The Blandford-Znajek Process 
as a Central Engine for a Gamma Ray Burst

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Abstract:
We investigate the possibility that gamma-ray bursts are powered by a central engine consisting of a black hole with an external magnetic field anchored in a surrounding disk or torus. The energy source is then the rotation of the black hole, and it is extracted electromagnetically via a Poynting flux, a mechanism first proposed by Blandford and Znajek (1997) for AGN.

Our reanalysis of the strength of the Blandford-Znajek power shows that the energy extraction rate of the black hole has been underestimated by a factor ten in previous works. Accounting both for the maximum rotation energy of the hole and for the efficiency of electromagnetic extraction, we find that a maximum of 9% of the rest mass of the hole can be converted to a Poynting flow, i.e. the energy available to produce a gamma-ray burst is \(1.6 \times 10^{53}(M/M_\odot)\) erg for a black hole of mass \(M\). We show that the black holes formed in a variety of gamma-ray burst scenarios probably contain the required high angular momentum.

To extract the energy from a black hole in the required time of \(\lesssim 1000\) s a field of \(10^{15}\) G near the black hole is needed. We give an example of a disk-plus-field structure that both delivers the required field and makes the Poynting flux from the hole dominate that of the disk. Thereby we demonstrate that the Poynting energy extracted need not be dominated by the disk, nor is limited to the binding energy of the disk. This means that the Blandford-Znajek mechanism remains a very good candidate for powering gamma-ray bursts.

Key Words: gamma ray bursts, black hole, accretion disk

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1 Introduction

Gamma ray bursts presently provide great excitement in astronomy and astrophysics as optical observations by way of many instruments give considerable detail of the history of each burst. We are concerned here with the prodigious energy in each burst, the estimate for GRB 971214 being \(\gtrsim 3 \times 10^{53} \text{ergs} \) [1], although this could be diminished if considerable beaming is involved in the central engine, as we will discuss.

Amazingly, \(2 \times 10^{54} \text{ergs}\) is just the rest mass energy of our sun, so it seems immediately clear that the central engine for the GRB must be able to extract a substantial fraction of the rest mass energy of a compact object, neutron star or black hole, and convert it into energy of GRB.

The second criterion for the central engine is that it must be able to deliver power over a long interval up to \(\sim 1000\) seconds, since some GRB’s last that long although other GRB’s last only a fraction of a second. It must also be able to account for the vast diversity in pulses, etc., or, alternatively, one must have a number of diverse mechanisms.

We believe the need to deliver power over the long time found in some bursters to be the most difficult requirement to fulfill, since the final merger time of the compact objects is only a fraction of a second and it is difficult to produce a high energy source of, e.g., \(\nu\bar{\nu}\)-collisions that goes on for more than two or three seconds.

For many years mergers of binary neutron stars were considered to be likely sources for the GRB’s. The estimated merger rate in our Galaxy of a few GEM \(^2\) is of the right order for the occurrence of GRB’s. The possible problem with binary mergers might be the ejected materials during the merging processes. Not more than \(\sim 10^{-5}M_\odot\) of baryons can be involved in the GRB, since it would not be possible to accelerate a higher mass of nucleons up to the Lorentz factors \(\Gamma \sim 100\) needed with the energies available.

We find the merger of a neutron star with a black hole to be a particularly attractive mechanism. The baryon number “pollution” problem can be solved by the main part of the baryons going over the event horizon. In the Blandford-Znajek mechanism[2] we wish to invoke, a substantial proportion of the rotational energy of the black hole, which will be sent into rapid rotation by swallowing up the neutron star matter, can be extracted through

\(^2\)1 GEM = 1 Galactic Event per Megayear
the Poynting vector. The rate of extraction is proportional to the square of magnetic field strength, $B^2$, as we shall discuss, so that power can be furnished over varying times, depending upon the value of $B$. With substantial beaming, we estimate that $B \sim 10^{15}$G would be sufficient to power the most energetic GRB’s with $\sim 10^{53}$ergs.

Recently at least three magnetars, neutron stars with fields $B \sim 10^{14} - 10^{15}$G, have been observed. Their visible lifetime is only a few thousand years, because neutron stars slow down by emission of magnetic dipole radiation and join the “graveyard” where they no longer emit pulses. The time of observability is proportional to $B^{-1}$, so the number of these high magnetic field stars may not be an order of magnitude less than the garden variety $10^{12}$G neutron stars. It is also possible that existing magnetic fields can be increased by the dynamo effect.

Failed supernovae were suggested by Woosley(1993)[3] as a source of GRB. In this case the black hole would be formed in the center of a massive star, and surrounding baryonic matter would accrete into it, spinning it up. This mechanism is often discussed under the title of hypernovae[4].

More recently Bethe and Brown(1998)[5] found that in binary neutron star evolutions, an order of magnitude more low-mass black-hole, neutron-star binaries were formed than binary neutron stars. The low-mass black-hole mass of $\sim 2.4M_\odot$ looks favorable for the Blandford-Znajek mechanism.

In some calculations which begin with a neutron star binary, one of the neutron stars evolves into a black hole in the process of accretion, and the resulting binary might also be a good candidate for GRB’s. In any case, there are various possibilities furnished by black-hole, neutron-star binaries.

In this paper we will discuss the Blandford-Znajek mechanism in quantitative detail. In section 2, an overview of the proposed central engine for gamma ray bursts using Blandford-Znajek process is given. The Blandford-Znajek process and the evolution of the black hole will be discussed in detail in section 3 and section 4 respectively. The structure of ambient magnetic field surrounding black hole and accretion disk is discussed in section 5. In section 6, we will give a rough estimation of the possible angular momentum of the black hole which might emerge as a final compact object during the merging or collapsing processes. The possible constraint from the surrounding accretion disk is discussed in section 7 and the results are summarized and discussed in section 8.
2 Overview of the Proposed Central Engine for GRB

Two decades ago Blandford and Znajek[2][6] proposed a process (BZ) in which rotational energy of black hole can be efficiently extracted. If there are sufficient charge distributions around the black hole to provide the force-free condition, then the magnetic field lines exert no force and corotate rigidly with the rotating black hole. The induced current loops loops which pass along the black hole’s stretched horizon feel the forces by the magnetic field supported by the environment. Hence these forces give magnetic braking of the black hole rotation. The maximum amount of energy which can be extracted out of the black hole without violating the second law of thermodynamics is the rotational energy, $E_{rot}$, which is defined as

$$E_{rot} = Mc^2 - M_{irr}c^2$$  \hspace{1cm} (1)

where

$$M_{irr} = \sqrt{\frac{A_Hc^4}{16\pi G^2}}$$  \hspace{1cm} (2)

$$= \sqrt{\frac{S_H}{4\pi k_B} M_{planck}}$$  \hspace{1cm} (3)

where $A_H$ and $S_H$ are surface area and entropy of a black hole respectively\(^3\) and $k_B$ is Boltzman’s constant. The rotational energy of a black hole with angular momentum $J$ is a fraction of the black hole mass $M$,

$$E_{rot} = f(\tilde{a})Mc^2$$  \hspace{1cm} (4)

$$f(\tilde{a}) = 1 - \sqrt{\frac{1}{2}[1 + \sqrt{1 - \tilde{a}^2}]}$$  \hspace{1cm} (5)

where $\tilde{a} = \frac{Jc}{McG}$ is the rotation parameter. For a maximally rotating black hole ($\tilde{a} = 1$), $f = 0.29$. In BZ, the efficiency of extracting the rotational

\(^3\)For a solar mass black hole, $A_H \sim 10^{12} cm^2$, $S_H \sim 10^{77} k_B$. The Planck mass is $M_{planck} = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-5} g$. 4
energy is determined by the ratio between the angular velocities of black hole, $\Omega_H$, and magnetic field angular velocity $\Omega_F$,

$$\epsilon_\Omega = \frac{\Omega_F}{\Omega_H}$$  \hspace{1cm} (7)

The rest of the rotational energy is dissipated into the black hole increasing the entropy or equivalently irreducible mass\(^4\). The total BZ energy available is

$$E_{BZ} = 1.8 \times 10^{54} \epsilon_\Omega f(\tilde{a}) \left( \frac{M}{M_\odot} \right) \text{erg}$$  \hspace{1cm} (8)

For the optimal processes $\epsilon_\Omega \sim 0.5$ [6].

Since the energy transport is in the form of Poynting flow in BZ the outgoing energy flux from the black hole is basically $B^2c$. An order of magnitude calculation is now in order. There are basically three parameters: Mass of the black hole $M$ and magnetic field on the horizon $B$, which are dimensionful, and angular momentum parameter of the black hole $\tilde{a}$. The time scale for the BZ process can be calculated by the ratio of the black hole mass to the output power from the black hole surface $B^2R^2c$

$$\tau_{BZ} \sim \frac{Mc^2}{B^2R^2c} \sim \frac{Mc^5}{B^2M^2G^2} = \frac{c^5}{B^2MG^2} = 2.7 \times 10^3 \left( \frac{10^{15} \text{gauss}}{B} \right)^2 \left( \frac{M_\odot}{M} \right) s$$  \hspace{1cm} (9)

where $M_\odot^{-1}(10^{15} \text{gauss})^{-2} \frac{c^5}{G^2} = 2.7 \times 10^3 s$ and the radius of horizon, $R$, is taken to be $\sim \frac{GM}{c^2}$. Also the outgoing Poynting power is

$$P_{BZ} \sim B^2R^2c = 6.7 \times 10^{50} \left( \frac{B}{10^{15} \text{gauss}} \right)^2 \left( \frac{M}{M_\odot} \right)^2 \text{erg/s}. \hspace{1cm} (10)$$

The fluence of the recently observed GRB971214[1] corresponds to $E_\gamma = 10^{53.5} \left( \frac{\Omega}{4\pi} \right) \text{erg}$ which is consistent with $E_{BZ}$. That of GRB990123 may be as large as $E_\gamma = 3.4 \times 10^{54} \left( \frac{\Omega}{4\pi} \right)$[7]. This suggests that if a strong enough magnetic field ($\sim 10^{15}$) on the black hole can be supported by the surrounding material(accretion disk) the BZ process is a good candidate to provide the

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\(^4\)The mass of a nonrotating black hole itself is its irreducible mass
powerful energy of the GRB in the observed time interval up to 1000s, which is comparable to the BZ time scale $\tau_{BZ}$.

