ABSTRACT

Recently, several white dwarfs with very strong surface magnetic fields have been observed. In this paper we explore the possibility that such stars could have sufficiently strong internal fields to alter their structure. We obtain a revised white dwarf mass-radius relation in the presence of strong internal magnetic fields. We first derive the equation of state for a fully degenerate ideal electron gas in a magnetic field using an Euler-MacLaurin expansion. We use this to obtain the mass-radius relation for magnetic $^4\text{He}$, $^{12}\text{C}$, and $^{56}\text{Fe}$ white dwarfs of uniform composition.

Subject headings: stars: white dwarfs - stars: magnetic fields - stars: interiors

1. INTRODUCTION

A number of white dwarfs with strong magnetic fields have been discovered (Kemp et al. 1970; Putney 1995; Schmidt & Smith 1995; Reimers et al. 1996) and extensively studied (Jordan 1992; Angel 1978; Channugam 1992 and references therein). Surface magnetic fields ranging from about $10^5$ G to $10^9$ G have been detected in about 50 (2%) of the $\approx 2100$ known white dwarfs (Jordan 1997 and references therein). As relics of stellar interiors, the study of the magnetic fields in and around degenerate stars should give important information on the role such fields play in star formation and stellar evolution. However, the origin and evolution of stellar magnetic fields remains obscure.

As early as Ginzburg (1964) and Woltjer (1964) it was proposed that the magnetic flux ($\Phi_B \sim BR^2$) of a star is conserved during its evolution and subsequent collapse to form a remnant white dwarf or neutron star. A main sequence star with radius on the order of $R \sim 10^{11}$ cm and surface magnetic field $B \sim 10^{-10}$ G [magnetic A-type stars have typical surface fields $\lesssim 10^5$ G (Shapiro & Teukolsky 1983)] would thus collapse to form a white dwarf with $R \sim 10^6$ cm and $B \sim 10^{-5} \sim 10^8$ G, or a neutron star with $R \sim 10^6$ cm and $B \sim 10^{-11} \sim 10^{-14}$ G. Indeed, shortly after their discovery (Hewish et al. 1968) pulsars were identified as rotating neutron stars (Gold 1968) with magnetic fields $B \sim 10^{-11} \sim 10^{-14}$ G consistent with magnetic field amplification by flux conservation. In addition, neutron stars with surface magnetic fields exceeding $10^{14}$ G [so-called magnetars] have been recently suggested as the source of soft gamma-ray repeaters (Duncan & Thompson 1992; Thompson & Duncan 1995).

Moreover, the surface magnetic field of a star does not necessarily reflect the internal field (Ruderman 1980). For example, the toroidal fields below the surface of the Sun are at least on the order of $\sim 10^2$ to $10^4$ times stronger than the average surface dipole field strength of $\sim 1$ G (Galloway, Proctor, & Weiss 1977). Furthermore, at the region of the convective zone, the strength of small scale magnetic fields could reach a value as high as $7 \times 10^2$ G (Chauhan, Pandey & Pandey 1999, Pulido 1998). This would correspond to an interior field strength on the order of $\sim 10^9$ to $10^{13}$ G in a white dwarf, or $\sim 10^{15}$ to $10^{18}$ G in a neutron star. Condensed objects of size $R$ and mass $M$ have an upper limit to their field strengths of $B \lesssim 4\pi G^{1/2}$. For neutron stars with $R \approx 10$ km and $M \approx M_\odot$, the limit is $B \lesssim 10^{19}$ G (Lerche & Schramm 1977).

Indeed, the existence of white dwarfs with interior magnetic fields as strong as $4 \times 10^{13}$ G is not ruled out with the present uncertainties in the mass-radius relation (Shapiro & Teukolsky 1983). The present high upper limit on the strength of internal fields in white dwarfs is obtained by simply setting the magnetic pressure equal to the internal pressure of the star. However, white dwarfs with internal fields at or around this strength could be constrained (Mestel 1965) by a perceptibly different mass-radius relation.

Although white dwarfs in binaries with well determined masses do not appear to have surface magnetic fields larger than $\sim 10^5$ G, internal fields of order $10^{13}$ G could be well hidden below the surface (Angel 1978). Newly discovered magnetic degenerate stars, especially those with surface field strengths near the range of $B \sim 10^9$ G, always show strong circularly and/or linearly polarized spectral energy distributions (Schmidt et al. 1999). Moreover, these stars reveal unique spectral features (Engelhardt & Bues 1994) due to quasi-Landau resonances in extremely high magnetic fields of $> 10^6$ G.

