes larger than about 10 GeV, the modulus decays soon after the BBN. If the lifetime of the modulus is longer than about $10^4 \text{ sec}$, the high energy photons emitted from the modulus may destroy or overproduce the light elements of the universe synthesized by the BBN. Not to spoil the success of the BBN, the modulus abundance is stringently constrained [e.g., see the resent work [5]]. Moreover, when the modulus lifetime becomes shorter than about $10^4 \text{ sec}$, the hadronic cascade processes associated with the modulus decay may modify the primordial abundances of light elements. This also puts the upper bound [27].

Although we have neglected the modulus decay into the SUSY standard model particles, the stable lightest SUSY particle (LSP) produced by the modulus decay is potentially dangerous, since its energy density might overclose the universe. However, from the present mass limit of the LSP, we find that this constraint, if exists, is weaker than those from the distortion of the CMBR and the photo (hadronic) dissociation of the light elements.

It should be noted that the modulus with $m_\phi > 10 \text{ TeV}$ is cosmologically viable [13]. From Eq. (1) the modulus with such a heavy mass decays before the BBN and reheats the universe as $T > 10 \text{ MeV}$ which gives the initial condition of the BBN. Then one can evade the previous constraints.\footnote{However, in order to avoid the overclosure by the LSP produced by the modulus decay $m_\phi > 100 \text{ TeV}$ is required [14]. Furthermore, if the modulus decays into lighter gravitinos of mass $m_{3/2} \sim 100 \text{ GeV--10 TeV}$, we are faced with the gravitino problem [28].}

We show these upper bounds on the modulus abundance $\Omega_\phi$\footnote{For the stable modulus of mass $m_\phi \lesssim 100 \text{ MeV}$, $\Omega_\phi$ is given by $\Omega_\phi = (\rho_\phi)_{\text{0}}/\rho_{\text{cr}}$ with the present energy density of the modulus $(\rho_\phi)_{\text{0}}$. On the other hand, for the modulus of mass $m_\phi > 100 \text{ MeV}$, $\Omega_\phi$ is limited by $\Omega_\phi \lesssim 0.01$ [31].} in Figs. 1, 2 and 3 taking the modulus coupling as $b = 1, 0.1$ and 10, respectively. The predicted modulus abundances
FIG. 1. Various cosmological upper bounds on the modulus abundance for the case that the modulus coupling is $b = 1$. The dotted line represents the upper bound from the overclosure limit on the abundance of the modulus (or its decay products) for $m_\phi < 200$ GeV, and the upper bound from the BBN speed up effects for $m_\phi > 200$ GeV. The dot-dashed line represents the upper bound from the $x(\gamma)$-ray backgrounds. The short dashed line represents the upper bound from the CMBR spectrum. The long dashed line represents the upper bound from the dissociation of the BBN light elements. Note that there exist two typical modulus masses; $m_\phi \simeq 100$ MeV (the modulus lifetime is equal to the age of the universe) and $m_\phi \simeq 1$ GeV (the modulus decay into two gluons starts to open.) We also show the predicted modulus abundances of $\phi_0 = M_G$ by the solid lines for the case $m_\phi < \Gamma_{\varphi_I}$ [Eq. (3)] and for the case $m_\phi > \Gamma_{\varphi_I}$ with the reheating temperature $T_{RI} = 10$ MeV [Eq. (4)].

Eqs. (3) and (4) with $T_{RI} = 10$ MeV are also found in those figures. One can easily see that the string modulus with a mass from 10 eV to 10 TeV is excluded by the various cosmological observations. Since $m_\phi \simeq m_{3/2}$, the whole gravitino mass region typically predicted by both GMSB and HSSB models is not cosmologically allowed. This difficulty is often referred as “cosmological moduli problem”. Here we would like to stress that this problem could not be solved by choosing the model of the primordial inflation, i.e., even if one assumes the extremely low reheating temperature $T_{RI} \sim 10$ MeV. This is very different from the gravitino problem. Therefore, we required some extra mechanism to dilute the modulus mass density sufficiently other than the primordial inflation. In the following, we consider the thermal inflation model proposed by Lyth and Stewart [18] as such a dilution mechanism.

MeV, $\Omega_\phi$ is regarded as the ratio, $(\rho_\phi/s)/s_0$, where $(\rho_\phi/s)_D$ denotes the ratio of the energy density of the modulus to the entropy density when the modulus decays.
FIG. 2. Same figure as Fig. 1 except for the value of the modulus coupling \( b = 0.1 \).

FIG. 3. Same figure as Fig. 1 except for the value of the modulus coupling \( b = 10 \).
III. ORIGINAL THERMAL INFLATION MODEL

In this section we review an original thermal inflation model proposed by Lyth and Stewart [18]. The thermal inflation is caused by a scalar field (called “flaton”), and the flaton supermultiplet $X$ which is a singlet under the standard model gauge groups has the following superpotential:

$$W = \sum_{k=1}^{\infty} \frac{\lambda_k}{(n + 3)^k} \frac{X^{(n+3)k}}{M^*_3(n+3)^{k-3}} + C,$$

where $\lambda_k$ denotes the coupling constant and we take $\lambda_1 = 1$. Here we impose the discrete $Z_{n+3}$-symmetry to guarantee the flatness of the potential. In order to cancel out the cosmological constant the constant term $C$ should satisfy

$$|C| \simeq \frac{m_3^2}{2} M^2 G.$$

The cutoff scale of the model is denoted by $M_*$ in Eq. (13).\(^9\) Note that we only have to consider the leading term of $k = 1$ since the higher power terms are highly suppressed.

Then we obtain the following effective potential at low energy ($|X| \ll M, M_*$) as

$$V_{\text{eff}}(X) = V_0 - m_0^2 |X|^2 + \frac{n}{n + 3} \frac{C}{M^*_n} \frac{X^{n+3} + X^{*n+3}}{M^*_n} + \frac{|X|^{2n+4}}{M^*_{2n}},$$

where we use the same letter $X$ for the scalar component of the superfield. In this potential we have assumed the negative mass squared $-m_0^2$ at the origin which is induced by the SUSY breaking effects and $m_0$ is of the order of the electroweak scale $\Lambda_{\text{EW}} \sim 100$ GeV. Then one finds the vacuum expectation value (vev) of the flaton ($\langle X \rangle \equiv M$) as

$$\frac{M^{n+1}}{M^*_n} = \frac{1}{2(n + 2)} \left[ nm_3/2 + \sqrt{n^2 m^2_{3/2} + 4(n + 2)m_0^2} \right].$$

Therefore, the flaton is found to have a very large vev $M \gg \Lambda_{\text{EW}}$ since $M_* \gg \Lambda_{\text{EW}}$. Here notice that we assumed the soft SUSY breaking mass ($m_0$) for the flaton, and hence the potential (16) is applicable only for the scale below masses of messenger fields when we consider the light gravitino (modulus) mass region in the GMSB models. Therefore, the vev of the flaton $M$ should be smaller than their masses. However, in the original thermal

\(^9\)As the cutoff scale of the theory it is natural to choose the gravitational scale. Thus the interaction term of the superpotential (13) should be considered as

$$W_{\text{int}} = \frac{1}{n + 3} \frac{X^{n+3}}{M^*_n} = \lambda \frac{X^{n+3}}{M^*_G},$$

Here $\lambda$ is introduced as a coupling which should be $\lambda \lesssim \mathcal{O}(1)$, thus $M_* \gtrsim M_*$ is naturally expected. Although we will take it as a free parameter in the following analysis, we will also mention about this lower bound on $M_*$. 

10
inflation model, this constraint is always satisfied when we estimate the minimum of the modulus abundance in the next section.

The vacuum energy of the flaton $V_0$ is written as

$$V_0 = \frac{n+1}{n+2} M^2 \left[ m_0^2 + \frac{nm_{3/2}}{2(n+2)(n+3)} \left( nm_{3/2} + \sqrt{n^2m_{3/2}^2 + 4(n+2)m_0^2} \right) \right]. \quad (18)$$

The masses of the scalar particles associated with $X$ can be estimated by using the variables $\chi$ and $a$ which is defined as

$$X = \left( M + \frac{\chi}{\sqrt{2}} \right) \exp \left[ \frac{ia}{\sqrt{2}M} \right]. \quad (19)$$

Here $\chi$ is the flaton particle which causes the thermal inflation. On the other hand, we call the imaginary part of $X$, $a$, as an “$R$-axion”. Since the superpotential (13) possesses an approximate $U(1)_R$ symmetry which explicitly breaks down to $Z_{n+3}$ symmetry by the constant term $C$, $a$ is considered as a NG boson with an explicit breaking mass proportional to $C$. The masses of the flaton and the $R$-axion are estimated as

$$m_{\chi}^2 = \frac{2(n+1)m_0^2 + \frac{n(n+1)m_{3/2}}{2(n+2)}}{2(n+2)} \left[ nm_{3/2} + \sqrt{n^2m_{3/2}^2 + 4(n+2)m_0^2} \right], \quad (20)$$

$$m_a^2 = \frac{n(n+3)m_{3/2}}{2(n+2)} \left[ nm_{3/2} + \sqrt{n^2m_{3/2}^2 + 4(n+2)m_0^2} \right]. \quad (21)$$

Notice that if one takes $m_0$ as

$$m_0^2 > \frac{n^2(3n+11)(5n+13)}{4(n+1)^2(n+2)} m_{3/2}^2,$$

$$= \frac{21}{4} m_{3/2}^2 \text{ for } n = 1, \quad (22)$$

the flaton decay into two $R$-axions is kinematically allowed. Therefore, in the GMSB models where the gravitino mass is in the range $m_{3/2} \sim 10$ eV–1 GeV, the decay process $\chi \rightarrow 2a$ seems almost to be open since $m_{3/2} \ll m_0 \sim \Lambda_{EW}$. On the other hand, in the HSSB models, we expect $m_0 \sim m_{3/2} \sim \Lambda_{EW}$ and the flaton might not decay into $R$-axions. Although the decay channel of the flaton is crucial to estimate the dilution factor of the string modulus energy, we assume that the decay process $\chi \rightarrow 2a$ be always open in the whole gravitino mass region and that $m_0$ always satisfy Eq. (22) in the original model. Furthermore, if $m_0 \gg m_{3/2}$, the vev of the flaton $M$ is determined independently on $m_{3/2}$ as

$$M \simeq \left( \frac{1}{n+2} \right) \frac{1}{\pi^{(n+1)}} (m_0 M_*^n)^{\frac{1}{n+1}}, \quad (23)$$

and we also obtain simple expressions for $V_0$, $m_{\chi}$ and $m_a$ as

$$V_0 \simeq \frac{n+1}{n+2} m_0^2 M^2, \quad (24)$$

$$m_{\chi}^2 \simeq 2(n+1)m_0^2, \quad (25)$$

$$m_a^2 \simeq \frac{n(n+3)}{\sqrt{n+2}} m_{3/2}m_0. \quad (26)$$
We are now at the point to see how the flaton with the potential (16) causes a mini-inflation (= thermal inflation) at the late time of the evolution of the universe. For the thermal inflation to work, the flaton does not sit at the true minimum (17), but sits around the origin due to the finite temperature effects in the early universe. To realize it, the flaton has to interact rapidly with fields in the thermal bath of the universe. The Yukawa interaction

\[ W = g_\xi X \xi \overline{\xi}, \]  

with \( \xi \) and \( \overline{\xi} \) in the thermal bath is sufficient. Here \( g_\xi \) is the coupling. When the flaton sits at the true minimum, the fields \( \xi \) and \( \overline{\xi} \) obtain heavy masses \( m_\xi \simeq g_\xi M \). However, for the case that the flaton is near the origin, \( \xi \) and \( \overline{\xi} \) are almost massless and can be in the thermal bath if they couple strongly to the fields in the thermal bath. This condition is satisfied if, for example, \( \xi \) and \( \overline{\xi} \) are 5 and 5* in SU(5) gauge group respecting the unification of the gauge coupling constants. Then the interaction (27) gives an additional mass to the flaton in the early universe as

\[ V_{\text{eff}}(X) = V_0 + (cT^2 - m_0^2) |X|^2 - \frac{n}{n + 3} \frac{m_3^{3/2}}{M_*^n} (X^{n+3} + X^{*n+3}) + \frac{|X|^{2n+4}}{M_*^{2n}}, \]  

where \( c \) is the order one coupling and we take \( c = 1 \) in the following analysis. Thus for the cosmic temperature \( T > T_c \approx m_0 \) the flaton sits at the origin.

Here it should be noted that the Yukawa interaction (27) with \( g_\xi \sim 1 \) is also required from another reason. In fact, the interaction (27) with a large \( g_\xi \) can induce the negative mass squared at the origin in the potential (16) by the renormalization group effects.

During the time that the flaton is trapped at the origin, its vacuum energy \( V_0 \) becomes dominant and the universe expands exponentially, i.e., the thermal inflation takes place \[18\]. There are three possibilities for the form of the energy which dominates the universe before the thermal inflation occurs. If the radiation energy dominates the universe before thermal inflation, the thermal inflation starts when \( V_0 \) becomes comparable to the energy of the radiation at \( T = T_{\text{STI}} \sim V_0^{1/4} \). Then the thermal inflation lasts for \( T_{\text{STI}} > T > T_c \). On the other hand, if the universe before the thermal inflation is dominated by the energy of the modulus oscillation, the temperature at the beginning of the thermal inflation is estimated as \( T_{\text{STI}} \sim \left( \frac{V_0^2}{m_\phi M_G} \right)^{1/6} \) with the modulus mass \( m_\phi \). The universe before the thermal inflation might be dominated by the oscillating energy of the inflaton of the primordial inflation. In this case, \( T_{\text{STI}} \sim \frac{V_0^{1/8} M_G^{1/4} \Gamma_{\phi I}^{1/4}}{\Gamma_{\phi I}} \), where \( \Gamma_{\phi I} \) is the decay rate of the inflaton \( \phi_I \) [see Eq. (B11) in Appendix B]. Anyway, the flaton causes the thermal inflation for \( T_{\text{STI}} > T > T_c \) and it occurs just before the electroweak phase transition since \( T_c = m_0 \sim \Lambda_{\text{EW}} \).

We turn to the thermal history after the thermal inflation ends and see how the modulus abundance is diluted. When the cosmic temperature becomes smaller than \( T_c \), the flaton rolls down to its true minimum and oscillates around it. And the flaton decay occurs when the Hubble parameter becomes comparable to the total width \( \Gamma_\chi \) of the flaton. By the

\[ \text{In the case } \xi \text{ and } \overline{\xi} \text{ are 5 and 5* in SU(5), the constant } c \text{ takes a value } c = 5g_\xi^2/3. \]
flaton decay and the successive decay of the $R$-axion, the vacuum energy of the thermal inflation is transferred into the thermal bath and the universe is reheated. At this epoch the tremendous entropy is produced and the string moduli is diluted significantly.

