Hypermagnetic Knots, Chern-Simons Waves and the Baryon Asymmetry

Massimo Giovannini∗

Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155, USA

At finite hyperconductivity and finite fermionic density the flux lines of long range hypermagnetic fields may not have a topologically trivial structure. The combined evolution of the chemical potentials and of pseudoscalar fields (like the axial Higgs), possibly present for temperatures in the TeV range, can twist the hypercharge flux lines, producing, ultimately, hypermagnetic knots (HK). The dynamical features of the HK depend upon the various particle physics parameters of the model (pseudoscalar masses and couplings, strength of the electroweak phase transition, hyperconductivity of the plasma) and upon the magnitude of the primordial flux sitting in topologically trivial configurations of the hypermagnetic field. We study different cosmological scenarios where HK can be generated. We argue that the fermionic number sitting in HK can be released producing a seed for the Baryon Asymmetry of the Universe (BAU) provided the typical scale of the knot is larger than the diffusivity length scale. We derive constraints on the primordial hypermagnetic flux required by our mechanism and we provide a measure of the parity breaking by connecting the degree of knottedness of the flux lines to the BAU. We rule out the ordinary axion as a possible candidate for production (around temperatures of the order of the GeV) of magnetic knots since the produced electromagnetic helicity is negligible (for cosmological standard) if the initial amplitude of the axion oscillations is of the order of the Peccei-Quinn breaking scale.

I. FORMULATION OF THE PROBLEM

In a globally neutral plasma, the transversality of the magnetic fields implies, at high conductivity, the conservation of both magnetic flux and magnetic helicity. The plasma element evolves always glued together with magnetic flux lines whose number of associated knots and twists does not change. The energetical and the topological properties of the magnetic flux lines are conserved in the superconducting (or ideal) regime. Moreover, the linearity of the magnetohydrodynamical equations in the mean electric and magnetic fields implies that there are no chances that large scale magnetic fields could be generated only using electromagnetic plasma effects.

As noticed long ago [1,2] magnetohydrodynamics (MHD) represents a quite powerful method in order to analyze the dynamics of the interstellar plasma. In particular the topological properties of the magnetic field lines and of the bulk velocity field play a significant role in the dynamo theory (often invoked in order to explain the typical amplitudes of large scale magnetic fields in spiral galaxies [3]) and it seems well established that for successfully implementing the dynamo mechanism we are led to postulate a global parity violation at the scale at which the dynamo action is turned on [4,5]. Again, because of the linearity of MHD, in order to explain the galactic magnetic fields [3,5] we have to assume that at some finite moment of time (after the Big-Bang) some magnetic seeds [6] had to be present.

Up to now, the attention has been mainly focused on the energetical properties of the seeds and very little has been said concerning their topological features. Apart from the dynamo mechanism, the topology of the magnetic flux lines seems less relevant (for the seed problem) than their associated energy density. However, going backward in time, we find that the topological properties of long range (Abelian) gauge fields can have unexpected physical implications.

The importance of the topological properties of long range (Abelian) hypercharge magnetic fields has been indeed emphasized in the past [7,8]. In [7] the interesting possibility of macroscopic parity breaking at finite fermionic density and finite temperature was studied and it was suggested that if parity is globally broken an electromagnetic current density directed along the magnetic field can arise. A similar observation has been made in the context of the standard electroweak theory at finite temperature and chemical potential [8], where it has been observed that, thanks to the anomalous coupling of the hypercharge, the fermionic density can be converted into infra-red modes of the hypercharge field.

If the spectrum of hypermagnetic fields is dominated by parity non-invariant configurations the baryon asymmetry of the Universe could be the result of the decay of these condensates [9]. The purpose of the present paper is to connect the topological properties of the hypermagnetic flux tubes to the Baryon Asymmetry of the Universe (BAU) by discussing a class of dynamical mechanisms leading to the generation of HK.

∗Electronic address: giovan@cosmos2.phy.tufts.edu
The evolution of mean hypermagnetic fields in an electroweak plasma at finite conductivity and finite fermionic densities can be derived [10] and it turns out that a good gauge-invariant measure of global parity invariance is the hypermagnetic helicity, i.e. \( \mathcal{H}_Y \cdot \mathbf{\nabla} \times \mathcal{H}_Y \). If the hypermagnetic field distribution is topologically trivial (i.e. \( \langle \mathcal{H}_Y \cdot \mathbf{\nabla} \times \mathcal{H}_Y \rangle = 0 \)) matter–antimatter fluctuations can be generated at the electroweak scale [11]. Provided the typical correlation scale of the magnetic helicity is larger than (or of the order of) the neutron diffusion scale blue-shifted at the electroweak epoch [i.e. \( L_n(T_{ew}) \sim 0.3 \text{ cm} \) at \( T_{ew} \sim 100 \text{ GeV} \)], matter–antimatter domains can survive until the onset of Big-Bang nucleosynthesis (BBN) leading to non standard initial conditions for the light element abundances calculations. In the presence of a fraction of antimatter at BBN a reduction of the neutron to proton ratio can be foreseen with consequent reduction of the \(^4\text{He} \) abundance. This effect was recently pointed out in BBN calculations [12] in the presence of spherical antimatter domains and further work is in progress [13].

An example of topologically trivial distribution of hypermagnetic fields leading to some of the above mentioned effects is a stochastic background whose (parity-invariant) two-point function can be expressed, in Fourier space, as

\[
G_{ij}(k) = k^2 f(k) \delta^{(3)}(\mathbf{k} - \mathbf{k}') \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right), \quad f(k) = \frac{A(k)}{k} \tag{1.1}
\]

\( (A(k) \) is a dimension-less factor measuring the intensity of the stochastic hypermagnetic background). Since hypermagnetic fields are strictly divergence free (i.e. no hypermagnetic monopoles are present) the magnetic flux lines form a mixture of closed loops evolving glued to their electroweak plasma element traveling with velocity \( \mathbf{v} \).

If the Universe is filled, at some scale, by topologically non-trivial HK the flux lines form a network of stable knots and twists which cannot be broken provided the conductivity is high. Therefore \( \langle \mathcal{H}_Y \cdot \mathbf{\nabla} \times \mathcal{H}_Y \rangle \neq 0 \). Thus, completely homogeneous hypermagnetic configurations cannot be topologically non-trivial. If we want to have knots and twists in the flux lines, we are necessarily driven towards field configurations which are not only time but also space dependent like the Chern-Simons waves [8,9] analogous to the magnetic knot configurations [10] frequently employed in the analysis of dynamo instabilities [5,14].

If the hypermagnetic configurations are topologically non-trivial (and consequently inhomogeneous) some knots in the lines of the velocity field can be also expected. Indeed, by solving the hypermagnetic diffusivity equation to leading order in the resistivity expansion, the velocity field turns out to be proportional to the hypermagnetic field, namely \( \mathbf{v}_Y \sim \mathcal{H}_Y / (\sqrt{N_{eff}} T_{ew}^2) \). Notice that this is nothing but the electroweak analogous of the Alfvén velocity [15] so important in the spectrum of (ordinary) plasma waves [16]. Since the hypermagnetic fields are divergence-free, the fluid will also be incompressible (i.e. \( \mathbf{\nabla} \cdot \mathbf{v}_Y = 0 \)) as required, by equations of MHD. It is now clear that the topological properties of the magnetic flux lines can be connected with the topological properties of the velocity field lines.

In this paper we want to understand if it is at all possible to generate a topologically non-trivial distribution of hypercharge fluctuations from a topologically trivial collection of the same fields. More physically we are going to investigate how to create knots and twists in a stochastic collection of disconnected magnetic flux loops. We would also like to understand if this kind of mechanism can or cannot be implemented in some reasonably motivated cosmological framework.

A way of twisting the topology of a collection of independent (hypermagnetic) flux loops, is to have dynamical pseudoscalar fields at finite fermionic density. The reason is twofold. On one hand dynamical pseudoscalars fields are coupled to gauge fields through the anomaly and, as a result, they induce a coupling between the two polarizations of photons and/or hyperphotons. On the other hand, unlike the ordinary electromagnetic fields, the coupling of the hypercharge field to fermions is chiral [8]. The generation of hypermagnetic knots can influence the evolution of the chemical potential leading to the production of the BAU [9]. The BAU can be produced provided the electroweak phase transition (EWPT) is strongly first order, provided the typical scale of the HK exceeds the diffusivity scale associated with the finite value of the hyperconductivity, and provided the rate of the slowest processes in the plasma is sufficiently high. Unfortunately, in the minimal standard model (MSM) the EWPT cannot be strongly first order for Higgs boson masses larger than the W boson mass [17]. It is true that the presence of a (topologically trivial) hypermagnetic background [18,19] can modify the phase diagram but cannot make the EWPT strongly first order. We will argue that our scenario can have some chances in the context of the minimal supersymmetric standard model (MSSM) where, incidentally, the rate of the right-electron chirality flip processes can also be larger than in the MSM.

Dynamical pseudoscalars have been used in the past in order to generate seeds for the magnetohydrodynamical evolution [20,21]. Our concern here is different and it is mainly connected with the BAU. Suppose that at some moment in the life of the Universe dynamical pseudoscalars fields are present. Then, at finite conductivity and finite fermionic density (i.e. finite chemical potential) both, the chemical potential and the pseudoscalar particles, can be coupled to the anomaly.
where \( c \) is the pseudoscalar coupling constant, \( M \) the typical energy scale and \( \mathcal{Y}_{\alpha\beta} = \partial_\alpha \mathcal{Y}_\beta \) is the gauge field strength expressed in terms of the vector potentials. In our analysis we will be mainly concerned with the case of the hypercharge field and with the case of the ordinary electromagnetic field. Imagine now that the pseudoscalar field is uniform but time dependent. If our initial state is a (topologically trivial) stochastic distribution of gauge field fluctuations the effect of the interaction described in Eq. (1.2) will be to couple the two (transverse) polarizations of the gauge fields. Since the evolution equations of the two (independent) polarizations are different, there are no reasons to expect that the topological structure of the initial gauge field distribution will be left unchanged by the presence of pseudoscalar interactions. The inclusion of a (macroscopic) ohmic conductivity will automatically select a preferred reference frame (the so called “plasma frame”) where the flux and helicity associated with the magnetic component of the the Abelian background are both (approximately) conserved if the resistivity is small.

In the plasma frame, the initial stochastic distribution of gauge field fluctuations can be viewed as a collection of unknotted magnetic flux tubes (closed by transversality) which evolve independently without breaking or intersecting each others. The inclusion of pseudoscalar interactions will then alter the topological properties of the magnetic flux lines by introducing links between independent loops and by also twisting a single loop. Part of the results reported in this paper have been recently presented (in a more compact form) in [22].

The plan of our paper is then the following. In Section II we introduce the basic description of hypermagnetic fields coupled to dynamical pseudoscalars in an empty (but expanding) space. We will then solve the time evolution of the field operators in the Heisenberg representation. In Section III we will formally address the problem of emission of hypermagnetic knots induced by dynamical pseudoscalars. In Section IV we will concentrate our attention on the hypermagnetic helicity production during an inflationary phase whereas in Section V we will discuss the radiation dominated phase. In Section VI we will connect the production of (mean) hypermagnetic helicity to the BAU and in Section VII we will investigate the regions of parameter space where the generate BAU can be phenomenologically relevant. In Section VIII we will examine, for sake of completeness, the problem of global parity breaking induced by axionic particles. Section IX contains our concluding remarks. In the Appendix we collected some useful technical results.

### II. HYPERMAGNETIC FIELDS AND CHERN-SIMONS WAVES IN EMPTY SPACE

The Abelian nature of the hypercharge field does not imply that the hypermagnetic flux lines should have a trivial topological structure. We will name topologically trivial the configurations whose field lines are closed (by transversality of the field) and unknotted. Conversely, in the gauge \( \mathcal{Y}_0 = 0, \vec{\nabla} \cdot \vec{\mathcal{Y}} = 0 \), an example of topologically non-trivial configuration of the hypercharge field is the Chern-Simons wave [8–10]

\[
\begin{align*}
\mathcal{Y}_x(z,t) &= \mathcal{Y}(t) \sin k_0 z, \\
\mathcal{Y}_y(z,t) &= \mathcal{Y}(t) \cos k_0 z, \\
\mathcal{Y}_z(z,t) &= 0.
\end{align*}
\]

This particular configuration is not homogeneous but it describes a hypermagnetic knot with homogeneous helicity and Chern-Simons number density

\[
\mathcal{H}_Y \cdot \vec{\nabla} \times \mathcal{H}_Y = k_0 \mathcal{H}^2(t),
\]

\[
n_{CS} = -\frac{g^2}{32\pi^2} \mathcal{H}_Y \cdot \vec{\mathcal{Y}} = \frac{g^2}{32\pi^2 k_0} \mathcal{H}^2(t),
\]

where \( \mathcal{H}_Y = \vec{\nabla} \times \vec{\mathcal{Y}}, \mathcal{H}(t) = k_0 \mathcal{Y}(t); g' \) is the \( U(1)_Y \) coupling. Other examples can be found. For instance, it is possible to construct hypermagnetic knot configurations with finite energy and helicity which are localized in space within a typical size \( L_z \). These examples are reported in Appendix A.

Suppose now that the Universe can be described, at some epoch below the Planck scale, by a homogeneous, isotropic and spatially flat Friedmann-Robertson-Walker (FRW) metric with line element

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g_{\mu\nu} = a^2(\tau) \eta_{\mu\nu}
\]

(2.3)
(τ is the conformal time coordinate and \( \eta_{\mu\nu} \) is the ordinary flat metric with signature \([+,-,-,-]\)). Let us assume that dynamical pseudoscalar particles are evolving in the background geometry given by Eq. (2.3). The pseudoscalars are not a source of the background (i.e. they do not affect the time evolution of the scale factor) but, nonetheless, they evolve according to their specific dynamics and can excite other degrees of freedom.

The effective action describing the interaction of a dynamical pseudoscalars with hypercharge fields can be written, in curved space as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - V(\psi) - \frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} Y_{\alpha\beta} Y_{\gamma\delta} + \frac{c \psi}{4M} Y_{\alpha\beta} Y_{\gamma\delta} \right],
\]

This action is quite generic. In the case \( V(\psi) = (m^2/2)\psi^2 \) Eq. (2.4) is nothing but the curved space generalization of the model usually employed in direct searches of axionic particles [23]. The constant in front of the anomaly is a model-dependent factor. For example, in the case of axionic particles, for large temperatures \( T \gtrsim m_W \), the Abelian gauge fields present in Eq. (2.4) will be hypercharge fields and \( c = c_\psi \alpha'/2\pi \) where \( \alpha' = g^2/4\pi \) and \( c_\psi \) is a numerical factor of order one which can be computed (in a specific axion scenario) by knowing the Peccei-Quinn charges of all the fermions present in the model [24]. For small temperatures \( T \lessgtr m_W \), we have that the Abelian fields present in the action (2.4) will coincide with ordinary electromagnetic fields and \( c = c_\psi \alpha_{em}/2\pi \) where \( \alpha_{em} \) is the fine structure constant and \( c_\psi \) is again a numerical factor.