In this work, we explicitly compute the mass-radius relation of white dwarfs with internal magnetic fields. Previously, Ostriker & Hartwick (1968) have estimated effects of interior magnetic fields by considering a correction in terms of the ratio of magnetic to gravitational energy. They showed that a relatively small ratio of magnetic to gravitational energy would be sufficient to explain an observational discrepancy in the classical mass-radius relation for Sirius B. However, if white dwarfs could indeed have central magnetic fields as strong as $4.4 \times 10^{11} - 4.4 \times 10^{13}$ G, the revised mass-radius relation must be explicitly determined by taking the magnetic field into account in the equation of state. The present work thus expands upon that earlier study by explicitly computing the equation of state for a completely degenerate, noninteracting electron gas in a magnetic field. This equation of state is then applied to the Tolman-Oppenheimer-Volkoff (TOV) equation of stellar hydrostatic equilibrium.

The equation of state in a magnetic field should reduce to a normal equation of state in the absence of a magnetic field. Therefore, we use an Euler-MacLaurin expansion (Kerman, Starkman, & Vachaspati 1996) of the thermodynamic variables to recover the weak field limit. In integrating the TOV equation...
we simply follow the procedure of Hamada & Salpeter (1961) for degenerate matter of uniform composition of $^4$He, $^{12}$C, or $^{56}$Fe. Although $^4$He and $^{12}$C white dwarfs are expected to have a similar (though not identical) mass-radius relation, we explicitly consider each for completeness.

2. EQUATION OF STATE FOR AN ELECTRON GAS IN A MAGNETIC FIELD

The properties of an electron in an external magnetic field have been studied extensively (Landau & Lifshitz 1938; Johnson & Lippmann 1949; Canuto & Chiu 1968; Schwinger 1988). In brief, the energy states of an electron in a magnetic field are quantized and its properties are modified accordingly. In order to investigate these effects, we first solve the Dirac equation in a magnetic field. We make the convenient choice of gauge for the vector potential $A_z$.

The properties of an electron in an external magnetic field is then obtained by quantizing cyclotron states begin to exist. The maximum Landau level $n_f$ for a given Fermi energy $\epsilon_f$ and magnetic field strength $\gamma$ is given by

$$n_f \equiv \frac{\epsilon_f^2 - 1}{2\gamma} \geq n.$$  

The pressure of an ideal electron gas in a magnetic field is then

$$P_e = \frac{2}{3\pi^2}mc^2\left(\frac{mc}{\hbar}\right)^3\Phi(\epsilon_f, n),$$  

where

$$\Phi(\epsilon_f, n) = \frac{1}{2}\sum_{n=0}^{n_f} [2 - \delta_{n0}] \left[ \epsilon_f \sqrt{\epsilon_f^2 - (1+2\gamma n)} - (1+2\gamma n) \ln \left( \frac{\epsilon_f + \sqrt{\epsilon_f^2 - (1+2\gamma n)}}{\sqrt{1+2\gamma n}} \right) \right].$$

Similarly, the energy density is

$$\mathcal{E}(\epsilon_f, n) = \frac{2}{3\pi^2}mc^2\left(\frac{mc}{\hbar}\right)^3\chi(\epsilon_f, n),$$  

where

$$\chi(\epsilon_f, n) = \frac{1}{2}\sum_{n=0}^{n_f} [2 - \delta_{n0}] \left[ \epsilon_f \sqrt{\epsilon_f^2 - (1+2\gamma n)} + (1+2\gamma n) \ln \left( \frac{\epsilon_f + \sqrt{\epsilon_f^2 - (1+2\gamma n)}}{\sqrt{1+2\gamma n}} \right) \right].$$

From these, we obtain the energy per electron

$$E_e(\epsilon_f, n) = mc^2\frac{\chi(\epsilon_f, n)}{\zeta(\epsilon_f, n)}.$$  

In order to recover the usual equation of state in the absence of a magnetic field, we utilize an Euler-MacLaurin expansion of Eqs. (3) - (6) in the weak field limit. Then, the number density is given by

$$n_e \simeq \frac{1}{3\pi^2} \left(\frac{mc}{\hbar}\right)^3 \zeta(x),$$  

where

$$\zeta(x) = x^3 + \gamma^3 x + \mathcal{O}(\gamma^4) + \cdots,$$

and $x \equiv p_f/mc$ is the relativity parameter. Note that the electron number density increases as the magnetic field increases for a given $x$.

