Symmetry conserving non-perturbative s-wave renormalization of the pion in hot and baryon dense medium

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Abstract

A non-perturbative s-wave renormalization of the pion in a hot and baryon rich medium is presented. This approach proceeds via a mapping of the canonical pion into the axial Noether’s charge. The mapping was made dynamical in the Hartree-Fock-Bogoliubov random phase approximation (HFB-RPA). It is shown that this approach, while order mixing, is still symmetry conserving both in the baryon free and baryon rich sectors, at zero as well as finite temperature. The systematic character of this approach is emphasized and it is particularly argued that it may constitute an interesting alternative for the non-perturbative assessment of the nuclear matter saturation properties.

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I. INTRODUCTION

The understanding of the nuclear equation of state for hot and dense matter is one of the fiercely followed objective in the nuclear problem. Early studies although successful in quantitatively assessing the nuclear matter ground-state properties, rely heavily on phenomenological input, as the nuclear two body interaction for instance \cite{1,2}. These bear very little to the nowadays admitted fundamental theory of the strong interaction, the quantum chromodynamics (QCD). Due to its strong coupling limit in the low energy regimes, QCD is in fact intractable as a theory of its fundamental degrees of freedom. On the other hand, QCD has also symmetries which are as much fundamental. Their realization through effective hadronic theories seems to be a wise compromise and a strategy adopted in the so-called quantum hadrodynamics (QHD) (see \cite{3} for a review).

Of particular interest is chiral symmetry. Its nonlinear realization, through chiral perturbation theory (ChPT) for instance, has proven successful all over the past decade in understanding the low energy pion physics. This, however, falls short of answering some important questions as how to deal with highly collective structures like bound states or resonances and how to fulfill the fundamental unitarity condition. Thus one realizes that besides QCD’s symmetries, the QHD inherits as well the difficult aspects of the strongly interacting fields. Any reasonable handling of these theories requires therefore an adequate treatment of the complicated vacuum structure. This is addressed in general through non-perturbative methods. The symmetry, however, imposes stringent constraints on the way the evaluation of the dynamics is conducted. Therefore one is called to be very selective in choosing non-perturbative approaches that help in gathering the dynamics without destroying the symmetry.

This task, although trivial in the case of the coupling constant perturbation (CCP), is full of subtleties and requires in fact much more effort from within the strong interaction physics community. In this regard, as an educated non-perturbative approach, the concept of symmetry conserving dynamical mapping (SCDM) was introduced in \cite{4}. The idea is to map the original Fock-space, created at the quantization and which supports the CCP approach, to some ideal Fock-space selected via a given symmetry conserving mapping (SCM). The next step is to make a projection onto the physical Fock-space which is a subspace of the ideal Fock-space. The so-called projection is realized by making the SCM dynamical\(^1\).

There are various advantages in performing non-perturbative calculations following this scheme. The most interesting aspect is certainly the possibility of tracking all kind of transformations taking place in the Fock-space while gathering the dynamics \cite{4}. In particular the asymptotic particles, which are highly dressed by the dynamics, have also well defined normalizable states in a well defined Fock-space. This is a clear departure from some traditional non-perturbative approaches in which one loses this precious insight. Probably the difficult task in this picture is to actually find the right symmetry conserving mapping.

Beside the \(1/N\) expansion approach which was sorted-out using Holstein-Primakoff mapping, an other SCM, based on a field-to-current mapping, was presented in \cite{4,6}. That this

\(^1\)Further precautions need, of course, to be observed in projecting out unphysical states. The necessary technics to address this question are well available in the literature \cite{5}.
second mapping is systematic in handling complicated situations was proven in [7] within the chirally invariant $SU(3) \times SU(3)$ Gellmann-Levy lagrangian. It will be further consolidated here in considering a baryon rich environment at finite temperature within the $SU(2) \times SU(2)$ version of the model. It will be also made clear later on that the field-to-current mapping is a robust SCDM which is able to handle any complicated dynamics as long as the symmetry is linearly realized in the lagrangian. In what follows the field-current mapping will be made dynamical in the Hartree-Fock-Bogoliubov-RPA scheme. This was the approximation adopted previously in [6,7]. Higher approximations are also possible and these will soon be reported upon [8].

This paper is organized as follows: We briefly review in section III the effective chiral model used all over the paper and spell out the basic rule of the game, namely the preservation of the lowest chiral ward identity throughout our non-perturbative treatment. This will be realized via an HFB-RPA scheme. Therefore, we first introduce in subsection-A a Bogoliubov rotation and perform a self-consistent mean field calculation which prepares the HFB basis. The RPA fluctuations build on the HFB basis will be evaluated in subsection-B. It will be shown that the Goldstone theorem is fulfilled exactly. In section IV, the finite temperature and finite chemical potential extensions of this approach are made. Finally, the conclusions are drawn in section V.

II. THE MODEL

As a starting point, we recall the lagrangian density of the $SU(2) \times SU(2)$ linear $\sigma$-model with nucleonic degrees of freedom. To realize an explicit chiral symmetry breaking in the PCAC sense, one adds the usual linear term in the sigma field.

$$L_{\sigma} = \bar{\psi} \left[ i \partial + g (\hat{\sigma} + i\vec{\pi} \gamma_5) \right] \psi + \frac{1}{2} \left( (\partial_{\mu} \vec{\pi})^2 + (\partial_{\mu} \hat{\sigma})^2 \right) - \frac{\mu^2}{2} \left( \vec{\pi}^2 + \hat{\sigma}^2 \right) - \frac{\lambda^2}{4} \left( \vec{\pi}^2 + \hat{\sigma}^2 \right)^2 + c \hat{\sigma}$$

(1)

Here $\psi$, $\hat{\sigma}$ and $\pi$ represent the bar nucleon, sigma, and pion fields, respectively. To fix our notations for latter use, these are given below, in terms of their respective creation and annihilation operators, by the usual plane wave expansions

$$\psi(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}E_p} \sum_r \left[ u(p, r)e^{-i\vec{p}\cdot\vec{r}}c_{pr} + v(p, r)e^{i\vec{p}\cdot\vec{r}}a_{pr}^+ \right],$$

$$\pi_i(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}2\omega_p} \left( e^{i\vec{p}\cdot\vec{r}}\phi_{pi} + e^{-i\vec{p}\cdot\vec{r}}\phi_{pi}^+ \right), \quad \hat{\sigma}(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}2\omega_p} \left( e^{i\vec{p}\cdot\vec{r}}b_p + e^{-i\vec{p}\cdot\vec{r}}b_p^+ \right)$$

(2)

The on-shell energies are taken to be $\omega_p = \sqrt{\mu^2 + p^2}$ for the bosons, and $E_p = p$ for the massless fermions. The definition of the bi-spinors $u(p, r)$ and $v(p, r)$ considered here is the one which takes care of this limiting case [9].

As stated in the introduction, the aim here is to conduct a non-perturbative treatment of the model that goes beyond the usual coupling constant perturbation (CCP) as well as the semi-classical approaches. These two are, presently, the most extensively used approaches in the study of the dynamics of these QCD inspired hadronic models. Before embarking
in this, we recall the commonly used classical mean-field approximation. In this case, the masses of the different particles are given by

\[ m_\pi^2 = \mu^2 + \lambda^2 \langle \hat{\sigma} \rangle^2, \quad m_\sigma^2 = \mu^2 + 3\lambda^2 \langle \hat{\sigma} \rangle^2, \quad m_F^2 = -g \langle \hat{\sigma} \rangle, \quad c = \mu^2 \langle \hat{\sigma} \rangle + \lambda^2 \langle \hat{\sigma} \rangle^3 \]  

(3)

where \( \langle \hat{\sigma} \rangle \) denotes the vacuum expectation value of the sigma field. At this level, the Goldstone nature of the pion, in the chiral limit (\( c = 0 \)), is manifest. Indeed, one can also verify that the Ward identity

\[-D^{-1}_\pi(0) = \frac{c}{\langle \hat{\sigma} \rangle} \]  

(4)

is fulfilled. Of course, all higher Ward identities (WI) are also preserved by the CCP approach at each order of the perturbation, and thus, at this lowest order too. The present WI holds as well in the case of the semi-classical non-perturbative approach à la \( 1/N \)-expansion\(^2\). In fact, in the large \( N \)-limit, the condensate and the pion mass are obtained as solutions of two coupled BCS equations in the Hartree-Bogoliubov approximation (see for instance [4,10]).

The fluctuating part of the sigma field is found to be not involved at all in building the variational ground state for the whole \( 1/N \)-expansion approach. As such the sigma mass has an exclusively perturbative character. Here too, the whole hierarchy of WI’s, as in the case of the CCP, is preserved. In the present paper, however, our ambition is rather modest. Our aim is simply to set up a non-perturbative approach, which transcends the two approximations cited above, but still preserves in a systematic way the lowest WI in Eq.(4), at all conditions of temperature and baryon density.