In recent years a black hole plus debris torus system (or accretion disk) has been considered to be a plausible structure for the GRB central engine[8]. The presence of the accretion disk is important for the BZ process because it is the supporting system of the strong magnetic field on the black hole, which would disperse without the pressure from the fields anchored in the accretion disk. Recent numerical calculations[9] show that accretion disks formed by various merging processes are found to have large enough pressure such that they can support $\sim 10^{15}G$ assuming a value of the disk viscosity parameter $\alpha \sim 0.1$, where $\alpha$ is the usual parameter in scaling the viscosity. This gives a relevant order of magnitude magnetic field on the black hole, which is, however, not considered to be much larger than magnetic field of the inner accretion disk [10][11]. The discovery that soft gamma ray bursts are magnetars [12] also supports the presence of the strong magnetic fields of $\sim 10^{15}G$ in nature. The identification by now of three soft gamma repeaters as strong-field pulsars indicates that there may be a large population of such objects: since the pulsar spindown times scale as $B^{-1}$, we would expect to observe only 1 magnetar for every 1000 normal pulsars if they were formed at the same rate, and if selection effects were the same for the two populations. We see 3 magnetars and about 700 normal pulsars, but since they are found in very different ways the selection effects are hard to quantify. It is nonetheless clear that magnetars may be formed in our Galaxy at a rate not very different from that of normal pulsars.

The life time of the accretion disk is also very important for the GRB time scale because it supports the magnetic field on the black hole. According to numerical simulations of merging systems which evolve eventually into black hole - accretion disk configuration[9] the viscous life times are 0.1 - 150 s, which are not inconsistent with the GRB time scale. Also it has been pointed out[8] that a residual cold disk of $\sim 10^{-3}M_\odot$ can support $10^{15}G$, even after the major part of the accretion disk has been drained into the black hole or dispersed away.

Recent hydrodynamic simulations of merging neutron stars and black holes[13] show that along the rotation axis of the black hole an almost baryon-free funnel is possible. This can be easily understood since the material above the hole axis has not much angular momentum so that it can be drained quickly, leaving a baryon-free funnel. Hence relativistically expanding jets
along the funnel, fueled by Poynting outflow which is collimated along the rotation axis[14], can give rise to gamma ray bursts effectively. It has been observed that the BZ process is also possible from the disk since the magnetic field on the disk is not much less than that on the black hole[10]. However the energy outflow from the disk is mostly directed vertically from the disk where the baryon loading is supposed to be relatively high enough to keep the baryons from being highly relativistic. Therefore the BZ process from the disk can be considered to have not much to do with gamma ray burst phenomena. However the BZ from the disk could power an outflow with lower $\Gamma$, but nonetheless high energy, which could cause an afterglow at large angles. That would lead to more afterglows being visible than GRBs, because the afterglows are less beamed. The BZ mechanism can also play a very important role in the disk accretion because it carries out angular momentum from the disk. We will discuss this in more detail later.

The structure we are proposing as a central engine of the GRB is a system of black hole - accretion disk(debris of the torus):

The rotating black hole is threaded by a strong magnetic field. Along the baryon-free funnel relativistic jets fueled by Poynting outflow give rise to the GRB. The interaction between disk and black hole is characterized by accretion and magnetic coupling. We consider the BZ process only after the main accretion process is completed, leaving an accretion disk of cold residual material, which can support a strong enough magnetic field.

\section{Blandford-Znajek Process}

Consider a half hemisphere(radius $R$) rotating with angular velocity $\Omega$ and a circle on the surface at fixed $\theta$ (in the spherical polar coordinate system) across which a surface current $I$ flows down from the pole. When the external magnetic field $B$ is imposed to thread the surface outward normally, the surface current feels a force and the torque due to the Lorentz force exerted by the annular ring of width $Rd\theta$ is

$$\Delta T = -R \sin \theta \times IR \Delta B \theta$$

(11)

$$= -\frac{I}{2\pi} B \Delta A_{ann.}$$

(12)

$$= -\frac{I}{2\pi} \Delta \Psi$$

(13)
where $\Delta \Psi$ is the magnetic flux through annular ring extended by $d\theta$ with surface area $A_{\text{ann}}$. We consider an axially symmetric situation. From this magnetic braking, we can calculate the rotational energy loss rate

$$\Delta P_{\text{rot}} = \Omega \times \Delta T$$

$$= \Omega \frac{I}{2\pi} \Delta \Psi. \quad (15)$$

Blandford and Znajek\[2\] demonstrated that such a magnetic braking is possible, provided that the external charge distribution can support an electric current with the magnetic field threading the horizon. The original formulation of Blandford and Znajek[2] is summerized in Appendix 3. Macdonald and Thorne[15] reformulated the Blandford-Znajek process using a 3+1 dimensional formalism, in which the complicated physics beyond the horizon can be expressed in terms of physical quantities defined on the stretched horizon[6]. It can be shown that the rotational energy loss due to magnetic braking can be obtained from eq.(15) by simply replacing $\Omega$ and $B$

$$\Omega \rightarrow \Omega_H, \quad B \rightarrow B_H \quad (16)$$

where $H$ denotes the quantity on the stretched horizon [6]:

$$\Delta P_{\text{rot}} = \Omega_H \times \Delta T$$

$$= \Omega_H \frac{I}{2\pi} \Delta \Psi. \quad (18)$$

The current $I$ induced by the black hole rotation and the angular velocity $\Omega_F(\theta)$ of the rigidly rotating magnetic field which is dragged by the rotating black hole can be determined together with the magnetic field by solving Maxwell’s equations:

$$F^{\mu\nu} = 4\pi J^\mu \quad (19)$$

with the force free-condition

$$F^{\mu\nu}J_\nu = 0, \quad (20)$$

where $J^\mu$ is a current density vector. The detailed structure of the magnetic field will be discussed in section 5. In the BZ process, however, the power which can be carried out as Poynting outflow along the field lines is

$$\Delta P_{\text{mag}} = \Omega_F \frac{I}{2\pi} \Delta \Psi \quad (21)$$
Then the rest of the rotational energy is used to increase the entropy (or equivalently irreducible mass) of the black hole. Therefore the efficiency of the BZ process which is defined as the ratio of $P_{\text{mag}}$ to $P_{\text{rot}}$ is

$$
\epsilon_\Omega = \frac{P_{\text{mag}}}{P_{\text{rot}}} = \frac{\Omega_F}{\Omega_H}
$$

The ideal efficiency, $\epsilon_\Omega = 1$, is meaningless because for $\Omega_F = \Omega_H$ the Poynting outflow itself is zero as can be seen below in eq.(26). The optimal power can be obtained at $\epsilon_\Omega = 1/2[15]$. Then the rest of the rotational energy is used to increase the entropy (or equivalently irreducible mass) of the black hole. Now consider the loading region far from the black hole, onto which magnetic fields out of the black hole anchor. In most of the cases we are interested in, the inertia of the loading region can be considered to be so large that the transported angular momentum cannot give rise to any substantial increase of the angular velocity of the loading region. Therefore the angular velocity of the loading region can be assumed to be zero and the power delivered by the torque, eq.(13), along the field line is given

$$
\Delta P_L = \Omega_F \Delta T = \Omega_F \frac{I}{2\pi} \Delta \Psi
$$

which is identified as the BZ power for GRB:

$$
\Delta P_{\text{BZ}} = \Delta P_{\text{mag}} = \Delta P_L
$$

Since the current $I$ induced by black hole rotation is given by[6]

$$
I(\theta) = \frac{1}{2c}(\Omega_H - \Omega_F(\theta)) \tilde{\omega}^2 B_H
$$

the rotational energy loss and the BZ power can be given by

$$
\Delta P_{\text{rot}} = \frac{\Omega_H(\Omega_H - \Omega_F)}{4\pi} \tilde{\omega}^2 B_H \Delta \Psi
$$

$$
\Delta P_{\text{BZ}} = \frac{\Omega_F(\Omega_H - \Omega_F)}{4\pi c} \tilde{\omega}^2 B_H \Delta \Psi,
$$

where $\tilde{\omega}$ is a kind of cylindrical radius defined in Boyer-Lindquist coordinates (Appendix 1).
To get the total power, eq.(28) should be integrated from $\theta \sim 0$ to $\theta_{BZ}$, up to which the magnetic field lines from the black hole anchored on to the loading region for GRB. As a first approximation, we put

$$\theta_{BZ} \sim \pi/2$$  \hspace{1cm} (29)
$$B_H(\theta) \sim <B_H>$$  \hspace{1cm} (30)

and we get an optimal power

$$P_{BZ} = 1.7 \times 10^{50} \tilde{a}^2 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{<B_H>}{10^{15} \text{ gauss}}\right)^2 f(h) \text{ erg/s.}$$  \hspace{1cm} (31)

The details are given in Appendix 2. The rest of the rotational energy given by

$$P_H = P_{rot} - P_{BZ}$$  \hspace{1cm} (32)
$$= \frac{\Omega_H - \Omega_F}{\Omega_F} P_{BZ}$$  \hspace{1cm} (33)
$$= P_{BZ}, \text{ for optimal case}$$  \hspace{1cm} (34)

is dissipated into the black hole increasing the irreducible part of the black hole mass (equivalently increasing the entropy of the black hole).