Since we have assumed in the original thermal inflation model that $m_0$ satisfy Eq. (22), the flaton always decays into $R$-axions. This decay rate is given by

$$\Gamma(\chi \rightarrow 2a) = \frac{1}{64\pi} \frac{m^3_\chi}{M^2},$$

(29)

where we have neglected the mass of the $R$-axion. In addition, the flaton decay into two photons occurs through the Yukawa interaction (27) via one-loop diagrams of $\xi$ and $\bar{\xi}$. For example, $\xi$ and $\bar{\xi}$ are $5$ and $5^*$ in SU(5), its rate is estimated as

$$\Gamma(\chi \rightarrow 2\gamma) = \frac{2}{9\pi} \left(\frac{\alpha_{em}}{4\pi}\right)^2 \frac{m^3_\chi}{M^2}.$$  

(30)

Here we have neglected the effects of the SUSY breaking. In a similar way, the flaton can decay into two gluons if kinematically allowed, i.e., $m_\chi \gtrsim 1$ GeV, and the decay rate is given by

$$\Gamma(\chi \rightarrow 2g) = \frac{1}{4\pi} \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{m^3_\chi}{M^2}.$$  

(31)

Furthermore, the flaton might also decay into the gaugino pair. However, we forbid them since such a decay overproduces the LSPs and hence is cosmologically dangerous.

On the other hand, the flaton has a chance to couple directly to the SUSY standard model particles. Within the renormalizable interactions the suitable charges of the Higgs supermultiplets $H$ and $\bar{H}$ under the $Z_{n+3}$-symmetry allow the superpotential

$$W = \lambda_\mu X H \bar{H},$$  

(32)

where $\lambda_\mu$ is a dimensionless coupling. In this case the ordinary “$\mu$ term” is forbidden but is induced by the flaton vev as $\mu = \lambda_\mu M$, and hence the coupling constant $\lambda_\mu$ should be extremely small so that $\mu$ becomes the electroweak scale.\footnote{The electroweak scale of $\mu$ can be naturally understood by the nonrenormalizable interaction [31] as

$$W \sim \frac{X^2 H \bar{H}}{M_G},$$  

(33)

other than Eq. (32). However, the following discussion is almost same in both cases.} Through the interaction Eq. (32) the flaton can decay into Higgs bosons or Higgsinos if allowed. Their decay rate can be written as

$$\Gamma(\chi \rightarrow 2h) = C_h \frac{1}{16\pi} \frac{m^3_\chi}{M^2},$$  

(34)
where we have ignored the masses of final states. Here $C_h$ is a constant parameter and $C_h = (\lambda_\mu M/m_\chi)^4 = (\mu/m_\chi)^4$ for the decay into the Higgs bosons and $C_h = (\lambda_\mu/m_\chi)^2$ for the decay into Higgsino pair. In order that $\mu$ does not exceeds the electroweak scale $\mu \lesssim \Lambda_{EW} \sim m_\chi$, it should be $C_h \lesssim 1$. Since Higgsinos produced by the flaton decay might also lead to the overclosure of the LSP, we only consider the flaton decay into Higgs bosons. Note that when we take $\lambda_\mu \neq 0$, the charged Higgs and Higgsino also contribute to the flaton decay into two photons and its rate becomes

$$
\Gamma(\chi \to 2\gamma) = \frac{49}{72\pi} \left( \frac{\alpha_{\text{em}}}{4\pi} \right)^2 \frac{m_\chi^3}{M^2}.
$$

(35)

The $R$-axion produced by the flaton decay has the decay processes similar to the flaton. The $R$-axion can decay into two photons with the rate

$$
\Gamma(a \to 2\gamma) = \frac{2}{9\pi} \left( \frac{\alpha_{\text{em}}}{4\pi} \right)^2 \frac{m_a^3}{M^2},
$$

(36)

for $\lambda_\mu = 0$ and

$$
\Gamma(a \to 2\gamma) = \frac{49}{72\pi} \left( \frac{\alpha_{\text{em}}}{4\pi} \right)^2 \frac{m_a^3}{M^2},
$$

(37)

for $\lambda_\mu \neq 0$. In addition, it also decays into two gluons for $m_a \gtrsim 1$ GeV with the rate

$$
\Gamma(a \to 2g) = \frac{1}{4\pi} \left( \frac{\alpha_s}{4\pi} \right)^2 \frac{m_a^3}{M^2}.
$$

(38)

Note that the $R$-axion can not decay into Higgs boson pair even for $\lambda_\mu \neq 0$. Although it might decay into Higgsino pair, this process is assumed to be forbidden in the same way as the flaton decay. Therefore, the $R$-axion is assumed to have only the radiative decay modes.

Now we are ready to estimate how an entropy is produced by the decays of the flaton and the $R$-axion. When the Hubble parameter becomes comparable to the flaton’s total width $\Gamma_\chi$, the flaton decays into both the SM particles and the $R$-axions. Since the $R$-axion only has the interaction with the thermal bath suppressed by $1/M$, the flaton energy transferred into the $R$-axion could not reheat the universe at this time. Then the only the energy transferred into the SM particles reheats the universe at $T = T_{SM}$ by the flaton decay. The ratio of the entropy densities just before to after the flaton decay is estimated as

$$
\Delta_{SM} = 1 + (1 - \epsilon_a) \frac{4}{3 (2\pi^2/45) g_* T_c^3 T_{SM}},
$$

(39)

where $\epsilon_a$ denotes the branching ratio of the flaton decay into two $R$-axions.

After that, when the Hubble parameter becomes comparable to the total width of the $R$-axion, the $R$-axion decays into the SM particles occurs and reheats the universe at $T = T_R$. At this time the entropy of the universe increases by the factor $\Delta_a$:

$$
\Delta_a = 1 + \epsilon_a \frac{4}{3 (2\pi^2/45) g_* T_c^3 T_R} \left( \frac{2m_a}{m_\chi} \right) \frac{1}{\Delta_{SM}}.
$$

(40)
Here note that the energy transferred into the \( R \)-axions is diluted at the rate \( R^{-4} \) for \( T > m_a \) while diluted at \( R^{-3} \) for \( T < m_a \).

Then, all of the vacuum energy of the flaton \( V_0 \) is released into the thermal bath and the thermal inflation does increase the entropy of the universe by the factor \( \Delta \):

\[
\Delta = \Delta_{SM} \times \Delta_a,
\]

\[
= 1 + (1 - \epsilon_a) \frac{4}{3} \frac{V_0}{(2\pi^2/45)g^*T_{SM}^3} + \epsilon_a \frac{4}{3} \frac{V_0}{(2\pi^2/45)g^*T_R^3} \left( \frac{2m_a}{m_\chi} \right) .
\]  

(41)

Here the branching ratio of \( \chi \rightarrow 2a \) is \( \epsilon_a \approx 1 \) for the case that the flaton can not decay into Higgs bosons, because the radiative decay channels of the flaton are only induced by one-loop diagrams and their rates are significantly suppressed. On the other hand, if the flaton decay into Higgs bosons is allowed, \( \epsilon_a \approx 1/(1 + C_h/4) \). Even in this case, however, \( \Delta \) takes its maximum value when \( \epsilon_a \approx 1 \). This is because the \( R \)-axion decay rate is much smaller than the flaton decay rate and hence \( m_\chi T_R/(2m_a T_{SM}) \ll 1 \). Therefore in order to obtain the maximum entropy production, the flaton decay into Higgs bosons should be suppressed. In the following, in order to make a conservative analysis we take \( \epsilon_a = 1 \) and use

\[
\Delta \approx \frac{4}{3} \frac{V_0}{(2\pi^2/45)g^*T_R^3} \left( \frac{2m_a}{m_\chi} \right) .
\]  

(42)

For this (maximum) entropy production the reheating temperature \( T_R \) is determined by the total decay width of the \( R \)-axion which can be written as

\[
\Gamma_a = C_a m_a^3 M^2,
\]

(43)

where the parameter \( C_a \) depends on the decay mode and is given by [see Eqs. (36), (37), and (38)]

\[
C_a = \begin{cases} 
\frac{2}{9\pi} \left( \frac{\alpha_{em}}{4\pi} \right)^2 & \text{for } \lambda_\mu = 0 \\
\frac{49}{25\pi} \left( \frac{\alpha_{em}}{4\pi} \right)^2 & \text{for } \lambda_\mu \neq 0
\end{cases}
\]

(44)

for the case that the \( R \)-axion decays only into photons \( (m_a \lesssim 1 \text{ GeV}) \), and since a heavier \( R \)-axion dominately decays into two gluons

\[
C_a \approx \frac{1}{4\pi} \left( \frac{\alpha_s}{4\pi} \right)^2,
\]

(45)

for \( m_a \gtrsim 1 \text{ GeV} \). Then, the reheating temperature \( T_R \) is obtained as

\[
T_R \approx 0.96 \sqrt{\Gamma_a M_G} \approx 0.96 C_a^{1/2} m_a^{3/2} M_G^{1/2} M^{-1/2}.
\]

(46)

From Eqs. (24), (25), and (26) we can write the vev and the vacuum energy of the flaton as
\[ M \simeq 0.96\ C_a^{1/2} \frac{n^{3/4}(n+3)^{3/4} m_0^{3/4} m_{3/2}^{3/2} M_G^{1/2}}{(n+2)^{3/8}} \frac{1}{T_R}, \]  
(47)

\[ V_0 \simeq 0.92\ C_a \frac{n^{3/2}(n+1)(n+3)^{3/2}}{(n+2)^{7/4}} \frac{m_0^2 m_{3/2}^2 M_G^{1/2}}{T_R^2}. \]  
(48)

The entropy production factor (42), therefore, is given by

\[ \Delta \simeq 2.0 \times 10^{-2} \frac{n^2(n+1)^{1/2}(n+3)^2}{(n+2)^2} C_a \left( \frac{T_c}{m_0} \right)^{-3} \frac{m_{3/2}^2 M_G}{T_R^3}. \]  
(49)

Note that \( \Delta \) is independent on \( m_0 \) since \( m_0 \simeq T_c \). It can be seen that the lowest reheating temperature \( T_R \sim 10 \text{ MeV} \) gives the maximum entropy production as

\[ \Delta \simeq 8.4 \times 10^{17} \left( \frac{T_c}{m_0} \right)^{-3} \left( \frac{m_{3/2}}{1 \text{ GeV}} \right)^2 \left( \frac{T_R}{10 \text{ MeV}} \right)^{-3}, \]  
(50)

for the case \( n = 1 \) and the \( R \)-axion decays dominantly decays into gluons \( (m_a \gtrsim 1 \text{ GeV}) \). Therefore, the thermal inflation is found to produce a tremendous entropy at late time of the universe and can dilute all of the unwanted particles which are long-lived like the string moduli.

To end this section, we briefly discuss about the initial condition of the thermal inflation. In order to realize the thermal inflation, the flaton should be trapped at the origin of the potential by the thermal effects. Thus we have required the Yukawa interaction (27). If the flaton sits around the origin just after the primordial inflation, \( \xi \) and \( \xi \) become massless and give the flaton a mass comparable to \( T \). Therefore the key point of the initial condition of the thermal inflation is where the flaton sits just after the primordial inflation.

One may think that the flaton sits at the true minimum \( (\langle X \rangle = M) \) just after the primordial inflation. In this case, since \( \xi \) and \( \xi \) obtain very heavy masses, the maximum temperature \( T_{\text{MAX}} \) achieved after the inflation should be higher than their mass, i.e., \( T_{\text{MAX}} \gtrsim m_{\xi \xi} \simeq M \) in order to thermalize them. The maximum temperature is estimated as [see Appendix A]

\[ T_{\text{MAX}} \simeq 0.702 \ g_* (T_{\text{MAX}})^{-1/4} g_* (T_{RI})^{1/8} T_{RI}^{1/2} V_I^{1/8}, \]  
(51)

where the reheating temperature of the primordial inflation is \( T_{RI} \) and its vacuum energy is \( V_I \). Requiring \( T_{\text{MAX}} \gtrsim m_{\xi \xi} \), we obtain

\[ V_I^{1/4} \simeq 2.03 \ g_* (T_{\text{MAX}})^{1/2} g_* (T_{RI})^{-1/4} M^2 T_{RI}^{-1}, \]

\[ \simeq 7.64 \times 10^{17} \text{ GeV} \left( \frac{M}{10^{10} \text{ GeV}} \right)^2 \left( \frac{T_{RI}}{1 \text{ TeV}} \right)^{-1}. \]  
(52)

Thus the inflation model with very low \( T_{RI} \) should have very large vacuum energy.

However, one can avoid this difficulty by considering the effects of the supergravity. The flaton can sit at the origin due to the additional SUSY breaking effect from the large vacuum energy of the inflaton. In this case the flaton starts to roll down toward its true
minimum when $H \simeq m_0 \sim \Lambda_{EW}$. Then the flaton can be trapped at the origin, if the cosmic temperature at $H \simeq m_0$ is higher than the negative curvature at the origin, i.e., $T(H \simeq m_0) \gtrsim m_0$. This gives the lower bound on the reheating temperature $T_{RI}$ as

$$T_{RI} \gtrsim 7.10 \frac{m_0^{3/2}}{M_G^{3/2}},$$

$$\simeq 4.59 \times 10^{-6} \text{ GeV} \left( \frac{m_0}{100 \text{ GeV}} \right)^{3/2}. \quad (53)$$

This condition is always hold since $T_{RI}$ should be higher than about 10 MeV from the BBN observations. Therefore, the initial condition of the thermal inflation can be explained naturally by the supergravity effects.

IV. MODULI PROBLEM WITH ORIGINAL THERMAL INFLATION MODEL

In this section, we estimate the modulus abundance in the presence of the original thermal inflation explained in the previous section and examine whether it could solve the cosmological moduli problem or not.