A. HARTREE-FOCK-BOGOLIUBOV MEAN-FIELD

The starting point for our mean field considerations is the following Bogoliubov transformation for both nucleon and meson operators

\[
\begin{align*}
\hat{a}_p^+ &= u_\pi(p) \hat{a}_p^+ + v_\pi(p) \hat{a}_p^-; \quad \beta_p^+ &= u_\sigma(p) \hat{b}_p^+ - v_\sigma(p) \hat{b}_p^- - w_0 \delta(p) \\
C_{pr}^+ &= u_F(p) \hat{c}_{pr}^+ - rv_F(p) \hat{d}_{pr}^-; \quad D_{pr}^+ &= u_F(p) \hat{d}_{pr}^+ + rv_F(p) \hat{c}_{pr}^-
\end{align*}
\]  

(5)

where \( u_\pi(p), u_\sigma(p), v_\pi(p), v_\sigma(p), u_F(p) \) and \( v_F(p) \) are even functions of their arguments. To account for a finite \( \langle \hat{\sigma} \rangle \) condensate in the Goldstone phase, the Bogoliubov transformation is made inhomogeneous for the sigma field by introducing a \( c \)-number shift \( w_0 \). The canonicity of the above transformations is enforced by means of the following conditions:

\[ u_\pi(p)^2 - v_\pi(p)^2 = 1, \quad u_\sigma(p)^2 - v_\sigma(p)^2 = 1, \quad u_F(p)^2 + v_F(p)^2 = 1 \]  

(6)

Accordingly, the vacuum |0\rangle of the theory at the classical mean field, defined by: \( \hat{a}_\pi|0\rangle = \hat{b}_\sigma|0\rangle = \hat{c}|0\rangle = \hat{d}|0\rangle = 0 \), is unitarily transformed to a new one |\Phi\rangle, such that

\[ \hat{\sigma}|\Phi\rangle = \beta|\Phi\rangle = C|\Phi\rangle = D|\Phi\rangle = 0, \]

\(^2N\) is the number of pion charges
and explicitly given in terms of the following squeezed vacuum

\[ |\Phi\rangle = \exp \left[ \int d\vec{p} \left\{ z_\pi(p)\tilde{a}_p^+\tilde{a}_{-p}^+ + z_\sigma(p)b_{p}b_{-p} + z_F(p)\sum_r rc^+_p d^+_{-pr} \right\} + \frac{u_0}{u_\sigma(0)}b^+_0 - \text{h.c.} \right] |0\rangle . \]

(7)

where \( z_\pi(p) = \text{arccosh}\left[u_\pi(p)\right] \), \( z_\sigma(p) = \text{arccosh}\left[u_\sigma(p)\right] \) and \( z_F(p) = \text{arccosh}\left[u_F(p)\right] \). The Hamiltonian of the model derived from Eq.(1) and written in the quasi-particles basis reads:

\[
\mathcal{H} = \mathcal{H}_0 + \eta \left[ \beta_0 + \beta_0^+ \right] + \int d\vec{p} \mathcal{E}_\pi(p) \sum_r \left[ C^+_p C^{pr}_r + D^+_p D^{pr}_r \right] + \int d\vec{p} \mathcal{E}_\sigma(p) \beta^+_p \beta_p
\]

\[
+ \int d\vec{p} \mathcal{E}_\sigma(p) \beta^+_p \beta_p + \int d\vec{p} c_F(p) \sum_r r \left[ C^+_p D^{pr}_r + D^+_p C^{pr}_r \right]
\]

\[
+ \int d\vec{p} c_F(p) \left[ \tilde{a}^+_p \tilde{a}^-_p + \tilde{a}^-_p \tilde{a}^+_p \right] + \int d\vec{p} c_F(p) \left[ \beta^+_p \beta^-_p + \beta^-_p \beta^+_p \right]
\]

\[
- \int dx \left[ g\tilde{\gamma}(\sigma + i\tilde{\pi}\gamma_5)\tilde{\psi} \right] + \int dx \left[ \lambda^2 (\lambda^2 - \sigma^2) + \frac{\lambda^2}{4} \left( \tilde{\pi}^2 + \sigma^2 \right)^2 \right] .
\]

(8)

where \( \sigma \) represents the shifted sigma-field: \( \sigma = \tilde{\sigma} - \langle \tilde{\sigma} \rangle \), and \( \langle \tilde{\sigma} \rangle \) is the sigma condensate which is given in term of the parameters of the Bogoliubov transformation in Eq.(5) by

\[
\langle \tilde{\sigma} \rangle = \frac{\langle b_0^+ \rangle + \langle b_0 \rangle}{\sqrt{(2\pi)^3 2\mu}} = \frac{(u_\sigma(0) + v_\sigma(0))(w_0 + w^*_0)}{\sqrt{(2\pi)^3 2\mu}} .
\]

The semicolons ";;" in the expression (8) of the Hamiltonian denote operators normal ordering. Finally, the coefficients \( \mathcal{H}_0, \eta, \mathcal{E}_{F,\pi,\sigma} \) and \( c_{F,\pi,\sigma} \) read explicitly

\[
\mathcal{H}_0 = -\gamma \int \frac{d\vec{p}}{(2\pi)^3} \left[ E_p \left( u_F(p)^2 - v_F(p)^2 \right) - 2g\langle \tilde{\sigma} \rangle u_F(p)v_F(p) \right]
\]

\[
+ \int \frac{d\vec{p}}{(2\pi)^3} \omega_p \left( 3v_\pi(p)^2 + v_\pi(p)^2 + 2 \right)
\]

\[
+ \frac{3\lambda^2}{4} \left[ I^2_\sigma + 5I^2_\pi + 2I_\sigma I_\pi \right] + \frac{3\lambda^2}{2} \left[ I_\sigma + I_\pi \right] + \frac{\mu^2}{2} \left( \langle \tilde{\sigma} \rangle \right)^2 + \frac{\lambda^2}{4} \left( \langle \tilde{\pi} \rangle \right)^2 - c\langle \tilde{\sigma} \rangle ,
\]

(9)

where \( \gamma = 4 \) is the spin-isospin degeneracy of the nucleon, and

\[
\eta = \sqrt{(2\pi)^3} \frac{u_\sigma(0) + v_\sigma(0)}{\sqrt{2\mu}} \left[ 3\lambda^2 \langle \tilde{\sigma} \rangle I_\sigma + 3\lambda^2 \langle \tilde{\pi} \rangle I_\pi + \lambda^2 \langle \tilde{\sigma} \rangle^3 + \mu^2 \langle \tilde{\sigma} \rangle^2 - c - gI_F \right] ,
\]

\[
c_F(p) = g\langle \tilde{\sigma} \rangle \left( u_F(p)^2 - v_F(p)^2 \right) + 2E_p u_F(p)v_F(p) ,
\]

\[
c_\pi(p) = \omega_p u_\pi(p)v_\pi(p) + \frac{\lambda^2 (u_\pi(p) + v_\pi(p))^2}{2\omega_p} \left[ 5I_\pi + I_\sigma + \langle \tilde{\sigma} \rangle \right] ,
\]

\[
c_\sigma(p) = \omega_p u_\sigma(p)v_\sigma(p) + \frac{3\lambda^2 (u_\sigma(p) + v_\sigma(p))^2}{2\omega_p} \left[ I_\pi + I_\sigma + \langle \tilde{\sigma} \rangle \right] ,
\]

\[
\mathcal{E}_F(p) = E_p \left( u_F(p)^2 - v_F(p)^2 \right) - 2g\langle \tilde{\sigma} \rangle u_F(p)v_F(p) ,
\]

\[
\mathcal{E}_\pi(p) = \omega_p \left( u_\pi(p)^2 + v_\pi(p)^2 \right) + \lambda^2 \left( u_\pi(p) + v_\pi(p) \right)^2 \left[ 5I_\pi + I_\sigma + \langle \tilde{\sigma} \rangle \right] ,
\]

\[
\mathcal{E}_\sigma(p) = \omega_p \left( u_\sigma(p)^2 + v_\sigma(p)^2 \right) + 3\lambda^2 \left( u_\sigma(p) + v_\sigma(p) \right)^2 \left[ I_\pi + I_\sigma + \langle \tilde{\sigma} \rangle \right] .
\]