The coupled system of equations describing the evolution of the pseudoscalars and of the Abelian gauge fields can be treated as completely homogeneous (i.e. that any background gauge field is absent. In the linearisation procedure we will also assume that the pseudoscalar field can be treated as completely homogeneous (i.e. \( \nabla |\psi| \ll \psi' \)). This seems to be natural if, prior to the radiation dominated epoch, an inflationary phase diluted the gradients of the pseudoscalar.

In this approximation, the result of the linearization can be simply written in terms of the vector potentials in the gauge \( Y^0 = 0 \) and \( \vec{\nabla} \cdot \vec{Y} = 0 \):
In terms of these linear combinations the decomposition of (2.12) becomes
\[ \dot{Y}^\alpha(\vec{x}, \tau) = \sum_\alpha \int \frac{d^3k}{\sqrt{(2\pi)^3}} \left( Y^\alpha_\alpha(k, \tau) \vec{e}_\alpha e^{i\vec{k} \cdot \vec{x}} + Y^\alpha_\alpha(k, \tau) \vec{e}_\alpha e^{-i\vec{k} \cdot \vec{x}} \right), \quad (2.12) \]
where \( \alpha = 1, 2 \) and runs over the two (real) linear polarization vectors \( \vec{e}_\alpha \) whose direction depends on the propagations direction \( \vec{k} \). According to this decomposition, \( \vec{e}_1, \vec{e}_2 \) and \( \vec{e}_3 = k/|k| \) form a set of three mutually orthonormal unit vectors (i.e. \( \vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \)). Inserting Eq. (2.12) into Eq. (2.9) we get a system of (still coupled) equations for the Fourier components
\[ Y_1'' + k^2 Y_1 - i k \frac{\psi'}{M} Y_2 = 0, \quad Y_2'' + k^2 Y_2 + i k \frac{\psi'}{M} Y_1 = 0, \quad (2.13) \]
In order to decouple this system of equations we can pass from linear polarization vectors to (complex) circular polarizations by defining:
\[ Y_\pm(k, \tau) = Y_1(k, \tau) \pm i Y_2(k, \tau), \quad Y_\mp^*(k, \tau) = Y_1^*(k, \tau) \mp i Y_2^*(k, \tau). \quad (2.14) \]
In terms of these linear combinations the decomposition of (2.12) becomes
\[ \dot{Y}^\beta(\vec{x}, \tau) = \frac{1}{\sqrt{2}} \sum_\beta \int \frac{d^3k}{\sqrt{(2\pi)^3}} \left( Y_\beta(k, \tau) \vec{e}_\beta e^{i\vec{k} \cdot \vec{x}} + Y_\beta^*(k, \tau) \vec{e}_\beta^* e^{-i\vec{k} \cdot \vec{x}} \right), \quad (2.15) \]
where now \( \beta = +, - \) labels the two circular polarizations
\[ \vec{e}_+ = \frac{\vec{e}_1 - i \vec{e}_2}{\sqrt{2}}, \quad \vec{e}_- = \frac{\vec{e}_1 + i \vec{e}_2}{\sqrt{2}}, \quad (2.16) \]
satisfying the usual identities
\[ \vec{e}_+ \cdot \vec{e}_- = 1, \quad \vec{e}_+^* = \vec{e}_-, \quad \vec{e}_-^* = \vec{e}_+, \quad \vec{e}_+ \cdot \vec{e}_+ = 0, \quad \vec{e}_- \cdot \vec{e}_- = 0, \quad \vec{k} \times \vec{e}_+ = i |\vec{k}| \vec{e}_-, \quad \vec{k} \times \vec{e}_- = -i |\vec{k}| \vec{e}_+. \quad (2.17) \]
In terms of \( Y_\pm \) Eqs. (2.13) can be written as
\[ Y_\pm'' + \omega^2_\pm Y_\pm = 0, \quad Y^\prime\prime_\pm + \omega^2_\pm Y^\prime_\pm = 0, \quad \omega^2_\pm = k^2 + \frac{c}{M} \psi'. \quad (2.18) \]
Notice that \( Y_\pm \) are the normal modes of the action describing the (classical) hypercharge fluctuations excited by the evolution of the dynamical pseudoscalar. In other words the action (2.4) is not diagonal in terms of \( Y_{1,2}(\vec{x}, \tau) \) but it can be diagonalised in terms of \( Y_\pm \). By inserting the decomposition given in Eq. (2.15) back into the action (2.4), by using Eqs. (2.17) and by integrating over \( d^3x \) we obtain that the action for the hypercharge fluctuations can be expressed as:
\[ S = \int d\tau L(\tau), \quad L(\tau) = \int d^3k L(k, \tau), \quad (2.19) \]

\[^1\]In order to avoid confusions we denoted with \( H \) the Hubble factor is cosmic time and with \( \mathcal{H} \) the Hubble factor in conformal time. These quantities cannot be confused with \( \dot{H}_Y = a^2 \dot{H}_Y \) since, when some ambiguity might arise, we will always keep the appropriate subscripts.
with
\[ L(k, \tau) = Y_0(k, \tau)Y_0^\dagger(k, \tau) + Y_\pm(k, \tau)Y_\pm^\dagger(k, \tau) - \omega_0^2 Y_+(k, \tau)Y_+^\dagger(k, \tau) - \omega_0^2 Y_-(k, \tau)Y_-^\dagger(k, \tau) \]

[notice that \( Y_+(k, \tau) = Y_+(k, \tau) \) and \( Y_-(k, \tau) = Y_-(k, \tau) \).

In order to make more explicit the meaning of Eqs. (2.9) and (2.10) let us start from some qualitative considerations. Suppose that the potential term appearing in the action (2.4) is exactly \((m^2/2)\psi^2\). Moreover, for sake of simplicity we will deal with the case where the conductivity is completely absent. If \( H \gg m \) the \( \psi \) field freely evolves, and, as a consequence, we will have, from Eqs. (2.8)-(2.18)

\[ \psi' \sim \frac{1}{a^2}, \quad Y''_\pm + \left[k^2 \pm \frac{c}{a^2M} \right] Y_\pm = 0, \]

and similarly for the other complex conjugate solutions. At \( H \sim m \), \( \psi \) will start oscillating with typical frequency \( m \) and typical amplitude \( \psi_0 \):

\[ \psi(t) \approx \frac{\psi_0}{a^{3/2}} \sin[\omega(t - t_0)]. \]

Since \( H \leq m \) the effect of the Universe expansion can be neglected during the oscillating phase as long as we limit ourselves to time intervals \( \Delta t \lesssim H^{-1} \). Moreover, since \( \psi' = a\psi \) from Eq. (2.18) we can see that for \( \psi > \omega c/M \), \( Y_+ \) will be amplified (we defined \( \omega = k/a \) as the physical frequency). The maximal amplified frequency will be \( \omega_{\text{max}} \approx \psi_{\text{max}}c/M \sim cm(\psi_0/M \) in the same regime \( Y_- \) is not amplified. Then it is not unreasonable to expect that pseudoscalar quantities \((\text{like } \vec{Y} \cdot \nabla \times \vec{Y}, \vec{H}_Y \cdot \nabla \times \vec{H}_Y \text{ etc.})\) can be produced. Of course, the fact that pseudoscalar configurations of the hypercharge fields are produced does not necessarily mean that they cannot be erased by the finite value of the conductivity. We will come back on this point later.

**III. HYPERMAGNETIC HELICITY AMPLIFICATION**

Starting from a stochastic mixture of topologically trivial fluctuations, the time evolution of a generic dynamical pseudoscalar excites gauge field configurations which are topologically non-trivial. Suppose that the gauge field fluctuations are in their vacuum state, and suppose that after a period of free-rolling the pseudoscalar field starts oscillating at a curvature scale \( H \sim m \). This can happen both in towards the end of an inflationary epoch and/or in a radiation dominated regime, as we will show in the next two Sections.

Particle creation in an external field [25] plays an important role in the analysis of graviton production due to the evolution of the gravitational background [26] and it is also crucial for a rigorous treatment of the normalization of density fluctuations in the context of ordinary inflationary models [27]. In quantum optics [28] similar techniques are used in order to describe the photon production in an external (pump) laser field interacting with a non-linear material. In condensed matter theory these techniques are also employed and their similarities with cosmological problems are certainly worth to emphasize [29].

In the Heisenberg description the process of amplification can be understood in terms of unitary transformations (the Bogoliubov-Valatin transformations) relating the field operators between two asymptotic vacua. In the Schroedinger picture the amplification process is described through the action of a time evolution operator connecting the initial vacuum state to the final (multiparticle) “squeezed” vacuum state [26,28,30].

The classical (canonical) Hamiltonian for the gauge field fluctuations can be obtained from the canonical action of Eq. (2.20)

\[ H(\tau) = \int d^3k L(k, \tau), \quad L(k, \tau) = \pi_+^2 + \pi_-^2 + \omega_0^2 Y_+^2 + \omega_0^2 Y_-^2, \]

where \( \pi_+ \) and \( \pi_- \) are the canonical momenta and where \( \omega_0^2 \) are not necessarily positive definite. From Eq. (3.1) the evolution equations (2.18) can be directly obtained in the Hamiltonian formalism. Our problem is analogous to the quantization of two decoupled (complex) harmonic oscillators and it is closely analogous to the quantization of the two circular polarization of the electromagnetic field in the radiation gauge. The only (but crucial) difference is that the two polarizations obey (in our case) two different evolution equations (i.e. \( \omega_+^2 \neq \omega_-^2 \)). In the limit \( \psi' = 0 \) (i.e. \( \omega_+ = \omega_- \)) the ordinary “electromagnetic” case is completely recovered [28].

Promoting the classical normal modes to quantum mechanical operators
we can impose the commutation relations
\[
[\hat{\beta}(k, \tau), \hat{\pi}\beta(p, \tau)] = i\delta_{\beta\beta}\delta^{(3)}(\vec{k} - \vec{p}), \quad [\hat{\beta}(k, \tau), \hat{Y}_\beta(p, \tau)] = 0, \quad [\hat{\pi}\beta(k, \tau), \hat{\pi}\beta(p, \tau)] = 0
\]  
where \(\beta = +, -.\) The dynamical evolution follows the Heisenberg equations of motion given by
\[
\dot{\hat{Y}}_\beta = i[\hat{H}, \hat{Y}_\beta], \quad \dot{\hat{\pi}}_\beta = i[\hat{H}, \hat{\pi}\beta],
\]  
where \(\hat{H}\) is the Hamiltonian operator. A consequence of these equations is that the operator \(\hat{\beta}(k, \tau)\) obey the same evolution equation of the related classical quantity derived in Eqs. (2.18).

We define the creation and annihilation operators as
\[
\hat{a}_\beta(\vec{k}, \tau) = \frac{1}{\sqrt{2\omega_+}} \left[ \hat{a}_\beta(-\vec{k}, \tau) + \hat{a}^\dagger_\beta(\vec{k}, \tau) \right], \quad \hat{\pi}_\beta(\vec{k}, \tau) = \frac{1}{\sqrt{2\omega_-}} \left[ \hat{\pi}_\beta(-\vec{k}, \tau) + \hat{a}^\dagger_\beta(\vec{k}, \tau) \right],
\]
and the Hamiltonian operator can then be written as
\[
\hat{H}(\tau) = \sum_{\beta=+, -} \int d^3k \left[ \omega_\beta(\tau) \left( \hat{a}^\dagger_\beta(\vec{k}, \tau)\hat{a}_\beta(\vec{k}, \tau) + \hat{a}^\dagger_\beta(-\vec{k}, \tau)\hat{a}_\beta(-\vec{k}, \tau) \right) + 1 \right].
\]  
The field operators before the amplification took place can then be written as:
\[
\hat{Y}_{+, in} = \left[ \hat{a}_+, in f_{in} + \hat{a}^\dagger_+, in f^*_in \right], \quad \hat{Y}_{-, in} = \left[ \hat{a}_-, in F_{in} + \hat{a}^\dagger_-, in F^*_in \right].
\]  
The same decomposition can be done for the operators after the amplification processes has taken place:
\[
\hat{Y}_{+, out} = \left[ \hat{a}_+, out g_{out} + \hat{a}^\dagger_+, out g^*_out \right], \quad \hat{Y}_{-, out} = \left[ \hat{a}_-, out G_{out} + \hat{a}^\dagger_-, out G^*_out \right].
\]  
We can formally write the out-going mode functions in terms of the in-going mode functions :
\[
g_{out}(\tau) = c_+ f_{in}(\tau) + c_- f^*_in(\tau), \quad G_{out}(\eta) = \tilde{c}_+ F_{in}(\tau) + \tilde{c}_- F^*_in(\tau).
\]  
Since \(\hat{Y}_{+, in}\) and \(\hat{Y}_{+, out}\) represent the same solution they must be equal:
\[
\hat{Y}_{+, in} = \hat{Y}_{+, out}, \quad \hat{Y}_{-, in} = \hat{Y}_{-, out}.
\]  
Inserting the explicit form of the solutions we have that
\[
\hat{a}_+, in f_{in} + \hat{a}^\dagger_+, in f^*_in = \hat{a}_+, out g_{out} + \hat{a}^\dagger_+, out g^*_out
\]
\[
\hat{a}_-, in F_{in} + \hat{a}^\dagger_-, in F^*_in = \hat{a}_-, in G_{out} + \hat{a}^\dagger_-, in G^*_out.
\]  
Using Eqs. (3.10) in Eq. (3.12) we obtain that the creation and the destruction operators after the amplification \((\hat{a}_{\pm, in})\) are related to the ones before the amplification \((\hat{a}_{\pm, in})\) by the following Bogoliubov-Valatin transformations:
\[ \hat{a}_{+,in} = c_+ \hat{a}_{+,out} + c_-^* \hat{a}_{-,out}^\dagger, \]
\[ \hat{a}_{-,in} = \bar{c}_+ \hat{a}_{-,out} + \bar{c}_-^* \hat{a}_{-,out}^\dagger, \]
\[ \text{(3.13)} \]
\[ \text{(3.14)} \]

Once the evolution of the pseudoscalar field \( \psi \) is completely specified, the coefficients \( c_\pm \) and \( \bar{c}_\pm \) can be explicitly computed. It is useful, at this point to re-write the full expression of the hypercharge after the amplification took place:

\[
\bar{Y}_{out}(x, \tau) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^3} \left\{ \left[ \left( \hat{a}_{+,out} g_{out} + \hat{a}_{+,out}^\dagger g_{out}^\ast \right) \bar{c}_+ + \left( \hat{a}_{-,out} G_{out} + \hat{a}_{-,out}^\dagger G_{out}^\ast \right) \bar{c}_- \right] e^{i\vec{k} \cdot \vec{x}} + \left[ \left( \hat{a}_{+,out}^\dagger g_{out}^\ast + \hat{a}_{+,out} g_{out} \right) \bar{c}_+ + \left( \hat{a}_{-,out}^\dagger G_{out}^\ast + \hat{a}_{-,out} G_{out} \right) \bar{c}_- \right] e^{-i\vec{k} \cdot \vec{x}} \right\}. \]  
\[ \text{(3.15)} \]

We have now all the ingredients in order to estimate the expectation values of the pseudoscalar quantities we are interested in. As it was previously stressed \([10,11]\) a good gauge-invariant measure of global parity violation is the spectrum of the hypermagnetic helicity can be written as:

\[
\langle 0 | \mathcal{H}_Y \cdot \vec{n} \times \mathcal{H}_Y | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} p^3 \left\{ (|c_+|^2 + |c_-|^2) f_{in}(\tau) f_{in}^\ast(\tau) - (|\bar{c}_+|^2 + |\bar{c}_-|^2) f_{in}(\tau) f_{in}^\ast(\tau) \right. \\
+ c_+^* c_- f_{in}^2(\tau) + c_+ c_-^* f_{in}^2(\tau) - \bar{c}_+^* \bar{c}_- F_{in}^2(\tau) - \bar{c}_+ \bar{c}_-^* F_{in}^2(\tau) \left\}, \right. \]  
\[ \text{(3.16)} \]

where \( |0\rangle \) denotes the vacuum state of the hypercharge field. Suppose that our initial field configuration is given by a stochastic background or by a thermal mixture \([30]\). Then \( f_{in}(\tau) \) and \( F_{in}(\tau) \) will be:

\[
f_{in}(\tau) = \frac{A(k)}{\sqrt{k}} e^{-i\vec{k} \cdot \vec{r}}, \quad F_{in}(\tau) = \frac{A(k)}{\sqrt{k}} e^{-i(kr + \varphi)}. \]  
\[ \text{(3.17)} \]

If there is no initial phase shift (as it should be) \( \varphi = 0 \) and then the two mode functions are equal. Thus, going to Eq. (3.16) we have that the whole expression is zero provided \( c_\pm \neq \bar{c}_\pm \). Indeed we showed that because \( Y_+ \) and \( Y_- \) obey different classical equation the Bogoliubov coefficients with and without tilde will be in general different.