For $\theta_{BZ} < \theta \leq \pi/2$, the magnetic field lines from the black hole can be anchored onto elsewhere than the loading region, for example, onto the inner accretion disk. Then the angular velocity of the field line is determined by the angular velocity of the disk $\Omega_D$, $\Omega_F = \Omega_D$. The innermost radius of the disk can be considered to be the marginally stable orbit, $r_{ms}$ [16]. For an orbit rotating in the same direction as the black hole rotation, we have:

$$r_{ms} = \frac{G}{c^2} M \left[3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\right],$$  \hspace{1cm} (35)
$$Z_1 = 1 + (1 - \tilde{a}^2)^{1/3}\left[(1 + \tilde{a})^{1/3} + (1 - \tilde{a})^{1/3}\right],$$  \hspace{1cm} (36)
$$Z_2 = \sqrt{3\tilde{a}^2 + Z_1^2}.$$  \hspace{1cm} (37)

Here $r_{ms}$ becomes $GM/c^2$ as $\tilde{a} \rightarrow 1$ (extreme rotation). In this limit, the angular velocity of disk $\Omega_D \sim \Omega_H$. Then one can expect $\Omega_F \sim \Omega_H$ so that there is no Poynting flow to the disk. However in this case there is no BZ Poynting outflow. For finite $\tilde{a} < 1$, we can have either $\Omega_H > \Omega_D$ or
\( \Omega_H < \Omega_D \), which can be determined by solving eq.(65). However if we can assume that the power from/to the disk can be much suppressed compared to that of the loading region, we can ignore the magnetic coupling between the black hole and the accretion disk. For example, suppose the portion of black hole - threading magnetic fields anchored onto the disk is somehow suppressed, \( \theta_{BZ} \sim \pi/2 \), then we can assume there is no significant energy and angular momentum feedback into the disk due to magnetic coupling. If it is not so we can discuss only the limiting case.

4 Evolution of a black hole via the Blandford-Znajek process

While the black hole is slowed down and a part of the rotational energy is carried out as Poynting outflow, the rest of the rotational energy increases the entropy of the black hole or its irreducible mass. The increasing rate of the irreducible mass is given by using eq.(27) and (28)

\[
\frac{dM_{irr}}{dt} = P_{rot} - P_L
\]

\[
= \int \frac{\left( \Omega_H - \Omega_F \right)^2}{4\pi c} \hat{\omega}^2 B_H \Delta \Psi.
\]

The irreducible mass eventually becomes the mass of Schwarzshild black hole when it stops rotating. The difference between the initial Kerr black hole mass, \( M_o \), and the final Schwarzshild mass, \( M \), is the energy output from black hole via the Blandford-Znajek process. The evolution of a Kerr black hole is determined by the evolution of its mass and angular momentum given by

\[
\frac{dM_k^2}{dt} = -P_L
\]

\[
\frac{dJ}{dt} = T
\]

Using eq.(23) and (41) in the Blandford-Znajek process, the evolution of the mass and the angular momentum are related by

\[
\frac{dM}{dt} = \Omega_F \frac{dJ}{dt}
\]
For the optimal case, \( \epsilon_\Omega = 1/2(\Omega_F = \Omega_H/2) \), we can obtain an analytic expression for the mass in terms of the angular momentum. With the angular velocity of a black hole expressed in the angular momentum given by in the natural units \( G = c = 1 \)

\[
\Omega_H = \frac{J}{2M^3(1 + \sqrt{1 - J^2/M^4})}
\]  

(43)

we get

\[
\frac{dM}{dt} = \frac{J}{4M^3(1 + \sqrt{1 - J^2/M^4})} \frac{dJ}{dt}
\]  

(44)

\[
2\frac{dM^4}{dt} = \frac{1}{1 + \sqrt{1 - J^2/M^4}} \frac{dJ^2}{dt}.
\]  

(45)

Introducing a new variable \( H \) defined as

\[
H = \frac{r_H}{M} = 1 + \sqrt{1 - \frac{J^2}{M^4}},
\]  

(46)

eq.(45) can be written

\[
2\frac{dM^4}{dt} = \frac{1}{H} \frac{dJ^2}{dt}.
\]  

(47)

Using the identities

\[
\frac{d}{dt} \left( \frac{J^2}{M^2} \right) = 2(1 - H) \frac{dH}{dt}
\]  

(48)

\[
\frac{dJ^2}{dt} = \frac{J^2}{M^4} \frac{dM^4}{dt} + M^4 2(1 - H) \frac{dH}{dt}
\]  

(49)

\[
\frac{J^2}{M^4} = 2H - H^2
\]  

(50)

eq.(47) can be written

\[
\frac{1}{M} \frac{dM}{dt} = \frac{(1 - H) dH}{2H^2} dt.
\]  

(51)

which can be integrated analytically to give[17]

\[
M = M_o e^{\frac{u - u_o}{c}} \sqrt{\frac{H_o}{H}}.
\]  

(52)
where $H_o$ and $M_o$ represent the initial angular momentum and mass of the black hole. From eq.(46) one can see that $H = 1$ for the maximally rotating ($\tilde{a} = 1$) black hole and $H = 2$ for the nonrotating one. Consider the black hole initially maximally rotating, which is slowed down by the Blandford-Znajek mechanism in the optimal mode ($\Omega_F = \Omega_H/2$)\(^5\). The final black hole mass is given by

$$M = \frac{e^{1/4}}{\sqrt{2}} M_o$$ \hspace{1cm} (53)

$$= 0.91 M_o.$$ \hspace{1cm} (54)

We see that 9\% of the initial mass or 31\% of the rotational energy can be taken out to power the gamma ray burst from the maximally rotating black hole. The extracted energy is therefore less than a half of the initial rotational energy. This can be easily understood by noticing that the fraction of the rotational energy drops faster because of the decreasing total mass and at the same time increasing irreducible mass in eq.(1). For $\tilde{a} = 0.5$, $M = 0.98 M_o$ or 2\% of the initial mass can be used to power the gamma ray burst. The time dependence of the power can be obtained from eq.(247) using eq.(50):

$$P = -\frac{dM c^2}{dt} = \frac{f(h)}{4} (2H - H^2) M^2 B_H^2 G^2 / c^5,$$ \hspace{1cm} (55)

where $f(h)$ is defined in Appendix 2 and $H = \frac{2}{1 + H^2}$. The initial rate for the maximally rotating black hole can be obtained by taking $H = 1$ and $f = \pi - 2$. Eq.(55) can be written as

$$\frac{1}{M^2} \frac{dM}{dt} = -\frac{\pi - 2}{4} B_H^2 G^2 / c^5.$$ \hspace{1cm} (56)

Assuming there is no change in magnetic field which is supported by the environment, we get

$$M = \frac{M_o}{1 + (\pi - 2) M_o B_H^2 G^2 t / 4 c^5}.$$ \hspace{1cm} (57)

If we extrapolate eq.(57) until the black hole stops rotating, $M \to 0.91 M_o$, we get the time scale of Blandford-Znajek process as

$$\tau = \frac{0.35 c^5}{M_o B_H^2 G^2}$$ \hspace{1cm} (58)

\(^5\)The more general discussion has been given by I. Okamoto[17] where $\zeta = 1/\epsilon_{\Omega} - 1$.\]
\[ \sim 10^3 \left( \frac{10^{15} \text{ gauss}}{B_H} \right)^2 \left( \frac{M_\odot}{M} \right) \text{s}. \]  

(59)

Since there is no considerable change of the weighting factor \( f(h) \) in eq.(55) from the maximally rotating black hole to the nonrotating black hole (\( f = 2/3 \)), Eq.(59) can be considered as a reasonable estimate of a time scale, which is consistent with the rough estimate given in eq.(9).

5 Magnetic field and Force-free plasma

The rotating black hole immersed in the magnetic field induces an electric field around the black hole[6]. The electromagnetic field in the vicinity of rotating black hole immersed in the uniform magnetic field in free space has been obtained [6][18][19] as a solution of the source-free Maxwell equation

\[ F_{\mu\nu} = 0. \]  

(60)

From the analytic expression(Appendix 2) which gives an asymptotically uniform magnetic field at infinity (\( r \to \infty \))

\[ \vec{B} = B \hat{z}, \ r \to \infty, \]  

(61)

one can see that the radial component of the magnetic field on the horizon, \( B^H_r \),

\[ B^H_r = \frac{B \cos \theta}{(r^2_H + (a/c)^2 \cos^2 \theta)^2} \left[ (r^2_H - (a/c)^2)(r^2_H - (a/c)^2 \cos^2 \theta) \right. \]

\[ + 2(a/c)^2 r_H (r_H - M)(1 + \cos^2 \theta) \]  

(62)

vanishes as the rotation of black hole approaches extreme rotation

\[ a \to GM/c, \ r_H \to GM/c^2. \]  

(64)

This is what has been observed [19][20] as the absence of the magnetic flux across the maximally rotating black hole. One can also see that there is no outward Pointing flow, which means no outflow of energy to infinity. Essentially it is because of the absence of the toroidal component of the
magnetic field due to the vacuum environment which is charge and current free space. In other words, there is no current on the stretched horizon on which the external magnetic field exerts torques to slow the black hole down so as to extract its rotational energy. Therefore it is necessary to have a magnetosphere with charges and currents to extract the rotational energy of the black hole. The force-free magnetosphere around a rotating black hole has been proposed [2]:

$$F_{\mu\nu} = 4\pi J^\mu$$ \hspace{1cm} (65)

where $J^\mu$ is a current density vector and

$$F_{\mu\nu} J^\nu = 0$$ \hspace{1cm} (66)

which is the force-free condition. From eq.(66), we get the degenerate condition

$$\vec{E} \cdot \vec{B} = 0$$ \hspace{1cm} (67)

In the case of a rotating black hole in charge-free space one can see using eqs. in Appendix 2

$$\vec{E}^H \cdot \vec{B}^H \neq 0$$ \hspace{1cm} (68)

It should be remarked that since the force-free condition is essential for the BZ process the expulsion of magnetic field on the rotating black hole demonstrated for the charge-free space cannot be directly addressed in the BZ process[20].