(10)
\( I_\pi, I_\sigma \) and \( I_F \) are expectation values on the squeezed vacuum of bilinear forms of the pion, the sigma and the baryon fields, respectively. They read

\[
I_\pi = \int \! dx \, \langle \Phi \mid \pi_i(x) \pi_i(x) \mid \Phi \rangle = \int \! \frac{d\mathbf{p}}{(2\pi)^3} \frac{(u_\pi(p) + v_\pi(p))^2}{2\omega_p},
\]

\[
I_\sigma = \int \! dx \, \langle \Phi \mid \sigma(x) \sigma(x) \mid \Phi \rangle = \int \! \frac{d\mathbf{p}}{(2\pi)^3} \frac{(u_\sigma(p) + v_\sigma(p))^2}{2\omega_p},
\]

\[
I_F = \int \! dx \, \langle \Phi \mid \psi(x) \psi(x) \mid \Phi \rangle = -\gamma \int \! \frac{d\mathbf{p}}{(2\pi)^3} 2u_F(p)v_F(p) \cdot \quad (11)
\]

To fully fix the amplitudes \( u_\pi, v_\pi, u_\sigma, v_\sigma, u_F, v_F \), as well as the value of the sigma condensate \( \langle \hat{\sigma} \rangle \), we make use of the Rayleigh-Ritz variational principle. Minimizing the ground state energy \( \mathcal{H}_0 = \langle \Phi \mid H \mid \Phi \rangle / \langle \Phi \mid \Phi \rangle \) while maintaining the canonicity condition of the Bogoliubov transformation Eq.(6), one gets, after a straightforward algebra, the following set of equations

\[
\left. \frac{\delta \mathcal{H}_0}{\delta u_\pi(p)} \right|_{u^2 - v^2 = 1} = 0 \quad \Rightarrow \quad c_\pi(p) = 0 ,
\]

\[
\left. \frac{\delta \mathcal{H}_0}{\delta u_\sigma(p)} \right|_{u^2 - v^2 = 1} = 0 \quad \Rightarrow \quad c_\sigma(p) = 0 ,
\]

\[
\left. \frac{\delta \mathcal{H}_0}{\delta u_F(p)} \right|_{u^2 + v^2 = 1} = 0 \quad \Rightarrow \quad c_F(p) = 0 ,
\]

\[
\left. \frac{\delta \mathcal{H}_0}{\delta \langle \hat{\sigma} \rangle} \right| = 0 \quad \Rightarrow \quad \eta = 0 . \quad (12)
\]

From the above, one sees that at the minimum of the HFB ground state, the Hamiltonian in Eq.(8) reduces to a sum of un-coupled quasi-particle modes (bilinear parts) and residual interactions which are normal ordered with respect to the squeezed vacuum. The set of equations in Eq.(12) constitutes four coupled and self-consistent gap equations. Inserting the solutions of these into the definitions of the quasi-particle energies, one gets the following:

\[
\mathcal{E}_\pi(p) = \omega_p u_\pi(p) - v_\pi(p))^2 = \sqrt{\omega_p^2 + \lambda^2 [5I_\pi + I_\sigma + \langle \hat{\sigma} \rangle^2]} ,
\]

\[
\mathcal{E}_\sigma(p) = \omega_p u_\sigma(p) - v_\sigma(p))^2 = \sqrt{\omega_p^2 + 3\lambda^2 [I_\pi + I_\sigma + \langle \hat{\sigma} \rangle^2]} ,
\]

\[
\mathcal{E}_F(p) = \frac{(p^2 + g^2\langle \hat{\sigma} \rangle^2)(u_F(p)^2 - v_F(p)^2)}{E_p} = \sqrt{p^2 + g^2\langle \hat{\sigma} \rangle^2} ,
\]

\[
\frac{c}{\langle \hat{\sigma} \rangle} = \mu^2 - \frac{g}{\langle \hat{\sigma} \rangle}(I_F + 3\lambda^2(I_\pi + I_\sigma) + \lambda^2\langle \hat{\sigma} \rangle)^2 . \quad (13)
\]

Finally, one deduces the BCS gap equations for the quasi-particle masses as well as the condensate

\[
\mathcal{E}_\pi^2(0) = \mu^2 + \lambda^2 [5I_\pi + I_\sigma + \langle \hat{\sigma} \rangle^2] ,
\]

\[
\mathcal{E}_\sigma^2(0) = \mu^2 + 3\lambda^2 [I_\pi + I_\sigma + \langle \hat{\sigma} \rangle^2] ,
\]

\[
\langle \hat{\sigma} \rangle = -\frac{c}{-\mu^2} - \frac{g}{-\mu^2}I_F + \frac{3\lambda^2\langle \hat{\sigma} \rangle}{-\mu^2}I_\pi + \frac{3\lambda^2\langle \hat{\sigma} \rangle}{-\mu^2}I_\sigma + \frac{\lambda^2\langle \hat{\sigma} \rangle^3}{-\mu^2} ,
\]

\[
\mathcal{E}_F(0) = -g\langle \hat{\sigma} \rangle . \quad (14)
\]
Here $I_\pi$, $I_\sigma$ and $I_F$ are nothing but the one loop tadpoles of the fully dressed quasi-pion, quasi-sigma and quasi-fermion propagators, respectively.

\[
I_\pi = \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + \mathcal{E}_\pi^2(0)}}, \\
I_\sigma = \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{2\sqrt{p^2 + \mathcal{E}_\sigma^2(0)}}, \\
I_F = -2\gamma \int \frac{d\vec{p}}{(2\pi)^3} \frac{\mathcal{E}_F(0)}{2\sqrt{p^2 + \mathcal{E}_F^2(0)}}
\]  

The Feynmann diagrammatic representation of the gap equations in Eq.(14) is displayed in Figure 1. Through the Yukawa coupling, the fermion is in fact minimally coupled to the rest of the gap equations. The gap equation for the fermion could be more tightly coupled if one considers, for instance, a chirally invariant quartic interaction among fermions à la Nambu Jona-Lasinio. This will be briefly discussed at the end of the paper.

At this point, a comment is in order. After inspection of the BCS gap equations Eq.(14), it is clear that a solution with finite condensate ($\langle \hat{\sigma} \rangle \neq 0$) will lead to the following values for the masses of the quasi-pion and quasi-sigma:

\[
\mathcal{E}_\pi^2(0) = \frac{c + gI_F}{\langle \hat{\sigma} \rangle} + 2\lambda^2 [I_\pi - I_\sigma] \\
\mathcal{E}_\sigma^2(0) = \frac{c + gI_F}{\langle \hat{\sigma} \rangle} + 2\lambda^2 \langle \hat{\sigma} \rangle^2
\]  

In this Nambu-Goldstone phase, these two modes are not degenerate as it should be. Therefore, the difference $I_\pi - I_\sigma$ as well as the fermionic tadpole $I_F$ are in fact finite. Thus in the chiral limit ($c \to 0$), the Goldstone mode seems to be absent from the spectrum of the theory, since $\mathcal{E}_\pi$ does not vanish. Consequently, one can check that the Ward identity in Eq.(4) is also violated. Since the condensate and the Goldstone mass are the building blocks for all higher Ward identities, it is clear that the quasi-sigma mass is also unphysical in this approach. It appears then that the squeezed vacuum, apart from the fact that it can accommodate a finite condensate, does neither have the right curvature for the sigma-like excitation, nor a valley along the pion-like excitation. In the following, we can only propose a way to resolve the second problem, namely correcting for the pion-like excitation so as to fulfill the Goldstone theorem. This will have quantitative consequences, as there will be important corrections to the ground state energy.

### B. RPA FLUCTUATIONS

To proceed further, we recall the Hamiltonian in Eq.(8). In the HFB basis, the latter takes the form:

\[
\mathcal{H} = \mathcal{H}_0 + \int d\vec{p} \mathcal{E}_F(p) \sum_r [C^+_r C_r + D^+_r D_r] + \int d\vec{p} \mathcal{E}_\pi(p) \tilde{\alpha}^+_p \tilde{\alpha}_p + \int d\vec{p} \mathcal{E}_\sigma(p) \beta^+_p \beta_p \\
- \int d\mathbf{x} : [\bar{\psi} (\sigma + i\vec{\pi} \vec{\gamma}_5) \psi] + \int d\mathbf{x} : \left[ \lambda^2 \langle \hat{\sigma} \rangle \sigma \left( \vec{\pi}^2 + \sigma^2 \right) + \frac{\lambda^2}{4} \left( \vec{\pi}^2 + \sigma^2 \right)^2 \right]
\]  

(17)
Here the fields $\psi(x)$, $\pi_i(x)$ and $\sigma(x)$ have now the following plane wave expansions in the quasi-particle basis

$$\psi(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^32E_F(p)}} \sum_r [u(p,r)e^{-ipx}C_{pr}^r + v(p,r)e^{ipx}D_{pr}^+]$$,

$$\pi_i(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^32E_\pi(p)}} \left(e^{ipx}\alpha_{pi}^r + e^{-ipx}\alpha_{pi}^+\right)$$,

$$\sigma(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^32E_\sigma(p)}} \left(e^{ipx}\beta_p + e^{-ipx}\beta_p^+\right),$$  \hspace{1cm} (18)

where the harmonic modes are the respective quasi-particle energies.