If the Hubble parameter during the thermal inflation, $H_{TI}^{12}$, is larger than the modulus mass $m_\phi$, the modulus oscillation starts after the end of the thermal inflation. The ratio of the energy density of this oscillation to the entropy density $s$ after the reheating process of the thermal inflation is estimated as

$$\frac{\rho_\phi}{s} \simeq \frac{1}{8} T_R \left( \frac{m_\chi}{2 m_a} \right) \left( \frac{\phi_0}{M_G} \right)^2 \gtrsim 1.25 \times 10^{-3} \text{ GeV} \left( \frac{\phi_0}{M_G} \right)^2. \quad (54)$$

where $T_R$ is the final reheating temperature of the thermal inflation. Here we have used the fact that $T_R \gtrsim 10 \text{ MeV}$ from the BBN observations and $m_\chi > 2 m_a$ since the flaton always decays into to the R-axions [see Eq. (22)]. We have also assumed that the modulus oscillation starts at least before the reheating of the thermal inflation, i.e., we have assumed the modulus mass as

$$m_\phi > 1.1 \frac{T_R^2}{M_G} \simeq 4.5 \times 10^{-23} \text{ GeV} \left( \frac{T_R}{10 \text{ MeV}} \right)^2. \quad (55)$$

The abundance (54) becomes larger than the result Eq.(4) for $T_{RI} = T_R$. Therefore, the cosmological moduli problem could not solved in this case and we should consider the case $H_{TI} < m_\phi$.

In the following we will discuss the present modulus abundance in the presence of the original thermal inflation model with $H_{TI} < m_\phi$. Most of the previous works assumed that the oscillation of the modulus start after the reheating process of the primordial inflation completes. Thus the thermal inflation should occur after the end of the reheating of the

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12The Hubble parameter during the thermal inflation $H_{TI}$ is estimated as $H_{TI} = V_0^{1/2}/(\sqrt{3} M_G)$ with the vacuum energy of the flaton $V_0$. 

17
primordial inflation. However, various models of the primordial inflation do not meet this assumption and give completely different consequences. In the present article, we will consider more general cases. The key time scales of the discussion are the following three scales: (I) $H \simeq m_\phi$: the cosmic time when the modulus starts to oscillate. (II) $H \simeq \Gamma_{\varphi_I}$: the cosmic time when the reheating process of the primordial inflation ends. Here $\Gamma_{\varphi_I}$ denotes the decay rate of the inflaton $\varphi_I$ of the primordial inflation. (III) $H \simeq H_{TI}$: the cosmic time at which the thermal inflation occurs. Since we are considering the case $m_\phi > H_{TI}$ as mentioned above, the thermal history of the universe is classified into the following three cases in general.

- Case I: $\Gamma_{\varphi_I} \geq m_\phi \geq H_{TI}$.
- Case II: $m_\phi \geq \Gamma_{\varphi_I} \geq H_{TI}$.
- Case III: $m_\phi \geq H_{TI} \geq \Gamma_{\varphi_I}$.

For these three cases we will estimate the modulus abundance with the original thermal inflation model.\textsuperscript{13}

A. Case I: For the case $\Gamma_{\varphi_I} \geq m_\phi \geq H_{TI}$

First of all, we consider the case $\Gamma_{\varphi_I} \geq m_\phi \geq H_{TI}$ where a modulus field $\phi^{14}$ starts to oscillate after the reheating process of the primordial inflation completes. In this case, the reheating temperature of the primordial inflation, $T_{RI}$, which can be written as Eq. (5), should be

$$T_{RI} \geq T_\phi \simeq 7.2 \times 10^8 \text{ GeV} \left(\frac{m_\phi}{1 \text{ GeV}}\right)^{1/2},$$

where $T_\phi$ denotes the cosmic temperature when the modulus starts to oscillate [see Eq. (2)]. Therefore, a relatively higher reheating temperature is required and in some models of the inflation, such as chaotic or hybrid inflation, we can easily obtain such high $T_{RI}$ [29].

At $T = T_\phi$ the ratio between the energy density of the modulus oscillation to the entropy density is given by Eq. (3). After that, the energy density of the modulus is carried by the coherent oscillation. Hereafter, we call this modulus as “big-bang modulus”. In the presence of the original thermal inflation the present abundance\textsuperscript{15} of this big-bang modulus is diluted by the entropy production factor $\Delta$ [see Eq. (42)] and becomes

\textsuperscript{13}The modulus oscillation is considered to start after the primordial inflation, i.e., $m_\phi < H_I = V_{I}^{1/2}/(\sqrt{3}M_G)$.

\textsuperscript{14}Here we also assume one modulus field with $m_\phi \simeq m_{3/2}$ to make a conservative analysis.

\textsuperscript{15}For the unstable modulus of mass $m_\phi > 100$ MeV, it corresponds to the abundance just before the modulus decay.
\[
\left( \frac{\rho_\phi}{s} \right)_{BB} = \frac{1}{2} m_\phi^2 \delta \phi_0^2 \frac{2 \pi^2 g_* T^3_\phi}{45} \times \frac{1}{\Delta}. \tag{57}
\]

Moreover, it should be noted that the modulus energy is also produced after the thermal inflation. During the thermal inflation the modulus does not sit at the true minimum but is displaced from it by an amount \( \delta \phi_\sim (V_0/m_\phi^2 M_G^2) \phi_0 \), and this causes the secondary oscillation of the modulus [18]. We call this modulus as the “thermal-inflation” modulus. The abundance of the thermal-inflation modulus is given by

\[
\left( \frac{\rho_\phi}{s} \right)_{TI} = \frac{1}{2} m_\phi^2 \delta \phi_0^2 \frac{2 \pi^2 g_* T^3_\phi}{45} \times \frac{1}{\Delta} = \frac{1}{2} \frac{V_0^2}{m_\phi^2 M_G^2} \left( \phi_0 \right)^2 \times \frac{1}{\Delta}. \tag{58}
\]

Here notice that the thermal-inflation modulus starts to oscillate before the entropy production of the thermal inflation takes place and can be diluted. From Eq.(42) the both modulus abundances are expressed as

\[
\left( \frac{\rho_\phi}{s} \right)_{BB} \approx 3.8 \frac{m_\phi}{V_0} \left( \frac{T_\phi}{m_0} \right)^{3} \left( \frac{m_\phi}{2 m_a} \right) \left( \frac{\phi_0}{M_G} \right)^2, \tag{59}
\]

\[
\left( \frac{\rho_\phi}{s} \right)_{TI} \approx 0.38 \frac{V_0 T_R}{m_\phi^2 M^2_G} \left( \frac{m_\phi}{2 m_a} \right) \left( \frac{\phi_0}{M_G} \right)^2. \tag{60}
\]

Therefore, the total abundance of the modulus is given by

\[
\left( \frac{\rho_\phi}{s} \right)_0 \approx \text{MAX} \left[ \left( \frac{\rho_\phi}{s} \right)_{BB}, \left( \frac{\rho_\phi}{s} \right)_{TI} \right]. \tag{61}
\]

We turn to estimate the lower bound on this total abundance of the modulus and compare with various cosmological constraints. The original thermal inflation model is parameterized by two mass scales \( m_0 \) and \( M_\star \) besides the gravitino mass \( m_{3/2} (\sim m_\phi) \). Here we take \( m_0 \) and \( T_R \) as two free parameters since \( M_\star \) is the function of \( m_0, T_R \text{ and } m_{3/2} \). These two parameters are constrained as Eq. (22) for \( m_0 \) since the flaton is assumed to decay into \( R \)-axions in the original thermal inflation model, and \( T_R \gtrsim 10 \text{ MeV} \) from the BBN observations.

The lower bound on the total abundance of the modulus (61) can be estimated by using the fact

\[
\left( \frac{\rho_\phi}{s} \right)_0 \geq \sqrt{ \left( \frac{\rho_\phi}{s} \right)_{BB} \left( \frac{\rho_\phi}{s} \right)_{TI}}, \]

\[
\approx 0.84 \frac{(n + 1)^{1/2}(n + 2)^{1/4}}{n^{1/2}(n + 3)^{1/2}} \frac{m_0^2 T_R}{m_{3/2}^4 M_G^3} \left( \frac{m_{3/2}}{m_\phi} \right)^{1/2} \left( \frac{T_\phi}{m_0} \right)^{3/2} \left( \frac{\phi_0}{M_G} \right)^2, \tag{62}
\]

where the equality holds when \( \left( \frac{\rho_\phi}{s} \right)_{BB} = \left( \frac{\rho_\phi}{s} \right)_{TI} \), i.e.,

\[
m_0 \approx \frac{1.9}{C_\alpha^{1/2}} \frac{(n + 2)^{7/8}}{n^{3/4}(n + 1)^{1/2}(n + 3)^{3/4}} \frac{T_R M_G^{1/8}}{m_\phi^{1/8}} \left( \frac{m_{3/2}}{m_\phi} \right)^{-3/4} \left( \frac{T_\phi}{m_0} \right)^{3/4}. \tag{63}
\]

Therefore, we obtain
\[
\left( \frac{\rho_\phi}{s} \right)_0 \gtrsim \frac{2.9}{C_a} \frac{(n + 2)^2}{n^2(n + 1)^{1/2}(n + 3)^2} \frac{T_R^3}{m_{\phi}^{3/2} M_G^{1/2}} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 \tag{64}
\]

and the lowest reheating temperature \( T_R \simeq 10 \text{ MeV} \) gives the lower bound on the total abundance of the modulus. It should be noted that the dependence of the index \( n \) in the flaton superpotential (13) appears only in the pre-factor. For \( n = 1 \) we write down it in terms of the density parameter as

\[
\Omega_\phi h^2 \gtrsim 3.0 \times 10^{-2} \left( \frac{m_\phi}{1 \text{ GeV}} \right)^{-3/2} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 \tag{65}
\]

for the case that a \( R \)-axion can decay into gluons \((m_\phi \gtrsim 10 \text{ keV})\), and

\[
\Omega_\phi h^2 \gtrsim \begin{cases} 
8.7 \times 10^9 \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-3/2} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu = 0 \\
2.8 \times 10^9 \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-3/2} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu \neq 0
\end{cases} \tag{66}
\]

for the case that a \( R \)-axion can only decay into photons \((m_\phi \lesssim 10 \text{ keV})\).

However, the above estimation should be changed when the modulus mass becomes

\[
m_\phi \gtrsim 1.2 \text{ TeV} \frac{(n + 1)^4/9(n + 2)^{11/9}}{n^{4/9}(n + 3)^{2/3}(3n + 11)^{4/9}(5n + 13)^{4/9}} \left( \frac{m_{3/2}}{m_\phi} \right)^{-14/9} \left( \frac{T_c}{m_0} \right)^{2/3}
\]

\[
\approx 200 \text{ GeV} \left( \frac{m_{3/2}}{m_\phi} \right)^{-14/9} \left( \frac{T_c}{m_0} \right)^{2/3} \text{ for } n = 1,
\tag{67}
\]

because \( m_0 \) in Eq. (63) lies outside of the region (22) if one takes \( T_R \simeq 10 \text{ MeV} \). In such modulus mass region the reheating temperature which gives the lower bound on \( \left( \frac{\rho_\phi}{s} \right)_0 \) should be higher than 10 MeV and is given by

\[
T_R \simeq 7.1 \times 10^{-4} \frac{n^{7/4}(n + 3)^{3/4}(3n + 11)^{1/2}(5n + 13)^{1/2}}{(n + 1)^{1/2}(n + 3)^{1/2}} \frac{m_{3/2}^{9/8}}{M_G^{1/8}} \left( \frac{m_{3/2}}{m_\phi} \right)^{7/4} \left( \frac{T_c}{m_0} \right)^{-3/4}
\]

\[
\approx 60 \text{ MeV} \left( \frac{m_\phi}{1 \text{ TeV}} \right)^{9/8} \left( \frac{m_{3/2}}{m_\phi} \right)^{7/4} \left( \frac{T_c}{m_0} \right)^{-3/4} \text{ for } n = 1,
\tag{68}
\]

and then we obtain from Eq. (64)

\[
\Omega_\phi h^2 \gtrsim 2.0 \times 10^{-4} \left( \frac{m_\phi}{1 \text{ TeV}} \right)^{15/8} \left( \frac{m_{3/2}}{m_\phi} \right)^{13/4} \left( \frac{T_c}{m_0} \right)^{3/4} \left( \frac{\phi_0}{M_G} \right)^2 \text{ for } n = 1.
\tag{69}
\]

Here note that the \( R \)-axion is heavy enough to decay into gluons in the modulus mass region (67).

We show in Fig. 4 the lower bound on the total modulus abundance. It is found that only the modulus with mass \( m_\phi(\simeq m_{3/2}) \gtrsim 100 \text{ GeV} \) is cosmologically allowed. Thus the moduli problem in the HSSB models can be solved by the original thermal inflation model.
FIG. 4. The lower bounds on the modulus abundance in the presence of the original thermal inflation for the case I: $\Gamma_{\phi I} \geq m_\phi \geq H_{TI}$. We assume $m_\phi = m_{3/2}$, $T_c = m_0$ and $\phi_0 = M_G$ and take $n = 1$ and $b = 1$. The solid (dashed) line denotes the lower bound when $\lambda_\mu = 0$ ($\lambda_\mu \neq 0$). Upper bounds from various cosmological constraints are all shown by the dotted lines.

However, the light modulus $m_\phi \lesssim 1$ GeV predicted by the GMSB models is still faced with serious cosmological difficulties even if one assumes the original thermal inflation in this case I: $\Gamma_{\phi I} \geq m_\phi \geq H_{TI}$ [23].

We have considered the parameters of the original thermal inflation model, $m_0$ and $M_*$, (i.e., $m_0$ and $T_R$), as free parameters. Here we discuss how the lower bound on the modulus abundance obtained here is changed, when the cutoff scale of the original thermal inflation model is bounded from below as $M_* > M_{cr} \sim M_G$ [see the footnote in Sec. III]. This lower bound on $M_*$ leads to the lower bound on $m_0$ as

$$m_0 > (n + 2)^{1/2} \left[ \frac{1.2}{C^2 a^3 (n + 3)^3} \right]^{n+1 \over n-1} T_R^{4(n+1) \over 2n-1} m_{3/2}^{3(n+1) \over 2n-1} M_G^{2(n-1) \over 2n-1} \left( \frac{M_{cr}}{M_G} \right)^{4n \over n-1}. \quad (70)$$

Since this lower bound becomes more stringent as $n$ becomes large, which results in the larger abundance of the modulus, we take $n = 1$ in the following:

$$m_0 > 3.2 \times 10^{-2} \frac{T_R^4}{C^2 a^3} \frac{T_R}{m_{3/2}} \left( \frac{M_{cr}}{M_G} \right)^2. \quad (71)$$

Due to this lower bound on $m_0$ we find that in the modulus mass region

$$m_\phi < 0.18 \text{ GeV} \left( \frac{T_R}{10 \text{ MeV}} \right)^{24/23} \left( \frac{m_{3/2}}{m_\phi} \right)^{-18/23} \left( \frac{T_c}{m_0} \right)^{-6/23} \left( \frac{M_{cr}}{M_G} \right)^{16/23}, \quad (72)$$
FIG. 5. Same figure as Fig. 4 except for $M_\ast > M_G$. We also show the results in Fig. 4 by the thin lines.

the abundance of the thermal-inflation modulus becomes always larger than that of the big-bang modulus. Therefore the minimum of the total abundance of the modulus is given by using Eq. (71) as

$$
\left( \frac{\rho_\phi}{s} \right)_0 = \left( \frac{\rho_\phi}{s} \right)_{TI} \simeq \frac{4.9 \times 10^{-7}}{C_a^7} \frac{T_R^{15}}{m_\phi^{13} M_G} \left( \frac{m_{3/2}}{m_\phi} \right)^{-11} \left( \frac{M_{cr}}{M_G} \right)^8 \left( \frac{\phi_0}{M_G} \right)^2,
$$

and the lowest reheating temperature $T_R = 10 \text{ MeV}$ leads to

$$
\Omega_\phi h^2 \gtrsim 680 \left( \frac{m_\phi}{100 \text{ MeV}} \right)^{-13} \left( \frac{m_{3/2}}{m_\phi} \right)^{-11} \left( \frac{M_{cr}}{M_G} \right)^8 \left( \frac{\phi_0}{M_G} \right)^2,
$$

for $m_\phi \gtrsim 1 \text{ GeV}$. On the other hand, for the heavier mass region, the minimum of the modulus abundance is same as the previous one Eqs.(65) and (69), even if one takes $M_\ast \gtrsim M_{cr} \sim M_G$.