To maintain the current flows charged particles (electrons and positrons) should be supplied by a source outside the horizon since the black hole itself cannot provide the outgoing particle from inside the horizon. Blanford and Znajek[2] proposed that the strong electric field induced by the rotating black hole can give rise to a spark gap in which sufficient charged particles are created to supply the currents[21]. Another mechanism provides charged particles around black hole: electron-positron pair creation by neutrino annihilation[22]. Recent numerical studies of merging binary systems [9][23] which result in black hole-accretion disk configurations show that the power of electron-positron pairs by the annihilation of neutrinos and antineutrinos which are radiated out of the disk is

$$\dot{E}_{\nu\bar{\nu}} \sim 10^{50} \text{erg/s}.$$ \hspace{1cm} (69)
This power is being poured into the space above the black hole for $0.01 - 1s$. If we divide it by the average neutrino energy $\langle \epsilon_{\nu} \rangle \sim 10MeV$ [23], we get a rough estimate of the numbers of $e^+e^-$ pairs

$$\dot{N}_{pair} \sim 10^{56}/s \quad (70)$$

or the charge producing rate for $e^+$ (or $e^-$; the sum is always zero) is

$$\dot{Q}_e = e\dot{N}_{pair} \sim 10^{37}C/s \quad (71)$$

The magnitude of the currents involved in the optimal BZ process can be obtained from eq.(250)

$$I \sim \sqrt{\frac{10^{30}erg/s}{R_H}} \sim 10^{20}C/s \quad (72)$$

for a black hole with solar mass threaded by a $10^{15}G$ magnetic field. $R_H$ is the surface resistance of the horizon, 377 Ohm. From the comparison of eq.(71) with (72), the neutrino annihilation process produces orders of magnitude more than enough pairs to keep the necessary currents for the optimal BZ process. The possible effects of neutrino annihilation during the active neutrino cooling of the accretion disk is that the magnetosphere for the BZ process might be disturbed so much that the BZ process is suspended until the burst of $e^+e^-$ pairs clears out. However, since the pairs are produced with strong directionality along the rotation axis, most of the produced pairs expand along the axis in less than a second[23]. This is reasonable because the electric field along the magnetic field lines is almost negligible and therefore it will take too long for the $e^+$ to reverse its velocity to make the same current as the $e^-$. Also the strong magnetic field keeps them from moving perpendicular to the magnetic field lines so that the separations between them cannot be effective. Therefore the pair contribution to the net current flow can be negligible and it might not disturb the magnetosphere for the BZ process so violently. However, the effects of the neutrino cooling process of the accretion disk can be considered as an additional disturbing burst by the $\nu\bar{\nu}$ driven $e^+e^-$ burst, which lasts less than few seconds of the BZ burst.

\footnote{Although very hard to estimate, a very small fraction of pairs which has very small momentum component along the axis can contribute to BZ currents along the magnetic field lines, which might be sufficient for the force-free configuration.}
The structure of a force-free magnetosphere can be described by a stream function \( \psi(r, \theta) \) and two functions of \( \psi \); \( \Omega_F(\psi) \) and \( B_\phi(\psi) \) (or equivalently \( I = -\frac{1}{2}\alpha \dot{\omega} B_\phi \) \[2\][24]. Hence \( \psi \) at a point of \((r, \theta)\) is equal to the total magnetic flux upward through the azimuthal loop at \((r, \theta)\). On the other hand from the equation of motion, the stream function is proportional to the toroidal component of the vector potential, \( \psi = 2\pi A_\phi \). On the horizon, it determines the total magnetic flux, \( \Psi(\theta) \), through the horizon up to \( \theta \), \( \Psi(\theta) = \psi(r_H, \theta) \). The poloidal and toroidal components of the electromagnetic field are

\[
\vec{E}_P = -\frac{\Omega_F - \omega}{2\pi \alpha} \vec{\nabla} \psi, \quad \vec{E}_T = 0
\]

\[
\vec{B}_P = \frac{\vec{\nabla} \psi \times \hat{\phi}}{2\pi \dot{\omega}}, \quad \vec{B}_T = -\frac{2I}{\alpha \dot{\omega}} \hat{\phi}
\]

where \( I(r, \theta) \) is the total current downward through the loop at \((r, \theta)\). \( \Omega_F(\psi) \) is the angular velocity of the magnetic field relative to absolute space. Blandford and Znajek\[2\] derived the differential equation (stream equation) for the stream function, which takes the form\[24\]

\[
\vec{\nabla} \left\{ \frac{\alpha}{\omega^2} \left[ 1 - \frac{(\Omega_F^2 - \omega^2)\omega^2}{\alpha^2} \right] \vec{\nabla} \psi \right\} + \frac{(\Omega_F - \omega)}{\alpha} \frac{d\Omega_F}{d\psi} (\vec{\nabla} \psi)^2 + \frac{16\pi^2}{\alpha \omega^2} I \frac{dI}{d\psi} = 0. \quad (75)
\]

It includes two functions of \( \psi \), which are not known \textit{a priori}. In solving eq.(75) the boundaries of the force-free region and the boundary conditions on \( \psi, \Omega_F, \) and \( I \) should be specified. There is no known analytical solution of eq.(75). Blandford and Znajek obtained perturbative solutions for small \( a/M \) using the analytic solutions in charge free space around the non-rotating Schwarzschild black hole. With \( \Omega_F = 0, \omega = 0 \) and \( I = 0 \), the stream equation reduces to

\[
\vec{\nabla} \left\{ \frac{\alpha}{\omega^2} \vec{\nabla} \psi \right\} = 0. \quad (76)
\]

MacDonald\[24\] developed a numerical method to obtain solutions with finite \( a/M \), in which the solutions of eq.(76) with appropriate boundary conditions are spun up numerically. It has been found that the poloidal field structure does not change greatly as the holes and fields are spun up. This result implies that the flux of the poloidal magnetic field threading black hole does
not decrease greatly as $a \to M$ in contrast to the rotating black hole in free space.

One of the interesting solutions is the poloidal field structure generated by the paraboloidal magnetic field solution of eq.(76):

$$\psi = \frac{\psi_0}{4 \ln 2} \left\{ (r - 2)(1 - \cos \theta) + 2[2 \ln 2 - (1 + \cos \theta) \ln(1 + \cos \theta)] \right\}$$  (77)

where $\psi_0$ is the total flux threading the hole. Since this solution extends not only onto the horizon but also onto the equatorial plane where the accretion disk is supposed to be placed, the boundary conditions on the disk should be satisfied by the spun up solutions generated by eq.(77). The boundary conditions depend strongly on the accretion disk model, which will be discussed in the next section. The presence of the accretion disk may be considered to be the main source of the difficulty in obtaining a solution with these complicated boundary conditions. However, the solutions with proper boundary conditions provide us a way as to how we can infer the magnetic field from those developed in the disk. It has been demonstrated[10][11] that the strength of the magnetic field on the black hole is not much stronger than those on the inner disk.

6 Rotation of the Black Hole

It has been suggested that merging compact binary systems[8] or hypernovae [4] may result in a disk with rapidly rotating black hole at the center. In this section we will demonstrate how this is possible using semiquantitative arguments. The basic idea is that a substantial part of the orbital angular momentum of the binary system or the spin angular momentum of the progenitor of hypernova can be imparted onto the black hole.

Consider first the BH-NS merger. The typical distance for the merging system in this case is the tidal radius, $R_t$, at which the neutron star(radius $R_{NS}$) fills the Roche lobe[25]:

$$R_t = \frac{R_{NS}}{0.46} \left( \frac{1 + q}{q} \right)^{1/3},$$  (78)

where

$$q = \frac{M_{NS}}{M_{BH}}$$  (79)
If the tidal radius is greater than the last stable orbit radius, we can calculate the Keplerian orbital angular velocity

\[ \Omega_K = \sqrt{\frac{GM}{R_t^3}}, \]  

(80)

where

\[ M = M_{BH} + M_{NS}. \]  

(81)

Then the orbital angular momentum of the binary system can be written

\[ J_{binary} = \mu R_t^2 \Omega_K \]  

(82)

\[ = M_{BH} M_{NS} \sqrt{\frac{GM}{M}}, \]  

(83)

where \( \mu \) is the reduced mass, \( \mu = \frac{M_{BH} M_{NS}}{M} \). Using eq.(78),

\[ J_{binary} = 1.47 M_{BH} M_{NS} \sqrt{\frac{GM_{NS}}{M}} \left( \frac{M}{M_{NS}} \right)^{1/6} \]  

(84)

During the collapse, a part of angular momentum is carried off by gravitational waves (or possibly in the later stage by neutrino cooling) and a part of the total mass explodes away or remains in the torus around the black hole. Assuming that a fraction(\( xy \)) of the orbital angular momentum goes into the black hole, which keeps a fraction(\( y \)) of the total mass \( M_8 \), we can calculate the angular momentum parameter of the black hole, \( \tilde{a} \):

\[ J_{BH} = \tilde{a} (yM)^2 \frac{G}{c} = xyJ_{binary} \]  

(85)

\[ \tilde{a} = 1.47 \frac{x M_{BH} M_{NS}}{y M^2} \sqrt{\frac{R_{NS} c^2}{GM}} \left( \frac{M}{M_{NS}} \right)^{1/6} \]  

(86)

For \( M_{BH} = 2.5M_\odot, M_{NS} = 1.5M_\odot, R_t = 10^6 cm \),

\[ \tilde{a} = 0.53 \frac{x}{y} \]  

(87)

\(^7\)This leads to the condition for the black hole mass[26]: \( M_{BH} < M_{NS} \left( 0.4 \frac{R_{NS} c^2}{GM_{NS}} \right)^{3/2} - 1 \) \( \sim 2.3M_\odot \).

\(^8\)\( x \) is a fraction of the specific angular momentum.
which is a quite reasonable value for an efficient Blandford-Znajek process.

For the NS-NS merger, the radius of physical contact ($2R_{NS}$) is smaller than the tidal radius,

$$R_t = 2.46R_{NS}^{2^{1/3}} = 3R_{NS}. \quad (88)$$

Hence it is reasonable to consider the orbital angular momentum at the tidal radius. Following the same procedure as in the BH-NS merger, we get

$$\tilde{\alpha} = 0.31 \frac{x}{y} \sqrt{\frac{R_{NS} c^2}{G M_{NS}}}$$

$$= 0.67 \frac{x}{y} \quad (89)$$

where we replaced $M_{BH}$ by $M_{NS}$ in eq.(86).