For a further gathering of the dynamics, we want to proceed by a perturbative diagonalization of the Hamiltonian\(^3\). This can now be done via two kinds of RPA’s; one with sigma-like excitations and an other with pion-like excitations. For symmetry considerations, we rather favor the second. Indeed it was shown in [6] that a RPA with the quantum numbers of the pion field can be made symmetry conserving by enlarging the RPA excitation operator in such a way to accommodate the whole set of excitations formally present in the two-body axial charge operator. On the other hand, Noether theorem defines the axial current to be

$$A_{\mu}^a = i\bar{\psi}\gamma_{\mu}\gamma_5\frac{\tau^a}{2}\psi + \partial_{\mu}\sigma_\pi^a - \partial_{\mu}\pi^a(\sigma + \langle \hat{\sigma} \rangle).$$  \hspace{1cm} (19)

The symmetry operator $Q_5^a$, being the volume integral of the time component $A_{50}^a$, takes in second quantization a structure in which one can identify linear forms standing for the creation and annihilation of quasi-pion modes at rest, as well as bilinear forms corresponding to creation and annihilation of pairs of either fermions or bosons with opposite parities and vanishing total momentum.

According to the Goldstone theorem, the action of the axial charge on the full correlated vacuum with a spontaneously broken phase creates in the chiral limit a non normalizable state which is nothing but the pion state

$$Q^a_5|\text{vac}\rangle \propto |\pi^a\rangle.$$

Here one can mimic this situation and build, on an approximate vacuum which is considered here to be a RPA ground state ($|RPA\rangle$), a normalizable pion state, away from the chiral limit, using the following RPA excitation operator

$$Q_\nu^{a+} = X_\nu^{(1)}\alpha_0^a + Y_\nu^{(1)}\alpha_0^a + \sum_k \left[X_\nu^{(2)}(k)\alpha_k^a + Y_\nu^{(2)}(k)\alpha_k^a\right] + \sum_k \left[X_\nu^{(3)}(k)\alpha_k^a + Y_\nu^{(3)}(k)\alpha_k^a\right] + \sum_{kr\pi} \left[X_\nu^{(4)}(k)C_{kr}^a + Y_\nu^{(4)}(k)D_{kr}^a\right] + \sum_{kr\pi} \left[X_\nu^{(5)}(k)C_{kr}^a + Y_\nu^{(5)}(k)D_{kr}^a\right]$$.  \hspace{1cm} (20)

\(^3\)What is meant here is not the coupling constant perturbation. This word is used here to highlight the additive character of this approach as opposed to a variational self-consistent approach.
The latter is defined to contain the same pair-excitations as the axial charge but with amplitudes which remains to be fixed dynamically. The RPA ground state is defined as the vacuum for the above operator $Q_a^\pi|\textit{RPA}\rangle = 0$, while the single pion state, given by $|\pi^a\rangle$, is normalized as follows

$$\langle \textit{RPA}| [Q_\mu^a, Q_{\nu^a}^+] |\textit{RPA}\rangle = \delta_{\mu\nu} . \quad (21)$$

Using Rowe’s equation of motion method [11], one is lead to the usual RPA secular equations

$$\langle \textit{RPA}| \left[ \delta Q^\mu_{\nu^a}, [H, Q_{\nu^a}^a] \right] |\textit{RPA}\rangle = \omega_{\nu} \langle \textit{RPA}| \left[ \delta Q^\mu_{\nu^a}, Q^a_{\nu^a} \right] |\textit{RPA}\rangle . \quad (22)$$

These are solved within the usual quasi-boson approximation in which the $|\textit{RPA}\rangle$ ground state is replaced by the HFB ground state $|\Phi\rangle$. This procedure which seems brutal and inconsistent is nevertheless commonly used and fully under control. It can be shown that it is equivalent to a lowest order truncated bosonization of all the bilinear operator products which form the Hamiltonian and the excitation operator. In the ansatz above, only the number non-conserving bilinear terms contribute effectively to the RPA equation Eq.(22). The remaining number conserving terms decouples totally. They should appear in higher non harmonic orders in the bosonization. This point will not be developed further here. The equations of motion Eq.(22) are in fact a set of coupled channel equations where all possible states with opposite parities scatter according to the vertices allowed by the Lagrangian Eq.(1). These coupled-channel Lippmann-Schwinger equations are further coupled to a Dyson equation for the pion propagator. These processes are represented schematically below

$$\begin{pmatrix}
\pi \rightarrow \pi & \overline{\pi} \rightarrow \overline{\pi} \\
\overline{\pi} \sigma \rightarrow \overline{\pi} & \overline{\pi} \sigma \rightarrow 0 \\
\overline{\psi} \gamma_5 \frac{\tau}{2} \psi \rightarrow \overline{\psi} \gamma_5 \frac{\tau}{2} \psi & \overline{\psi} \gamma_5 \frac{\tau}{2} \psi \rightarrow 0
\end{pmatrix}$$

Since the linear $\sigma$-model does not provide for a coupling between pairs of fermions and bosons the matrix which governs the subspace for pure fermionic re-scattering is actually diagonal. Including quartic interaction terms à la Nambu Jona-Lasinio leads to a fermionic RPA type of re-scattering. This is briefly shown at the end of the paper.

Let us turn now to the proper resolution of the RPA equations. These can be recast in the following eigenvalue problem form

$$\int d\vec{q}_2 \begin{pmatrix} \mathcal{A}(\vec{q}_1, \vec{q}_2) & \mathcal{B}(\vec{q}_1, \vec{q}_2) \\ \mathcal{B}(\vec{q}_1, \vec{q}_2) & \mathcal{A}(\vec{q}_1, \vec{q}_2) \end{pmatrix} \begin{pmatrix} U_{\nu}(\vec{q}_2) \\ V_{\nu}(\vec{q}_2) \end{pmatrix} = \omega_{\nu} N \begin{pmatrix} U_{\nu}(\vec{q}_1) \\ V_{\nu}(\vec{q}_1) \end{pmatrix} . \quad (23)$$

As anticipated above, the amplitudes $X^{(3)}, Y^{(3)}, X^{(5)}$ and $Y^{(5)}$ in Eq.(20), corresponding to the excitations generated by all number conserving pairs of operators, are found to decouple in the present case. It will be seen later on that this situation does not persist at finite temperature for the number conserving bosonic pairs. The number conserving fermionic pairs, on the other hand, start contributing only at finite total three momentum and finite baryon density. Thus in the eigenvalue RPA equation above, $\mathcal{A}$ and $\mathcal{B}$ are $3 \times 3$ matrices. The $\mathcal{A}$ matrix is the sum of a diagonal matrix which represents the free propagation of re-scattering states, and the $\mathcal{B}$ matrix such that
\[ A(\vec{q}_1, \vec{q}_2) = \begin{pmatrix} E_\pi(0) & 0 & 0 \\ 0 & [E_\pi(\vec{q}_1) + E_\sigma(\vec{q}_1)] & 0 \\ 0 & 2E_F(\vec{q}_1) & \delta(\vec{q}_1 - \vec{q}_2) + B(\vec{q}_1, \vec{q}_2) \end{pmatrix} \]  

(24)

The interaction is fully encoded in the \( B \) matrix which reads

\[ B(\vec{q}_1, \vec{q}_2) = \frac{1}{\sqrt{2E_\pi(0)}} \begin{pmatrix} 0 & 2\lambda^2(\hat{\sigma}) R(\vec{q}_1) \gamma g(2\pi)^{-\frac{3}{2}} \\ 2\lambda^2(\hat{\sigma}) R(\vec{q}_1) \gamma g(2\pi)^{-\frac{3}{2}} & 0 \\ 0 & 0 \end{pmatrix} \]  

(25)

with

\[ R(\vec{q}) = [(2\pi)^34E_\pi(\vec{q})E_\sigma(\vec{q})]^{-1/2} \]  

(26)

Finally the norm matrix \( N \) as well as the RPA eigenvectors \( U_\nu(\vec{q}) \) and \( V_\nu(\vec{q}) \) are given by:

\[ N = \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}, \quad U_\nu(\vec{q}) = \begin{pmatrix} X^{(1)}_{\nu}(\vec{q}) \\ X^{(2)}_{\nu}(\vec{q}) \\ X^{(4)}_{\nu}(\vec{q}) \end{pmatrix}, \quad V_\nu(\vec{q}) = \begin{pmatrix} Y^{(1)}_{\nu}(\vec{q}) \\ Y^{(2)}_{\nu}(\vec{q}) \\ Y^{(4)}_{\nu}(\vec{q}) \end{pmatrix} \]  

(27)

where \( I_d \) is the \( 3 \times 3 \) identity matrix.