The specific form of the Bogoliubov coefficients will of course depend upon the specific cosmological model, upon the specific particle physics candidate for \( \psi \). However, even in the general case we can carry on the calculation one step further. The amplification coefficients are represented, in our formalism, by \( c_\pm \) and \( \bar{c}_\pm \). Thus, the logarithmic energy spectrum of the hypermagnetic helicity can be written as:

\[
\frac{d\mathcal{H}_Y}{d\log \omega} = \frac{1}{\pi^2} \omega^5 \mathcal{G}(\omega, \tau) |A(\omega)|^2, \]  
\[ \text{(3.18)} \]

with:

\[
\mathcal{G}(\omega, \tau) = \left| [c_-(\omega)]^2 - [\bar{c}_-(\omega)]^2 \right| - c_+^* c_- f_{in}^2(\tau) + c_+ c_-^* f_{in}^2(\tau) - \bar{c}_+^* \bar{c}_- F_{in}^2(\tau) - \bar{c}_+ \bar{c}_-^* F_{in}^2(\tau) \right|, \]  
\[ \text{(3.19)} \]

where we defined, as usual, the physical frequency \( \omega = k/a \). Notice that in deriving Eq. (3.18) we used the relation \(|c_+|^2 - |c_-|^2 = 1\) (and analogously for the tilded quantities) which is a trivial consequence of the unitarity.

By defining the logarithmic energy spectrum of the initial distribution of hypermagnetic fields \([11]\)

\[
\rho(\omega) = \frac{d\rho_Y}{d\log \omega} = \frac{\omega^4}{\pi^2} |A(\omega)|^2, \]  
\[ \text{(3.20)} \]

Eq. (3.18) becomes:

\[
\frac{d\mathcal{H}_Y}{d\log \omega} = \omega \mathcal{G}(\omega, \tau) \rho(\omega). \]  
\[ \text{(3.21)} \]
Even if the amplification related to the dynamical pseudoscalars is of order of 1, this last expression tells us that the obtained helicity can be large provided the initial state is not the vacuum. We will see that the result of Eq. (3.21) can be quite relevant for our applications.

Therefore, in this section we derived a number of results. As previously noted [20] we obtained that dynamical pseudoscalars induce an interaction between the polarizations of Abelian gauge fields. This coupling has the effect of modifying the evolution of the two circular polarizations in such a way that the magnetic helicity (initially zero) can be dynamically generated.

Up to now our results do not take into account the possible effect, in the final state, of some effective Ohmic current which can account (macroscopically) of the dissipative effects associated with a plasma at finite conductivity. This will be an important point in the next two sections. On one hand the finite value of the conductivity provides a useful (and physically motivated) ultraviolet cut-off for our helicity and energy spectra. On the other hand we will stress that, also at finite conductivity, the Chern-Simons density can be directly related to the helicity.

### IV. HYPERMAGNETIC KNOTS PRODUCTION DURING AN INFLATIONARY PHASE

The ways our mechanism can be implemented depend upon the early history of the Universe. We will firstly consider a simplified model where an inflationary phase occurs $H$. The inflationary phase terminates at the time $\tau_1$ and the ordinary (radiation-dominated) phase settles in.

Notice that $\psi$ can oscillate both in the inflationary phase and in the radiation dominated phase. These two logical possibilities involve two different physical pictures. In the case of inflationary production of hypermagnetic helicity the oscillations will only be damped by the Universe expansion. If, on the contrary, $\psi$ oscillates during the radiation dominated phase we can expect that the evolution of the hypercharge fields will be damped by the finite value of the conductivity. In the present and in the following Sections we will analyse these two regimes.

Suppose that our (massive) pseudoscalar field evolves during an inflationary phase following its equation of motion which we write (in cosmic) time as

$$\ddot{\psi} + 3H\dot{\psi} + m^2\psi = 0,$$  \hspace{1cm} (4.1)

where $H$ is (approximately) constant and it corresponds to the maximal (curvature) scale reached during inflation. In this model we have essentially three parameters. One is $H_i/M_P$ fixing the maximal curvature scale during inflation. From the contribution of gravitational waves to the Cosmic Microwave Background (CMB) anisotropy we know that

$$\frac{H_i}{M_P} \lesssim 10^{-6},$$  \hspace{1cm} (4.2)

where $H_i$ is the curvature scale at the end of the inflationary phase. Moreover, we also know that the EWPT takes place when $T > 100$ GeV. Since, by that time, the Universe was dominated by radiation we also have to assume that

$$\frac{H_i}{M_P} > 10^{-33}. $$  \hspace{1cm} (4.3)

The second parameter of our model is $m/H_i$ which essentially fixes the amount of amplification of the hypercharge. In Eq. (2.18) the instability develops for $k \sim k_{\text{max}} \sim c(\psi_0/M)ma$. The amplification occurring for $k \approx k_{\text{max}}$ in the mode function will be of the order of $Y_\perp(k, \tau) \sim \exp[c(\psi_0/M)ma\Delta \tau]$ where $a\Delta \tau \sim \Delta t \sim H_i^{-1}$ is the typical amplification time. The typical growth of $Y_\perp$ will then be proportional to $\exp[c(\psi_0/M)(m/H_i)]$. As we can see the amplification depends also upon $(\psi_0/M)$ which is the third parameter of our analysis. The coupling constant $c$ appearing in the action (2.4), is in principle, determined by the specific particle physics model. We want immediately to notice that the amplification occurs for $m \gg H_i$. At the same time the coherence length of the amplification processes is determined by $L_{\text{max}} \sim \omega_{\text{max}}^{-1} \sim c^{-1}(M/\psi_0)m^{-1}$ and, therefore, the larger the amplification is the shorter is the coherence length of the amplified field in horizon units. If $m > H_i$ effect of the Universe expansion can be neglected at least in the first approximation.

This qualitative analysis can be refined by a numerical study which includes the Universe expansion. For this purpose we can write the hypercharge evolution in cosmic time. The system which should be solved numerically is, in general,

$$\dot{y}_\pm + \left[\omega^2 - \frac{c}{M}H^2 - \frac{\dot{H}}{2}\right]y_\pm = 0,$$
\[ \dot{\Psi} + \left[ m^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} \right] \Psi = 0, \]
\[ y_{\pm} = \sqrt{\alpha} Y_{\pm}, \quad \Psi = a^{3/2} \psi. \]  

(4.4)

In our specific case this second order differential system can be simplified since \( \dot{H} = 0 \). We studied numerically this set of equations by reducing it to a first order linear differential system. Some examples of our results are reported in Fig. 1. The crucial point in our analysis is the determination of the maximally amplified mode \( \omega_{\text{max}} \). Once this is obtained we can solve the system given in Eq. (4.4) to include the Universe expansion. In the approximation where \( m \gg H \) (i.e., expansion negligible compared to the oscillations) we have that the Eqs. (4.4) can be further simplified since \( \psi \sim a^{-3/2} \sin [m(t - t_0)] \). We get a Mathieu-type equation [35,36]

\[
\frac{d^2 y_{\pm}}{dz^2} + \left[ \delta \mp \epsilon \cos 2z \right] y_{\pm} = 0, 
\]

(4.5)

where \( \delta = (2\omega/m)^2, \epsilon = c(\psi_0/M)(2\omega/m) \) and \( z = (mt)/2 \). Note that when \( \epsilon = 0 \) and \( \delta = (2k + 1)^2 \) (where \( k = 0, 1, 2, \ldots \)) there is solution of Eq. (4.5) with period \( 2\pi \) but when \( \epsilon = 0 \) and \( \delta = (2k)^2 \) there is a solution of Eq. (4.5) with period \( \pi \). Therefore, in the plane \( (\delta,\epsilon) \) the instability boundaries will cross the \( \epsilon = 0 \) at the points defined by \( \delta = (2k)^2 \) or \( \delta = (2k+1)^2 \). By studying the instability of Eq. (4.5) in the case of \( \epsilon \neq 0 \) (but still \( \epsilon < 1 \)) we get that the first instability region (corresponding to \( \delta \sim 1 \)) occurs for a region in the plane \( (\delta,\epsilon) \) bounded by \( \delta - 1 \sim \epsilon/2 \). Inserting the values of \( \delta \) and \( \epsilon \) we get that, for small \( \epsilon \) this region is centered around \( \omega_{\text{max}} \sim (\epsilon/2)(\psi_0/M)m \). Of course there will be also other bands defined by their intersections on the \( \epsilon = 0 \) axis (\( \delta = 4, 9, 16, 20, \ldots \)). Those bands correspond to higher frequencies but they are narrower than the first one defined by the \( \delta = 1 \) intersection [35,36].

![Fig. 1](image)

FIG. 1. We report two numerical solutions of the system given in Eqs. (4.4). We assume \( \omega = \omega_{\text{max}} \). At the left we report the case of \( c = 0.01 \) and \( H/m = 0.001 \). At the right we report the case with \( c = 0.001 \) and \( H/m \sim 0.00013 \). For the two cases we would get, from our considerations following Eq. (4.5), amplifications of the order of \( e^5 \) (left) and \( e^8 \) (right). The damping due to expansion settles in quickly (roughly after \( H^{-1} \) in mass units). The agreement our our estimates with the numerical evaluation gets worse for frequencies far from the maximal amplified frequency.

According to this analysis we can say that \( \omega_{\text{max}} \) leads to an amplification of the order of \( \exp \left[ (\epsilon/2)(\psi_0/M)m\Delta t \right] \), where \( \Delta t \sim H^{-1} \). As noted in previous studies of a similar equation, the Universe expansion will soon shut-off the instability occurring in the \( \epsilon < 1 \) case [34].

In Fig. 1 we report the solution of the system given by Eq. (4.4) where we took \( \omega = \omega_{\text{max}} \). The Universe expansion has been included. If we take the case with \( c = 0.01 \) and \( m/H = 1000 \) we get that the amplification should be \( e^5 = 148 \). Numerically we get that during one Hubble time the maximal amplification is roughly 150 (left picture in Fig. 1). However the amplification is quickly shadowed by the Universe expansion which becomes effective roughly after a time of the order of \( H^{-1} \) in mass units.

Thus pseudoscalar oscillations are quite effective in changing the topology of a stochastic background by inducing correlations between the two independent polarizations of the gauge field. However, they are not effective in inflating the amplitude of the background. Therefore, as we will see even better in the case of oscillations occurring in a relativistic plasma, one of the crucial assumption of our considerations will be the existence of a primordial stochastic background of hypermagnetic fields whose topology can be eventually twisted by the pseudoscalar oscillations.

In Fig. 1 we completely ignored the role played by the finite value of the conductivity. During the inflationary dominated regime the effective conductivity is equal to zero since no charged particles are present in the Universe. When the Universe reheats charged particles are generated and, therefore, an Ohmic current is developed. This effect
can be modeled by thinking about the situation where the conductivity is zero prior to the onset of the radiation dominated phase but it gets larger and larger as soon as charged particles are created.

Needless to say that the smooth transition of the conductivity from zero to a finite (large) value is quite important for our considerations. Suppose, as a warm-up, that we are in flat space and suppose that the conductivity jumps from zero to a finite value at some time $t_*$. Let us also assume, for simplicity, that there are no pseudoscalar interactions in the game. Then the evolution equation of the hypercharge fields will be the usual D’Alembert equation for $t < t_*$ but it will be modified as:

$$\dddot{\mathbf{Y}} + \sigma_c \dot{\mathbf{Y}} - \nabla^2 \ddot{\mathbf{Y}} = 0,$$

(4.6)

for $t > t_*$. If we match the two solution at $t = t_*$ we simply get that the hypermagnetic fields are left unchanged by the transition whereas the hyperelectric fields are damped [43]:

$$\mathbf{H}(t_*) = \mathbf{H}(t_*)', \quad \mathbf{E}(t_*) = e^{-\sigma_c t_1} \mathbf{E}(t_*)',$$

(4.7)

where the subscript $>$ ($<$) denotes the value of the fields after (before) the transition occurring at $t_*$. Since pseudoscalar interactions are absent in this example we also have that $(Y_+ - Y_-)$ will always be zero.