In the hypernova model[4][27], a massive rapidly spinning progenitor ($M_o \sim 40M_\odot$) is supposed to collapse into a rapidly rotating black hole. If we assume a critically rotating progenitor ($\Omega = \Omega_K$ at the surface, $R_o$) and solid body rotation throughout, the angular momentum of the core which eventually collapses into the black hole is

$$J_{core} = 2 \frac{2}{5} M_{core} R_{core}^2 \Omega_K(R_o) \quad (90)$$

where the moment of inertia of the core is assumed be that of a uniformly distributed spherical object$^9$, $I = \frac{2}{5} M_{core} R_{core}^2$. Using

$$\Omega_K = \sqrt{\frac{G M_o}{R_o^3}} \quad (91)$$

we get

$$J_{core} = 2 \frac{2}{5} M_{core} R_{core}^2 \sqrt{\frac{G M_o}{R_o^3}} \quad (92)$$

As in the previous merger case, a part of the core angular momentum($x$) and the core mass($y$) collapses into the black hole ($M_{BH} = y M_{core}$). Then we get

$$J_{BH} = xy J_{core} \quad (93)$$

$^9$Since in general $I = k^2 M_{core} R_{core}^2$, and $k^2 \ll 1$ for radiative stars[28], this is an upper limit to the true angular momentum.
\[
\tilde{a} = \frac{2}{5} \frac{y}{2-M_{\text{core}}R_{\text{core}}^2 \sqrt{\frac{GM_o}{R_o^3}}} \tag{94}
\]
\[
= \tilde{a} (yM_{\text{core}})^2 \frac{G}{c}. \tag{95}
\]

Then the black hole angular momentum parameter is given by
\[
\tilde{a} = \frac{2x}{5y} \frac{R_{\text{core}}^2 c}{GM_{\text{core}}} \sqrt{\frac{GM_o}{R_o^3}} \tag{96}
\]
\[
= \frac{2x}{5y} \left( \frac{R_{\text{core}}}{R_o} \right)^{3/2} \sqrt{\frac{R_{\text{core}}^2 c^2}{GM_{\text{core}}} \sqrt{\frac{M_o}{M_{\text{core}}}}} \tag{97}
\]

With
\[
R_o \sim 10^5 \text{km}, \quad R_{\text{core}} \sim 10^3 \text{km}
\]
\[
M_o \sim 15M_\odot, \quad M_{\text{core}} \sim 2M_\odot (\sim 3 \text{ km}) \tag{98}
\]
we get
\[
\tilde{a} \sim 0.1 \frac{x}{y} \tag{99}
\]

This depends strongly on the numbers taken in eq.(98) and on how the core angular momentum is determined. The specific angular momentum of the core can be calculated using eq.(92) and eq.(98) as
\[
a_{\text{core}} = \frac{J_{\text{core}}}{M_{\text{core}}} = \frac{2}{5} \frac{R_{\text{core}}^2}{y} \sqrt{\frac{GM_o}{R_o^3}} \tag{100}
\]
\[
= 1.8 \times 10^{14} \text{cm}^2 \text{s}^{-1} \tag{101}
\]
which is much smaller than that from the numerical simulation of collapsars[9]:
\[
a_{\text{core}} \sim 10^{16} \text{cm}^2 \text{s}^{-1}. \quad \text{Of course, if the core angular momentum is not completely redistributed during the precollapse evolution the core is likely to be spinning faster than } \Omega_K; \quad \text{The maximum possible value comes from replacing } \Omega_K \text{ determined at the progenitor radius in eq.(92) by the one determined at the core radius}
\]
\[
\Omega_{K_{\text{core}}} = \sqrt{\frac{GM_{\text{core}}}{R_{\text{core}}^3}} \tag{102}
\]
\[
= (1.3 \times 10^3) \Omega_K, \tag{103}
\]
and we get

\[ a_{\text{core}} = 2.34 \times 10^{16} \text{cm}^2 \text{s}^{-1}. \]  

(104)

Then the angular momentum parameter can be given by

\[ \tilde{a} = \frac{x}{y} \frac{a_{\text{core}}}{a_{\text{max}}} = 2.3 \frac{x}{y}. \]

(105)

In summary, it is very plausible to have a rapidly rotating black hole (\(\tilde{a} > 0.1\)) as a resulting object in the center in the merging systems and also in hypernovae of large angular momentum progenitors, but a precise value of \(\tilde{a}\) will be difficult to calculate.

7 Magnetized accretion disks

A black hole by itself cannot keep magnetic fields on it for a long time. Magnetic fields diffuse away in a short time \(\sim R_H/c[6][10]\). The most plausible environments which can support a magnetic field threading the black hole are accretion disks surrounding the black hole. There are two issues about accretion disks that we need to consider in order to decide whether the Blandford-Znajek process is a viable power source for gamma ray bursts: The life time should be long enough to extract the bulk of the black hole spin energy and also the magnetic field on the disk should be strong enough to power the gamma ray bursts from the spinning black hole. A strong magnetic field on the inner part of the accretion disk is necessary to keep the magnetic pressure comparable to that of magnetic fields on the black hole\([6]\). It has been also demonstrated\([11]\) from the axisymmetric solutions discussed in section 4. Since the magnetic field on the disk affects the angular momentum transfer of the accretion disk via magnetic braking\([29]\) and/or magnetic viscosity, the accretion process is not independent of the magnetic field on the disk. The magnetic field on the disk should not be larger than the value from the equipartition argument:

\[ \frac{B_{\text{eqp}}^2}{8\pi} \sim P_{\text{disk}}; \]

(106)

and \(B_{\text{eqp}}\) can be considered as an upper limit to the magnetic field which can be supported by the accretion disk. Recent numerical calculations on the
hyper-accreting black hole by Popham et al.[9] show

\begin{align}
P_{\text{disk}} & \sim 10^{30}\text{erg/cm}^3 \\
B_{\text{eqp}} & \sim 5 \times 10^{15}\text{gauss},
\end{align}

which implies the accretion disk of the hyper-accreting black hole may support a magnetic field strong enough for the gamma ray burst. However it depends on the detailed mechanism how strong a magnetic field can be built up on the disk. One of the approaches is that the magnetic field is evolved magnetohydrodynamically during the accreting process, which depends on the magnetic viscosity.

The magnetic viscosity, $\nu^{\text{mag}}$, is defined[30] as

\begin{equation}
\frac{B_r B_\phi}{4\pi} = -\nu^{\text{mag}} (r \frac{d\Omega_{\text{disk}}}{dr}) \rho,
\end{equation}

which can be parametrized by the viscosity parameter, $\alpha$, in terms of dimensionful quantities of the disk as[32] [33],

\begin{equation}
\nu^{\text{mag}} = \alpha^{\text{mag}} c_s H
\end{equation}

where $c_s$ is the sound velocity of the disk($c_s = \sqrt{\gamma P_{\text{disk}}/\rho}$) and $H$ is a half of the disk thickness. Using hydrostatic equilibrium perpendicular to the disk plane,

\begin{align}
H &= \sqrt{\frac{P_{\text{disk}}}{\rho}/\Omega_{\text{disk}}} \\
&= \frac{c_s}{\Omega_{\text{disk}} \sqrt{\gamma}}
\end{align}

we get

\begin{equation}
\nu^{\text{mag}} = \alpha^{\text{mag}} \frac{c_s^2}{\Omega_{\text{disk}} \sqrt{\gamma}}.
\end{equation}

For a Keplerian orbit

\begin{equation}
r \frac{d\Omega_{\text{disk}}}{dr} = -\frac{3}{2} \Omega_{\text{disk}},
\end{equation}

23
eq.(109) can be written as

\[
\frac{B_{r}^{dyn} B_{\phi}^{dyn}}{4\pi} = \frac{3}{2} \alpha^{mag} P_{disk} \sqrt{\gamma}
\]  

(115)

We can see that the Maxwell stress of the magnetic field which has been built up by the accretion dynamo is also proportional to the disk pressure but for a different reason from that of the equipartition argument, eq.(106). Numerical estimations of the viscosity parameter, \(\alpha^{mag}\), obtained for various boundary conditions range from 0.001 to 0.005\[30\]. This means that the magnetic pressure is only a small fraction of the disk pressure\[10\]. Using the estimation of disk pressure by Popham et al., we can estimate the dynamically generated magnetic field by the accretion:

\[
B^{dyn} \sim 10^{13} \text{ gauss}
\]

(116)

which may not be strong enough for a gamma ray bursts powered by the Blandford-Znajek process.

However, an accretion dynamo might not be the only process responsible for the magnetic fields on the disk. We know from the recent observations that there are a number of pulsars, magnetars, which are believed to have a strong magnetic field of \(\sim 10^{15}\) G. Although the origin\[31\] of such strong magnetic fields is not well known at the moment, one may consider the case that a debris torus or disk around the black hole which was formed by the disruption of a neutron star retain the high ordered field of that neutron star.