The solution of the eigenvalue problem proceeds through straightforward calculation. This can be carried analytically here via a so-called Feshbach projection from the two-particle states subspaces onto the single pion-state subspace. We get the following expression for the RPA eigenvalues

\[ \omega^2_\nu = E^2_\pi(0) + g^2 \Sigma_{FF}(\omega^2_\nu) + \frac{4\lambda^4(\hat{\sigma})^2 \Sigma_{\pi\sigma}(\omega^2_\nu)}{1 - 2\lambda^2 \Sigma_{\pi\sigma}(\omega^2_\nu)}, \]  

(28)

where \( \Sigma_{FF} \) and \( \Sigma_{\pi\sigma} \) are the RPA bubbles given by

\[ \Sigma_{FF}(\omega^2_\nu) = \gamma \int \frac{d\vec{p}}{(2\pi)^3} \frac{4E_F(p)}{\omega^2_\nu - 4E_F(p)^2} \]  

\[ \Sigma_{\pi\sigma}(\omega^2_\nu) = \int \frac{d\vec{p}}{(2\pi)^3} \frac{E_\pi(p) + E_\sigma(p)}{2E_\pi(p)E_\sigma(p)} \frac{1}{\omega^2_\nu - (E_\pi(p) + E_\sigma(p))^2}. \]  

(29)

Eqs. (28,29) are diagrammatically represented in Fig. (2). As it was motivated earlier, one of the RPA solutions has to have the Goldstone character. This solution is commonly known, in the nuclear problem, as the spurious solution. This negative connotation expresses the fact that this solution does not correspond to a real excitation of the system, but rather is an expression of a dynamical breaking of a symmetry in the system as, for instance, the space rotation or translation of a nucleus. In the present case, we hold the spurious solution for a real excitation of the vacuum and it corresponds to the pionic mode. To retrieve explicitly this solution, we first notice the following identities

\[ \Sigma_{\pi\sigma}(0) = \frac{I_\pi - I_\sigma}{E^2_\pi(0) - E^2_\sigma(0)}, \quad \Sigma_{FF}(0) = \frac{I_F}{E_F(0)}. \]  

(30)
These, together with the gap equations Eq.(14), allow to rewrite the RPA frequencies in Eq.(28) into the following form
\[ \omega^2 = \frac{c}{\langle \hat{\sigma} \rangle} + g^2 \left[ \Sigma_{FF}(\omega^2) - \Sigma_{FF}(0) \right] + \frac{2\lambda^2 [\Sigma_{\pi\sigma}(\omega^2) - \Sigma_{\pi\sigma}(0)]}{1 - 2\lambda^2 \Sigma_{\pi\sigma}(\omega^2)} \] (31)

It is clear, from the expression above, that the Goldstone solution is manifestly present in the RPA spectrum. Besides, there exist evidently a continuum of solutions which correspond to two cuts; one is the free propagation of pairs of fermions with opposite parities, and the other corresponds to the quasi-sigma and quasi-pion scattering process.

It is well known from the nuclear many-body problem that the random phase approximation is the perturbative procedure which further diagonalizes the residual interaction inherited from the mean field calculation. In the present case, the considered RPA did not affect the whole residual interaction. Actually, the part of the Hamiltonian which was diagonalized here is the one which allows the transition between states of mixed parities. This part is given by
\[ \mathcal{H}_{RPA} = \int d\vec{p} \mathcal{E}_{\pi}(\vec{p}) \sum_r \left[ C^+_p C^c_p + D^+_p D^c_p \right] + \int d\vec{p} \mathcal{E}_{\pi}(\vec{p}) \bar{\alpha}^+_p \alpha^c_p + \int d\vec{p} \mathcal{E}_{\sigma}(\vec{p}) \beta^+_p \beta^c_p \]
\[ + \int d\vec{x} : \left[ \lambda^2 \langle \hat{\sigma} \rangle \sigma^2 + \frac{\lambda^2}{2} \sigma^2 \pi^2 - ig \bar{\psi} \gamma_5 \gamma_3 \psi \right] : , \] (32)
and the full Hamiltonian reads
\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{RPA} + \mathcal{H}_{\text{residual}} \]
\[ \mathcal{H}_{\text{residual}} = \int d\vec{x} : \left[ \lambda^2 \langle \hat{\sigma} \rangle \sigma^2 + \frac{\lambda^2}{4} \left( (\pi^2)^2 + \sigma^4 \right) - g \bar{\psi} \sigma \psi \right] : \] (33)
where \( \mathcal{H}_0 \) is the mean field contribution to the ground state energy and the semicolons stand for the normal ordering with respect to the HFB squeezed state. The RPA diagonalization brings as well a finite contribution to the vacuum energy. In the RPA basis, the \( \mathcal{H}_{RPA} \) piece of the Hamiltonian has, therefore, the following form
\[ \mathcal{H}_{RPA} = E_{RPA} + : \mathcal{H}_{RPA} : \]
\[ : \mathcal{H}_{RPA} : = \sum_{q,\nu} \omega_\nu(q) Q^+_\nu(q) Q_\nu(q) \]
\[ E_{RPA} = \langle RPA | \mathcal{H}_{RPA} | RPA \rangle \] (34)
where the semicolons indicate this time a normal ordering with respect to the RPA ground state. The evaluation of the RPA contribution to the ground state requires in fact a finite three momentum RPA calculation which was not addressed in this paper. Therefore the \( E_{RPA} \) will not be explicitly given here. This point as well as the nuclear matter equation of state will be addressed soon in a forthcoming publication. As a prerequisite, let us next see how to extend the present approach to the case of finite temperature and finite baryon density.
III. FINITE TEMPERATURE AND BARYON DENSITY EXTENSIONS

The extension of the above approach to a baryon rich system at finite temperature follows through very standard technics. One of these is the Thermo Field Dynamics (TFD) \[12\]. This approach embraces very well the concept of dynamical mappings since it keeps transparent the notion of the vacuum and thus of the Fock space. However, it remains very much involved in the present type of calculations because it proceeds via a doubling of the dynamical variables (\textit{i.e.} doubling of the Fock space). In what follows we adopt the more practical and rather standard approach which consists in evaluating the Grand Canonical Potential \(\Omega\) of the grand canonical ensemble. The conditions of stationarity (vanishing of the first derivative of \(\Omega\)), and stability (positivity of its second derivative) lead to the thermodynamical equilibrium of the system. In the present case, \(\Omega\) takes the form

\[
\Omega = \langle H \rangle - TS - \mu \langle N \rangle .
\]  

(35)

Given the grand canonical partition function \(Z = \text{Tr}[e^{-\beta(H-\mu N)}]\), and density operator \(D = Z^{-1}[e^{-\beta(H-\mu N)}]\), with \(\beta = 1/k_B T\), the expectation values in Eq.(35) are short hand notations which express the following traces

\[
\langle H \rangle = \text{Tr}[DH], \quad S = -k_B \sum_{\vec{p},\rho = \pi,\sigma} [f_\rho(\vec{p})\ln f_\rho(\vec{p}) - (1 + f_\rho(\vec{p}))\ln(1 + f_\rho(\vec{p}))]
\]

\[
\langle N \rangle = \text{Tr}[DN].
\]  

(36)

Here \(k_B\) is the Boltzmann constant, \(T\) the temperature, \(\mu\) the baryon chemical potential, \(H\) the model Hamiltonian of the system, and \(N\) the baryonic particles number operator, given as usual by:

\[
N = \int dx : \bar{\psi} \gamma^0 \psi : .
\]  

(37)

For a theory of free fields, \textit{i.e.} when \(H\) reduces to a diagonal bilinear form in the field operators, computing the thermodynamics of the present system is a common textbook exercise \[13\]. The partition function \(Z\) takes then a simple form which allows to express, for instance, the entropy \(S\) as \[14\]

\[
S = -k_B \sum_{\vec{p},\rho = \pi,\sigma} [f_\rho(\vec{p})\ln f_\rho(\vec{p}) - (1 + f_\rho(\vec{p}))\ln(1 + f_\rho(\vec{p}))]
\]

\[
- k_B \sum_{\vec{p},\rho = F,\bar{F}} [f_\rho(\vec{p})\ln f_\rho(\vec{p}) - (1 - f_\rho(\vec{p}))\ln(1 - f_\rho(\vec{p}))] ,
\]  