From a physical point of view we can expect that the effect of the finite value of the conductivity will be to fix a definite sign for $(Y_+ - Y_-)$. By ignoring the effect of the conductivity we can argue (see Fig. 1) that $(Y_+ - Y_-)$ is certainly amplified but, at the same time, the sign of this quantity is not well defined. In order to clarify this point let us study the full system of equations at finite conductivity

$$Y''_\pm + \sigma Y'_\pm + \left[ k^2 \pm k c M \psi' \right] Y_\pm = 0,$$

$$\psi'' + 2H \psi' + m^2 a^2 \psi = 0,$$

(4.8)

where we remind that the conductivity, in curved space, evolves as $\sigma(\tau) = \sigma_c a(\tau)$. In order to model a smooth transition from the inflationary epoch to the radiation dominated epoch let us assume that the geometry evolves continuously from a de Sitter (or quasi de Sitter) stage to a radiation dominated stage of expansion. This can be achieved with a specific choice of the scale factor, namely

$$a(\tau) = \left( \frac{\tau}{\tau_1} \right) + \sqrt{\left( \frac{\tau}{\tau_1} \right)^2 + 1}.$$

(4.9)

Notice that for $\tau \to +\infty$ we have that $a(\tau) \sim \tau$. For $\tau \to -\infty$ we have that $a(\tau) \sim \tau^{-1}$. For $\tau \to 0$ we have that $a(\tau) \sim$ constant. So we can say that this geometry evolves smoothly from a de Sitter phase to a radiation dominated phase. The Hubble parameter in conformal time can be easily computed and it is also a smooth function of $\tau$

$$\mathcal{H}(\tau) \equiv \frac{a'}{a} = \frac{1}{\sqrt{\tau^2 + \tau_1^2}}.$$

(4.10)

The duration of the transition period between the de Sitter phase and the radiation dominated phase is controled by $\tau_1$. By increasing $\tau_1$ the transition period gets longer. By decreasing $\tau_1$ the transition period gets shorter. This aspect can be appreciated by looking at the time evolution of $\mathcal{H}(\tau)$ (see Fig. 2).

![FIG. 2. We illustrate the Hubble parameter $\mathcal{H} = a'/a$ computed from Eq. (4.9). By increasing $\tau_1$ the transition regime gets longer. For $\tau_1 = 1$ the Hubble parameter is reported with the full (black) line. For larger values of $\tau_1$ (i.e. $\tau_1 = 5$ [thick dashed line] and $\tau_1 = 10$ [full thin line]) the height of the Hubble parameter decreases and the transition regime gets broader.](image.png)
The behavior of the conductivity follows from the evolution of the geometry. Therefore, in this model, we will have that the conductivity is vanishingly small during the de Sitter phase and it grows linearly in conformal time during the radiation dominated phase. By rescaling with respect to the mass in order to have fully dimensionless quantities we can easily re-write Eqs. (4.8) in the form of a first order differential system

\[
\begin{align*}
\frac{dq_+}{dx} &= p_+(x), & \frac{dq_-}{dx} &= p_-(x), & \frac{du}{dx} &= v(x), \\
\frac{dp_+}{dx} &= -\left(\frac{\sigma_c}{m}\right)a(x)p_+ - \left[\varepsilon^2 - \frac{c}{M}u(x)\right]q_+(x), \\
\frac{dp_-}{dx} &= -\left(\frac{\sigma_c}{m}\right)a(x)p_- - \left[\varepsilon^2 + \frac{c}{M}u(x)\right]q_-(x), \\
\frac{dv}{dx} &= -\frac{2v(x)}{\sqrt{x^2 + x_1^2}} - a^2(x)u(x),
\end{align*}
\]

where we denoted \(q_{\pm} \equiv Y_{\pm}, p_{\pm} = Y'_{\pm}\) and \(u = \psi\) and \(v = \psi'\). Notice, moreover that \(x = m\tau, x_1 = m\tau_1\) and \(\varepsilon = k/m\). We can observe that there are three important parameters for our discussion. The first one is, \(\mathcal{H}_1 = \mathcal{H}(\tau_1) \sim \tau_1^{-1}\) the Hubble parameter at the end of the inflationary phase. The Hubble parameter in cosmic time is simply related to \(\mathcal{H}_1\) since \(\mathcal{H}_1 = a_1H_1 \sim (1 + \sqrt{2})H_1\). Moreover, as we noticed before, in order to be compatible with the large scale observations in the microwave sky we have to require \(H_1/M_P < 10^{-6}\). Notice that \(H_1\) always appears in the combination \(x_1 = m\tau_1 \sim m/H_1\). Consequently, if we want large pseudoscalar oscillations prior to \(\tau_1\) we have to require \(x_1 \ll 1\). The last parameter appearing in Eqs. (4.11) is \(\sigma_c/m\). Numerically \(\sigma_c = \sigma_0T_0\) where \(\sigma_0 = 70-100\) and \(T_0\) is the temperature of the Universe at the beginning of the radiation dominated epoch. Notice that \(\sigma_c/m \equiv (\sigma_c/H_1)(H_1/m)\). Therefore, taking into account that \(H_1 < 10^{-6}M_P\) and that we want \(T_0\) as high as \(10^{10}\) GeV we have to require that \(\sigma_c/m < 1\). Suppose, for instance, that \(H_1 \sim 10^{-9}M_P, T_0 \sim 10^{10}\) GeV, \(m/H_1 = 100\). Then, we get that \(\sigma_c/m = \sigma_0/100 \sim 0.7\). We will keep \(x_1\) and \(\sigma_c/m\) as free parameters in the integration. Notice finally that we will focus our attention on the maximally amplified modes and, therefore, we will fix \(\varepsilon \sim c/2\).

![Fig. 3](image1.png)

**FIG. 3.** We illustrate the numerical integration of the system given in Eqs. (4.11). At the left we report the cases where \(\sigma_c/m = 0.1\) and \(x_1 = 100\) (full thick line) and \(x_1 = 10^4\) (dot-dashed line). At the right we fix \(x_1 = 100\) and we choose, respectively, \(\sigma_c/m = 0.1\) (full thick line), \(\sigma_c/m = 0.01\) (full thin line) and \(\sigma_c/m = 10^{-4}\) (dot-dashed line).

In Fig. 3 and 4 we report the results of the numerical integration for different sets of parameters. In Fig. 3 (left plot) we fix \(\sigma_c/m \sim 0.1\) and we choose \(x_1 = 100\) (full line), \(x_1 = 10^4\). We see that \((Y_+ - Y_-)\) acquires a definite sign as we expected. Moreover the transition regime gets broader as soon as we increase \(x_1\). This can be understood by recalling that by increasing \(\tau_1\) the transition regime gets more significant. What happens if we change \(\sigma_c/m\)? In Fig. 3 (left plot) we fix \(x_1 = 100\) and we decrease the value of \(\sigma_c/m\). The the full (thick) line corresponds to the case \(\sigma_c/m = 0.1\), whereas the dot-dashed line corresponds to the case \(\sigma_c/m = 10^{-4}\). The full (thin) line corresponds to the case \(\sigma_c/m = 0.01\). We clearly see that in the limit \(\sigma_c/m \to 0\) we recover partially the case of Fig. 1 since \((Y_+ - Y_-)\) does not have definite sign when the radiation dominated phase settles in. The fact that \(\sigma_c/m\) cannot be too small can be understood physically by recalling that the small \(\sigma_c/m\) limit is equivalent to a small reheating temperature. What happens when \(\sigma_c/m > 1\)? We can expect that if the conductivity is too large at the beginning of the radiation dominated epoch the damping term will be dominant and the amplification will be reduced. This aspect is illustrated in Fig. 4.
The limit $\sigma_c/m \gg 1$ simply means that we are in the damped regime which will be specifically addressed in Sections V and VII.

In the sudden approximation, the Bogoliubov coefficients describing the amplification of the mode function when the background changes from the inflationary phase to the radiation dominate phase can be analytically computed in the framework of our model. When the modes of the fields evolve in the radiation dominated phase we will also have to take into account the effect of the conductivity and this will be the second step of our calculation. We will solve the evolution of the mode function for modes close to $\omega_{\text{max}}$. For $\tau < \tau_0$ the pseudoscalar does not oscillate but when its mass is of the order of $H_i$, the inflationary oscillations will begin until the radiation dominated phase starts at $\tau \sim \tau_1$. The mode function corresponding to $Y_+$ for $k \geq k_{\text{max}}$ evolves in the three regions as

$$
\begin{align*}
g_I(\tau) &= \frac{1}{\sqrt{k}} e^{-ik\tau}, \quad \tau < \tau_0, \\
g_{II}(\tau) &= \frac{1}{\sqrt{k}} \left[ B_+ e^{\beta\tau} + B_- e^{-\beta\tau} \right], \quad \tau_0 < \tau < \tau_1, \\
g_{III}(\tau) &= \frac{1}{\sqrt{k}} \left[ e^{ik\tau} + e^{-ik\tau} \right], \quad \tau > \tau_1,
\end{align*}
$$

(4.12)

where $\beta = c/2(\psi_0/M)ma$. We want to stress that the solution for $\tau_0 < \tau < \tau_1$ holds for modes $k \sim k_{\text{max}}$ and for times $a\Delta \tau \lesssim H^{-1}$ (where $\Delta \tau \sim \tau_1 - \tau_0$) but it should not be trusted outside of this range as discussed in Fig. 1. Notice that $c_\pm$ are exactly the Bogoliubov coefficients we are looking for since they relate the incoming mode function with the outgoing mode function. They can be determined by matching the three solutions (and their first derivatives) in the three regions defined by $\tau_0$ and $\tau_1$. This can be easily done and the result is

$$
\begin{align*}
c_+ &= \frac{1}{2} e^{-i k \Delta \tau} \left[ 2 \cosh [\beta \Delta \tau] - i \left( k \frac{\beta}{k} \right) \sinh [\beta \Delta \tau] \right], \\
c_- &= -\frac{i}{2} e^{-i k \Delta \tau} \left( \left( \frac{k}{\beta} + \frac{\beta}{k} \right) \sin [\beta \Delta \tau] \right),
\end{align*}
$$

(4.13)

where $\Delta \tau = \tau_1 - \tau_0$ defines the duration of the oscillatory phase. During the time the $Y_+$ is amplified we also have that the $Y_-$ will oscillate and the corresponding Bogoliubov coefficients will not describe amplification but mixing between the positive and negative frequencies. They are

$$
\begin{align*}
\tilde{c}_+ &= \frac{1}{2} e^{i k \Delta \tau} \left[ 2 \cos [\beta \Delta \tau] + i \left( k \frac{\beta}{k} \right) \sin [\beta \Delta \tau] \right], \\
\tilde{c}_- &= -\frac{i}{2} e^{i k \Delta \tau} \left( \left( \frac{k}{\beta} - \frac{\beta}{k} \right) \sin [\beta \Delta \tau] \right).
\end{align*}
$$

(4.14)

Our expressions of the Bogoliubov coefficients satisfy $|c_+|^2 - |c_-|^2 = 1$, and $|\tilde{c}_+|^2 - |\tilde{c}_-|^2 = 1$ as required by the unitarity of the Bogoliubov transformation which preserves the commutation relations between the creation and annihilation operators in the asymptotic regions.

According to Eq. (3.18), the mean helicity spectrum in the case $\mathcal{A}(k) = 1$ will be:

$$
\frac{dH_{\gamma \gamma}}{d\log \omega} \sim \frac{\omega_{\text{max}}^5}{\pi^2} \epsilon^{(m/H_i)}(\psi_0/M).
$$

(4.15)
Thus according to Eq. (3.21) we will have:

\[
\frac{dH_Y}{d\log \omega} \simeq \frac{e m}{2} \left( \frac{\psi_0}{M} \right) e^{c(m/H_i)(\psi_0/M)} \rho(\omega_{\text{max}}), \quad \rho(\omega_{\text{max}}) = \frac{1}{\pi^2} (\omega_{\text{max}})^4 |A(\omega_{\text{max}})|^2.
\]

In Eq. (4.15) we assumed that the hypermagnetic field modes are re-entering in the radiation dominated epoch but we did not take into account properly the damping effect of the conductivity. If the field modes are amplified (as in our case) during a phase where the conductivity is strictly zero (since no charged particles are present) the only effect of the conductivity (arising when the field modes enter in the radiation dominated phase) is to cut-off (exponentially) all the momenta larger than the typical momentum associated with the conductivity. However, if the pseudoscalar oscillations occur in the radiation dominated phase the effect of the conductivity can be even more important, as we will show in the next section.

V. HYPERMAGNETIC KNOTS PRODUCTION DURING A RADIATION PHASE

If pseudoscalar particles are present prior to the onset of the electroweak epoch they can offer a mechanism for the production of Chern-Simons condensates whose decay can seed the BAU. An example of such a particle is the axial Higgs whose mass enters, after supersymmetry breaking, in the expression of the neutral Higgs boson mass [31,32]. Inspired by the axial Higgs we can require the pseudoscalar mass to be larger than 300 GeV. If \( m > 300 \text{ GeV} \) the MSSM Higgs sector is not too different from the one of the MSM.

We want to extend the discussion of hypermagnetic helicity generation to the case of a radiation dominated epoch. Suppose now that at some higher curvature scale a stochastic distribution of hypermagnetic field has been created. Then, suppose that at some lower energy scale the pseudoscalar oscillations will start developing. In this case the evolution of the hypercharge fields is

\[
Y''_\pm + \sigma Y'_\pm + \left[ k^2 \mp kc \frac{\psi'}{M} \right] Y_\pm = 0.
\]

(5.1)

By eliminating the first derivative through a field rescaling we get

\[
y''_\pm + \left[ k^2 \mp \frac{c^2}{4 M} \sigma^2 \mp \frac{\sigma'}{2} \right] y_\pm = 0, \quad Y_\pm = y_\pm \exp \left[ -\frac{1}{2} \int^\tau \sigma(\tau') d\tau' \right].
\]

(5.2)

If an initial (stochastic) hypermagnetic field is present, then the pseudoscalar interactions will be able to twist the topologically trivial distribution. If no initial stochastic distribution is present, the obtained hypermagnetic helicity will be quite small. In other words we find that at finite conductivity the pseudoscalar interactions are only efficient in twisting a topologically trivial hypermagnetic distribution. Indeed, as we previously discussed, we can always define a conductivity scale at any time in the radiation dominated epoch. All the modes of the fields with typical frequency larger than the typical conductivity frequency are essentially washed away whereas the other have some chance of being amplified. Eq. (5.2) becomes, in the large conductivity limit,

\[
\sigma Y'_\pm + \left[ k^2 \mp \frac{c \psi'}{M} \right] Y_\pm = 0,
\]

(5.3)

and the corresponding solution can be written as

\[
Y_\pm \sim e^{-\int \frac{k^2}{\sigma^2} d\tau'} \left[ e^{\mp \int \frac{c}{\sigma} \psi(\tau') d\tau'} \right] A(\omega)
\]

(5.4)

where the initial amplitude \( A(\omega) \) coincides with the typical amplitude of the stochastic hypermagnetic distribution. The maximal value of the hypermagnetic helicity in critical units can be obtained from Eq. (5.4) with the result that,

\[
\frac{\langle \vec{H}_Y \cdot \vec{\nabla} \times \vec{H}_Y \rangle}{\frac{\sigma \rho_c}{\rho_c}} \sim \frac{e}{2\sigma_0} \frac{\Delta \psi}{M_0} T_e \frac{\Delta \psi}{M_0} (\omega_{\text{max}}),
\]

(5.5)

\[^{2}\text{Assuming no invisible decays, the direct limit is } m > 65 \text{ GeV (at 95\% confidence level and in the case tan } \beta > 1) \text{ [33].}\]
where $\sigma_0 = \sigma_c/T_c$ and $\sigma = \sigma_c a$ and $\Delta \psi$ is the increment of $\psi$ from its initial value. Typically (without fine-tuning the initial amplitude of $\psi$ at the beginning of the oscillations) we will have that $\Delta \psi \sim M$. Notice that in Eq. (5.5) we also defined

$$r(\omega_{\text{max}}) = \frac{\langle H^2 \rangle}{\rho_c} = \frac{\omega_{\text{max}}^4}{\pi^2 \rho_c} |A(\omega_{\text{max}})|^2, \quad \omega_{\text{max}} = \frac{c}{2 a M M_0},$$

as the critical fraction of energy density present in the form of a stochastic background of hypermagnetic fields for $\omega \sim \omega_{\text{max}}$. Eq. (5.5) holds for frequencies smaller than typical conductivity frequency

$$\omega_{\sigma}(\tau) \sim \sqrt{\sigma_0} \sqrt{N_{\text{eff}}} \sqrt{\frac{T}{M_P}} T.$$ 

It is quite relevant to notice that in Eq. (5.5) the exponential damping holds for all the modes $\omega > \omega_{\sigma}$. The only way of getting a huge amplification (at finite conductivity) would be to require $\omega_{\text{max}} \gg \omega_{\sigma}$. However, in this limit, the conductivity erases all the field modes. Therefore we are forced to discuss the case $\omega_{\text{max}} < \omega_{\sigma}$. This limitation on the amplified physical scales applies for all the temperatures (in the symmetric phase of the electroweak theory) where helicity generation takes place.