Now consider the axisymmetric solution around a disk with a force-free magnetosphere which has been discussed by Blandford\[29\]. Here we adopt cylindrical coordinates, \((r, \phi, z)\), where the \(z\)−direction is perpendicular to the disk. The sum of current flows into the disk up to radius \(r\) defines the surface current density, \(J_{r}\), which is proportional to the poloidal component of the magnetic field on the disk, \(B_{\phi}\):

\[
\frac{4\pi J_{r}}{2} = -B_{\phi}
\]

(117)

where 2 in the denominator comes from the fact that the radial current density \(J_{r}\) includes the currents into the disk both from above and below. The torque exerted by the annular ring with width \(\Delta r\) of the disk due to the
Lorentz force is given by

$$\Delta T = -r \, 2\pi r J_r B_z \Delta r$$

(118)

$$= 2r \frac{B_\phi B_z}{4\pi} \Delta S, \Delta S = 2\pi r \Delta r$$

(119)

For the steady state accretion disk with surface density, $\Sigma$, the angular momentum conservation can be written by

$$\Sigma v_r \frac{\partial (r^2 \Omega)}{\partial r} 2\pi r \Delta r = \Delta T + \frac{\partial G}{\partial r} \Delta r$$

(120)

$$\dot{M} \frac{\partial (r^2 \Omega)}{\partial r} \Delta r = \Delta T + \frac{\partial G}{\partial r} \Delta r$$

(121)

where the torque due to the shear force of differential rotation, $G[33]$, is given by

$$G = 2\pi r \nu \Sigma \left( r \frac{\partial \Omega}{\partial r} \right) r$$

(122)

To see the effect of magnetic field, we consider only the magnetic viscosity for the moment. Using eq.(109), we get

$$G_{\text{mag}} = 2\pi r^2 \Sigma \frac{\rho B_\phi B_r}{4\pi}$$

(123)

$$= 4\pi r^2 H \frac{B_\phi B_r}{4\pi}$$

(124)

$$= 4\pi r^2 \frac{c_s}{\Omega_{\text{disk}}} \frac{B_\phi B_r}{4\pi}$$

(125)

where $2H \rho = \Sigma$. Assuming a simple power dependence of $H \frac{B_\phi B_r}{4\pi} \propto r^n$,

$$\frac{\partial G_{\text{mag}}}{\partial r} = 4(2 + n)\pi r H \frac{B_\phi B_r}{4\pi}$$

(126)

and we get from eq.(121)

$$\dot{M} \frac{\partial (r^2 \Omega_{\text{disk}})}{\partial r} = r^2 B_\phi B_z [1 + (2 + n) \frac{H}{r} \frac{B_r}{B_z}]$$

(127)

Since $H \ll r$, the accretion rate is determined by the magnetic braking as far as $B_r$ is not much larger than $B_z$ and the second term in the right hand side
of the above equation can be neglected. For \( r \gg GM_{BH}/c^2 \) where the disk angular velocity can be approximated by a Keplerian velocity, \( \partial(r^2\Omega)/\partial r = r\Omega_{disk}/2 \),

\[
\dot{M} = 2rB_\phi B_z/\Omega_{disk}
\]  

(128)

Using the axisymmetric solution[29],

\[
B_\phi = 2r\Omega_{disk} B_z/c
\]  

(129)

we get

\[
\dot{M} = 4r^2B_z^2/c
\]  

(130)

Since for a steady accretion \( \dot{M} \) is independent of \( r \), \( B_z \sim 1/r \) at large distance. For a numerical estimation, we take \( r = 10^6 cm \) and \( B_z = 10^{14} gauss \) Then we get

\[
\dot{M} = 6 \times 10^{-4} \left( \frac{B_z}{10^{14} gauss} \right)^2 M_\odot s^{-1}
\]  

(131)

which is much larger than Eddington luminosity, \( \dot{M}_{Ed} \sim 10^{-16} \frac{M_{BH}}{M_\odot} M_\odot s^{-1} \). The total accretion during the gamma ray burst period is at most \( 10^{-1}M_\odot \), which may not change the discussions on Blandford-Znajek process in the previous sections. Hence black holes surrounded by the hyper-accretion disks might be a good candidate for the central engine of the gamma ray bursts. The magnetic field on the disk extracts not only the angular momentum as described above but also a substantial energy out of disk as it does on the black hole. The power of the disk magnetic braking can be calculated using eq.(119)

\[
\Delta P_{disk} = \Omega_{disk}\Delta T
\]  

(132)

\[
= 2 \frac{B_\phi B_z}{4\pi}(r\Omega_{disk})\Delta S
\]  

(133)

Using eq.(129) with Keplerian angular velocity of the disk, we get

\[
\Delta P_{disk} = 4c \frac{B_z^2}{4\pi} \left( \frac{GM_{BH}}{rc^2} \right)\Delta S
\]  

(134)

\[
dP_{disk} = \frac{1}{\pi c} B_z^2 r^2 \frac{GM_{BH}}{r^3} dS,
\]  

(135)
then setting \(dS = 2\pi r dr\) and using the steady-state relation \(B_z^2 r^2 = B_z(r_{in})^2 r_{in}^2\), we can obtain the total power by integrating eq.(135) from \(r_{in}\) to infinity,

\[
P_{\text{disk}} = 2B_z(r_{in})^2 r_{in} \frac{GM_{BH}}{c}
\]

where \(r_{in}\) is the distance from the black hole to the inner edge of the accretion disk. Compared to the BZ power from the spinning black hole, eq.(246)), the ratio can be given by

\[
\frac{P_{\text{disk}}}{P_{\text{BH}}} = 8 \frac{r_{in} c^2}{GM f(h) \tilde{a}^2} \left( \frac{B_z(r_{in})}{B_H} \right)^2
\]

The current conservation condition, namely that the total current flows onto the black hole should go into the inner edge(\(r_{in}\)) of the accretion disk, implies

\[
2MB_H \phi(\theta = \pi/2) = \tilde{\omega}(r_{in}) B_{\phi}^{\text{disk}}(r_{in}).
\]

Since the cylindrical radius in Kerr geometry \(\tilde{\omega}(r_{in}) > 2M\), we can see that \(B_H^H\) is larger than \(B_{\phi}^{\text{disk}}\). From the boundary conditions on the horizon[6] in the optimal case,

\[
B_H^H = -\Omega_H MB_H,
\]

and on the accretion disk[29] with angular velocity \(\Omega_D\),

\[
B_{\phi}^{\text{disk}} = -2\Omega_D r B_z^{\text{disk}},
\]

we get

\[
\frac{B_z(r_{in})}{B_H} = \sqrt{\frac{GM \tilde{a} GM}{r_{in} c^2 2 \Omega_D r c^2}} < 1,
\]

and the power of the disk magnetic braking \(P_{\text{disk}} = \frac{2}{f(\tilde{a})} \left( \frac{GM}{r_{in} c^2} \right)^2 P_{BZ}\). It shows that the magnetic field on the horizon cannot be smaller than that on the inner edge of the accretion disk[34] and the disk power need not be substantially larger than that from the black hole[10] [11].

8 Conclusion

We have evaluated the power and energy that can be extracted from a rotating black hole immersed in a magnetic field, the Blandford-Znajek effect. We
improve on earlier calculations to find that the power from a black hole of
given mass immersed in an external field is ten times greater than previously
thought. The amount of energy that can be extracted from a black hole in
this way is limited by the fact that only 29% of the rest mass of a black hole
can be in rotational energy, and that the maximum efficiency with which
energy can be extracted from the hole via the Blanford-Znajek effect is 31%.
The net amount of energy that can be extracted is therefore 9% of the rest
energy of the black hole. We consider various scenarios for the formation of
rotating black holes in gamma-ray burst engines, and while the resulting an-
gular momenta are quite uncertain in some cases, it seems that the required
values of the rotation parameter, $\tilde{a} \gtrsim 0.5$, are achievable.

The rate at which angular momentum is extracted depends on the mag-
netic field applied to the hole. A field of $10^{15}$ G will extract the energy in
less than 1000 s, so time scales typical of gamma-ray bursts can be obtained.
Since the black hole cannot carry a field, there must be an ambient gas in
which the field is anchored that drives the Poynting flux from the black hole.
The most obvious place for it is the accretion disk or debris torus surrounding
the black hole just after it formed.

It has been argued that a field turbulently generated in the disk would not
give a strong Blandford-Znajek flux[10], because the disk would dominate the
total Poynting output, and no more than the disk’s binding energy could be
extracted. We show explicitly that a field in the disk could be much greater,
for example if it is derived from the large, ordered field of a neutron star that
was disrupted. The field distribution proposed by Blandford[29] for such a
case would allow the field on the hole to be much greater than on the disk,
such that the Poynting flow would not be dominated by the disk and not
subject to any obvious limits imposed by the disk.

We also note that a Poynting flow may provide an alternative way of
providing a very large magnetic field for the shocked material that radiates
the afterglow and the gamma-ray burst itself: the standard assumption is
that the required high fields grow turbulently in the shocked gas, up to near-
equipartition values. But the field in the Poynting flow only decreases as $1/r$,
so if it is $10^{15}$ G at $r = 10^5$ cm, it could be as high as $10^4$ G at the deceleration
radius ($10^{16}$ cm), ample to cause an energetic gamma-ray burst.

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Appendix 1. Rotating Black Hole

A rotating black hole is defined as its angular momentum \(J\) and mass \(M\). The angular momentum is measured by the non-newtonian gravitational effect (gravitomagnetic effect) far from the black hole and the mass is measured by the newtonian gravitational field far from the black hole. The solution of Einstein’s equation[35] defines the metric around the rotating black hole with specific angular momentum \(a = J/M\). Using the Boyer-Lindquist coordinates[36], in the natural unit \(G = c = 1\) the metric can be written as

\[
ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).\tag{142}
\]

\(\alpha\) and \(\beta\) are lapse function and shift function[6] defined respectively as

\[
\alpha = \frac{\rho\sqrt{\Delta}}{\Sigma},\tag{143}
\]

\[
\beta^\phi = -\frac{2aMr}{\Sigma^2},\tag{144}
\]

\[
\beta^\theta = \beta^r = 0.\tag{145}
\]

where

\[
\Delta = r^2 + a^2 - 2Mr,\tag{147}
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta,\tag{148}
\]

\[
\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.\tag{149}
\]

The metric tensor \(g_{ij}\) is given by

\[
g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2, \quad g_{\phi\phi} = \tilde{\omega}^2\tag{151}
\]

where

\[
\tilde{\omega} = \frac{\Sigma}{\rho} \sin \theta.\tag{152}
\]

The volume and area elements \(dV\) and \(d\mathcal{S}\) are defined in the standard way[6] by

\[
dV = \sqrt{\text{det}(g_{ij})} drd\theta d\phi.\tag{153}
\]

30
\[
\frac{\rho^2 \omega}{\sqrt{\Delta}} dr d\theta d\phi
\]
(154)

\[
dS_i = \sqrt{\det(g_{ij})} \epsilon_{ijk} \frac{\partial x^j}{\partial b} \frac{\partial x^k}{\partial c} db dc
\]
(155)

The circumference of a circle around the axis of symmetry is \(2\pi \sqrt{g_{\phi \phi}} = 2\pi \tilde{\omega}\).