We show the result in Fig. 5. One finds that the minimum of the modulus abundance becomes much larger than the previous results in the mass region given by Eq. (72). On the other hand, the allowed region of the modulus mass $m_\phi \sim 100 \text{ GeV}$ still survives even if one takes $M_\ast \gtrsim M_{cr} \sim M_G$. Thus, for the case I: $\Gamma_\phi \geq m_\phi \geq H_{TI}$ the original thermal inflation can naturally solve the cosmological difficulties of the string moduli particles if the gravitino mass is $m_{3/2}(\sim m_\phi) \gtrsim 100 \text{ GeV}$ which is predicted in the HSSB models, while cannot solve the moduli problem in the lighter gravitino mass region of the GMSB models.
B. Case II: For the case $m_\phi \geq \Gamma_{\varphi_I} \geq H_{TI}$

Next we consider the case that $m_\phi \geq \Gamma_{\varphi_I} \geq H_{TI}$. In this case, the big-bang modulus starts to oscillate before the reheating process of the primordial inflation completes. Thus the energy density of the modulus is diluted by the primordial inflation as Eq. (4). Then the original thermal inflation takes place after that, and the mass density of the big-bang modulus is further reduced by the thermal inflation as

$$\left(\frac{\rho_\phi}{s}\right)_{BB} \approx \frac{1}{8} T_{RI} \left(\frac{\phi_0}{M_G}\right)^2 \times \frac{1}{\Delta} \approx 5.8 \times 10^{-2} \Gamma_{\varphi_I}^{1/2} M_G^{1/2} \left(\frac{\phi_0}{M_G}\right)^2 \times \frac{1}{\Delta},$$

with the reheating temperature

$$T_\phi \geq T_{RI} \geq 0.351 \frac{1}{1/4},$$

where $T_\phi$ is the temperature when the big-bang modulus starts to oscillate and is given by Eq. (2). This abundance takes its minimum value when $\Gamma_{\varphi_I} = H_{TI}$ as

$$\left(\frac{\rho_\phi}{s}\right)_{BB} \geq \left(\frac{\rho_\phi}{s}\right)_{BBm} = 4.4 \times 10^{-2} \frac{1}{1/4} \left(\frac{\phi_0}{M_G}\right)^2 \times \frac{1}{\Delta}. \quad (77)$$

On the other hand, the abundance of the thermal-inflation modulus is the same as Eq. (58) in the previous case I.

Let us estimate the lower bound on the total abundance of the modulus (61) in this case. One can write $\left(\frac{\rho_\phi}{s}\right)_{BBm}$ and $\left(\frac{\rho_\phi}{s}\right)_{TI}$ in terms of $m_0$ and $T_R$ as

$$\left(\frac{\rho_\phi}{s}\right)_{BBm} \approx \frac{2.2}{C a_{3/4}^{5/4}} \frac{(n + 2)^{25/16}}{n^{13/8}(n + 1)^{1/4}(n + 3)^{13/8}} \frac{m_0^{7/8} T_R^{5/2}}{m_{3/2}^{13/8} M_G^{3/4}} \left(\frac{T_c}{m_0}\right)^3 \left(\frac{\phi_0}{M_G}\right)^2, \quad (78)$$

$$\left(\frac{\rho_\phi}{s}\right)_{TI} \approx \frac{0.24}{C a} \frac{n(n + 1)^{3/2}(n + 3)}{(n + 2)^{3/2}} \frac{m_0^4 m_{3/2}^2}{m_{\phi}^2 M_G T_R} \left(\frac{\phi_0}{M_G}\right)^2. \quad (79)$$

Both abundances become larger as $m_0$ becomes larger, and $\left(\frac{\rho_\phi}{s}\right)_{BBm} = \left(\frac{\rho_\phi}{s}\right)_{TI}$ is achieved when

$$m_0 = (m_0)_{eq} \approx \frac{2.01}{C a_{14/25}^{14/25}} \frac{(n + 2)^{49/50}}{n^{31/25}(n + 1)^{14/25}(n + 3)^{21/25}} \frac{m_0^{16/25} M_G^{2/25} T_R^{28/25}}{m_{3/2}^{21/25}} \left(\frac{T_c}{m_0}\right)^{24/25}. \quad (80)$$

Therefore the minimum of the total modulus abundance is given by $\left(\frac{\rho_\phi}{s}\right)_{BBm}$ with the lowest values of $T_R$ and $m_0 (\leq (m_0)_{eq})$. The lower bound on $m_0$ Eq. (22) gives

$$\left(\frac{\rho_\phi}{s}\right)_0 \geq \left(\frac{\rho_\phi}{s}\right)_{BBm} \geq \frac{1.2}{C a_{3/4}^{3/4}} \frac{(n + 2)^{9/8}(3n + 11)^{7/16}(5n + 13)^{7/16}}{n^{9/8}(n + 1)^{9/8}(n + 3)^{13/8}} \frac{T_R^{5/2}}{m_{3/2}^{3/4} M_G^{3/4}} \left(\frac{T_c}{m_0}\right)^3 \left(\frac{\phi_0}{M_G}\right)^2. \quad (81)$$
For \( n = 1 \) the lowest reheating temperature \( T_R = 10 \text{ MeV} \) leads to

\[
\Omega_\phi h^2 \gtrsim 7.3 \times 10^{-7} \left( \frac{m_\phi}{1 \text{ GeV}} \right)^{-3/4} \left( \frac{m_{3/2}}{m_\phi} \right)^{-3/4} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2, \tag{82}
\]

for the case that the \( R \)-axion can decay into gluons \( (m_\phi \gtrsim 100 \text{ MeV}) \), and

\[
\Omega_\phi h^2 \gtrsim \begin{cases} 
5.2 \times 10^{-2} \left( \frac{m_\phi}{100 \text{ keV}} \right)^{-3/4} \left( \frac{m_{3/2}}{m_\phi} \right)^{-3/4} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu = 0 \\
2.3 \times 10^{-2} \left( \frac{m_\phi}{100 \text{ keV}} \right)^{-3/4} \left( \frac{m_{3/2}}{m_\phi} \right)^{-3/4} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu \neq 0
\end{cases}, \tag{83}
\]

for the case that the \( R \)-axion cannot decay into gluons \( (m_\phi \lesssim 100 \text{ MeV}) \).

However, for the modulus mass region \( m_\phi \lesssim 10^{-5} \text{ GeV} \), when the total abundance takes its minimum value Eq. (83), \( m_0 \) becomes so small that the vacuum energy of the original thermal inflation \( V_0 \) is less than the energy of the radiation at \( T = T_R \). To avoid this failure, \( m_0 \) should be

\[
m_0 > \frac{1.5}{C_a^{2/7}} \frac{(n + 2)^{1/2}}{n^{3/7} (n + 1)^{2/7} (n + 3)^{3/7}} \frac{T_R^{12/7}}{m_{3/2}^3 M_G} \left( \frac{m_{3/2}}{m_\phi} \right)^{-3/7}. \tag{84}
\]

Therefore the minimum of the total abundance is given by

\[
\left( \frac{\rho_\phi}{s} \right)_0 \gtrsim \frac{3.0}{C_a} \frac{(n + 2)^2}{n^2 (n + 1)^{1/2} (n + 3)^2} \frac{T_R^4}{m_{3/2}^3 M_G} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2. \tag{85}
\]

For \( n = 1 \) we obtain

\[
\Omega_\phi h^2 \gtrsim \begin{cases} 
59 \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-2} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu = 0 \\
19 \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-2} \left( \frac{m_{3/2}}{m_\phi} \right)^{-2} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu \neq 0
\end{cases}. \tag{86}
\]

On the other hand, for the modulus mass region \( m_\phi \gtrsim 10 \text{ GeV} \), in order that \( m_0 \) satisfies both \( m_0 \leq (m_0)_{eq} \) and Eq. (22), the reheating temperature should be higher than 10 MeV as

\[
T_R \gtrsim 7.7 \times 10^{-4} \frac{n^{23/14} (n + 3)^{3/4} (3n + 11)^{25/56} (5n + 13)^{25/56}}{(n + 1)^{11/28} (n + 2)^{37/28}} \frac{m_{15/14}}{m_{14}^{15/14}} \frac{m_{3/2}^{23/14}}{m_{3/2}^{23/14}} \left( \frac{T_c}{m_0} \right)^{-6/7},
\]

\[
\simeq 31 \text{ MeV} \left( \frac{m_\phi}{100 \text{ GeV}} \right)^{15/14} \left( \frac{m_{3/2}}{m_\phi} \right)^{23/14} \left( \frac{T_c}{m_0} \right)^{-6/7} \text{ for } n = 1. \tag{87}
\]

Therefore we obtain from Eq. (81)

\[
\Omega_\phi h^2 \gtrsim 4.0 \times 10^{-7} \left( \frac{m_\phi}{100 \text{ GeV}} \right)^{27/14} \left( \frac{m_{3/2}}{m_\phi} \right)^{47/14} \left( \frac{T_c}{m_0} \right)^{6/7} \left( \frac{\phi_0}{M_G} \right)^2 \text{ for } n = 1. \tag{88}
\]
We show the obtained lower bound on the total abundance of the modulus in Fig. 6. It is found that since the primordial inflation does dilute the energy of the big-bang modulus, the lower bound becomes weaker than the previous case I and the allowed modulus mass regions are \( m_\phi \gtrsim 1 \text{ GeV} \) and \( m_\phi \simeq 10 \text{ keV}–1 \text{ MeV} \). Therefore, the gravitino mass region predicted by the HSSB scenario can be cosmologically viable and it should be noted that small window for the gravitino mass range in the GMSB models does appear. This feature is crucially different from the previous results.

However, when we take the cutoff scale of the original thermal inflation model as \( M_* > M_{cr} \sim M_G \), this new window is disappeared as shown in Fig. 7. The lower bound on the total abundance of the modulus becomes more stringent for \( m_\phi \lesssim 1 \text{ GeV} \) where the abundance of the thermal-inflation modulus is always larger than the big-bang modulus one and then the limit is the same as Eq. (74) in the previous case. Therefore, in order that the original thermal inflation dilutes sufficiently the light modulus of mass \( m_\phi \simeq 10 \text{ keV}–1 \text{ MeV} \) in GMSB models, the extremely low cut off scale as \( M_* \sim 10^6–10^{10} \text{ GeV} \) is required. On the other hand, the allowed region for \( m_\phi \gtrsim 1 \text{ GeV} \) still exists even for the case \( M_* \gtrsim M_G \).

C. Case III: For the case \( m_\phi \geq H_{TI} \geq \Gamma_{\varphi_I} \)

Finally we consider the case \( m_\phi \geq H_{TI} \geq \Gamma_{\varphi_I} \), where the reheating process of the primordial inflation completes after the thermal inflation ends and its reheating temperature becomes extremely low. In this case the present abundance of the big-bang modulus is given by [see Appendix B]
FIG. 7. Same figure as Fig. 6 except for $M_\ast > M_G$. We also show the results in Fig. 6 by the thin lines.

\[
\left(\frac{\rho_\phi}{s}\right)_{BB} \simeq 4.8\frac{m_0^4 T_R}{\Gamma_{\varphi I} V_0} \left(\frac{m_\chi}{2 m_a}\right) \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2. 
\]

This ratio takes its minimum value when $\Gamma_{\varphi I} = H_{T I}$ as

\[
\left(\frac{\rho_\phi}{s}\right)_{BB} \simeq \left(\frac{\rho_\phi}{s}\right)_{BBm} = 8.2\frac{m_0^4 T_R}{V_0} \left(\frac{m_\chi}{2 m_a}\right) \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2. 
\]

As well as the previous cases we can rewrite in terms of $m_0$ and $T_R$ as

\[
\left(\frac{\rho_\phi}{s}\right)_{BBm} \simeq 6.3 \frac{(n+2)^2}{C_a n^2(n+1)^{1/2}(n+3)^2} \frac{m_0 T_R^3}{m_0^{2/3} M_G} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2. 
\]

On the other hand the abundance of the thermal-inflation modulus is the same as the previous two cases [see Eq. (79)]. Now the condition $\left(\frac{\rho_\phi}{s}\right)_{BBm} = \left(\frac{\rho_\phi}{s}\right)_{T I}$ holds when

\[
m_0 = (m_0)_{eq} \simeq 3.0 \frac{C_a^{2/3}}{n(n+1)^{2/3}(n+3)} \frac{(n+2)^{7/6}}{m_0^{2/3} T_R^{4/3}} \frac{m_\phi^{2/3} T_R^{4/3}}{m_0^{2/3}} \left(\frac{T_c}{m_0}\right)^{4/3}. 
\]

Therefore the abundance of the big-bang modulus (91) with the lowest values of $T_R$ and $m_0(\leq (m_0)_{eq})$ gives the lower bound on the total modulus abundance (61). From the lower bound on $m_0$ [Eq. (22)] in order to allow the flaton decay into $R$-axions we find
\begin{align}
\frac{(\rho_\phi/s)_0}{B B m} & \gtrsim \frac{3.2}{C_\alpha} \frac{(n + 2)^{3/2}(3n + 11)^{1/2}(5n + 13)^{1/2}}{n(n + 1)^{3/2}(n + 3)^2} \frac{T^2_R}{m_{3/2}M_G} \frac{T_c}{m_0}^4 \left(\frac{\phi_0}{M_G}\right)^2. \tag{93}
\end{align}

For \( n = 1 \) the lowest reheating temperature \( T_R = 10 \text{ MeV} \) gives
\begin{align}
\Omega_\phi h^2 \gtrsim 0.96 \times 10^{-10} \left(\frac{m_\phi}{1 \text{ GeV}}\right)^{-1} \left(\frac{m_{3/2}}{m_\phi}\right)^{-1} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2, \tag{94}
\end{align}
when the \( R \)-axion can decay into gluons \( (m_\phi \gtrsim 100 \text{ MeV}) \), and
\begin{align}
\Omega_\phi h^2 \gtrsim \begin{cases}
2.8 \times 10^{-5} \left(\frac{m_\phi}{1 \text{ MeV}}\right)^{-1} \left(\frac{m_{3/2}}{m_\phi}\right)^{-1} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu = 0, \\
0.92 \times 10^{-5} \left(\frac{m_\phi}{1 \text{ MeV}}\right)^{-1} \left(\frac{m_{3/2}}{m_\phi}\right)^{-1} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu \neq 0,
\end{cases} \tag{95}
\end{align}
when the \( R \)-axion cannot decay into two gluons \( (m_\phi \lesssim 100 \text{ MeV}) \).