If the conductivity is finite it does not make sense to consider the case of amplification of the hypermagnetic helicity from the vacuum fluctuations. The reason is that, in this case, $r(\omega_{\text{max}})$ is extremely minute. This can be easily seen in general. Suppose, in fact that $\omega_{\text{max}}$ can be made as large as $0.1 \omega_{\sigma}$. Then we get that the vacuum fluctuations will be in this case

$$r(\omega_{\text{max}}) \sim \frac{10^{-2} \sigma_0}{\pi^2} \left( \frac{T}{M_P} \right)^2$$

which is very small for in the symmetric phase of the electroweak theory and in the TeV range.

**VI. BAU FROM HYPERMAGNETIC HELICITY**

We are interested in the possible effects of the hypermagnetic fields on the electroweak scale and then we will mainly consider the plasma effects associated with the dynamics of Abelian gauge fields in the symmetric phase of the electroweak theory. This is consistent with the assumptions made in the previous section where we focused our attention on hypermagnetic fields amplified and/or re-entered during a radiation dominated phase.

The description of the plasma effects arising in the symmetric phase of the electroweak theory has to rely on some generalization of the MHD equations. In [11] it was argued that such a generalization can be provided by the equations of anomalous magnetohydrodynamics (AMHD) where the effect of finite conductivity and finite chemical potential can be simultaneously described. In the present analysis we have to generalize AMHD to the case of pseudoscalar interactions and non vanishing bulk velocity of the plasma. Our variables are the hypermagnetic and hyperelectric fields, the right electron chemical potential and the pseudoscalar field $\psi$. Of course these equations have to be supplemented by the evolution equation for the bulk velocity of the plasma. The complete set of equations reads

\[
\begin{align*}
\left[(p + \rho)\dot{v}\right] + \nabla \cdot \left( (p + \rho) v \right) + \nabla \cdot \left( (p + \rho) \dot{v} \right) & = - \nabla p + \vec{J}_Y \times \vec{H}_Y + \eta (\nabla^2 \dot{v} + \frac{1}{3} \nabla (\nabla \cdot \dot{v})), \quad \eta = (p + \rho) \nu \\
\vec{H}_Y & = - \nabla \times \vec{E}_Y, \quad \nabla \cdot \vec{H}_Y = 0, \\
\vec{E}_Y + \vec{J}_Y & = \frac{q^2}{\pi^2} \mu a \vec{H}_Y + \frac{c}{M} [\psi' \vec{H}_Y + \nabla \psi \times \vec{E}_Y] + \nabla \times \vec{H}_Y, \quad (6.1) \\
\nabla \cdot \vec{E}_Y & = \frac{c}{M} \nabla \psi \cdot \vec{H}_Y, \quad \vec{J}_Y = \sigma (\vec{E}_Y + \nabla \times \vec{H}_Y), \quad (6.2) \\
\psi'' + 3H \psi' + m^2 a^2 \psi - \nabla^2 \psi & = - \frac{c}{M} \frac{\vec{E}_Y \cdot \vec{H}_Y}{a^2}, \quad \sigma = a \sigma_c, \\
\frac{1}{a} \frac{\partial (\mu r a)}{\partial \tau} & = - \frac{q^2}{4 \pi^2} \frac{783}{88} \frac{1}{a^3 T^3} \vec{H}_Y \cdot \nabla \times \vec{H}_Y - (\Gamma + \Gamma_H) (\mu r a) + D R \nabla^2 \mu_R, \quad \Gamma_H = \frac{783}{22} \frac{a^2}{\sigma a^2} \frac{|\vec{H}_Y|}{a^2 T^3}, \quad (6.3) \\
\frac{\partial \rho}{\partial \tau} + \nabla \cdot [(p + \rho) \dot{v}] & = \vec{E}_Y \cdot \vec{J}_Y, \quad (6.4)
\end{align*}
\]
where \( \bar{v} \) is the bulk velocity of the plasma, \( \rho = a^4\rho_r \) and \( p = a^4p_r \) are the (rescaled) energy and pressure densities of the radiation dominated fluid. Notice that \( \nu \) and \( 1/\sigma \) are the thermal and magnetic diffusivity coefficients whereas \( D_R \) accounts for the dilution of right electrons in the case where the chemical potential is not completely homogeneous.

Using the fact that \( \bar{J}_Y \sim \bar{\nabla} \times \bar{H}_Y \) the Navier-Stokes equation, for scales larger than the thermal diffusivity scale, becomes

\[
[(p + \rho)\bar{v}^\prime + \bar{v} \cdot \bar{\nabla}[(p + \rho)\bar{v}] + \bar{v} \bar{\nabla} \cdot [(p + \rho)\bar{v}] = -\bar{\nabla}[p + \frac{1}{2}|\bar{H}_Y|^2] + [\bar{H}_Y \cdot \bar{\nabla}]\bar{H}_Y, \tag{6.5}
\]

where we used the fact that \( \bar{\nabla} \times \bar{H}_Y \times \bar{H}_Y = -\frac{1}{2} \bar{\nabla}|\bar{H}_Y|^2 + [\bar{H}_Y \cdot \bar{\nabla}]\bar{H}_Y \). Notice that we will be always concerned with the case where \( p = \rho/3 \gg |\bar{H}_Y|^2 \) (i.e. vanishing hypermagnetic pressure). Similarly, the continuity equation will be modified as

\[
\frac{\partial \rho}{\partial \tau} + \bar{v} \cdot [\rho + \rho] \bar{v} = \frac{1}{\sigma} |\bar{\nabla} \times \bar{H}_Y|^2. \tag{6.6}
\]

Since we are dealing with modes of the hypercharge field with momentum \( k \) smaller than

\[
k_{\sigma} \sim \sqrt{\frac{\sigma}{M_0} T}, \quad M_0 = \frac{M_{Pl}}{1.66\sqrt{N_{eff}}} \sim 7.1 \times 10^{17}\text{GeV}, \tag{6.7}
\]

the source term appearing in the continuity equation can be ignored and we can indeed assume that our plasma is incompressible (i.e. \( \bar{\nabla} \cdot \bar{v} = 0 \)). In this case, consistency with the continuity equation requires that \( \rho \) and \( p \) depend only upon \( \tau \). Then, Eq. (6.5) can be further simplified

\[
\frac{\partial \bar{v}}{\partial \tau} + \bar{v} \cdot [\bar{\nabla} \bar{v}] = [\bar{H}_Y \cdot \bar{\nabla}]\bar{H}_Y. \tag{6.8}
\]

With the same set of assumptions we can also obtain a generalized version of the hypermagnetic diffusivity equation, namely

\[
\frac{\partial \bar{H}_Y}{\partial \tau} = -\frac{4\alpha \psi}{\pi \sigma} \bar{\nabla} \times \left[ \mu_r \bar{H}_Y \right] - \frac{c}{M} \bar{\nabla} \times [\psi^\prime \bar{H}_Y] + \bar{\nabla} \times (\bar{v} \times \bar{H}) + \frac{1}{\sigma} \nabla^2 \bar{H}_Y. \tag{6.9}
\]

where, as in the previous Sections, we focus our attention on the case where \( |\bar{\nabla} \psi| \ll \psi^\prime \). Again, if \( k < k_{\sigma} \) and if the pseudoscalar oscillations already took place, Eq. (6.9) can be written as

\[
\frac{\partial \bar{H}_Y}{\partial \tau} + [\bar{v} \cdot \bar{\nabla}]\bar{H}_Y = [\bar{H}_Y \cdot \bar{\nabla}]\bar{v}. \tag{6.10}
\]

where we re-expressed \( \bar{\nabla} \times (\bar{v} \times \bar{H}_Y) \) according to the usual vector identities. Under these approximations a fully nonlinear solution of Eqs. (6.10) and (6.5), is given by the Alfvén velocity [15]

\[
\bar{v} = \pm \frac{\bar{H}_Y}{\sqrt{\rho + p}} \sim \pm \frac{\bar{H}_Y}{\sqrt{\rho}}. \tag{6.11}
\]

This result seems to imply that if the hypermagnetic field distribution is tangled (for example as a result of the pseudoscalar oscillations), then also the velocity field might be tangled in the same way. The arguments we presented up to now hold in the case where \( \bar{J}_Y \times \bar{H}_Y \neq 0 \). We want to point out that this is not exactly our case. Indeed, as we showed in the previous Sections, the maximally amplified hypercharge modes are \( k \sim k_{\max} \). For these modes \( \langle \bar{H}_Y \cdot \bar{\nabla} \times \bar{H}_Y \rangle \) is maximal and, therefore we can also argue that \( \langle \bar{\nabla} \times \bar{H}_Y \times \bar{H}_Y \rangle \sim 0 \). Thus, we can argue that our system is (approximately) force free [59]. Now, if the hypermagnetic field is not completely homogeneous we could argue from the general expression of the Navier-Stokes equation, that the hypermagnetic inhomogeneities could drive inhomogeneities in the energy density according to a well known observation by Wassermann [37]. Even if the hypermagnetic field would not be force-free we could argue that since the electroweak epoch occurs, in our scenario, deep in the radiation epoch, the force term can only seed a decaying mode for the density fluctuations [38]. In the case of a force free field the Navier-Stokes and diffusivity equations acquire a symmetric form [15].
\[
\frac{\partial \tilde{H}_Y}{\partial \tau} = \tilde{\nabla} \times (\tilde{v} \times \tilde{H}_Y) + \frac{1}{\sigma} \nabla^2 \tilde{H}_Y \\
\frac{\partial \tilde{\omega}}{\partial \tau} = \tilde{\nabla} \times (\tilde{\sigma} \times \tilde{\omega}) + \nu \nabla^2 \tilde{\omega}.
\] (6.12)

Let us now move to the analysis of the kinetic equation for the chemical potential. We could introduce, in principle, a chemical potential for all the perturbative strong and weak processes, Yukawa interactions of leptons and quarks which are in thermal equilibrium for temperatures \( T > T_c \). However, the processes really crucial for our purposes are the slowest perturbative processes related to the \( U(1)_Y \) anomaly, namely the processes flipping the chirality of the right electron which are in thermal equilibrium until sufficiently late because of the smallness of their Yukawa coupling [39]. Therefore the right electron number density will be diluted thanks to perturbative processes (scattering of right electrons with the Higgs and gauge bosons and with the top quarks because of their large Yukawa coupling) but it will also have a source term coming from the anomaly contribution. Notice that \( \Gamma \) is the (perturbative) chirality changing rate. Moreover, there is a second rate appearing in Eq. (6.3) which is proportional to the hypermagnetic energy density. The main effect of \( \Gamma_H \) is to dilute the fermion number even in the absence of perturbative processes flipping the chirality of right electrons (i.e. \( \Gamma = 0 \)).

In Eq. (6.3) we see that the right electron chemical potential can be diluted also by diffusion. The diffusion constant \( D_R \) appearing in Eq. (6.3) defines the relation between the diffusion current \( \vec{J}_R \) and the gradient of the right electron number density

\[
\vec{J}_R = -D_R \nabla n_R.
\] (6.13)

The right electron number dilution equation can then be written, in flat space, as

\[
\frac{\partial n_R}{\partial t} = -\frac{g^2}{4\pi^2} \vec{E}_Y \cdot \vec{H}_Y - \Gamma(n_R - n_R^{eq}) - \vec{\nabla} \cdot \vec{J}_R.
\] (6.14)

Using Eqs. (6.1)-(6.2) together with Eq. (6.13) into Eq. (6.14) we can obtain Eq. (6.3). The right electron diffusion coefficient can be roughly estimated by taking into account the interactions of right electrons with hypercharge fields and it turns out that the typical momentum for which diffusion becomes effective is of the order of

\[
k_D \sim \alpha'_T \sqrt{\frac{T}{M_0} T}.
\] (6.15)

For field modes \( k < k_D \) the diffusion term can be consistently neglected in the study of our problem. The hypermagnetic helicity appears to be the source term in the right electrons dilution equation. This is a quite relevant point for our considerations since a sizable mean magnetic helicity (possibly generated through pseudoscalar oscillations) can influence the BAU present in the symmetric phase of the electroweak theory. In our considerations we will assume that the hypermagnetic diffusivity and the thermal diffusivity are of the same order. More precisely one can define the Prandtl number which usually measures, in the context of ordinary MHD, the relative balance between thermal and magnetic diffusivity

\[
Pr_m = \nu \sigma = \frac{\sigma_0}{\alpha'_T}.
\] (6.16)

The approximation of unitary Prandtl number is often employed in 3D MHD simulations [15]. The approximation of unitary Prandtl number is often employed in 3D MHD simulations [15]. In this limit, hypercharge modes \( k < k_r \) are not damped by thermal diffusion. In our case (as in the case of realistic MHD simulations) \( Pr_m > 1 \).

Eq. (6.3) can be discussed in two physically different regimes. The first one is the case where \( \Gamma > \Gamma_H \), whereas the second case is the one where \( \Gamma < \Gamma_H \). In the first case the rate of dilution of the chemical potential is dictated by the perturbative processes, whereas, in the second case, it is determined mainly by the Abelian anomaly.

The preliminary analysis of BAU generation through Chern-Simons waves [11] showed that the case where \( \Gamma > \Gamma_H \) may lead, potentially, to a higher BAU and therefore we will focus our attention on the case where \( \Gamma > \Gamma_H \) commenting, when needed, on the other possibility. Large values of \( \Gamma \) seem also consistent with the MSSM \(^3\) where, for large values

\(^3\)In the case of the MSSM the Yukawa couplings enhancement is certainly not the only effect which might increase the rate. Since the particle spectrum changes completely from the case of the MSM new diagrams will contribute to the rate producing a further enhancement. In particular one should perhaps assume that the right electron number is now shared between electrons and selectrons by supergauge interactions. Therefore it will be necessary to consider also processes which change selectron number.
of tan $\beta$ (giving the ratio of the two v.e.v. of the two doublets), the right electron Yukawa couplings can be enhanced by a factor $(\cos \beta)^{-1}$.