The horizon, \(r_H\), is defined as a larger root of \(\Delta_{r=r_H} = 0\),

\[
r_H = M + (M^2 - a^2)^{1/2}
\]
(156)

\[
2M r_H = r_H^2 + a^2
\]
(157)

The surface area of the horizon, \(A_H\), is given by

\[
A_H = 4\pi (r_H^2 + a^2)
\]
(158)

and the entropy and the irreducible mass of the black hole are given respectively by

\[
S_H = \frac{k_B}{4\hbar} A_H
\]
(159)

\[
= \frac{\pi k_B}{\hbar} (r_H^2 + a^2) = \frac{2\pi k_B}{\hbar} M r_H.
\]
(160)

\[
M_{irr} = \sqrt{\frac{A_H}{16\pi}} = \frac{1}{2} \sqrt{r_H^2 + a^2} = \sqrt{\frac{S_H}{4\pi}}
\]
(161)

The mass of the black hole can be rewritten in terms of \(J, M_{irr}, \) and \(S_H\),

\[
M = \sqrt{\frac{S_H}{4\pi} + \frac{J^2}{S_H}} = \sqrt{M_{irr}^2 + \frac{J^2}{4M_{irr}^2}}
\]
(162)

The angular velocity of the black hole is given by

\[
\Omega_H = -\beta_H^\phi = \frac{a}{2Mr_H} = \frac{J}{2M^2 r_H}
\]
(163)

\[
= \frac{J}{2M^3 (1 + \sqrt{1 - J^2/M^4})} = \frac{1}{2M} \frac{\tilde{a}}{1 + \sqrt{1 - \tilde{a}^2}}
\]
(164)

where \(\tilde{a}\) is the angular momentum parameter defined as \(\tilde{a} = J/M^2\).
Appendix 2. Electromagnetic fields in vacuum around rotating black hole

For a rotating black hole immersed in an asymptotically uniform magnetic field

$$\vec{B} = B\hat{z}, \ r \to \infty,$$

(165)

the poloidal components of electric and magnetic fields can be written analytically [6]

$$B_r = \frac{B}{2\Sigma \sin \theta} \frac{\partial X}{\partial \theta}$$

(166)

$$B_\theta = -\frac{B\sqrt{\Delta}}{2\Sigma \sin \theta} \frac{\partial X}{\partial r}$$

(167)

$$E_r = -\frac{Ba\Sigma}{\rho^2} \left[ \frac{\partial \alpha^2}{\partial r} + \frac{M \sin^2 \theta}{\rho^2} \left( \Sigma^2 - 4a^2 Mr \right) \frac{\partial}{\partial r} \left( \frac{r}{\Sigma} \right) \right]$$

(168)

$$E_\theta = -\frac{Ba\Sigma}{\rho^2 \sqrt{\Delta}} \left[ \frac{\partial \alpha^2}{\partial \theta} + \frac{\sin^2 \theta}{\rho^2} \left( \Sigma^2 - 4a^2 Mr \right) \frac{\partial}{\partial \theta} \left( \frac{1}{\Sigma^2} \right) \right]$$

(169)

where $X = \frac{\sin^2 \theta}{\rho^2} \left( \Sigma^2 - 4a^2 Mr \right)$ and $a$ is the specific angular momentum of the black hole, $J = aM$. Here we adopt the natural units $G = c = 1$.

The radial components of magnetic field on the horizon, $B^H_r$, can be written explicitly by

$$B^H_r = \frac{B \cos \theta}{(r^2_H + (a/c)^2 \cos^2 \theta)^2} \left[ (r^2_H - (a/c)^2)(r^2_H - (a/c)^2 \cos^2 \theta) + 2(a/c)^2 r_H(r_H - M)(1 + \cos^2 \theta) \right]$$

(170)

$$E^H_r = -\frac{aB}{2r_H c^2 (r^2_H + (a/c)^2 \cos^2 \theta)^2} \left[ (r^2_H - (a/c)^2)(3r^2_H + (a/c)^2(1 + \cos^2 \theta)) \cos^2 \theta - r^2_H \right]$$

(171)

(172)

It is clear that $B^H_r$ vanishes as the rotation of black hole approaches extreme rotation

$$a \to GM/c, \ r_H \to GM/c^2$$

(173)

This is what has been observed [19][20] as the absence of the magnetic flux across the maximally rotating black hole. There are no $\phi$ components of
electric and magnetic fields in eq.(169), and the $\theta$ components vanish as $\sqrt{\Delta}$ as the horizon is approached. The main reason for these vanishing tangential components is that there are no currents outside the horizon in charge-free space. In the absence of in/out currents there is no toroidal magnetic field on the horizon. This also requires a vanishing tangential component of electric field on the horizon, which induces a charge separation on the horizon[6]. Therefore there is no current on which the external magnetic field exerts torques to slow the black hole down so as to extract its rotational energy. One can verify from eq.(166) - eq.(169) that there is no outward component of the Poynting vector (essentially it is due to the absence of the $\phi$ component of the magnetic field), which means no outflow of energy to infinity.
Appendix 3. Axial-symmetric Force-free Magnetosphere

In this appendix, the original formulation of the Blandford-Znajek process is summarized. Consider the stationary and axial symmetric situation where all the partial derivatives of the physical quantities with respect to time and azimuthal angle $\phi$ are vanishing. The electromagnetic field tensor is given by the vector potential $A_\mu$ as

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

and the magnetic field is given by

$$B_r = F_{\theta\phi} = A_{\phi,\theta}$$
$$B_\theta = F_{\phi r} = -A_{\phi,r}$$
$$B_\phi = F_{r\theta} = A_{\theta,r} - A_{r,\theta}$$

The magnetic flux, $\Psi$, through a circuit encircling $\phi = 0 \rightarrow 2\pi$,

$$\Psi = \oint A_{\phi} d\phi = 2\pi A_{\phi}$$

It defines a magnetic surface on which $A_{\phi}(r, \theta)$ is constant and therefore is characterized by the magnetic flux $\Psi$ contained inside it. Magnetic field lines are spiraling on the surface and no magnetic field lines can cross the magnetic surface.

(1) Force-free magnetosphere

The force-free condition for the magnetosphere with the current density $J^\mu$,

$$F_{\mu\nu} J^\mu = 0,$$

can be written by

$$A_{0,r} J^r + A_{0,\theta} J^\theta = 0$$
$$A_{\phi,r} J^r + A_{\phi,\theta} J^\theta = 0$$
$$-A_{0,r} J^\theta - A_{\phi,r} J^\phi = B_{\phi} J^\theta$$
$$A_{0,\theta} J^r + A_{\phi,\theta} J^\phi = B_{\phi} J^r.$$
From eq.(181) and (182) we get

$$\frac{A_{0,r}}{A_{0,\theta}} = \frac{A_{\phi,r}}{A_{\phi,\theta}} = -\frac{J^\theta}{J^r} \quad (185)$$

We can see that $A_0$ is also constant along the magnetic field lines and the electric field is always perpendicular to the magnetic surface. We can define a function $\omega(r, \theta)$,

$$\frac{A_{0,r}}{A_{\phi,r}} = \frac{A_{0,\theta}}{A_{\phi,\theta}} = -\omega \quad (186)$$

$$dA_0 = -\omega dA_{\phi} \quad (187)$$

which is also constant along the magnetic surface. $\omega$ can be identified as an electromagnetic angular velocity (an angular velocity of magnetic field line on a magnetic surface). We can also suppose a function $\mu(r, \theta)$ satisfying

$$4\pi J^r = \mu \sqrt{-\det(g_{\mu\nu})} \quad (188)$$

where $\sqrt{g} = \sqrt{-\det(g_{\mu\nu})} = \rho^2 \sin \theta$. The current conservation

$$\frac{1}{\sqrt{g}} (\sqrt{g}J^\mu)_{,\mu} = 0 \quad (189)$$

leads to

$$(\mu A_{\phi,\theta})_r = (\mu A_{\phi,r})_{,\theta} \quad (190)$$

which implies that $\mu$ is also constant on the magnetic surface.

From eq.(183)(or equivalently from eq.(184)), we get

$$J^\phi = \omega J^0 + \frac{\mu}{4\pi \sqrt{g}} B_{\phi} \quad (191)$$

Together with eq.(188), we now have an expression of the current in terms of the magnetic field and charge density with yet undetermined function $\omega$ and $\mu$. The outward current $I$ can be calculated from the $r$ and $\theta$ components of
the current. Since the outward current is proportional to the magnetic field, the current between the magnetic surface, $dI$, can be written in terms of the magnetic flux as

$$\begin{align*}
    dI &= \frac{1}{4\pi} \mu d\Psi \\
    &= \frac{1}{2} \mu dA_\phi
\end{align*}$$

(2) Energy and Angular Momentum Outflow

For a stationary and axial symmetric system, the conserved fluxes can be defined about the axis of symmetry. The force-free condition ensures the conserved electromagnetic energy flux

$$\mathcal{E}^\mu = T_{0}^\mu$$

(194)

and angular momentum flux

$$\mathcal{L}^\mu = -T_\phi^\mu$$

(195)

where the electromagnetic energy momentum tensor is given by

$$T^{\mu\nu} = \frac{1}{4\pi} \left( F_\mu^{\rho} F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right).$$

(196)

The energy flux and angular momentum flux have a simple relation,

$$\mathcal{E}^\mu = \omega \mathcal{L}^\mu$$

(197)

and the poloidal components can be written as

$$\begin{align*}
    \mathcal{E}^r &= -\frac{1}{4\pi\omega} \frac{\Delta}{\rho^4} A_{\phi,\theta} B_\phi \\
    \mathcal{E}^\theta &= \frac{1}{4\pi\omega} \frac{\Delta}{\rho^4} A_{\phi,r} B_\phi
\end{align*}$$

(198, 199)

Using the physical component, $\mathcal{E}_r$,

$$\mathcal{E}_r = \sqrt{g_{00}/g_{rr}} \mathcal{E}_r = \sqrt{g_{00}g_{rr}} \mathcal{E}_r$$