(38)

where \(f_{\pi,\sigma}\) are Bose occupation numbers for the pion and the sigma modes, respectively, while \(f_{F,\bar{F}}\) are Fermi occupation numbers for the baryons. For an interacting system of particles, the solution for the partition function \(Z\) as well as for the entropy \(S\), as sketched above, persists in its simple and sympathetic form only in the case of an independent quasi-particle picture. The Hamiltonian, in this case, is split into a diagonal bilinear part which is fully admitted into the thermodynamics, and a residual interaction which is subsequently treated in a perturbation. There exist several ways of realizing a quasi-particle picture. These are based, for instance, on the self-consistent approximations à la Hartree-Bogoliubov
or à la Hartree-Fock-Bogoliubov\textsuperscript{4}. Since we have established in the previous sections that, for symmetry requirements, the HFB basis is needed, we then chose to realize the independent quasi-particle picture in this latter. For this purpose, a thermal Bogoliubov rotation is applied to each field in the Hamiltonian according to:

\begin{align}
\bar{\alpha}^+_p(T) &= u_\pi(p,T)\alpha^+_p - v_\pi(p,T)\alpha^-_p ; & \beta^+_p(T) &= u_\sigma(p,T)b^+_p - v_\sigma(p,T)b^-_p - w_0(T)\delta(p) \\
C^+_{pr}(T) &= u_F(p,T)c^+_{pr} - rv_F(p,T)d^-_{pr} ; & D^+_{pr}(T) &= u_F(p,T)d^+_{pr} + rv_F(p,T)c^-_{pr}
\end{align}

(39)

Here too the canonical normalization of the thermal amplitudes \( u_\pi(p,T), u_\sigma(p,T), v_\pi(p,T), v_\sigma(p,T), u_F(p,T) \) and \( v_F(p,T) \) is assumed

\begin{align}
\langle\pi(p,T)\rangle^2 - \langle\pi(p,T)\rangle^2 &= 1, & \langle\sigma(p,T)\rangle^2 - \langle\sigma(p,T)\rangle^2 &= 1, & \langle\pi(p,T)\rangle^2 + \langle\pi(p,T)\rangle^2 &= 1. \quad (40)
\end{align}

Except for the temperature dependence, the above transformations strictly follow the zero temperature ones performed previously. Using Bloch - De Dominicis theorem [15], it is straightforward to compute the traces \( \langle H \rangle \) and \( \langle N \rangle \). The entropy \( S \) keeps its form in Eq.(38) with the occupation factors given now in terms of the self-consistent thermal quasi-energies by:

\begin{align}
f_\pi(q) &= \left[ \exp\left(\beta\mathcal{E}_\pi^T(q)\right) - 1 \right]^{-1}, & f_\sigma(q) &= \left[ \exp\left(\beta\mathcal{E}_\sigma^T(q)\right) - 1 \right]^{-1}, \\
f_F(q) &= \left[ \exp\left(\beta \left(\mathcal{E}_F^T(q) - \mu\right)\right) + 1 \right]^{-1}, & f_F(q) &= \left[ \exp\left(\beta \left(\mathcal{E}_F^T(q) + \mu\right)\right) + 1 \right]^{-1} \quad (41)
\end{align}

The thermal average of \( H \) reads:

\begin{align}
\langle H \rangle &= \int \frac{d\vec{q}}{(2\pi)^3} \frac{\omega_q}{2} \left[ 3 \left(1 + 2f_\pi(q)\right) \left(\langle\pi(q,T)\rangle^2 + \langle\pi(q,T)\rangle^2\right) + \left(1 + 2f_\sigma(q)\right) \left(\langle\sigma(q,T)\rangle^2 + \langle\sigma(q,T)\rangle^2\right) \right] \\
&+ \gamma \int \frac{d\vec{q}}{(2\pi)^3} \left( f_F(q) + f_F(q) - 1 \right) \left[ E_q \left(\langle\pi(q,T)\rangle^2 - \langle\pi(q,T)\rangle^2\right) - 2\langle\hat{\sigma}\rangle_T u_F(q,T)v_F(q,T) \right] \\
&+ \frac{3\lambda^2}{4} \left[ I^T_\pi I^T_\pi + 5I^T_\pi I^T_\pi + 2I^T_\pi I^T_\pi + 2\langle\hat{\sigma}\rangle_T^2 \left(I^T_\pi + I^T_\pi\right) \right] - c\langle\hat{\sigma}\rangle_T + \frac{\mu^2\langle\hat{\sigma}\rangle_T^2}{2} + \frac{\lambda^2\langle\hat{\sigma}\rangle_T^3}{4},
\end{align}

(42)

With the definitions

\begin{align}
I^T_\pi &= \int d\mathbf{x} \langle\pi_i(\mathbf{x})\pi_i(\mathbf{x})\rangle, & I^T_\sigma &= \int d\mathbf{x} \langle\sigma(\mathbf{x})\sigma(\mathbf{x})\rangle, & I^T_F &= \int d\mathbf{x} \langle\bar{\psi}(\mathbf{x})\psi(\mathbf{x})\rangle.
\end{align}

(43)

Minimizing the grand potential \( \Omega \) with respect to the shift parameter \( w_0(T) \) gives an equation for the chiral condensate

\begin{align}
\frac{\delta\Omega}{\delta\langle\hat{\sigma}\rangle_T} &= \frac{u_\sigma(0,T) + v_\sigma(0,T)}{\sqrt{\mu}} \left[ 3\lambda^2\langle\hat{\sigma}\rangle_T (I^T_\pi + I^T_\pi) - gI^T_F + \lambda^2\langle\hat{\sigma}\rangle_T^3 + \mu^2\langle\hat{\sigma}\rangle_T - c \right] = 0 \quad (44)
\end{align}

On the other hand, the variations of \( \Omega \) with respect to the thermal Bogoliubov amplitudes \( u_\pi(q,T), u_\sigma(q,T) \) and \( v_F(q,T) \), while keeping the unitarity constraints Eq.(40) satisfied, lead to

\textsuperscript{4}Indeed both of them deliver self-consistent operator bases which can keep the bilinear part of the Hamiltonian diagonal, thus allowing an independent quasi-particle picture.
\[
\frac{\delta \Omega}{\delta u_\pi(q, T)} \bigg|_{u_\pi^2-v_\pi^2=1} = \mathcal{E}_\pi^T(q) \\
= \omega_q u_\pi(q, T) v_\pi(q, T) + \frac{\lambda^2}{2} \frac{(u_\pi(q, T) + v_\pi(q, T))^2}{\omega_q} \left[ 5I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right] = 0
\]
\[
\frac{\delta \Omega}{\delta u_\sigma(q, T)} \bigg|_{u_\sigma^2-v_\sigma^2=1} = \mathcal{E}_\sigma^T(q) \\
= \omega_q u_\sigma(q, T) v_\sigma(q, T) + \frac{3\lambda^2}{2} \frac{(u_\sigma(q, T) + v_\sigma(q, T))^2}{\omega_q} \left[ I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right] = 0
\]
\[
\frac{\delta \Omega}{\delta u_F(q, T)} \bigg|_{u_F^2+v_F^2=1} = \mathcal{E}_F^T(q) \\
= 2E_p u_F(q, T) v_F(q, T) + g\langle \hat{\sigma} \rangle_T \left[ u_F(q, T)^2 - v_F(q, T)^2 \right]
\]

It is interesting to notice that the stationarity conditions listed above ensure at the same time the independent particle picture for the finite temperature HFB solution, giving by this a consistency to the whole approach

\[
\langle [\alpha_i(q), [H, \alpha_j^+(p)]] \rangle \propto \mathcal{E}_\pi^T(q) \delta_{q,p} \delta_{ij} = 0 \\
\langle [\beta(q), [H, \beta(p)]] \rangle \propto \mathcal{E}_\sigma^T(q) \delta_{q,p} = 0 \\
\langle \{ D_i(q), \{ H, C_j(p) \} \} \rangle \propto \mathcal{E}_F^T(q) \delta_{q,p} \delta_{rs} = 0
\]