Since for $T > T_c$ the fermion number seats both in fermions and in hypermagnetic fields (which can carry fermionic number) we can expect that for $T < T_c$ the fermion number sitting in the hypermagnetic fields should be released in the form real fermions because the ordinary magnetic field (surviving after the phase transition) does not carry fermion number. This argument holds provided if we have a homogeneous hypermagnetic field at the time of the EWPT [19], the cross-over region [17] of the phase diagram is still present for sufficiently large values of the phase transition itself [9,18]. It was recently pointed out that, if we have a homogeneous hypermagnetic field at $H$ is large with respect to vacuum fluctuations. Suppose in fact that $H > \Gamma$.

As we can see, a substantial BAU can be achieved provided the primordial value of the initial hypermagnetic field be strongly first order for Higgs boson masses larger than the $W$ boson mass. This result is not crucially modified in the minimal standard model (MSM) $\Gamma \sim T(T/R/M_0)$ where $T_R \sim 80$ TeV. If $\Gamma > \Gamma_H$ the hypermagnetic field has to be sufficiently weak and/or $\Gamma$ is quite large. For example in the minimal supersymmetric standard model (MSSM) the right electron Yukawa coupling is larger than in the MSM case so that $T_R$ can be larger by a factor of the order of $10^4$ for value of tan $\beta > 50$. If the hypermagnetic background is sufficiently intense, then, $\Gamma_H > \Gamma$. This last case can arise in the MSM. If we suppose that $\Gamma_H > \Gamma$ we run, however, into a contradiction among our assumptions. In fact if we are just in the context of the MSM the phase transition cannot be strongly first order for Higgs boson masses larger than the $W$ boson mass. This result is not crucially modified by the presence of a strong hypermagnetic background [19,17]. Therefore, if $\Gamma_H > \Gamma$ the BAU possibly generated for $T > T_c$ by our mechanism will be washed out by a subsequent stage of thermal equilibrium. However, even if we would assume that, thanks to some presently unknown phenomena, the EWPT would be strongly first order in the MSM, we will be able to show in the next section that the BAU obtained in the case $\Gamma_H > \Gamma$ is too small to be phenomenologically relevant.

In the case where $\Gamma > \Gamma_H$ we have that the BAU can be expressed as

$$\frac{n_B}{s}(\vec{x},t_c) = \frac{\alpha'}{2\pi\sigma_c} \frac{n_f}{s} \frac{\mathcal{H}_Y \cdot \nabla \times \mathcal{H}_Y}{\Gamma + \Gamma_H} \frac{\Gamma M_0}{T_c^2}, \quad (6.17)$$

$[s = (2/45)\pi^2 N_{eff} T^3]$ is the entropy density and $N_{eff}$ is the effective number of massless degrees of freedom (106.75 in the minimal standard model)]. In the minimal standard model (MSM) $\Gamma \sim T(T_R/M_0)$ where $T_R \sim 80$ TeV. If $\Gamma > \Gamma_H$ the hypermagnetic field has to be sufficiently weak and/or $\Gamma$ is quite large. For example in the minimal supersymmetric standard model (MSSM) the right electron Yukawa coupling is larger than in the MSM case so that $T_R$ can be larger by a factor of the order of $10^4$ for value of tan $\beta > 50$. If the hypermagnetic background is sufficiently intense, then, $\Gamma_H > \Gamma$. This last case can arise in the MSM. If we suppose that $\Gamma_H > \Gamma$ we run, however, into a contradiction among our assumptions. In fact if we are just in the context of the MSM the phase transition cannot be strongly first order for Higgs boson masses larger than the $W$ boson mass. This result is not crucially modified by the presence of a strong hypermagnetic background [19,17]. Therefore, if $\Gamma_H > \Gamma$ the BAU possibly generated for $T > T_c$ by our mechanism will be washed out by a subsequent stage of thermal equilibrium. However, even if we would assume that, thanks to some presently unknown phenomena, the EWPT would be strongly first order in the MSM, we will be able to show in the next section that the BAU obtained in the case $\Gamma_H > \Gamma$ is too small to be phenomenologically relevant.

In the case where $\Gamma > \Gamma_H$ we have that the BAU can be expressed as

$$\frac{n_B}{s}(\vec{x},t_c) = \frac{\alpha'}{2\pi\sigma_c} \frac{n_f}{s} \frac{\mathcal{H}_Y \cdot \nabla \times \mathcal{H}_Y}{\Gamma + \Gamma_H} \frac{\Gamma M_0}{T_c^2}, \quad (6.18)$$

There are two (physically complementary) situations where Eqs. (6.17)–(6.18) can be applied. The first one is the case of a stochastic hypermagnetic background. In this case $\langle n_B/s \rangle$ is strictly zero but its fluctuations (i.e. $\langle (n_B/s)^2 \rangle$) are non vanishing [11]. On the other hand in our present case the hypermagnetic field distribution is not topologically trivial because of the action of the pseudoscalar oscillations and, therefore, $\langle n_B/s \rangle \neq 0$.

VII. PHENOMENOLOGICAL CONSIDERATIONS

As a preliminary exercise let us consider the case of a Chern-Simons wave configuration [8,9] (see also [11]). Suppose that at some moment of time the universe is populated by parity non-invariant configurations of the type discussed in Eq. (2.1). using the hypermagnetic field configuration obtainable from eq. (2.1) into Eq. (6.17) we get that, in the limit $\Gamma > \Gamma_H$ (we work, for simplicity, in flat space), the BAU can be written as

$$n_B(s) = \frac{\alpha'}{2\pi\sigma_c} \frac{n_f}{s} \frac{\mathcal{H}_Y \cdot \nabla \times \mathcal{H}_Y}{\Gamma + \Gamma_H} \frac{k_0}{T_c} \mathcal{H}^2(t_c) \simeq 10^{10} \frac{k_0}{T_c} \left( \frac{\mathcal{H}^2}{T_c^2} \right), \quad (7.1)$$

As we can see, a substantial BAU can be achieved provided the primordial value of the initial hypermagnetic field is large with respect to vacuum fluctuations. Suppose in fact that $\mathcal{H}^2$ is only built up from vacuum fluctuations. Then $\mathcal{H}^2 \sim k_\sigma^4$. Taking now into account that, at the electroweak scale, the conductivity momentum is, roughly, $k_\sigma \sim 10^{-7} T_c$ we have that the obtained BAU for the highest possible mode which could survive in the plasma is of the order of $10^{-25}$. Too small to be relevant. In the framework of this example we see that there exist always a minimal hypermagnetic energy density which is compatible with the BAU. The reason for this fact is the existence of
We can obtain the explicit form of the logarithmic spectrum of the BAU, namely \( \omega \sim \omega_{\text{max}} \), the mean value of the hypermagnetic helicity is

\[
\langle \vec{H}_Y \cdot \nabla \times \vec{H}_Y \rangle = \frac{\omega_{\text{max}}^5}{\pi^2} e^{\left(|\omega_0|/|\omega_{\text{max}}|\right)} e^{-2(5\omega_{\text{max}})^2}.
\] (7.2)

Inserting Eq. (7.2) into Eq. (6.18) and taking into account that the maximally amplified frequency \( \omega(t_i) \sim \omega_{\text{max}}(t_i) = m(c/2)(\psi_0/M) \), red-shifted at the electroweak epoch, reads

\[
\omega_{\text{max}}(t_{\text{ew}}) = \frac{c}{2} \left( \frac{\psi_0}{M} \right) \left( \frac{m}{H_i} \right) \sqrt{\frac{H_i}{M_P}} (N_{\text{eff}})^{1/4} T_c.
\] (7.3)

we can obtain the explicit form of the logarithmic spectrum of the BAU, namely

\[
\frac{d(n_B/s)}{d \log \omega} = \frac{g^2 c^5}{512} \frac{45}{\sigma_0 \pi^6} \frac{N_{\text{eff}}}{M_P} \left( \frac{\psi_0}{M} \right)^5 \left( \frac{m}{H_i} \right)^5 \left( \frac{H_i}{M_P} \right)^{\frac{3}{2}} e^{\left(|\psi_0|/|\omega_{\text{max}}|\right)} e^{-2(5\omega_{\text{max}})^2}.
\] (7.4)
We want now to compare Eq. (7.2) with the required value of the baryon asymmetry. Defining 
\[ x = \log_{10}(m/H_i), \]
\[ y = \log_{10}(H_i/M_P), \]
and \[ z = \log_{10}(\psi_0/M) \]
we have that, if we want \( n_B/s \gtrsim 10^{-10} \), we have to demand, from Eqs. (6.17)–(6.18), that

\[ y \geq -8.5 - \frac{1}{10} \log_{10} N_{\text{eff}} + \frac{2}{5} \log_{10} \sigma_0 - 2 \log_{10} c - 2(x + z) - \frac{2}{5} c 10^{(x+z)} \log_{10} e, \]  

(7.5)

where \( \sigma_0 = \sigma_c/T \sim 70–100 \) [41]. Eq. (7.5) illustrated in Fig. 5 and 6. In the thick lines we represent curves of constant BAU (of the order of \( 10^{-10} \)) as function of the two parameters of the model, namely the maximal inflationary scale and the mass of the pseudoscalar. We can see that for inflationary curvature scales \( H_i \sim 10^{-6} - 10^{-10} M_P \) the mass has to be of the order of \( 10^2 H_i \). At the same time, provided the parameter space does not hit the electroweak curvature scale (lower dot-dashed line in Fig 5), the inflationary scale can be reduced and then, for \( H_i \sim 10^{-15} - 10^{-20} M_P \), we should require masses of the order of \( 10^3 - 10^4 H_i \) in order to achieve the required value of the BAU.

If, initially, the field is not in its vacuum state the hypermagnetic helicity is given by

\[ \langle \vec{H}_Y \cdot \vec{\nabla} \times \vec{H}_Y \rangle = \frac{\omega_{\text{max}}^5}{\pi^2} |A(\omega_{\text{max}})|^2 e^{\left(\frac{\omega_{\text{max}}}{\beta}\right)} (\psi_0/M). \]  

(7.6)

Inserting Eq. (7.6) into Eq. (6.18) the logarithmic BAU spectrum will depend explicitly on \( r(\omega_{\text{max}}) \) which is the critical fraction of energy density stored in the initial (topologically trivial) hypermagnetic field distribution.

In order to be compatible with value of the BAU obtained in the context of the homogeneous and isotropic BBN scenario we have to require

\[ y \gtrsim -46.88 - \frac{1}{2} \log_{10} N_{\text{eff}} - 2 \log_{10} \sigma_0 - 2 \log_{10} r(\omega_{\text{max}}) - 2 \log_{10} c - 2(x + z) - 2 c 10^{x+z} \log_{10} e. \]  

(7.7)

By looking at Eqs. (7.2)–(7.5) together with Eq. (7.7) we see that in order to achieve the observed value of the BAU with reasonable values of the pseudoscalar mass in curvature units (i.e. \( m/H_i \sim 10^2 \)) we have to postulate an inflationary phase at curvature much higher than the electroweak epoch \( H_i \sim 10^{-10} M_P \). If, initially, the hypercharge modes are not in the vacuum we can see that if \( m \gtrsim H_i \) and \( H_i \sim 10^{-20} M_P \) the observed value of the BAU can be achieved provided \( r(\omega_{\text{max}}) \sim 10^{-10} \). This aspect is illustrated in Fig. 7. By lowering the energy density of the primordial stochastic background we are pushed towards higher inflationary scales and/or higher pseudoscalar masses.
The stochastic background is formed by vacuum fluctuations. Therefore \(\text{BAU} \sim 10^{-10}\). The acceptable region of parameter space lies, as usual, between the two dot-dashed lines and above the full (thick) lines. If the initial state of the hypercharge fields is the vacuum the maximal inflationary scale compatible with a BAU of the order of \(10^{-10}\) gets lower if compared with the case reported in Fig. 5. We also illustrate (full thin lines) the same case reported in the thick lines but for a lower value of \(r(\omega_{\text{max}})\), namely, \(10^{-15}\).

Therefore, the generation of hypermagnetic helicity is quite likely if the pseudoscalar oscillations take place during an inflationary phase for a wide range of masses. If we want the BAU generation to occur at very high curvature scales (i.e. \(H_i \sim 10^{-30} - 10^{-10} M_P\) we are also led towards high mass scales (i.e. \(m \gtrsim 10^2 H_i \sim 10^8 - 10^{12}\) TeV). The exponential amplification appearing in the Bogoliubov coefficients is responsible of the sharp cut in the plots reported in Figs. 5-8. Thus, if we take the curvature scale to be sufficiently low (say, for instance, \(10^{-21} M_P\), then we can get a sensible BAU for masses \(m \gtrsim 10^4 H_i\), i.e. masses of the order of few hundred GeV at the time the amplification occurred. Notice that the inclusion of stimulated emission goes exactly in this direction, namely it allows, more easily, low curvature scales.

We finally notice that, from Eq. (7.3), the the maximal amplified frequency (red-shifted at the electroweak scale ) is always smaller than the diffusivity frequency at the electroweak scale (i.e. \(\omega_e(T_{\text{ew}}) \sim 10^{-7} T_{\text{ew}}\)). This guarantees that the obtained BAU will not be washed out by hypermagnetic diffusion.

B. BAU from Hypermagnetic Knots Generated During a Radiation Epoch

We are now going to consider the phenomenological implications of pseudoscalar oscillations occurring in a radiation dominated phase. In order to get a more explicit expression of the BAU in this case let us insert Eq. (5.5) into Eq. (6.17). Since \(\omega_{\text{max}}/\omega_a < 1\) we have that the BAU can be written as

\[
n_B/s \approx \frac{45 n_f}{8 \pi^2 \sigma_0} c \alpha' \frac{\Delta \psi}{M} r(\omega_{\text{max}}).
\]

(7.8)

It is interesting to notice that, as far as the maximal frequency is concerned, the BAU does not depend upon the pseudoscalar mass but only upon the pseudoscalar coupling constant. Let us estimate, very roughly, the size of the

\[
\frac{\text{BAU}}{\text{BAU}^{\text{max}}} = P_{\text{BAU}}(\omega_{\text{max}}) \sim 10^{-10}\text{.}
\]

Notice that the inclusion of stimulated emission goes exactly in this direction, namely it allows, more easily, low curvature scales.