(200)
we get the power at infinity given by

\[ P = \int \mathcal{E}_i \cdot dS_i \]  \hspace{1cm} (201)

\[ = -\frac{1}{4\pi} \int \omega \sqrt{\frac{g_{00}}{g_{\theta\theta}g_{rr}}} B_\phi A_{\phi,\theta} \sqrt{g_{\theta\theta}g_{\phi\phi}} d\theta d\phi \]  \hspace{1cm} (202)

\[ = -\frac{1}{4\pi} \int \omega \sqrt{\frac{g_{00}g_{\phi\phi}}{g_{rr}g_{\theta\theta}}} B_\phi A_{\phi,\theta} d\theta d\phi \]  \hspace{1cm} (203)

\[ = -\frac{1}{4\pi} \int \omega (\Delta \sin \theta \rho^2 B_\phi) A_{\phi,\theta} d\theta d\phi \]  \hspace{1cm} (204)

Defining $B_T[2]$ as

\[ B_T = \frac{\Delta \sin \theta}{\rho^2} B_\phi \]  \hspace{1cm} (205)

the power can be written as

\[ P = -\frac{1}{2} \int \omega B_T A_{\phi,\theta} d\theta \]  \hspace{1cm} (206)

\[ = -\frac{1}{2} \int \omega B_T dA_{\phi}. \]  \hspace{1cm} (207)

From the inhomogeneous Maxwell equations

\[ F_{\mu\nu}^{\mu\nu} = 4\pi J^\nu \]  \hspace{1cm} (208)

we get

\[ B_{T,\theta} = 4\pi \sqrt{g} J^\theta \]  \hspace{1cm} (209)

\[ B_{T,r} = -4\pi \sqrt{g} J^r \]  \hspace{1cm} (210)

Using eq.(188),

\[ \mu A_{\phi,\theta} = B_{T,\theta} \]

\[ \mu A_{\phi,r} = B_{T,r} \]  \hspace{1cm} (211)

which shows that $B_T$ is also constant along the magnetic surface. Since

\[ \mu dA_\phi = dB_T, \]  \hspace{1cm} (212)
the outward current can be calculated as

\[ dI = \frac{1}{2} \mu dA_\phi \]

and we get

\[ I(\theta) = \frac{1}{2} \int dB_T = \frac{1}{2} B_T(\theta) \]

where \( B_T(\theta = 0) = 0 \) has been used. Using the physical component, \( B_\phi = \frac{B_\phi}{\sqrt{g_{rr} g_{\theta\theta}}} \), eq.(215) can be written

\[ I = \frac{1}{2} \omega \alpha B_\phi \]

which is nothing but the Ampere's law on the stretched horizon

\[ 2\pi \omega B_\phi^H = 4\pi I \]

using the tangential field on the stretched horizon[6]

\[ B_\phi^H = \alpha B_\phi \]

Using eq. (207), (213) and (215) we get

\[ dP = -\omega I dA_\phi \]

\[ = -\frac{1}{2\pi} \omega I d\Psi \]

\[ P = -\frac{1}{2\pi} \int \omega I d\Psi \]

Since \( \omega, I \) and \( \Psi \) are constant along the magnetic surface we can evaluate the integral on the horizon of the black hole:

\[ P = -\frac{1}{2\pi} \int_{\text{horizon}} \omega I d\Psi \]

Hence we can see that it is a proof of the simple-minded derivation of eq.(21) in the text with \( \omega \) identified with \( \Omega_F \).
Appendix 4. Power from Rotating Black Hole

Using the boundary condition\[37\] that $B_r$ is finite and

\[
B_T = \frac{\sin \theta \omega (r_H^2 + a^2) - a}{r_H^2 + a^2 \cos \theta} B_r
\]

\[
= \frac{\sin \theta (\omega - \Omega_H) \Sigma_H}{\rho_H^2} B_r
\]

(223)

(224)

which can be written for $B_\phi$ as

\[
B_\phi = \frac{\rho^2 (\omega - \Omega_H) \Sigma_H}{\Delta} B_r
\]

\[
= \frac{(\omega - \Omega_H) \Sigma_H}{\Delta} B_r
\]

(225)

(226)

and the toroidal component on the stretched horizon is

\[
B_H = (\omega - \Omega_H) \tilde{\omega} B_H
\]

(228)

where $B_H = B_r$. Then the current up to $\theta = \theta_H$ can be written using eq.(215)

\[
I(\theta_H) = \frac{1}{2} \frac{\Delta}{\rho^2} \sin \theta B_\phi = \frac{1}{2} (\omega - \Omega_H) \tilde{\omega}^2 B_H
\]

(229)

which is exactly the same current derived in the membrane paradigm\[6\].

The total power out of the black hole can be calculated using eq. (206), (215), and (229),

\[
P = \frac{1}{2} \int \omega B_T B_r d\theta
\]

\[
= \int \omega I \rho \tilde{\omega} B_H d\theta
\]

(230)

(231)

\[
= \frac{1}{2} \int \omega (\omega - \Omega_H) B_H^2 \rho \tilde{\omega}^3 d\theta
\]

(232)

Adopting the optimal condition\[6\], $\omega \sim \Omega_H / 2$, the power can be written as

\[
P = \frac{1}{2} \int \omega (\omega - \Omega_H) B_H^2 \rho \tilde{\omega}^3 d\theta
\]

(233)
\[
\frac{1}{8} \left( \frac{a}{2Mr_H} \right)^2 \int \frac{(r_H^2 + a^2)^3 \sin^3 \theta}{r_H^2 + a^2 \cos^2 \theta} B^2_H d\theta \\
= \frac{1}{32} \left( \frac{a}{M} \right)^2 \frac{(2Mr_H)^2}{r_H^2} \left( 1 + \left( \frac{r_H}{a} \right)^2 \right) \int \frac{\sin^3 \theta}{\left( \frac{r_H}{a} \right)^2 + \cos^2 \theta} B^2_H d\theta
\]

(234)

Using the integral identity,
\[
\int_0^{\pi/2} \frac{\sin^3 \theta}{\left( \frac{r_H}{a} \right)^2 + \cos^2 \theta} d\theta = [(h + 1/h) \arctan h - 1]
\]

(236)

where \( h \) is defined in ref [17] as
\[
h = \frac{a}{r_H} = \frac{J/M^2}{1 + \sqrt{1 - J^2/M^4}} = \frac{\tilde{a}}{1 + \sqrt{1 - \tilde{a}^2}},
\]

(237)

\[
\frac{M}{r_H} = \frac{M a}{a r_H} = \tilde{h}
\]

(238)

we integrate the angular integral assuming the average magnetic field, \(< B^2_H >\),
\[
P = \frac{1}{32} \left( \frac{a}{M} \right)^2 \frac{(2Mr_H)^2}{r_H^2} \frac{(1 + \tilde{h}^2)}{h^2} < B^2_H > 2[(h + 1/h) \arctan h - 1]
\]

(240)

\[
= \frac{1}{4} \left( \frac{a}{M} \right)^2 M^2 < B^2_H > f(h)
\]

(241)

where
\[
f(h) = \frac{(1 + h^2)}{h^2} [(h + 1/h) \arctan h - 1]
\]

(242)

\[
\rightarrow 2/3, \quad \text{for } h \rightarrow 0,
\]

(243)

\[
= \pi - 2, \quad \text{for } h = 1 (\text{extreme rotation, } a = M)
\]

(244)

Now inserting \( G \) and \( c \) properly, we get
\[
P = \frac{1}{4} \left( \frac{ac}{GM} \right)^2 M^2 < B^2_H > f(h) \frac{G^2}{c^3}
\]

(245)

\[
= \frac{1}{4} \tilde{a}^2 M^2 < B^2_H > f(h) \frac{G^2}{c^3}
\]

(246)

\[
= \frac{1}{4c} \frac{J^2}{M^4} M^2 < B^2_H > f(h)
\]

(247)
Numerically,
\[
\frac{G^2}{c^3} M_\odot^2 (10^{15} \text{ gauss})^2 = (2 \times 10^{33} \times 10^{15})^2 \times \frac{(6.7 \times 10^{-8})^2}{(3 \times 10^{10})^3} \quad (248)
\]
\[
= 6.7 \times 10^{50} \text{ erg/s} \quad (249)
\]
Then the total power of Poynting flux is
\[
P = 1.7 \times 10^{50} \left( \frac{ac}{GM} \right)^2 (\frac{M}{M_\odot})^2 (\frac{<B_H>}{10^{15} \text{ gauss}})^2 f(h) \text{ erg/s} \quad (250)
\]
One of the frequently-quoted numbers for the total power of BZ process from ref.[6] is
\[
P \sim 10^{50} \left( \frac{ac}{GM} \right)^2 (\frac{M}{M_\odot})^2 (\frac{<B_H>}{10^{15} \text{ gauss}})^2 \text{ erg/s} \quad (251)
\]
which is one order of magnitude lower than eq.(245). In ref[6], the angular integration of eq.(222) is approximated by replacing
\[
\tilde{\omega}^2 \rightarrow r_H^2/2 \quad (253)
\]
\[
\Delta \Psi \rightarrow \pi r_H^2 B_H \quad (254)
\]
\[
P \sim \frac{1}{128} \left( \frac{ac}{GM} \right)^2 M^2 \frac{<B_H>}{10^{15} \text{ gauss}}^2 g^2(a/M) \frac{G^2}{c^3} \quad (255)
\]
\[
g^2(a/GM) = 1 + (1 - (ac/GM)^2)^{1/2} \quad (256)
\]
\[
\rightarrow 1 \text{ for } a \rightarrow 0 \quad (257)
\]
\[
= 2, \text{ for } a = M(\text{extreme rotation}) \quad (258)
\]
Then
\[
\frac{P(\text{eq.255})}{P(\text{eq.245})} = \frac{g^2(ac/GM)}{32f(h)} \quad (259)
\]
\[
\rightarrow 3/64, \text{ for } a \rightarrow 0 \quad (260)
\]
\[
\sim 5/56, \text{ for } a = M(\text{extreme rotation}) \quad (261)
\]
which explains the difference of an order-of-magnitude.

\[\text{Recently Popham et al. [9] use}\]
\[
P \sim 10^{50} \left( \frac{ac}{GM} \right)^2 (\frac{<B_H>}{10^{15} \text{ gauss}})^2 \text{ erg/s} \quad (252)
\]
for \(M = 3M_\odot\) hence equivalent to eq.(251).
References

[31] W. Kluzniak and M. Ruderman, astro-ph/9712320
[34] L-X. Li, astro-ph/9902352