The thermal quasi-particle energies which correspond to the harmonic excitations of the system can be computed from

\[
\langle [\alpha_i(q), [H, \alpha_j^+(p)]] \rangle = \mathcal{E}_\pi^T(q) \delta_{q,p} \delta_{ij} \\
= \omega_q \left[ u_\pi(q, T)^2 + v_\pi(q, T)^2 \right] + \lambda^2 \frac{(u_\pi(q, T) + v_\pi(q, T))^2}{\omega_q} \left[ 5I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right] \\
\langle [\beta(q), [H, \beta(p)]] \rangle = \mathcal{E}_\sigma^T(q) \delta_{q,p} \\
= \omega_q \left[ u_\sigma(q, T)^2 + v_\sigma(q, T)^2 \right] + 3\lambda^2 \frac{(u_\sigma(q, T) + v_\sigma(q, T))^2}{\omega_q} \left[ I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right] \\
\langle \{ D_i(q), \{ H, C_j^+(p) \} \} \rangle = \mathcal{E}_F^T(q) \delta_{q,p} \delta_{rs} \\
= E_q \left( u_F(q, T)^2 - v_F(q, T)^2 \right) - 2g\langle \hat{\sigma} \rangle_T u_F(q, T)v_F(q, T)
\]

Comparing the above results with those obtained earlier for the zero temperature case, one can see that the equations describing both stationarity conditions, namely the ground state energy for the first and the thermal equilibrium for the second, have kept the same form, apart of course from the presence of the thermal occupation factors in the latter. One can also check that at the minimum of \( \Omega \), the quasi-particle energies read

\[
\mathcal{E}_{\pi, \sigma}^T(q) = \omega_q \left[ u_{\pi, \sigma}(q, T) - v_{\pi, \sigma}(q, T) \right]^2 , \quad \mathcal{E}_F^T(q) = \sqrt{q^2 + g^2 \langle \hat{\sigma} \rangle^2_T} ,
\]

which lead as well to the familiar expressions of the thermal tadpoles

\[
I_{\pi, \sigma}^T = \int \frac{d^3q}{(2\pi)^3} \frac{1 + 2f_{\pi, \sigma}(q)}{2 \mathcal{E}_{\pi, \sigma}^T(q)} , \quad I_F^T = 2\gamma \mathcal{E}_F(0) \int \frac{d^3q}{(2\pi)^3} \frac{f_F(q) + f_F(q) - 1}{2\mathcal{E}_F(q)} .
\]
Finally the finite temperature BCS solutions take the form of the following four coupled and self-consistent equations:

\[
\begin{align*}
\mathcal{E}_\pi^T(0) &= \mu^2 + \lambda^2 \left[ 5 I_\pi^T + I_\sigma^T + \langle \dot{\sigma} \rangle_T^2 \right], \\
\mathcal{E}_\sigma^T(0) &= \mu^2 + 3\lambda^2 \left[ I_\pi^T + I_\sigma^T + \langle \dot{\sigma} \rangle_T^2 \right], \\
\langle \dot{\sigma} \rangle_T &= -\frac{c}{-\mu^2} - \frac{g I_\pi^T}{-\mu^2} + \frac{3\lambda^2 \langle \dot{\sigma} \rangle_T}{-\mu^2} I_\pi^T + \frac{3\lambda^2 \langle \dot{\sigma} \rangle_T}{-\mu^2} I_\sigma^T + \frac{\lambda^2 \langle \dot{\sigma} \rangle_T^3}{-\mu^2}, \\
\mathcal{E}_F^T(0) &= -g \langle \dot{\sigma} \rangle_T. 
\end{align*}
\]

(50)

As in the zero temperature case, the finite temperature BCS equations lead to a dynamical mass generation and thus to a violation of the Goldstone theorem. Therefore, one needs to go one step further and consider those necessary thermal fluctuations present in the residual interaction. One way of bringing about these effects consists in taking the limit of a weakly interacting system which leads to a linearization of the thermal TDHF equations. Such a solution is known to correspond to the thermal RPA approximation (see ref. [16] for details). The TRPA equation are very similar in form to the zero temperature Rowe equations of motion Eq. (22). The expectation values are, however, not realized anymore on the RPA ground state. They correspond, instead, to traces taken on the grand canonical ensemble, such that

\[
\langle [\delta \hat{Q}_\nu, [H, \hat{Q}_\nu^+]] \rangle = \omega_\nu^T \langle [\delta \hat{Q}_\nu, \hat{Q}_\nu^+] \rangle. 
\]

(51)

The evaluation of such traces is a task which again can be handled very well by means of the Bloch-De Dominicis theorem. The net result is the following eigenvalue problem

\[
\int d\vec{q}_2 T^{\nu}(\vec{q}_1) \begin{pmatrix} A^T(\vec{q}_1, \vec{q}_2) & B^T(\vec{q}_1, \vec{q}_2) \\ B^T(\vec{q}_1, \vec{q}_2) & A^T(\vec{q}_1, \vec{q}_2) \end{pmatrix} T^\nu(\vec{q}_2) \begin{pmatrix} U_{\nu}(\vec{q}_2, T) \\ V_{\nu}(\vec{q}_2, T) \end{pmatrix} = \omega_\nu(T) \begin{pmatrix} U_{\nu}(\vec{q}_1, T) \\ V_{\nu}(\vec{q}_1, T) \end{pmatrix},
\]

(52)

where \( A^T \) and \( B^T \) are this time 4 \times 4 matrices which have only an implicit dependence on temperature inherited from the temperature dependence of the thermal HFB basis. They are given by

\[
A^T(\vec{q}_1, \vec{q}_2) = \begin{pmatrix} \mathcal{E}_\pi^T(0) & 0 & 0 & 0 \\ 0 & \mathcal{E}_\pi^T(\vec{q}_1) + \mathcal{E}_\sigma^T(\vec{q}_1) & 0 & 0 \\ 0 & 0 & \mathcal{E}_\pi^T(\vec{q}_1) - \mathcal{E}_\sigma^T(\vec{q}_1) & 0 \\ 0 & 0 & 0 & 2\mathcal{E}_F^T(\vec{q}_1) \end{pmatrix} \delta(\vec{q}_1 - \vec{q}_2) + B^T(\vec{q}_1, \vec{q}_2)
\]

(53)

The effect of the residual interaction which is responsible for the RPA re-scattering is encoded in the \( B^T \) matrix which takes the form

\[
B^T(\vec{q}_1, \vec{q}_2) = \frac{1}{\sqrt{2\mathcal{E}_\pi^T(0)}} \begin{pmatrix} 0 & 2\lambda^2 \langle \dot{\sigma} \rangle_T R^T(\vec{q}_2) & 2\lambda^2 \langle \dot{\sigma} \rangle_T R^T(\vec{q}_2) & g(2\pi)^{-\frac{3}{2}} \\ 2\lambda^2 \langle \dot{\sigma} \rangle_T R^T(\vec{q}_1) & 0 & 2\lambda^2 \langle \dot{\sigma} \rangle_T R^T(\vec{q}_1) & 2\lambda^2 \langle \dot{\sigma} \rangle_T R^T(\vec{q}_1) \\ 2\lambda^2 \langle \dot{\sigma} \rangle_T R(\vec{q}_1) & 2\lambda^2 \langle \dot{\sigma} \rangle_T R(\vec{q}_1) & 0 & 0 \\ g(2\pi)^{-\frac{3}{2}} & 0 & 0 & 0 \end{pmatrix}
\]

(54)

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Here $R^T(q)$ has an implicit temperature dependence and is of the same form as in Eq.(26).

The explicit temperature dependence of the RPA equations is carried by the norm matrix $N^T$. This as well as the RPA amplitudes $U_\nu(q,T)$ and $V_\nu(q,T)$ are given by:

$$N^T(q) = \begin{pmatrix} N^T_1(q) & 0 \\ 0 & -N^T_1(q) \end{pmatrix}, \quad U_\nu(q,T) = \begin{pmatrix} X^{(1)}_\nu(q,T) \\ X^{(2)}_\nu(q,T) \\ X^{(3)}_\nu(q,T) \\ X^{(4)}_\nu(q,T) \end{pmatrix}, \quad V_\nu(q,T) = \begin{pmatrix} Y^{(1)}_\nu(q,T) \\ Y^{(2)}_\nu(q,T) \\ Y^{(3)}_\nu(q,T) \\ Y^{(4)}_\nu(q,T) \end{pmatrix}.$$ (55)

Now $N^T_1$ is a $4 \times 4$ diagonal matrix, having as diagonal elements:

$$N^T_1(q)_{11} = 1,$$
$$N^T_1(q)_{22} = 1 + f_\sigma(q) + f_\sigma(q),$$
$$N^T_1(q)_{33} = f_x(q) - f_x(q),$$
$$N^T_1(q)_{44} = 1 - f_F(q) - f_F(q).$$ (56)