We are now going to consider the phenomenological implications of pseudoscalar oscillations occurring in a radiation dominated phase. In order to get a more explicit expression of the BAU in this case let us insert Eq. (5.5) into Eq. (6.17). Since \(\omega_{\text{max}}/\omega_a < 1\) we have that the BAU can be written as

\[
n_B/s \approx \frac{45 n_f}{8 \pi^2 \sigma_0} c \alpha' \frac{\Delta \psi}{M} r(\omega_{\text{max}}).
\]

(7.8)

It is interesting to notice that, as far as the maximal frequency is concerned, the BAU does not depend upon the pseudoscalar mass but only upon the pseudoscalar coupling constant. Let us estimate, very roughly, the size of the BAU emerging from Eq. (7.8). Suppose that \(N_{\epsilon_{\text{eff}}} = 106.75\) (same particle content of the standard model) and that \(c = 0.01\). Thus for \(\sigma_0 \approx 70\) we get that \(n_B/s \approx 10^{-7} r(\omega_{\text{max}})\). Thus the BAU can be reproduced if \(r(\omega_{\text{max}}) \sim 10^{-3}\). Therefore, the presence of an initial background of hypermagnetic fields is essential as in the case of the Chern-Simons wave discussed at the beginning of the present Section. Suppose in fact that that \(\lambda(\omega) = 1\). In this case the initial stochastic background is formed by vacuum fluctuations. Therefore \(r(\omega_{\text{max}}) = N_{\epsilon_{\text{eff}}} (\omega_{\text{max}}/T)^4 \sim 10^{-33}\) giving a BAU of the order of \(10^{-40}\).

The logarithmic variation of the BAU as a function of the parameters of the model can be expressed as

\[
y = -2.4 + \log_{10} c - \log_{10} \sigma_0 + \log_{10} \frac{\Delta \psi}{M} + x
\]

(7.9)

where now \(x = \log_{10} r(\omega_{\text{max}})\) and \(y = \log_{10} (n_B/s)\). From this expression illustrated in Fig. 8 we can draw two conclusions. The first one is that the BAU possibly generated during the radiation dominated epoch depends upon the pseudoscalar coupling but not upon the pseudoscalar mass. This happens because we estimated the BAU arising in the case of the maximally amplified frequency whose actual value is dictated by Eq. (5.4). The smaller is the coupling the larger has to be the primordial hypermagnetic background.
FIG. 8. We illustrate the logarithmic variation of the BAU computed in Eq. (7.9) for different sets of parameters. More precisely we have $c = 0.01$, $\sigma_0 = 70$ (full thick line), $c = 10^{-3}$, $\sigma_0 = 70$ (dot-dashed line), $c = 10^{-4}$, $\sigma_0 = 100$ (full thin line).

The second observation is that the possible explanation of the BAU through Eqs. (7.8)–(7.9) depends crucially upon the strength of the EWPT. In fact, in order to derive Eq. (7.8) we demanded the rate of right electron chirality flip processes (i.e. $\Gamma$) to be larger than $\Gamma_H$. As we discussed, this hypothesis is certainly not forbidden in the case of the MSSM. One could also wonder if the MSM physics would be enough to implement our scenario. Let us assume, for a moment, to be in the framework of the MSM. Then, as argued in [10,11], $\Gamma_H > \Gamma$. The principal problem with this assumption is the strength of the EWPT. In fact we know that even in the presence of a strong magnetic field the typical cross-over behavior [17] of the phase transition does not change.

However, even if the phase transition would be strongly first order our mechanism would be ineffective. Indeed, let us compute the BAU in the case where $\Gamma_H > \Gamma$. We have that the BAU can be expressed as [10]

$$
\frac{n_B}{s} = \frac{\alpha'}{2\pi\sigma_c} \frac{n_f}{s} \langle \vec{H}_Y \cdot \vec{\varphi} \times \vec{H}_Y \rangle \frac{\Gamma_H M_0}{T_R^2}.
$$

(7.10)

The expectation value appearing in the previous equation involves a highly non local operator which can be evaluated for scales larger than the diffusivity scale with the result that [11]

$$
\langle \vec{H}_Y \cdot \vec{\varphi} \times \vec{H}_Y \rangle \sim \frac{\langle \vec{H}_Y \cdot \vec{\varphi} \times \vec{H}_Y \rangle}{\langle |\vec{H}_Y|^2 \rangle}.
$$

(7.11)

Thus, we get that the BAU can be expressed as

$$
\frac{n_B}{s} \simeq 0.04 e \left( \frac{\Delta \psi}{M} \right) \left( \frac{T_R}{M_0} \right)
$$

(7.12)

where $T_R \sim 80$ TeV is the right-electron equilibration temperature [39]. For typical values of the parameters (the same we used in the case $\Gamma > \Gamma_H$ the BAU is of the order of $10^{-18}$. We would be tempted to say that for large $T_R$ the BAU could be recovered. Unfortunately this assumption would contradict our initial hypotheses since it would mean that we go beyond the MSM and that, consequently, we would have $\Gamma_H > \Gamma$. Notice that, as previously discussed [22], the results of our analysis are not in agreement with the statement that the BAU can be generated from dynamical pseudoscalars in the MSM, for hypermagnetic modes of the order of $k_{\text{max}} \sim T$ [46] and independently of the specific value of the perturbative rate of right electron chirality flip processes.

We want to observe, finally, that some mechanisms (like the breaking of conformal invariance) operating at some higher energy scale [42] can generate a topologically trivial hypermagnetic background whose energy density is small in critical units but anyway quite large if compared to the energy density of the vacuum fluctuations. Since we are at high conductivity, the dissipation of the seeds is protected from being washed out by the hypermagnetic flux conservation. Then, at some temperature (in the TeV range) pseudoscalar oscillations will begin. At finite conductivity the pseudoscalar oscillations do not crucially amplify the initial hypermagnetic background but they twist its topology leading to a non vanishing Chern-Simons number density and, ultimately, to the BAU.

---

4More precisely, for sufficiently small hypermagnetic backgrounds (i.e. $|\vec{H}_Y|/T^2 \leq 0.3$) the phase transition is stronger but it turns into a cross-over for sufficiently large Higgs masses. It would be interesting to understand if for sufficiently strong hypermagnetic backgrounds the Ambjorn-Olesen phase can be generated [45].
VIII. PARITY BREAKING FROM AXIONS?

The same mechanism studied in the case of HK can be indeed exploited in order to produce ordinary magnetic knots for temperatures smaller than 100 GeV. The natural candidate would then be, in this case, the ordinary QCD axion. We will start by recalling the possible implications of global parity breaking in ordinary MHD and we will then show that the amount of global parity breaking induced by axion oscillations is indeed quite minute.

A. Global Parity Breaking in MHD

If, at some scale \( \sim 5 \langle \vec{H} \cdot \vec{\nabla} \times \vec{H} \rangle \neq 0 \), than we also have that the Alfvén (stationary) flow will be tangled in the same way the related magnetic field is tangled since \( \langle \vec{v} \cdot \vec{\omega} \rangle = \langle \vec{v} \cdot \vec{\nabla} \times \vec{v} \rangle \neq 0 \). This last condition together with the incompressible closure \( \vec{\nabla} \cdot \vec{v} = 0 \) implies that also the velocity field is formed by a collection of closed loops whose degree of knottedness can be estimated from \( \vec{v} \cdot \vec{\omega} \). Notice that in the limit of negligible thermal and magnetic diffusivity both the magnetic and the velocity field can be treated with topological techniques [43,44]. When the thermal and magnetic diffusivity are adiabatically switched on (or, more physically, when we approach the magnetic and thermal diffusivity scales) various dissipative phenomena arise and turbulence sets in. We do not want even to attempt to address the very rich structure arising in MHD turbulence. However, we simply want to notice that there are crucial differences if the turbulence arises starting from a parity invariant or from a parity non invariant configuration of the bulk velocity field [4,5]. It has been actually observed long ago that [47], apparently, cosmic turbulence occurs usually in rotating bodies with density gradients. This results in a violation of the mirror symmetry of the plasma since it is possible, by taking the scalar product of the vorticity with the density gradient, to construct a pseudoscalar. The relevance of (global) parity breaking in MHD can be also appreciated by looking at the turbulent dynamo mechanism [5,47]. The idea is that in astrophysical and cosmological plasmas the correlation scale of the velocity field is normally shorter than the correlation scale of the magnetic field. Under this assumption from Eq. (6.10) it possible to obtain a dynamo equation by carefully averaging over the velocity field over scales and times exceeding the characteristic correlation scale and time \( \tau_0 \) of the velocity field. In this approximation the magnetic diffusivity equation can be written as

\[
\frac{\partial \vec{H}}{\partial \tau} = \alpha (\vec{\nabla} \times \vec{H}) + \frac{1}{\sigma} \nabla^2 \vec{H}, \quad \alpha = -\frac{\tau_0}{3} \langle \vec{v} \cdot \vec{\nabla} \times \vec{v} \rangle,
\]

where \( \alpha \) is the so-called dynamo term. In Eq. (8.1) \( \vec{H} \) is the magnetic field averaged over times larger than \( \tau_0 \), which is the typical correlation time of the velocity field. We can clearly see that the crucial requirement for the all averaging procedure we described is that the turbulent velocity field has to be “globally” non-mirror-symmetric. If the velocity field is parity invariant (i.e. no vorticity for scales comparable with the correlation length of the magnetic field), then the dynamics of the infrared modes is decoupled from the velocity field since, over those scales, \( \alpha = 0 \). Therefore, for our purposes the \( \alpha \) term will be a good measure of the degree of knottedness associated with the velocity field.

Needless to say that that there is a puzzle associated with the turbulent dynamo mechanism [5]. On one hand we see that our galaxy has an \( \alpha \) term which can be roughly estimated to be \( \alpha_G \sim 0.8 \times 10^{15} \text{ cm/sec} \) in the interval 3 Kpc \( \leq r \leq 12 \text{ Kpc} \) (it is assumed that \( |\vec{\omega}| \sim 10^{-15} \text{ sec}^{-1} \)) [5]. On the other hand we have to postulate that some \( \alpha \) term had to be present also at the time of the protogalactic collapse (occurred around \( t_{gc} \sim 10^{10} \text{ sec} \)). This is because we want the dynamo term to amplify, by differential rotation the primordial magnetic seeds up to the observed value of themagnetic field observed in spiral galaxies. Thus, the parity breaking of the galactic turbulence seems to be more a consequence of some primordial (global) parity breaking than a theory of the galactic dynamo action.

If one wants to postulate some global parity breaking in the plasma the obvious question which arises is when this phenomenon could occur in the early Universe. It should certainly be useful for the dynamo action to have some sort of global parity breaking at the decoupling epoch \( t_{dec} \sim 10^{11} \text{ sec} \) which is not so far (for cosmological standards) from the time of the protogalactic collapse. In principle, the parity breaking could occur even before but the only problem could be, in this case, that many of the known phenomena occurring in the early Universe are normally discussed in terms of a mirror-symmetric plasma. For example the present BBN calculations do not include, to the best of our knowledge, possible effects of global parity violation [48,49] and we would guess that to have global parity breaking might be a problem for nucleosynthesis.

\( ^5 \)We denote with \( \vec{H} \) the ordinary magnetic field as opposed to the hypermagnetic field.
The axion oscillations starting at a temperature of the order of 1 GeV can certainly produce a source of global parity breaking. The question is how large is the induced $\alpha$ term in the primordial plasma. Our aim is now to estimate the order of magnitude of the parity breaking induced through primordial electromagnetic helicity.

B. Parity Breaking induced by ordinary Magnetic Knots

As it is well known in axion models [24,51] an extra (global) $U_{PQ}(1)$ symmetry complements the MSM. This symmetry is broken at the Peccei-Quinn scale $F_a$ and leads to a dynamical pseudo Goldstone boson (the axion) which acquires a small mass because of soft instanton effects at the QCD phase transition. If an axionic density is present in the early Universe, bounds can be obtained for the Peccei-Quinn symmetry breaking scale. These bounds together with astrophysical bounds (coming mostly from red-giants cooling and from SN 1987a) leave a window opportunity $10^{10}$ GeV $< F_a < 10^{12}$ GeV [50].

There are different realizations of axion models. In some models [24,52] quarks and leptons do not carry the Peccei-Quinn charge, whereas in other realizations the usual standard model leptons are not singlet under the Peccei-Quinn symmetry [24,53]. We will not enter into the details of these two models since what matters for our consideration is, at least in the very first approximation, how the axion couples to the anomaly.

There are also different cosmological realizations of axionic models. If an inflationary epoch takes place prior to the usual radiation dominated regime, the Peccei-Quinn phase can be aligned by today [54]. In the absence of an inflationary phase it is likely that as the Universe passes through the Peccei-Quinn phase transition a network of cosmic strings will be generated via the Kibble mechanism [55]. We will assume that an inflationary phase took place well before ($H \lesssim 10^{-6} M_P$) the onset of the EWPT.

If the temperature of the Universe exceeds the critical temperature of the EWPT the effective action describing the interaction of axions with the Standard model gauge fields can be written as

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ -\frac{\psi}{F_a} \left( \frac{g_s^2}{32\pi^2} G_\mu^a G_a^{\mu\nu} + c_{\psi W} \frac{g^2}{32\pi^2} W_\mu^a W^{\mu\nu a} + c_{\psi Y} \frac{g^2}{32\pi^2} Y_\mu^a Y^{\mu\nu a} \right) \right]$$

(8.2)

where $g_s$, $g$ and $g'$ are, respectively, the SU$_c$(3), SU$_L$(2) and U$_Y$(1) coupling constants ($c_{\psi W}$ and $c_{\psi Y}$ are numerical constants of order 1 which depend upon the specific axion model).

For $T \leq T_c$ the effective action becomes

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ -\frac{\psi}{F_a} \left( \frac{g_s^2}{32\pi^2} G_\mu^a G_a^{\mu\nu} + c_{\psi \gamma} \frac{e^2}{32\pi^2} F_\mu^a F^{\mu\nu} \right) \right]$$

(8.3)

where $F_{\mu\nu}$ is the ordinary (electromagnetic) gauge field strength.

There are important differences between the axion evolution for temperatures larger and smaller than the EWPT temperature. In fact [56], for $T \geq T_c$ the axion evolution equation is not only modified by the Hubble damping factor, but also by damping induced by the QCD sphalerons which are not suppressed at high temperatures. This effect leads to an equation for $\psi$ which differs from the one introduced in the previous section and which can be written as

$$\ddot{\psi} + (3H + \gamma)\psi = 0$$

(8.4)

where we dropped for simplicity the potential term. It turns out that [56]

$$\gamma = \frac{\Gamma_{\text{sph}}}{F_a^2 T} \approx \frac{\alpha_s^4 T^3}{F_a^2}$$

(8.5)

(where $\alpha_s = g_s^2/4\pi$). Thus, if $F_a > 10^9$ GeV we have to conclude that the sphaleron induced damping dominates over the damping produced by the expansion of the Universe (i.e. $\sim T^3/M_P$). The axion time evolution will then be (approximately) suppressed as $\psi \propto e^{-\gamma t} \sim e^{-\gamma \tau^2}$ in a radiation dominated Universe (recall that $t \sim \tau^2$). Therefore, we have to accept that in Eq. (2.18) the polarization mixing will be exponentially suppressed as time goes by. The sphaleron induced damping does not exclude other possible consequences associated with the axions at the EWPT like the possible existence of strong CP violating effects [57].