In the same way as the previous case II, however, for the modulus mass \( m_\phi \lesssim 10 \text{ keV} \) we find from the lower bound on \( m_0 \) \( \text{[Eq. (84)]} \)
\begin{align}
\frac{(\rho_\phi/s)_0}{B B m} \gtrsim \frac{9.3}{C_\alpha^{9/7}} \frac{(n + 2)^{5/2}}{n^{17/7}(n + 1)^{11/4}(n + 3)^{17/7}} \frac{T_R^{33/7}}{m_{3/2}^{17/7}M_G^{9/7}} \frac{T_c}{m_0}^4 \left(\frac{\phi_0}{M_G}\right)^2, \tag{96}
\end{align}
and taking \( T_R = 10 \text{ MeV} \) we obtain for \( n = 1 \)
\begin{align}
\Omega_\phi h^2 \gtrsim \begin{cases}
1.6 \left(\frac{m_\phi}{1 \text{ keV}}\right)^{-17/7} \left(\frac{m_{3/2}}{m_\phi}\right)^{-17/7} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu = 0, \\
0.39 \left(\frac{m_\phi}{1 \text{ keV}}\right)^{-17/7} \left(\frac{m_{3/2}}{m_\phi}\right)^{-17/7} \left(\frac{T_c}{m_0}\right)^4 \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu \neq 0,
\end{cases} \tag{97}
\end{align}

When the total abundance takes its minimum value, the reheating temperature becomes higher than \( 10 \text{ MeV} \) for \( m_\phi \gtrsim 3 \text{ GeV} \) as
\begin{align}
T_R \gtrsim 7.0 \times 10^{-4} \frac{n^{3/2}(n + 3)^{3/4}(3n + 11)^{3/8}(5n + 13)^{3/8} m_{3/2}^{3/2}}{(n + 1)^{1/4}(n + 2)^{5/4}} \frac{T_c}{m_0}^{-1},
\end{align}
\begin{align}
\approx 3.4 \text{ GeV} \left(\frac{m_\phi}{1 \text{ TeV}}\right)^{3/2} \left(\frac{m_{3/2}}{m_\phi}\right)^{3/2} \left(\frac{T_c}{m_0}\right)^{-1} & \text{ for } n = 1,
\end{align}
and from Eq. (93) we obtain for \( n = 1 \)
\begin{align}
\Omega_\phi h^2 \gtrsim 3.7 \times 10^{-6} \left(\frac{m_\phi}{1 \text{ TeV}}\right)^2 \left(\frac{m_{3/2}}{m_\phi}\right)^{7/2} \left(\frac{T_c}{m_0}\right) \left(\frac{\phi_0}{M_G}\right)^2, \tag{99}
\end{align}
where the flaton dominately decays into gluons.
Original Thermal Inflation Model (Case III)

![Graph](Original Thermal Inflation Model (Case III)

\(n=1, b=1\)

\[\Omega_{\phi} h^2\] vs \(m_{\phi} [\text{GeV}]\)

**FIG. 8.** Same figure as Fig. 4 for the case III: \(m_{\phi} \geq H_{TI} \geq \Gamma_{\phi I}\).

We show the lower bound on the total abundance of the modulus in Fig. 8. It is found that the lower bound becomes smaller than those in the previous two cases, because the big-bang modulus is also diluted by the primordial inflation during the thermal inflation. In this case the allowed regions for the modulus mass are \(m_{\phi} \simeq 1\) keV–10 MeV and \(m_{\phi} \gtrsim 300\) MeV, and the allowed mass region in the GMSB models extends wider than the case II.

In these allowed modulus mass regions, in order to obtain the lowest modulus abundance the required reheating temperature of the “primordial inflation” is \(T_{RI} \simeq 10\) MeV for \(m_{\phi} \simeq 1\) keV–10 keV, and

\[
T_{RI} \simeq \begin{cases} 
1.6 \text{ GeV} \left( \frac{m_{\phi}}{1 \text{ MeV}} \right)^{5/4} \left( \frac{m_{3/2}}{m_{\phi}} \right)^{5/4} & \text{for } \lambda_{\mu} = 0 \\
2.1 \text{ GeV} \left( \frac{m_{\phi}}{1 \text{ MeV}} \right)^{5/4} \left( \frac{m_{3/2}}{m_{\phi}} \right)^{5/4} & \text{for } \lambda_{\mu} \neq 0 
\end{cases}
\]  

(100)

for \(m_{\phi} \simeq 10\) keV–10 MeV. On the other hand, for the heavier modulus of mass \(m_{\phi} \gtrsim 3\) GeV, \(T_{RI}\) is given by

\[
T_{RI} \simeq 5.5 \times 10^6 \text{ GeV} \left( \frac{m_{\phi}}{1 \text{ TeV}} \right)^{3/4} \left( \frac{m_{3/2}}{m_{\phi}} \right)^{1/2} \left( \frac{T_c}{m_0} \right)^{1/2}.
\]  

(101)

Therefore, extremely low reheating temperature \(T_{RI} \sim 10\) MeV–10 GeV is required to dilute sufficiently the light modulus of mass \(m_{\phi} \simeq 1\) keV–10 MeV, even if one assume the original thermal inflation model.
FIG. 9. Same figure as Fig. 8 except for $M_* > M_G$. We also show the results in Fig. 8 by the thin lines.

Furthermore, if one takes the cutoff scale as $M_* \gtrsim M_G$, the allowed region for the modulus mass $m_\phi \simeq 1$ keV–10 MeV vanishes as shown in Fig. 9, and only the modulus of mass $m_\phi \gtrsim 1$ GeV is allowed. As well as the previous cases I and II, the lower bound on the total modulus abundance when $M_* \gtrsim M_G$ is given by Eq. (74) for $m_\phi \lesssim 1$ GeV. Therefore, in order to dilute sufficiently the light modulus, the thermal inflation model with the extremely low cut-off scale $M_* \sim 10^5$–$10^{12}$ GeV is required for $m_\phi \sim 1$ keV–10 MeV, as well as the low reheating temperature of the primordial inflation as $T_{RI} \sim 10$ MeV–10 GeV.

V. MODIFIED THERMAL INFLATION MODEL

As showed in the previous section, the original thermal inflation can dilute the relic abundance of the string modulus significantly, and the modulus mass regions $m_\phi(\simeq m_{3/2}) \sim 1$ keV–10 MeV, and $m_\phi \gtrsim 300$ MeV survive the various cosmological constraints if we consider the model of the primordial inflation with an extremely low reheating temperature. However, if we take the gravitational scale as the natural cutoff scale of the thermal inflation model, i.e. $M_* \gtrsim M_G$, the former allowed region vanishes and only the modulus mass region $m_\phi \gtrsim 1$ GeV predicted by the HSSB models is allowed. To dilute sufficiently the light modulus particle predicted by the GMSB models, one needs the relatively low cutoff scale $M_* \sim 10^5$ GeV–$10^{12}$ GeV for $m_\phi \sim 1$ keV–10 MeV. This weak point of the original model comes from the fact that the flaton can always decay into $R$-axions. The vacuum energy of the thermal inflation is mostly transferred into relativistic $R$-axions by the flaton decay and the $R$-axions lose their energy faster than non-relativistic particles as the universe expands,
which leads to much less entropy production (i.e., less dilution of the modulus density). Therefore, one might expect to dilute the modulus density more effectively by the thermal inflation model which the decay process $\chi \rightarrow 2a$ is not kinematically allowed.

Furthermore, the original model is faced with another serious difficulty. Since the potential of the flaton possesses the exact discreet $Z_{n+3}$ symmetry to ensure the flatness of the potential, the degenerate minima leads to the domain wall problem. In order to avoid this problem one has to introduce a term which breaks the symmetry explicitly in the potential.

One of economical modifications of the original thermal inflation model, which solves above two difficulties simultaneously, has been proposed by Ref. [23]. A linear term which breaks the $Z_{n+3}$ completely is added to the original superpotential of the flaton Eq. (13) as

$$W \simeq \frac{1}{(n+3)} \frac{X^{n+3}}{M^2_n} + C + \alpha X,$$

with the dimensionfull parameter $\alpha$ which is required to be

$$|\alpha| \gtrsim \frac{m_{3/2}^2 M}{M_{pl}^2}$$

in order to eliminate the domain walls [30]. Then this superpotential gives the low energy potential to the flaton as

$$V_{\text{eff}}(X) = V_0 - \frac{2\alpha C}{M^2_G} (X + X^*) - m_0^2 |X|^2 + \frac{\alpha}{M^n} (X^{n+2} + X^{*n+2})$$

$$+ \frac{n}{n+3} \frac{C}{M^2_G M^n} (X^{n+3} + X^{*n+3}) + \frac{|X|^{2n+4}}{M^{2n}_n}.$$  (104)

The vev of the flaton is estimated as

$$\langle X \rangle = M \simeq \left[ \frac{1}{(n+2)(1-x)} \right]^{\frac{1}{2(n+1)}} (m_0 M^n_x)^{\frac{1}{2(n+1)}},$$

and the vacuum energy $V_0$ is

$$V_0 \simeq \frac{n(1-x) + 1}{(n+2)(1-x)} m_0^2 M^2.$$  (105)

with the dimensionless parameter $x < 1$ which is defined as $\alpha = -x M^{n+2}/M^n_x$. Here we assumed $m_0 \gg m_{3/2}$. It should be noted that even in the presence of the explicit breaking term in Eq. (102) the dynamics of the thermal inflation is shown not to change much if $m_0 \gg m_{3/2}$ and $x$ is not so close to one [23].

In the flaton potential (104) we assume the SUSY breaking mass $m_0$ at the origin as well as in the original model. Therefore, when we work in the light gravitino mass region of the GMSB models, we obtain the constraint that the vev of the flaton $M$ should be smaller than the masses of the messenger multiplets. In the modified model, this constraint plays a significant role when we take the cutoff scale as $M_\star \gtrsim M_G$ [see the discussion in Sec. VI].

Then, masses of the flaton and the $R$-axion are given by
\[
\begin{align*}
m^2_\chi & \simeq \frac{2(n+1)-nx}{1-x} m^2_0, \\
m^2_a & \simeq \frac{(n+2)x}{1-x} m^2_0.
\end{align*}
\]

Therefore, if \( x \) takes a value
\[
(1 > ) x \geq \frac{2(n+1)}{5n+8} \equiv x_{\text{min}},
\]
the flaton decay into two \( R \)-axions is kinematically forbidden.\(^{16}\) Therefore, this modified inflation model with \( m_0 \gg m_{3/2} \) and \( x \) in the region (109) gives one way to forbid the flaton decay into \( R \)-axions and as well as to solve the domain wall problem. In the following we take \( x = x_{\text{min}} \) for simplicity and \( m_0 \) as
\[
m^2_0 > \frac{n^2(3n+11)(5n+13)}{4(n+1)^2(n+2)} m^{3/2}_{3/2} = \frac{21}{4} m^{3/2}_{3/2} \text{ for } n = 1.
\]

for the purpose of the comparison with the previous results in the original model. This is the same condition for \( m_0 \) as assumed in the original thermal inflation model [Eq. (22)].

Then we turn to discuss the thermal history after the modified thermal inflation ends. In the modified model the vacuum energy of the flaton is completely transferred into the thermal bath when the Hubble parameter becomes comparable to the total width of the flaton \( \Gamma_\chi \) and reheats the universe at the temperature \( T = T_R \). As well as the case of the \( R \)-axion, the total width \( \Gamma_\chi \) can be written as [see Sec. III]
\[
\Gamma_\chi = C_\chi \frac{m^3_\chi}{M^2}.
\]

Here \( C_\chi \) is given by
\[
C_\chi = \begin{cases} 
\frac{2}{9\pi} \left( \frac{\alpha_{em}}{4\pi} \right)^2 & \text{for } \lambda_\mu = 0 \\
\frac{49}{72\pi} \left( \frac{\alpha_{em}}{4\pi} \right)^2 & \text{for } \lambda_\mu \neq 0
\end{cases},
\]

for \( m_\chi \lesssim 1 \text{ GeV}, \)
\[
C_\chi \simeq \frac{1}{4\pi} \left( \frac{\alpha_\chi}{4\pi} \right)^2,
\]

for \( 1 \text{ GeV} \lesssim m_\chi \leq 2 m_h \) (\( m_h \) is the Higgs boson mass.\(^{17}\)), and

\(^{16}\)In this region of \( x \), the domain wall problem is solved, i.e., \( \alpha \) satisfies Eq. (103).

\(^{17}\)It should be regarded as the lightest Higgs boson mass and we take \( m_h \simeq 70 \text{ GeV} \).
C_{\chi} \simeq \frac{C_h}{16\pi} + \frac{1}{4\pi} \left( \frac{\alpha_{em}}{4\pi} \right)^2, \quad (114)

for m_\chi > 2m_h. Then the reheating temperature is obtained as

\[ T_R = 0.96\sqrt{\Gamma_\chi M_G}. \quad (115) \]

By using this reheating temperature, the modified thermal inflation model increases the entropy of the universe by a factor

\[ \Delta \simeq 1 + \frac{4}{3} \frac{V_0}{(2\pi^2/45)g_\star T_c^3 T_R}, \quad (116) \]

which is obtained by putting \( \epsilon_a = 0 \) and substituting \( T_R \) for \( T_{R,SM} \) in Eq. (41). Comparing with the entropy production factor Eq. (42) in the original thermal inflation model you can see that the suppression factor \((2m_a/m_\chi)\) is dropped off so that the relic density of the string modulus is expected to be diluted more effectively.