The difference between $N^T_1$ and $I_4$, the analog matrix at zero temperature Eq.(27), represents the single formal departure of the thermal RPA equations from the zero temperature ones. Finally one can proceed to the resolution of the RPA eigenvalue problem. The RPA frequencies, as in the zero temperature case, can be read from the characteristic equation

$$\omega^2_\nu(T) = \mathcal{E}^T_{\pi}(0) + g^2 \Sigma^T_{FF}(\omega^2_\nu(T)) + \frac{4\lambda^4 \langle \hat{\sigma} \rangle_{\pi}^2 \Sigma^T_{\pi\sigma}(\omega^2_\nu(T))}{1 - 2\lambda^2 \Sigma^T_{\pi\sigma}(\omega^2_\nu(T))},$$ (57)

where

$$\Sigma^T_{FF}(\omega^2_\nu(T)) = \gamma \int \frac{d\hat{p}}{(2\pi)^3} 4\mathcal{E}^T_{\pi}(\hat{p}) \frac{1 - f_F(\hat{p}) - f_\pi(\hat{p})}{\omega^2_\nu(T) - 4\mathcal{E}^T_{\pi}(\hat{p})},$$
$$\Sigma^T_{\pi\sigma}(\omega^2_\nu(T)) = \int \frac{d\hat{p}}{(2\pi)^3} \left\{ \mathcal{E}^T_{\pi}(\hat{p}) + \mathcal{E}^T_{\sigma}(\hat{p}) \frac{1 + f_\pi(\hat{p}) + f_\sigma(\hat{p})}{2\mathcal{E}^T_{\pi}(\hat{p})\mathcal{E}^T_{\sigma}(\hat{p}) \omega^2_\nu(T) - (\mathcal{E}^T_{\pi}(\hat{p}) + \mathcal{E}^T_{\sigma}(\hat{p}))^2} \right.\left. + \frac{\mathcal{E}^T_{\pi}(\hat{p}) - \mathcal{E}^T_{\sigma}(\hat{p})}{2\mathcal{E}^T_{\pi}(\hat{p})\mathcal{E}^T_{\sigma}(\hat{p})} \frac{f_\pi(\hat{p}) - f_\sigma(\hat{p})}{\omega^2_\nu(T) - (\mathcal{E}^T_{\pi}(\hat{p}) - \mathcal{E}^T_{\sigma}(\hat{p}))^2} \right\}. $$ (58)

Now, like in the zero temperature case and making use of the identities

$$\Sigma^T_{\pi\sigma}(0) = \frac{\Sigma^T_{FF}(0)}{\mathcal{E}^T_{\pi}(0) - \mathcal{E}^T_{\pi}^{T2}(0)}, \quad \Sigma^T_{FF}(0) = \frac{I^T_{FF}(0)}{\mathcal{E}^T_{\pi}(0)},$$ (59)

one gets, after inspection of the gap equations, the following

$$\omega^2_\nu(T) = \frac{c}{\langle \hat{\sigma} \rangle_T} + g^2 \left[ \Sigma^T_{FF}(\omega^2_\nu(T)) - \Sigma^T_{FF}(0) \right] + \frac{2\lambda^2 \left[ \mathcal{E}^T_{\pi}(0) - \mathcal{E}^T_{\pi}^{T2}(0) \right] \left[ \Sigma^T_{\pi\sigma}(\omega^2_\nu(T)) - \Sigma^T_{\pi\sigma}(0) \right]}{1 - 2\lambda^2 \Sigma^T_{\pi\sigma}(\omega^2_\nu(T))}. $$ (60)

Equation (60) exhibits clearly a zero energy solution for all temperatures below the transition to the Wigner Weyl phase. At the transition and beyond \(\langle \hat{\sigma} \rangle = 0\), this solution is not normalizable, as can be easily seen from the norm matrix $N^T$.
Before closing this paper, it is certainly worth briefly showing how robust is the field-to-current mapping in systematically preserving the symmetry. A situation of particular interest in the nuclear problem is the renormalization of the fermionic four-point function by means of the so-called short range correlation. This can be addressed via a chiral invariant fermionic quartic interaction of Nambu-Jona-Lasinio (NJL) type

\[ \mathcal{L}_1 = \mathcal{L}_\sigma + G \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right], \]

where the NJL-coupling \( G \) plays the role of the \( g' \) Migdal-parameter. The BCS gap equations are now further coupled by the insertion of a fermionic tadpole contribution to the fermion quasi-mass such that

\[
\begin{align*}
E_T^2(0) &= \mu^2 + \lambda^2 \left[ 5I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right], \\
E^T_\sigma(0) &= \mu^2 + 3\lambda^2 \left[ I^T_\pi + I^T_\sigma + \langle \hat{\sigma} \rangle^2_T \right], \\
\langle \hat{\sigma} \rangle_T &= -\frac{c}{-\mu^2} - \frac{3\lambda^2\langle \hat{\sigma} \rangle_T}{\mu^2} I^T_\pi + \frac{3\lambda^2\langle \hat{\sigma} \rangle_T}{\mu^2} I^T_\sigma + \frac{\lambda^2\langle \hat{\sigma} \rangle^3_T}{-\mu^2}, \\
E_T^F(0) &= -g\langle \hat{\sigma} \rangle_T + \tilde{G} I^T_F,
\end{align*}
\]

where \( \tilde{G} = -2G(1 + \frac{1}{3}) \). Here again, the asymptotic Goldstone pion is generated by the previously introduced parity-mixing RPA. This time one gets the following

\[
\omega^2_{\nu}(T) = \frac{c}{\langle \hat{\sigma} \rangle_T} + \frac{g^2 \left[ \Sigma_{FF}^T(\omega^2_{\nu}(T)) - \Sigma_{FF}^T(0) \right]}{\left[ 1 - \tilde{G} \Sigma_{FF}^T(\omega^2_{\nu}(T)) \right] \left[ 1 - \tilde{G} \Sigma_{FF}^T(0) \right]} \left[ 1 - \frac{2\lambda^2 \left[ E^T_\sigma(0) - E^T_\pi(0) \right]}{\Sigma^T_{\pi\nu}(\omega^2_{\nu}(T)) - \Sigma^T_{\pi\pi}(0)} \right].
\]

Figures (3) give the diagrammatic representation of the HFB-RPA dynamics in this case. Of course, further extensions of the model to other chirally invariant boson-fermion couplings are possible.

IV. CONCLUSION

In conclusion, we have presented an extension of the field-to-current mapping, introduced earlier, to a baryon rich regime at finite temperature. The mapping was made dynamical in the Hartree-Fock-Bogoliubov-RPA approximation. The Goldstone theorem was fulfilled exactly, although the dynamics was not sorted-out according to neither an expansion in the available coupling constants \( (\lambda, g, G) \) nor in the arbitrary charge \( N \) and flavor \( N_F \) numbers. In fact, it can be easily appreciated, from the four coupled BCS equations for instance, that this approach is highly order mixing.

All over this work, we did not consider the p-wave renormalization of the pion since it is well known that it does not endanger the Goldstone nature of the pion. Instead, we fixed our attention on the subtle situation of the s-wave renormalization where potential artefacts, linked to the dynamical mass generation, may alter the Goldstone character of the pion. This
was indeed visible from the BCS solution in which all the quasi-particle states were massive. Therefore it is certainly legitimate to state here that the HFB ground-state is not a viable vacuum and should not be tolerated as an approximate ground-state for nuclear matter saturation studies for instance. This point, although conceptually crucial and certainly quantitatively important, is usually not observed in the literature. As an alternative, we have presented the RPA ground-state, as a qualitatively correct approximate ground-state, in which the Nambu-Goldstone phase is exactly realized. On the other hand, we have also indicated that this approach can support whatever chirally invariant refinement made to the interaction Lagrangian. We hope that this will give a solid platform for a program of a quantitative assessment of the nuclear equation of state, beyond the usually considered classical mean-field calculations in QHD models. It should be also mentioned that the present work is of some relevance to the undergoing quantum mean-field calculations, based on chirally-invariant contact nuclear-forces, undertaken by Nikolaus, Hoch and Madland [17]. According to these authors, the inclusion of the pion dynamical contributions is a step which one needs to envisage seriously in the future [18]. The non-linear chiral realization à la heavy baryon ChPT is certainly not well suited for a non-perturbative treatment of the dynamics. Therefore we see in the present prescription an interesting framework for such a program.

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FIG. 1. Diagrammatic representation of the four coupled BCS solutions. The dashed line denotes the self-consistent quasi-pion propagator, the solid line the quasi-sigma, the double solid line the quasi-fermion, and the wavy line the two point Green’s function of the bare field $\hat{\sigma}$ of the Lagrangian density in Eq.(1).
FIG. 2. Upper part: The Dyson equation for the physical pion (thick dashed lines) for which the mass operator has been extracted from the scattering of the quasi-particles in a RPA equation. Lower part: The scattering equation for a pair of quasi-sigma (thin full lines) and quasi-pion (thin dashed lines).
FIG. 3. The full class of diagrams for the asymptotic Goldstone pion in the context of the extension discussed in the text.