Notice that this argument does not apply to temperatures $T \ll T_c$ when the plasma has already cooled sufficiently for $\gamma$ to have turned off. Therefore, for our purposes it seems certainly theoretically more plausible to consider the possible occurrence of coherent axion oscillations at temperatures of the order of 1 GeV when the electroweak symmetry is broken.
In can be, in principle, dangerous to have production of parity non invariant configurations of the ordinary magnetic field right before the onset of BBN. Suppose that the axion oscillations start at \( T \sim 1 \text{ GeV} \) with a typical mass of \( 10^{-19} \) GeV. Therefore we have to consider the ordinary axion oscillations coupled to the evolution of the ordinary vector potential at finite conductivity

\[
\ddot{\psi} + 3H \dot{\psi} + m^2 \psi = 0, \\
\sigma A'_\pm + \left[ k^2 + \frac{k \alpha_{\text{em}} \psi'}{2 \pi F_a} \right] A_\pm = 0.
\] (8.6)

At high temperatures \( T \gg \Lambda_{\text{QCD}} \) the axion is massless, but, at lower temperatures it develops a mass due to QCD instanton effects. The effects of instantons is highly suppressed at high temperatures, however, the temperature dependence of the axion mass is approximately given by

\[
m(T) \sim 0.1 \left( \frac{\Lambda_{\text{QCD}}}{T} \right)^{3.7} m_0,
\] (8.7)

where \( m_0 \) is the axion mass at zero temperature.

Axion oscillations between a temperature of the order of 1 GeV and \( T \sim 10 \text{ MeV} \) (i.e. around the nucleosynthesis epoch) can generate electromagnetic helicity. Suppose, for sake of simplicity that the axion oscillations will start with a typical mass of the order of \( 10^{-19} \) GeV. Then, according to Eq. (8.6), we have that the maximal amplified frequency and the diffusivity frequencies are of the order of

\[
\omega_{\text{max}} \sim \frac{\alpha_{\text{em}} \Delta \psi}{4 \pi F_a} \frac{T^2}{M_0}, \quad \omega_\sigma \sim \sqrt{\frac{\sqrt{N_{\text{eff}}}}{\alpha_{\text{em}}}} \sqrt{\frac{T}{M_P}} T.
\] (8.8)

where \( T \) is of the order of 1 GeV. We can solve Eq. (8.6) similarly to what we did in hypermagnetic case. In the case \( \omega_{\text{max}} < \omega_\sigma \) the magnetic helicity produced via axion oscillations will be of the order of

\[
\frac{\langle \vec{H} \cdot \vec{\nabla} \times \vec{H} \rangle}{\sigma \rho_c} \sim \frac{\alpha_{\text{em}} \Delta \psi}{4 \pi F_a} \frac{T_a}{M_0} \sim 10^{-22}
\] (8.9)

(notice that \( \sigma_0 \sim \alpha_{\text{em}}^{-1} \) and \( T_a \sim 1 \text{ GeV} \)). Eq. (8.9) provides then a rather small parity breaking and certainly irrelevant for the dynamo mechanism.

**IX. CONCLUDING REMARKS**

In this paper we investigated the production of hypermagnetic knots. We found that HK can be efficiently produced provided some topologically trivial configuration of the hypermagnetic field is already present for scales larger than the hypermagnetic diffusivity scale. The BAU can be successfully reproduced, but not in the MSM. The reason is twofold. On one hand the phase transition is too weak (even taking into account the possible presence of the hypermagnetic background). On the other hand the (perturbative) rate of the slowest processes in the plasma is too small. In the MSSM the BAU generation seems more probable and our mechanism can be viable, provided the typical scale of the HK exceeds the hypermagnetic diffusivity scale prior to the onset of the EWPT. Inspired by the axial Higgs we gave examples of specific scenarios where this mechanism can be implemented. We also discussed the possible production of magnetic helicity by ordinary QCD axions: we found that its value is too small to be phenomenologically relevant.

**ACKNOWLEDGMENTS**

I am deeply indebted to M. Shaposhnikov for important discussions and valuable collaboration which motivated this investigation. I would also like to acknowledge interesting conversations with A. Vilenkin. Discussions with K. Jedamzik, H. Kurki-Suonio, J. Rehm and E. Sihvola were also appreciated. I would like to thank the Theory division of CERN and the Institute of Theoretical Physics of the Lausanne University where part of the work has been completed.
APPENDIX A: EXAMPLES OF HYPERMAGNETIC KNOTS FROM TOPOLOGICALLY NON TRIVIAL CONFIGURATIONS OF MAGNETIC FIELDS

In Section II we presented some topologically non trivial configurations of the hypercharge field [8,9] whose main feature is to have a Chern-Simons number density which is time dependent and uniformly distributed in space. It is also possible to construct hypermagnetic knot configurations with finite energy and helicity which are localized in space and within typical distance scale $L_s$. Let us consider in fact the following configuration in spherical coordinates [14,60]

$$\mathcal{Y}_t(R, \theta) = -\frac{2B_0}{\pi L_s} \frac{\cos \theta}{[R^2 + 1]^2},$$
$$\mathcal{Y}_r(R, \theta) = \frac{2B_0}{\pi L_s} \frac{\sin \theta}{[R^2 + 1]^2},$$
$$\mathcal{Y}_\phi(R, \theta) = -\frac{2B_0}{\pi L_s} \frac{nR \sin \theta}{[R^2 + 1]^2},$$

(B.1)

where $R = r/L_s$ is the rescaled radius and $B_0$ is some dimensionless amplitude and $n$ is just an integer number whose physical interpretation will become clear in a moment. The hypermagnetic field can be easily computed from the previous expression and it is

$$\mathcal{H}_r(R, \theta) = -\frac{4B_0}{\pi L_s^2} \frac{n \cos \theta}{[R^2 + 1]^2},$$
$$\mathcal{H}_\theta(R, \theta) = -\frac{4B_0}{\pi L_s^2} \frac{R^2 - 1}{[R^2 + 1]^3} n \sin \theta,$$
$$\mathcal{H}_\phi(R, \theta) = -\frac{8B_0}{\pi L_s} \frac{R \sin \theta}{[R^2 + 1]^2}.$$  

(B.2)

The poloidal and toroidal components of $\vec{H}$ can be usefully expressed as $\vec{H}_p = \mathcal{H}_r \vec{e}_r + \mathcal{H}_\theta \vec{e}_\theta$ and $\vec{H}_t = \mathcal{H}_\phi \vec{e}_\phi$. The Chern-Simons number is finite and it is given by

$$N_{CS} = \frac{g^2}{32\pi^2} \int_V \vec{V} \cdot \vec{H} d^3x = \frac{g^2}{32\pi^2} \int_0^\infty \frac{8nB_0^2}{\pi^2} \frac{R^2 dR}{[R^2 + 1]^3} = \frac{g^2 n B_0^2}{32\pi^2}.$$  

(B.3)

In Eq. (B.3) the integration is perfectly convergent over the whole space and, since the field is $\vec{H}_Y = 0$ in any part of $\partial V$ at infinity, this quantity is also gauge invariant. Thus, we can see that the integer index $n$ labels the number of knots of the configuration. We can also compute the total helicity of the configuration namely

$$\int_V \vec{H}_Y \cdot \vec{\nabla} \times \vec{H}_Y d^3x = \frac{256 B_0^2 n}{\pi L_s^3} \int_0^\infty \frac{\pi^2}{[1 + (R^2)^2]} dR = \frac{5B_0^2 n}{L_s^3}.$$  

(B.4)

There is an important difference between Eqs. (B.3) and (B.4). In Eq. (B.3) the integrand is not gauge invariant (but the integrated Chern-Simons number density is gauge-invariant), on the other hand in Eq. (B.4) also the integrand is gauge-invariant. We can compute also the total energy of the field

$$E = \frac{1}{2} \int_V d^3x |\vec{H}_Y|^2 = \frac{B_0^2}{2 L_s^3} (n^2 + 1).$$  

(B.5)

and we discover that it is proportional to $n^2$. This means that one way of increasing the total energy of the field is to increase the number of knots and twists in the flux lines. We can also have some real space pictures of the core of the knot (i.e. $R = r/L_s < 1$). In Fig. 9 we plot the cartesian components of the hypermagnetic field

$$\mathcal{H}_x(\vec{X}) = \frac{4B_0}{\pi L_s^2} \frac{2X - 2 \ 2 n \ X \ Z}{[1 + X^2 + Y^2 + Z^2]^2},$$
$$\mathcal{H}_y(\vec{X}) = \frac{4B_0}{\pi L_s^2} \frac{2X + 2 \ 2 n \ Y \ Z}{[1 + X^2 + Y^2 + Z^2]^3},$$
$$\mathcal{H}_z(\vec{X}) = \frac{4B_0}{\pi L_s^2} \frac{n [X^2 + Y^2 - Z^2 - 1]}{[1 + X^2 + Y^2 + Z^2]^2}.$$  

(B.6)
in the case of \( n = 0 \) and \( n = 5 \) in terms of the rescaled coordinates \( X = x/L_s, Y = y/L_s, Z = z/L_s \).

\[ \text{Fig. 9. We plot the hypermagnetic knot configuration in Cartesian components [Eqs. (B.6)] for the cases } n = 0 \text{ (zero helicity) and } n = 5 \text{. The direction of the arrow at each point represents the tangent to the flux lines of the magnetic field, whereas the length of the vector is proportional to the field intensity. We notice that for large } n \text{ the field is concentrated, in practice, in the core of the knot (i.e. } r/L_s < 1). \]

**APPENDIX B: EXPECTATION VALUE OF THE HYPERMAGNETIC HELICITY**

The aim of this Appendix is to compute the expectation value of the hypermagnetic helicity which represents a fair gauge-invariant estimate of parity breaking induced by the presence of dynamical pseudoscalars in the hypermagnetic background.

Let us expand the hypercharge field operator according to Eq. (3.15) but with slightly different notations which turn out to be useful for the calculation at hand:

\[
\hat{Y}_{\text{out}}(\vec{x}, \tau) = \frac{1}{\sqrt{2}} \sum_{\beta} \int \frac{d^3k}{(2\pi)^3} \left( \hat{Y}_{\beta,\text{out}}(k, \tau) \hat{e}_\beta e^{i\vec{k} \cdot \vec{x}} + \hat{Y}^{\dagger}_{\beta,\text{out}}(k, \tau) \hat{e}_\beta e^{-i\vec{k} \cdot \vec{x}} \right) \tag{B.1}
\]

(where, as usual, \( \beta = \pm, \mp \)).

Therefore the expectation value of the hypermagnetic helicity will be

\[
\langle 0 | \vec{\hat{H}} \cdot \vec{\nabla} \times \hat{H} | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} k p^2 \left[ \langle \hat{Y}_+(k, \tau) \hat{Y}_+(p, \tau) \rangle e^{i(\vec{k} - \vec{p}) \cdot \vec{x}} - \langle \hat{Y}_-(k, \tau) \hat{Y}_-(p, \tau) \rangle e^{i(\vec{k} - \vec{p}) \cdot \vec{x}} \right.
\]

\[
+ \langle \hat{Y}^{\dagger}_+(k, \tau) \hat{Y}_+(p, \tau) \rangle e^{-i(\vec{k} - \vec{p}) \cdot \vec{x}} - \langle \hat{Y}^{\dagger}_-(k, \tau) \hat{Y}_-(p, \tau) \rangle e^{-i(\vec{k} - \vec{p}) \cdot \vec{x}} \right]. \tag{B.2}
\]

(we dropped the “out” subscript for the mode functions). The various correlators appearing in the previous expression can be evaluated, after some algebra, using the explicit form of the Bogoliubov transformation given in section II:

\[
\langle 0 | \hat{Y}_+(k, \tau) \hat{Y}_+(p, \tau) | 0 \rangle = \left[ \left( |c_+(k)|^2 + |c_-(k)|^2 \right) F_{in}(\tau) F^{*}_{in}(\tau) + c_+(k) c^*_-(k) F_{in}(\tau)^2 + c_-(k) c^*_+(k) F_{in}(\tau)^2 \right] \delta^{(3)}(\vec{k} - \vec{p}),
\]

\[
\langle 0 | \hat{Y}^{\dagger}_+(k, \tau) \hat{Y}_+(p, \tau) | 0 \rangle = \left[ \left( |\tilde{c}_+(k)|^2 + |\tilde{c}_-(k)|^2 \right) F^{*}_{in}(\tau) F_{in}(\tau) + \tilde{c}_+(k) \tilde{c}^*_-(k) F_{in}(\tau)^2 + \tilde{c}_-(k) \tilde{c}^*_+(k) F_{in}(\tau)^2 \right] \delta^{(3)}(\vec{k} - \vec{p}),
\]

\[
\langle 0 | \hat{Y}^{\dagger}_-(k, \tau) \hat{Y}_-(p, \tau) | 0 \rangle = \left( \langle 0 | \hat{Y}_+(k, \tau) \hat{Y}^{\dagger}_+(p, \tau) | 0 \rangle \right)^*,
\]

\[
\langle 0 | \hat{Y}^{\dagger}_-(k, \tau) \hat{Y}_-(p, \tau) | 0 \rangle = \left( \langle 0 | \hat{Y}_-(k, \tau) \hat{Y}^{\dagger}_-(p, \tau) | 0 \rangle \right)^*. \tag{B.3}
\]

From these last expressions we get Eq. (3.16). With the same technique other expectation values of parity non-invariant operators can be computed like

27
\[ \langle 0 | \vec{H}_Y \cdot \vec{E}_Y | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \left( |c_+(k)|^2 + |c_-(k)|^2 \right) \left( f_{in}(\tau)f_{in}(\tau)^* + f'_{in}(\tau)f_{in}(\tau)^* \right) \ight.

\[ + 2c_+(k)c_-(k)^* f_{in}(\tau)' f_{in}(\tau) + 2c_-(k)c_+(k)^* f_{in}(\tau)' f_{in}(\tau)^* \

\[ - \left( |\tilde{c}_+(k)|^2 + |\tilde{c}_-(k)|^2 \right) \left( F_{in}(\tau)F_{in}(\tau)' + F'_{in}(\tau)F_{in}(\tau)^* \right) \

\[ - 2\tilde{c}_+(k)\tilde{c}_-(k)^* F_{in}(\tau)' F_{in}(\tau) - 2\tilde{c}_-(k)\tilde{c}_+(k)^* F_{in}(\tau)' F_{in}(\tau)^* \right]. \tag{B.4} \]

Notice that this second gauge-invariant operator can be also used in order to measure the amount of parity breaking. However, in the plasma it seems that the first operator computed in this Appendix is more suitable. Indeed, at finite conductivity, inhomogeneous hypermagnetic fields lead to the presence of induced (Ohm) hyperelectric fields of the order of \( \vec{E}_Y \sim \sigma^{-1} \vec{\nabla} \times \vec{H}_Y \).
[42] see M. Gasperini, M. Giovannini and G. Veneziano in Ref. [6].
[59] see third reference in [21].