VI. MODULI PROBLEM WITH MODIFIED THERMAL INFLATION MODEL

In this section we examine whether the modified thermal inflation model could solve the cosmological moduli problem or not. Crucial difference from the original model is that the flaton decay into two \( R \)-axions is kinematically forbidden. Then the reheating temperature \( T_R \) and also the entropy production factor \( \Delta \) of the modified thermal inflation are determined by the flaton decay.

A. Case I: For the case \( \Gamma_{\varphi_I} \geq m_\phi \geq H_{TI} \)

First, we consider the case I: \( \Gamma_{\varphi_I} \geq m_\phi \geq H_{TI} \). In the presence of the modified thermal inflation model with the entropy production factor (116), the abundances of the big-bang modulus and the thermal inflation modulus are given from Eqs. (57) and (58) as

\( \left( \frac{\rho_\phi}{s} \right)_{BB} \simeq 3.8 \frac{m_{\phi}^{1/2} M_G^{1/2} m_0^3 T_R}{V_0} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2, \quad (117) \)

\( \left( \frac{\rho_\phi}{s} \right)_{TI} \simeq 0.38 \frac{V_0 T_R}{m_{\phi}^2 M_G^2} \left( \frac{\phi_0}{M_G} \right)^2. \quad (118) \)

Then, as well as for the original thermal inflation model, we find the minimum of the total modulus abundance given by Eq. (61) in the parameter space of \( m_0 \) and \( T_R \).

The vacuum energy of the modified thermal inflation (106) can be written in terms of \( m_0 \) and \( T_R \) as

\[ V_0 \simeq C_{V0} \frac{m_0^5 M_G}{T_R^2}. \quad (119) \]

Here \( C_{V0} \) is defined as
\[ C_{V0} = 0.92 \, C_{\chi} \left[ \frac{2(n+1) - nx}{1-x} \right]^{3/2} \frac{n(1-x) + 1}{(n+2)(1-x)}, \]

\[ \simeq 9.3 \, C_{\chi} \text{ for } n = 1, x = x_{\text{min}}. \]  

From Eq. (119) we can express both abundances in term of \( m_0 \) and \( T_R \) as

\[ \left( \frac{\rho_\phi}{s} \right)_{BB} \approx \frac{3.8}{C_{V0}} \frac{m_{\phi}^{1/2} T^3}{m_0^2 M_G^{1/2}} \left( \frac{T_c}{m_0} \right)^3 \left( \frac{\phi_0}{M_G} \right)^2, \]  

\[ \left( \frac{\rho_\phi}{s} \right)_{TI} \approx 0.38 \frac{m_0^5}{T_R m_0^2 M_G} \left( \frac{\phi_0}{M_G} \right)^2. \]  

The dependence on \( m_0 \) tells that the total abundance takes its minimum value when \( \left( \frac{\rho_\phi}{s} \right)_{BB} = \left( \frac{\rho_\phi}{s} \right)_{TI} \) is achieved, i.e.,

\[ m_0 = \frac{1.4}{C_{V0}^{2/7}} m_{\phi}^{5/14} T_R^{4/7} M_G^{1/14} \left( \frac{T_c}{m_0} \right)^{3/7}, \]  

and we obtain

\[ \left( \frac{\rho_\phi}{s} \right)_0 \geq \sqrt{\left( \frac{\rho_\phi}{s} \right)_{BB} \left( \frac{\rho_\phi}{s} \right)_{TI}} \approx 1.2 \frac{m_0^{3/2} T_R^{3/2}}{m_{\phi}^{3/4} M_G^{3/4}} \left( \frac{T_c}{m_0} \right)^{3/2} \left( \frac{\phi_0}{M_G} \right)^2, \]  

\[ \simeq 2.0 \frac{m_{\phi}^{3/7} T_R^{13/7}}{m_0^{3/4} M_G^{9/14}} \left( \frac{T_c}{m_0} \right)^{15/7} \left( \frac{\phi_0}{M_G} \right)^2. \]  

Therefore the lowest reheating temperature \( T_R = 10 \text{ MeV} \) gives for \( n = 1 \)

\[ \Omega_\phi h^2 \gtrsim \begin{cases} 
6.0 \times 10^{-3} \left( \frac{m_\phi}{10^{-8} \text{ GeV}} \right)^{-3/14} \left( \frac{T_c}{m_0} \right)^{15/7} \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu = 0 \\
3.7 \times 10^{-3} \left( \frac{m_\phi}{10^{-8} \text{ GeV}} \right)^{-3/14} \left( \frac{T_c}{m_0} \right)^{15/7} \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu \neq 0 
\end{cases}, \]  

for \( m_\chi \lesssim 1 \text{ GeV} \) (the flaton decays only into two photons) and

\[ \Omega_\phi h^2 \gtrsim 1.01 \times 10^{-5} \left( \frac{m_\phi}{\text{1 GeV}} \right)^{-3/14} \left( \frac{T_c}{m_0} \right)^{15/7} \left( \frac{\phi_0}{M_G} \right)^2, \]  

for \( 2 m_h > m_\chi \gtrsim 1 \text{ GeV} \) (the flaton decays dominately into two gluons).

On the other hand, when the flaton can decay into Higgs bosons \( (m_\chi \geq 2 m_h) \), one more free parameter \( C_h \) appears in the flaton decay width and the condition \( \left( \frac{\rho_\phi}{s} \right)_{BB} = \left( \frac{\rho_\phi}{s} \right)_{TI} \), i.e., Eq. (123), could be achieved by taking a moderate \( C_h \). Then we find from Eq. (124) that the lowest values of \( m_0 \) which corresponds to \( m_\chi = 2 m_h \) and \( T_R = 10 \text{ MeV} \) give the minimum of the total abundance as

\[ \Omega_\phi h^2 \gtrsim 4.6 \times 10^{-6} \left( \frac{m_\phi}{10 \text{ GeV}} \right)^{-3/4} \left( \frac{m_h}{70 \text{ GeV}} \right)^{3/2} \left( \frac{T_c}{m_0} \right)^{3/2} \left( \frac{\phi_0}{M_G} \right)^2. \]  

\[ 33 \]
FIG. 10. The lower bounds on the modulus abundance in the presence of the modified thermal inflation for the case I: $\Gamma_{\phi_1} \geq m_\phi \geq H_{TI}$. We assume $m_\phi = m_{3/2}$, $T_c = m_0$ and $\phi_0 = M_G$ and take $n = 1$ and $b = 1$. The solid (dashed) line denotes the lower bound when $\lambda_\mu = 0$ ($\lambda_\mu \neq 0$). Upper bounds from various cosmological constraints are all shown by the dotted lines.

for $n = 1$. In fact, this result gives the absolute minimum abundance for the modulus with a mass $m_\phi \simeq 6 - 26$ GeV. (See Fig. 10.)

However, for the modulus mass region $m_\phi \gtrsim 60$ GeV, the reheating temperature $T_R = 10$ MeV could not satisfy $(\frac{\epsilon_\phi}{s})_{BB} = (\frac{\epsilon_\phi}{s})_{TI}$ because of the lower bound on $m_0$ [Eq. (110)]. From Eq. (123) $T_R$ should be

$$T_R \gtrsim 18 \text{ MeV} \left( \frac{m_\phi}{100 \text{ GeV}} \right)^{9/8} \left( \frac{T_c}{m_0} \right)^{-3/4} \left( \frac{m_{3/2}}{m_\phi} \right)^{7/4} \text{ for } n = 1.$$  \hspace{1cm} (129)

Then from Eq. (124) this gives the minimum abundance as

$$\Omega_\phi h^2 \gtrsim 1.1 \times 10^{-5} \left( \frac{m_\phi}{100 \text{ GeV}} \right)^{15/8} \left( \frac{T_c}{m_0} \right)^{3/4} \left( \frac{m_{3/2}}{m_\phi} \right)^{13/4} \left( \frac{\phi_0}{M_G} \right)^2 \text{ for } n = 1.$$  \hspace{1cm} (130)

Fig. 10 shows the lower bound on the total abundance of the modulus. We find that the modified thermal inflation can dilute extensively the modulus density than the original one in the light modulus mass region $m_\phi \lesssim 100$ GeV. On the other hand, the lower bound is almost the same as the original model for $m_\phi \gtrsim 100$ GeV \[^{18}\] [see Fig. 4]. We obtain

\[^{18}\]The difference of the order one factor comes from the difference of the pre-factor in the expression of $m_\chi$ [see Eqs. (25) and (107)] and so on.
Modified Thermal Inflation Model (Case I)
n=1, b=1

FIG. 11. Same figure as Fig. 10 except for \( M_s > M_G \). We also show the results in Fig. 10 by the thin lines. The lower bound on the modulus mass [Eq. (139)] is represented by the dot-dashed line.

the allowed regions for the modulus mass: \( m_\phi \lesssim 3 \) MeV and \( m_\phi \gtrsim 4 \) GeV. Therefore, the window for the gravitino mass predicted by the GMSB models does exist even for the case I: \( \Gamma_{\phi_I} \geq m_\phi \geq H_{TI} \) and this point is crucially different from the original thermal inflation model.

Let us discuss the effect of the lower bound on the cutoff scale \( M_s > M_{cr} \sim M_G \). In the same way as the original thermal inflation model, it leads to the lower bound on \( m_0 \):

\[
m_0 > \frac{0.25}{C_\chi^{1/2}} \frac{T_R}{M_G} \left( \frac{M_{cr}}{M_G} \right)^{1/2}, \tag{131}
\]

for \( n = 1 \). Here we consider only the case \( n = 1 \) since larger \( n \) induces more abundant mass density of the modulus. This lower bound on \( m_0 \) does change the above analysis for the modulus mass region

\[
 m_\phi < 4.9 \times 10^{-5} \text{ GeV} \left( \frac{T_R}{10 \text{ MeV}} \right)^{6/5} \left( \frac{T_c}{m_0} \right)^{-6/5} \left( \frac{M_{cr}}{M_G} \right)^{7/5}, \tag{132}
\]

because the condition (123) could not be satisfied and \( \left( \frac{\rho_\phi}{s} \right)_{TI} \) be always larger than \( \left( \frac{\rho_\phi}{s} \right)_{BB} \).

In this case we find from Eqs. (122) and (131)

\[
 \left( \frac{\rho_\phi}{s} \right)_0 = \left( \frac{\rho_\phi}{s} \right)_{TI} > 3.2 \times 10^{-3} \frac{T_R^4}{C_\chi^{1/2} m_\phi^2 M_G} \left( \frac{M_{cr}}{M_G} \right)^{5/2} \left( \frac{\phi_0}{M_G} \right)^2, \tag{133}
\]

35
and the lowest reheating temperature $T_R = 10$ MeV gives

$$\Omega_{\phi} h^2 \gtrsim 0.20 \left( \frac{m_\phi}{1 \text{ keV}} \right)^2 \left( \frac{M_{cr}}{M_G} \right)^{5/2} \left( \frac{\phi_0}{M_G} \right)^2,$$

(134) for $m_\chi \gtrsim 1$ GeV. We show the lower bound on the modulus abundance in Fig. 11 when $M_\ast > M_G$.

Here we neglect the modulus mass region $m_\phi \lesssim 3$ keV, since the vev of the flaton becomes always larger than masses of messenger fields to obtain the reheating temperature $T_R \gtrsim 10$ MeV. In the GMSB models masses of messenger fields $m_{mess}$ can be written as $m_{mess} \simeq \langle F_{mess} \rangle / \Lambda$ with the $F$-component vev $\langle F_{mess} \rangle$ in the messenger sector and $\Lambda \sim 10^4$–$10^5$ GeV [6], and we obtain the upper bound on masses of messenger fields as

$$m_{mess} \lesssim \frac{\sqrt{3} m_{3/2} M_G}{\Lambda}.$$  

(135)

On the other hand, the vev of the flaton can be written from Eqs. (107) and (115) as

$$M = 3.4 C^1/2 \frac{m_{3/2} M_G^{1/2}}{T_R},$$

(136) for $n = 1$. Therefore, when one consider the GMSB models, in order that the vev of the flaton should be smaller than $m_{mess}$ we obtain

$$m_0 \lesssim \frac{0.64 m_{3/2} M_G^{1/2} T_R^{3/2}}{C^1/3 \Lambda^{2/3}}.$$  

(137)

For the case $M_\ast > M_{cr} \sim M_G$ the upper bound on the reheating temperature can be obtained from Eq. (131) as

$$T_R \lesssim 18 C^1/2 \frac{m_{3/2} M_G}{\Lambda^2} \left( \frac{M_{cr}}{M_G} \right)^{-3/2}.$$  

(138)

Then, the reheating temperature $T_R \gtrsim 10$ MeV is achieved only when

$$m_\phi \simeq m_{3/2} \gtrsim 3.0 \text{ keV} \left( \frac{\Lambda}{10^4 \text{ GeV}} \right) \left( \frac{M_{cr}}{M_G} \right)^{3/4},$$

(139) where we consider the flaton dominately decays into gluons.

From Fig. 11 we see that the dilution of the modulus density becomes less effective for the modulus mass region Eq. (132) and the modulus with a mass $m_\phi \lesssim 3$ keV is excluded. However, the allowed region of the modulus (gravitino) mass does exist in the region where the GMSB models predicts, even if we take $M_\ast \gtrsim M_G$ in the modified thermal inflation model, while it does not exist in the original model even for the case III ($m_\phi > H_{TI} > \Gamma_{\phi I}$) [see Figs. 5, 7 and 9].
B. Case II: For the case $m_\phi \geq \Gamma_{\varphi I} \geq H_{TI}$

Next we consider the case $m_\phi \geq \Gamma_{\varphi I} \geq H_{TI}$. From Eq. (75) the modified thermal inflation model with the entropy production factor (116) gives the abundance of the big-bang modulus as

$$\left(\frac{\rho_\phi}{s}\right)_{BB} \simeq 3.8 \frac{\Gamma_{\varphi I}^{1/2} M_G^{1/2} m_0^3 T_R}{V_0} \left(\frac{T_c}{m_0}\right)^3 \left(\frac{\phi_0}{M_G}\right)^2.$$  (140)

On the other hand, the abundance of the thermal-inflation modulus is the same as Eqs. (118) and (122) in the previous case. Now the abundance of the big-bang modulus takes its minimum value when $\Gamma_{\varphi I} = H_{TI}$ as

$$\left(\frac{\rho_\phi}{s}\right)_{BBm} \simeq 2.9 \frac{m_0^3 T_R}{V_0^{3/4}} \left(\frac{T_c}{m_0}\right)^3 \left(\frac{\phi_0}{M_G}\right)^2.$$  (141)

Then the lower bound on the total abundance Eq. (61) is given by

$$\left(\frac{\rho_\phi}{s}\right)_0 \geq \sqrt{\left(\frac{\rho_\phi}{s}\right)_{BBm} \left(\frac{\rho_\phi}{s}\right)_{TI}} \simeq 2.2 \frac{T_R^{47/23}}{C_{V_0}^{12/23} m_\phi^{6/23} M_G^{18/23}} \left(\frac{T_c}{m_0}\right)^{60/23} \left(\frac{\phi_0}{M_G}\right)^2,$$  (142)

with

$$m_0 = \frac{1.4}{C_{V_0}^{7/25}} m_\phi^{8/23} M_G^{1/23} T_R^{14/23} \left(\frac{T_c}{m_0}\right)^{12/23}.$$  (143)

Therefore, we obtain for $n = 1$ with the lowest reheating temperature $T_R = 10$ MeV:

$$\Omega_\phi h^2 \gtrsim \begin{cases} 7.6 \times 10^{-5} \left(\frac{m_\phi}{10^{-8} \text{ GeV}}\right)^{-6/23} \left(\frac{T_c}{m_0}\right)^{60/23} \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu = 0, \\ 4.3 \times 10^{-5} \left(\frac{m_\phi}{10^{-8} \text{ GeV}}\right)^{-6/23} \left(\frac{T_c}{m_0}\right)^{60/23} \left(\frac{\phi_0}{M_G}\right)^2 & \text{for } \lambda_\mu \neq 0, \end{cases}$$  (144)

for $m_\chi \lesssim 1$ GeV and

$$\Omega_\phi h^2 \gtrsim 3.2 \times 10^{-8} \left(\frac{m_\phi}{1 \text{ GeV}}\right)^{-6/23} \left(\frac{T_c}{m_0}\right)^{60/23} \left(\frac{\phi_0}{M_G}\right)^2,$$  (145)

for $2m_h > m_\chi \gtrsim 1$ GeV.

Furthermore, if the modulus mass becomes larger than about 10 GeV, the total abundance takes its minimum value when the reheating temperature is not 10 MeV but

$$T_R \simeq 120 \text{ MeV} \left(\frac{m_\phi}{100 \text{ GeV}}\right)^{15/14} \left(\frac{m_\phi^{3/2}}{m_\phi}\right)^{23/14} \left(\frac{T_c}{m_0}\right)^{-6/7}$$  for $n = 1$,  (146)

due to the lower bound on $m_0$ [Eq. (110)], which leads to
FIG. 12. Same figure as Fig. 10 for the case II: $m_\phi \geq \Gamma_{\phi_I} \geq H_{TI}$.

$$\Omega_{\phi} h^2 \gtrsim 1.6 \times 10^{-6} \left( \frac{m_\phi}{100 \text{ GeV}} \right)^{27/14} \left( \frac{T_c}{m_0} \right)^{6/7} \left( \frac{m_{3/2}}{m_\phi} \right)^{47/14} \left( \frac{\phi_0}{M_G} \right)^2 \text{ for } n = 1. \quad (147)$$

This result is the same as Eq. (88) in the original thermal inflation model if we neglect the order one factor. We find that the dilution becomes ineffective when the flaton decays into Higgs bosons, and then the lower bounds listed above are the absolute ones.

In Fig. 12 the lower bound on the total abundance is displayed. Comparing the result in the original thermal inflation model showed in Fig. 6, the lower bound on the modulus abundance is extensively reduced for the lighter modulus mass region $m_\phi \lesssim 10$ GeV, while it is almost the same for $m_\phi \gtrsim 100$ GeV. In particular, the lower bound becomes much weaker than the previous case I for the lighter modulus mass region $m_\phi \lesssim 1$ GeV, so that only the modulus mass region $m_\phi \approx m_{3/2} / 2 \sim 10$ MeV–1 GeV is excluded by the stringent constraint from the cosmic $X(\gamma)$-ray backgrounds.

Moreover, when we take the cutoff scale as $M_s > M_{\phi} \sim M_G$, the lower bound behaves as shown in Fig. 13. This changes the lower bound for $m_\phi \lesssim 1$ MeV and the bound is given by Eq. (134) in the previous case I. Therefore, the modulus whose mass is $m_\phi \lesssim 3$ keV is excluded as well as the previous case I.

C. Case III: For the case $m_\phi \geq H_{TI} \geq \Gamma_{\phi_I}$

Finally we consider the case $m_\phi \geq H_{TI} \geq \Gamma_{\phi_I}$, where the abundance of the big-bang modulus in the presence of the modified thermal inflation model is given by [see Appendix B]
FIG. 13. Same figure as Fig. 12 except for $M_\star > M_G$. We also show the results in Fig. 12 by the thin lines. The lower bound on the modulus mass [Eq. (139)] is represented by the dot-dashed line.

$$\left( \frac{\rho_\phi}{s} \right)_\text{BB} \simeq 4.8 \frac{m_0^4 T_R}{\Gamma_{\varphi_I} V_0^{1/2} M_G} \left( \frac{T_c}{m_0} \right)^4 \left( \frac{\phi_0}{M_G} \right)^2.$$  

This ratio takes its minimum value when $\Gamma_{\varphi_I} = H_{TI}$ as

$$\left( \frac{\rho_\phi}{s} \right)_\text{BB} \geq \left( \frac{\rho_\phi}{s} \right)_\text{BBm} = 8.2 \frac{m_0^4 T_R}{V_0} \left( \frac{T_c}{m_0} \right)^4 \left( \frac{\phi_0}{M_G} \right)^2.$$  

While the abundance of the thermal-inflation modulus is the same as the previous two cases and given by Eqs. (118) and (122). Since $\left( \frac{\rho_\phi}{s} \right)_\text{BBm}$ becomes equal to $\left( \frac{\rho_\phi}{s} \right)_\text{TI}$ when

$$m_0 = \frac{1.7}{C_{V_0}^{1/3} m_\phi^{1/3}} T_R^{2/3} \left( \frac{T_c}{m_0} \right)^{2/3},$$

the lower bound on the total abundance (61) is given as

$$\left( \frac{\rho_\phi}{s} \right)_0 \succeq \frac{4.9}{C_{V_0}^{2/3} m_\phi^{1/3} M_G} \left( \frac{T_c}{m_0} \right)^{10/3} \left( \frac{\phi_0}{M_G} \right)^2.$$  

Therefore, the lowest reheating temperature $T_R = 10$ MeV leads to for $n = 1$

$$\Omega_\phi h^2 \simeq \begin{cases} 3.4 \times 10^{-8} \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-1/3} \left( \frac{T_c}{m_0} \right)^{10/3} \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu = 0, \\ 1.6 \times 10^{-8} \left( \frac{m_\phi}{1 \text{ keV}} \right)^{-1/3} \left( \frac{T_c}{m_0} \right)^{10/3} \left( \frac{\phi_0}{M_G} \right)^2 & \text{for } \lambda_\mu \neq 0. \end{cases}$$
FIG. 14. Same figure as Fig. 10 for the case III: $m_\phi \geq H_{TI} \geq \Gamma_{\phi}$. for the flaton can only decay into photons ($m_\phi \lesssim 1$ MeV), and
\[
\Omega_\phi h^2 \gtrsim 1.6 \times 10^{-11} \left( \frac{m_\phi}{100 \text{ MeV}} \right)^{-1/3} \left( \frac{T_c}{m_0} \right)^{10/3} \left( \frac{\phi_0}{M_G} \right)^2,
\]
(153) for the flaton dominately decays into two gluons ($m_\phi \gtrsim 1$ MeV). In the case III the flaton decay into Higgs bosons makes dilution ineffective and gives no absolute minimum of the total modulus abundance.

We show the lower bound on the total modulus abundance in Fig. 14. It is seen that in the case III the modulus mass density is sufficiently diluted so that the whole modulus (gravitino) mass region 10 eV–10 TeV predicted by both models of the GMSB and the HSSB survives the various cosmological constraints. This is a crucial result when one considers the primordial inflation model with quite low reheating temperature. When the total abundance takes its minimum value, the required reheating temperature of the primordial inflation is estimated as, for the modulus mass region $m_\phi \lesssim 1$ GeV,
\[
T_{RI} \simeq \begin{cases} 
230 \text{ GeV} \left( \frac{m_\phi}{1 \text{ keV}} \right)^{5/12} \left( \frac{T_c}{m_0} \right)^{5/6} & \text{ for } \lambda_\mu = 0 \\
190 \text{ GeV} \left( \frac{m_\phi}{1 \text{ keV}} \right)^{5/12} \left( \frac{T_c}{m_0} \right)^{5/6} & \text{ for } \lambda_\mu \neq 0 
\end{cases},
\]
(154) when the flaton decays only into two photons, and
\[
T_{RI} \simeq \ 1.1 \times 10^4 \text{ GeV} \left( \frac{m_\phi}{100 \text{ MeV}} \right)^{5/12} \left( \frac{T_c}{m_0} \right)^{5/6},
\]
(155)
when the flaton dominately decays into two gluons. On the other hand, for the mass region $m_\phi > \sim 1 \text{ GeV}$, the required reheating temperature is given by

$$T_{RI} \simeq 5.5 \times 10^6 \text{ GeV} \left( \frac{m_\phi}{1 \text{ TeV}} \right)^{3/4} \left( \frac{T_c}{m_{\Omega}} \right)^{1/2} \left( \frac{m_{3/2}}{m_\phi} \right)^{1/2},$$

where the flaton dominately decays into two gluons. Therefore, in order to sufficiently dilute the light modulus predicted by the GMSB models, the required reheating temperature of the primordial inflation is not so extremely low as that required in the original thermal inflation model [see Eq. (100)].

Furthermore, we show in Fig. 15 the lower bound on the modulus abundance for the case $M_* > M_{cr} \sim M_G$. The lower bound becomes more stringent for the modulus mass $m_\phi \lesssim 100 \text{ MeV}$ where the bound is given by Eq. (88) which is the same as the previous two cases. However, we find that in the wide region of the modulus mass $m_\phi \sim 3 \text{ keV} – 10 \text{ TeV}$ the cosmological moduli problem can be solved naturally by the modified thermal inflation, if the reheating temperature of the primordial inflation is low enough.

VII. DISCUSSION

The thermal inflation, whether it is original one or modified one, can dilute significantly unwanted long-lived particles (i.e., the string moduli). Especially, as shown in Subsec. VI C, the modified model can gives a solution to the cosmological moduli problem and the whole
modulus mass region $m_\phi(\simeq m_{3/2}) \sim 10 \text{ eV} - 10^4 \text{ GeV}$ predicted by both the GMSB and HSSB models survives from various cosmological constraints [see Fig. 14], if the reheating temperature of the primordial inflation is low enough for its reheating process to finish after the thermal inflation ends. Furthermore, in this case, we find that even if one takes the gravitational scale as the cutoff scale of the modified thermal inflation model ($M_* \gtrsim M_G$), the modulus mass region $m_\phi(\simeq m_{3/2}) \sim 3 \text{ keV} - 10^4 \text{ GeV}$ is allowed [see Fig. 15]. Note that gravitino is also diluted sufficiently and we have no gravitino problem.

However, since the abundances of all relic particles are also diluted by the thermal inflation, one might be faced with the following problems; how generate the present observed baryon asymmetry and what is the dark matter of our universe. Here we give some possible solutions to them.

We first discuss the baryon asymmetry of the universe. The generation of the observed baryon asymmetry $Y_B \equiv (n_B/s)_0 \sim 10^{-10} - 10^{-11}$ is one of the challenging question in particle cosmology. Furthermore, if one assume the thermal inflation in the history of the universe, the situation becomes worse. Because the primordial baryon asymmetry is also diluted after the thermal inflation and the temperature at that epoch is $T \sim 10 \text{ MeV}$ in order to dilute the moduli abundance maximally, the GUT baryogenesis and the electroweak baryogenesis do not work in this case.

However, as pointed out by Ref. [19], enough baryon number could be produced by the Affleck-Dine mechanism [15]. The generated asymmetry is related with the present modulus abundance as [19]

$$Y_B \lesssim \frac{1}{m_{3/2}} \left( \frac{\rho_\phi}{s} \right)_{BB} = 3.6 \times 10^{-9} \left( \frac{m_\phi}{1 \text{ GeV}} \right)^{-1} \left( \frac{m_{3/2}}{m_\phi} \right)^{-1},$$

(157)

where $\Omega_{\phi BB} = \rho_{\phi BB}/\rho_{\text{cr}}$. Therefore, a tremendous baryon number is produced primordially (i.e., $H \sim m_{3/2}$) and enough asymmetry could be left even after the entropy production by the thermal inflation. Then the lower bound on the modulus abundance can be obtained from the present value of $Y_B$ as

$$\Omega_\phi h^2 \geq \Omega_{\phi BB} h^2 \gtrsim 2.8 \times 10^{-3} \left( \frac{Y_B}{10^{-11}} \right) \left( \frac{m_\phi}{1 \text{ GeV}} \right) \left( \frac{m_{3/2}}{m_\phi} \right).$$

(158)

In Fig. 16 we show this bound together with the result obtained in Subsec. VI C [Fig. 14]. You can see that this lower bound (158) lies above various upper bounds on the modulus abundance for $m_\phi \gtrsim 4 \text{ MeV}$. Therefore the Affleck-Dine mechanism can generate enough baryon asymmetry in the lighter modulus mass region with $m_\phi \lesssim 4 \text{ MeV}$ [12] even in the presence of the thermal inflation, and, on the other hand, we require other mechanism of baryogenesis in the heavier modulus mass region.

In Ref. [31] a variant type of the Affleck-Dine mechanism was proposed, where an $D$-flat direction, $LH_u$, could produce the required baryon asymmetry after the thermal inflation ends. They assumed that $m_{H_u}^2 + m_L^2$ be negative ($m_{H_u}$ and $m_L$ are the soft SUSY breaking mass of a $H_u$ and a scalar lepton doublet $L$) in order that the $LH_u$ condensate rolls away from the origin. However, in this case the potential along a specific direction is unbounded from below [32]. Therefore, this interesting mechanism of baryogenesis is not phenomenologically
FIG. 16. Same figure as Fig. 14. We also show by the dot-dashed line the lower bound on the modulus abundance from the present baryon asymmetry $Y_B = 10^{-11}$ which is generated by the Affleck-Dine mechanism.

Although we have assumed the lowest reheating temperature $T_R \sim 10$ MeV of the thermal inflation to dilute the modulus density most effectively, one might have a chance to generate appropriate baryon asymmetry by the electroweak baryogenesis mechanism if one takes a reheating temperature as high as $\sim 100$ GeV. In Fig. 17, we show the lower bound on the modulus abundance in the modified thermal inflation with $m_\phi \geq H_{TI} \geq \Gamma_{\phi I}$ and $T_R \gtrsim 100$ GeV. We find that the moduli with $m_\phi \gtrsim 100$ GeV is cosmologically allowed even if $T_R \gtrsim 100$ GeV for $\lambda_\mu = 0$. Furthermore, for the case $\lambda_\mu \neq 0$, the allowed region becomes wider as $m_\phi \gtrsim 5$ GeV by the flaton decay into Higgs bosons. Therefore, in the heavy modulus (gravitino) mass region predicted by the HSSB models, one had a possibility to obtain the enough baryon asymmetry even in the presence of the thermal inflation, if the electroweak baryogenesis would work.

Therefore, in the presence of the thermal inflation, we have two possibilities, so far, to produce enough baryon asymmetry as well as to dilute the mass density of the moduli sufficiently: (i) $m_\phi \lesssim 4$ MeV by the Affleck-Dine mechanism and (ii) $m_\phi \gtrsim 5$ GeV by the electroweak baryogenesis mechanism.

Finally, we would like to mention about the dark matter candidates under the tremendous entropy production by the thermal inflation. Note that a dark matter of our universe is viable.

\[19\] We thank T. Yanagida for informing us of this point.
FIG. 17. The minimum abundance of the modulus in the presence of the modified thermal inflation for the case III: \( m_\phi \geq H_{TI} \geq \Gamma_{\phi I} \). The thick solid (dashed) line denotes the minimum abundance for the case \( \lambda_{\mu} = 0 \) (\( \lambda_{\mu} \neq 0 \)) when \( T_R > 100 \) GeV and the thin solid (dashed) line denotes that when \( T_R > 10 \) MeV. Upper bounds from various cosmological constraints are all shown by the dotted lines.

diluted away as well as the string moduli. One possibility is the stable modulus itself, if its mass is less than about 100 keV, since in this mass region we can obtain \( \Omega_\phi \sim 1 \) without conflicting with the x(γ)-ray background constraints. In this case, as shown in Ref. [33], the moduli dark matter with \( m_\phi \sim 100 \) keV will be tested by the future x-ray background experiments with high energy resolution. The axion is another candidate for the dark matter. If its decay constant is high enough as \( f_{PQ} \sim 10^{15}–10^{16} \) GeV, its energy becomes comparable to the whole energy of the present universe for the case that the reheating temperature is \( T_R \sim 10 \) MeV [34]. Recently, the author and Yanagida proposed another candidate for the dark matter in the presence of the thermal inflation [35]. The superheavy particle of mass \( 10^{12}–10^{14} \) GeV, which was primordially in the thermal equilibrium, could be the dark matter, if its lifetime was longer than the age of the universe.

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APPENDIX A: EVOLUTION OF ENERGY DENSITIES OF INFLATON AND RADIATION AFTER THE PRIMORDIAL INFLATION

Here we briefly estimate the evolution of energy densities of the inflaton $\varphi_I$ and the radiation after the primordial inflation. The model of the primordial inflation is basically fixed by two parameters: the vacuum energy of the inflation $V_I$ and the decay width of the inflation $\Gamma_{\varphi_I}$.

When the inflation ends at $t = t_{EI}$, the energy densities of the inflation $\rho_{\varphi_I}$ and the radiation $\rho_R$ are given by

$$\rho_{\varphi_I}(t_{EI}) = V_I,$$
$$\rho_R(t_{EI}) \simeq 0. \tag{A1}$$

The Hubble parameter of the universe is given by

$$H(t_{EI}) \equiv H_{EI} = \frac{V_I^{1/2}}{\sqrt{3}M_G}. \tag{A2}$$

Then the inflaton causes a coherent oscillation for $H_{EI} > H(t) > \Gamma_{\varphi_I}$. During this period the energy of the oscillation dominates the density of universe and the Hubble parameter is given as

$$H(t) \simeq \frac{\rho_{\varphi_I}(t)^{1/2}}{\sqrt{3}M_G}. \tag{A3}$$

The evolution of $\rho_{\varphi_I}$ and $\rho_R$ is governed by the following equations:

$$\dot{\rho}_{\varphi_I} + 3H\rho_{\varphi_I} = -\Gamma_{\varphi_I}\rho_{\varphi_I}, \tag{A4}$$
$$\dot{\rho}_R + 4H\rho_R = \Gamma_{\varphi_I}\rho_{\varphi_I}, \tag{A5}$$

where the dot represents the derivative with respect to the cosmic time. When $t \gg \Gamma_{\varphi_I}$ the solutions of them with the initial conditions (A1) and (A2) are given by

$$\rho_{\varphi_I}(t) \simeq V_I \left( \frac{R(t_{EI})}{R(t)} \right)^3,$$
$$\rho_R(t) \simeq \frac{2\sqrt{3}}{5} \Gamma_{\varphi_I} M_G V_I^{1/2} \left( \frac{R(t_{EI})}{R(t)} \right)^{3/2} \left[ 1 - \left( \frac{R(t_{EI})}{R(t)} \right)^{5/2} \right], \tag{A6}$$

where $R(t)$ is the scale factor of the universe at the time $t$. Thus the radiation energy density for $R(t) \gg R(t_{EI})$ is diluted by a rate $R^{-3/2}$ (not $R^{-4}$) as the universe expands. Note that $\rho_R$ takes its maximum value

$$\rho_R|_{MAX} \simeq 0.24 \Gamma_{\varphi_I} M_G V_I^{1/2}, \tag{A7}$$

at $R(t) \simeq 1.5R(t_{EI})$. Then the maximum temperature achieved after the primordial inflation is estimated as

$$T_{MAX} \simeq 0.93 g_s(T_{MAX})^{-1/4} \Gamma_{\varphi_I}^{1/4} M_G^{1/4} V_I^{1/8}, \tag{A8}$$
$$\simeq 0.70 g_s(T_{MAX})^{-1/4} g_s(T_{RI})^{1/4} T_{RI}^{1/8} V_I^{1/8}, \tag{A9}$$

where $T_{RI}$ denotes the reheating temperature of the primordial inflation and is given by

$$T_{RI} \simeq 1.7 g_s(T_{RI})^{-1/4} \sqrt{\Gamma_{\varphi_I} M_G}. \tag{A10}$$
Here we derive the abundance of the big-bang modulus for the case III: $m_\phi \geq H_{TI} \geq \Gamma_{\varphi I}$, where the reheating process of the primordial inflation completes after the thermal inflation ends and its reheating temperature becomes extremely low. The mass density of the modulus oscillation is diluted not only by the thermal inflation, but also by the primordial inflation even during the thermal inflation. Therefore, we expect the abundance of the modulus is extensively reduced than case I and II.

The big-bang modulus starts to oscillate with the initial amplitude $\phi_0 \sim M_G$ at $t = t_{BB}$ when $H \simeq m_\phi$. At this time, the energy densities of the modulus, the inflaton, and the radiation are estimated as

$$\rho_\phi(t_{BB}) = \frac{1}{2} m_\phi^2 M_G^2 \left( \frac{\phi_0}{M_G} \right)^2, \quad (B1)$$
$$\rho_{\varphi I}(t_{BB}) \simeq 3m_\phi^2 M_G^2, \quad (B2)$$
$$\rho_R(t_{BB}) \simeq \frac{6}{5} m_\phi \Gamma_{\varphi I} M_G^2. \quad (B3)$$

Here we used the fact that the energy of the inflaton’s oscillation dominates the universe for $H_{EI} > H(t) > m_\phi$ and

$$H(t_{BB})^2 \simeq m_\phi^2 \simeq H_{EI}^2 \left( \frac{R(t_{EI})}{R(t_{BB})} \right)^3 = \frac{V_I}{3M_G^2} \left( \frac{R(t_{EI})}{R(t_{BB})} \right)^3. \quad (B4)$$

The cosmic temperature at $t = t_{BB}$ is estimated as

$$T_{BB} \simeq \sqrt{\frac{6}{\pi}} g_*(T_{BB})^{-1/4} m_\phi^{1/4} \Gamma_{\varphi I}^{1/4} M_G^{1/2}. \quad (B5)$$

The thermal inflation starts at $t = t_{STI}$ when the vacuum energy $V_0$ of the flaton $\chi$ begins to dominate the energy of the universe. Since we are considering the case $m_\phi \geq H_{TI}$, $t_{STI}$ should be $t_{STI} > t_{BB}$. Then, for $t_{BB} < t < t_{STI}$ the energy of the universe is almost dominated by the inflaton energy as follows:

$$\rho(t) = \rho_{\varphi I}(t) + \rho_\phi(t) + \rho_R(t),$$
$$\simeq \rho_{\varphi I}(t) + \rho_\phi(t),$$
$$\simeq \left[ 3m_\phi^2 M_G^2 + \frac{1}{2} m_\phi M_G^2 \left( \frac{\phi_0}{M_G} \right)^2 \left( \frac{R(t_{BB})}{R(t)} \right)^3, \right]$$
$$\simeq 3m_\phi^2 M_G^2 \left( \frac{R(t_{BB})}{R(t)} \right)^3 = \rho_{\varphi I}(t). \quad (B6)$$

Therefore we obtain each energy density at $t = t_{STI}$ as
\[ \rho_\chi(t_{STI}) = V_0 \]  
\[ \rho_\phi(t_{STI}) \simeq \frac{1}{6} V_0 \left( \frac{\phi_0}{M_G} \right)^2 \]  
\[ \rho_{\varphi I}(t_{STI}) \simeq V_0 \]  
\[ \rho_R(t_{STI}) \simeq \frac{2\sqrt{3}}{5} \Gamma_{\varphi I} V_0^{1/2} M_G. \]

We find that the vacuum energy of the flaton when the thermal inflation starts is comparable to the energy of the inflaton’s coherent oscillation. The cosmic temperature at \( t = t_{STI} \) is

\[ T_{STI} \simeq \left( \frac{12\sqrt{3}}{\pi^2} \right)^{1/4} g_s(T_{STI})^{-1/4} \Gamma_{\varphi I}^{1/4} V_0^{1/8} M_G^{1/4}. \]  

While the thermal inflation lasts, the Hubble parameter takes the constant value given by

\[ H(t) = H_{TI} = \frac{V_0^{1/2}}{\sqrt{3} M_G}, \]  

and the evolution of the energy densities are governed by the following equations

\[ \dot{\rho}_{\varphi I} + 3H_{TI} \rho_{\varphi I} = -\Gamma_{\varphi I} \rho_{\varphi I}, \]  
\[ \dot{\rho}_\phi + 3H_{TI} \rho_\phi = 0, \]  
\[ \dot{\rho}_R + 4H_{TI} \rho_R = \Gamma_{\varphi I} \rho_{\varphi I}. \]

Here we have neglected the effect of the modulus decay. The solutions are

\[ \rho_{\varphi I}(t) \simeq V_0 \left( \frac{R(t_{STI})}{R(t)} \right)^3, \]  
\[ \rho_\phi(t) \simeq \frac{1}{6} V_0 \left( \frac{\phi_0}{M_G} \right)^2 \left( \frac{R(t_{STI})}{R(t)} \right)^3, \]  
\[ \rho_R(t) \simeq \sqrt{3} M_G \Gamma_{\varphi I} V_0^{1/2} \left( \frac{R(t_{STI})}{R(t)} \right)^3 \left[ 1 - \frac{3}{5} \left( \frac{R(t_{STI})}{R(t)} \right) \right]. \]

The thermal inflation ends when the the cosmic temperature becomes \( T = T_c \). At this time \( t = t_{ETI} \) the energy densities are given by

\[ \rho_\chi(t_{ETI}) = V_0, \]  
\[ \rho_\phi(t_{ETI}) \simeq \frac{1}{6} V_0 \left( \frac{\phi_0}{M_G} \right)^2 \frac{\pi^2}{30} g_s(T_c) T_c^4 \]  
\[ \sqrt{3} \Gamma_{\varphi I} V_0^{1/2} M_G, \]  
\[ \rho_{\varphi I}(t_{ETI}) \simeq V_0 \frac{\pi^2}{30} g_s(T_c) T_c^4 \]  
\[ \sqrt{3} \Gamma_{\varphi I} V_0^{1/2} M_G, \]  
\[ \rho_R(t_{ETI}) = \frac{\pi^2}{30} g_s(T_c) T_c^4. \]

47
Therefore, the ratio between the energy density of the big-bang modulus and the entropy density is estimated as

\[
\frac{\rho_\phi}{s} \simeq \frac{T_c V_0^{1/2}}{8\sqrt{3} \Gamma_{\varphi I} M_G} \left( \frac{\phi_0}{M_G} \right)^2.
\]  

(B23)

Below \( T = T_c \) the flaton rolls down to its true minimum and oscillates around it. By the flaton decay (and by the R-axion decay) the reheating process of the thermal inflation completes and the universe is finally reheated to the temperature \( T = T_R \). Then the huge entropy is produced by a factor \( \Delta \) given by Eqs. (42) and (116). Therefore, the present abundance of the big-bang modulus is given by

\[
\left( \frac{\rho_\phi}{s} \right)_{BB} \simeq \frac{T_c V_0^{1/2}}{8\sqrt{3} \Gamma_{\varphi I} M_G} \left( \frac{\phi_0}{M_G} \right)^2 \times \frac{1}{\Delta}.
\]  

(B24)

In the original thermal inflation model, the entropy production factor (42) leads to

\[
\left( \frac{\rho_\phi}{s} \right)_{BB} \simeq 4.8 \frac{m_0^4 T_R}{\Gamma_{\varphi I} V_0^{1/2} M_G} \left( \frac{m_\chi}{2m_a} \right) \left( \frac{T_c}{m_0} \right)^4 \left( \frac{\phi_0}{M_G} \right)^2.
\]  

(B25)

On the other hand, in the modified model, Eq. (116) gives

\[
\left( \frac{\rho_\phi}{s} \right)_{BB} \simeq 4.8 \frac{m_0^4 T_R}{\Gamma_{\varphi I} V_0^{1/2} M_G} \left( \frac{T_c}{m_0} \right)^4 \left( \frac{\phi_0}{M_G} \right)^2.
\]  

(B26)
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[29] See, for example, A.D. Linde, Particle Physics and Inflationary Cosmology, (Hawood, Chur, Switzerland, 1990).