The pressure becomes

$$P_e \simeq \frac{1}{24\pi^2}mc^2\left(\frac{mc}{\hbar}\right)^3\Phi(x),$$  

where

$$\Phi(x) = \Phi_0(x) + \gamma^2\Phi_B(x) + \mathcal{O}(\gamma^4) + \cdots.$$
\[ \Phi_0(x) = x\sqrt{x^2 + 1} + 1(2x^2 - 3) + 3\ln(x + \sqrt{x^2 + 1}), \]
\[ \Phi_\gamma(x) = \frac{\sqrt{x^2 + 1}}{x} + 2\ln(x + \sqrt{x^2 + 1}) - \left(1 + \frac{1}{x(x + \sqrt{x^2 + 1})}\right). \]

Note also that for a physically reasonable value of \(x\) the pressure always increases as \(\gamma\) increases.

The energy density can also be written
\[ \mathcal{E}(x) \simeq mc^2 \left(\frac{mc}{\hbar}\right)^3 \chi(x), \quad (9) \]
where
\[ \chi(x) = \chi_0(x) + \gamma^2 \chi_\gamma(x) + \mathcal{O}(\gamma^4) + \cdots, \]
\[ \chi_0(x) = \frac{1}{8\pi} \left[x(2x^2 + 1)\sqrt{x^2 + 1} - 1 - \ln(x + \sqrt{x^2 + 1})\right]. \]
\[ \chi_\gamma(x) = \frac{1}{24\pi^2} \left[1 + \frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x(x + \sqrt{x^2 + 1})} - 2\ln(x + \sqrt{x^2 + 1})\right]. \]

Finally, the energy per electron is given by
\[ E_e(x) \simeq \frac{3}{8} mc^2 \frac{\chi(x)}{\zeta(x)}, \quad (10) \]
Here we can see explicitly that as \(\gamma\) goes to zero, Eqs. (7) - (10) recover exactly the usual equation of state in the absence of a magnetic field.

3. MASS-RADIUS RELATION OF MAGNETIC WHITE DWARFS

The mass-radius relation of white dwarfs was first determined by Chandrasekhar (1939). Later Hamada & Salpeter (1961) obtained numerical models for various core compositions by considering a fully degenerate configuration at zero temperature. The theoretical relationship between the mass and radius of a white dwarf is important for the interpretation of observational results (see Koester & Channugam 1990 for a review). There are several recent studies and observations on the mass-radius relation of non-magnetic white dwarfs (Wood 1990; Vauclair, Schmidt, & Koester 1997; Vennes, Fontaine, & Brassard 1995; Provenchel et al. 1998).

In order to obtain the mass-radius relation for magnetic white dwarfs, we use Eqs. (7) - (10) for \(\gamma \leq 1 (B \leq 4.4 \times 10^{13} \text{G})\) and carry out stellar integrations for a uniform composition of \(^4\text{He}, ^{12}\text{C}, \text{and } ^{56}\text{Fe}\) as an illustrative model. In this uniform model (Hamada & Salpeter 1968; Fushiki et al. 1992; Rögnvaldsson et al. 1993) the total energy \(E\) of the plasma consists of a nearly uniform distribution of degenerate electrons with embedded ions,
\[ E = E_e + E_C, \quad (11) \]
where the first term is the energy of a uniform gas of free electrons and \(E_C\) corrects for the classical Coulomb energy. Although the noninteracting electron gas accounts for the dominant contribution to the equation of state at high density, the classical Coulomb correction is significant. Other corrections, such as the Thomas-Fermi, exchange, and correlation corrections give only a very small change in the mass-radius relation of white dwarfs. Actually the Thomas-Fermi correction in a strong magnetic field is important at low density (see Rögnvaldsson et al. 1993 and references therein), but as a whole it gives only a minor effect on the mass-radius relation. Hence, we ignore these minor effects for the present work.

Magnetic fields should not alter the spherical symmetry of the constituent atoms. Indeed, it is, perhaps, a remarkable fact that magnetic fields do not destroy even the approximate spherical symmetry of heavy atoms within the relevant range (Rögnvaldsson et al. 1993). Thus, we can use the ordinary electrostatic energy (Coulomb energy) per charge,
\[ E_C/Z = \frac{9}{5} Z^{2/3} R_y \frac{1}{r_e}, \quad (12) \]
where \(R_y = (\alpha/e^2 m_e c^2)\) is the Rydberg energy and from Eq. (7),
\[ r_e = \left(\frac{3\pi}{8\gamma\zeta(x)}\right)^{1/3}. \]
The corresponding pressure is
\[ P_C = -mc^2 \left(\frac{mc}{\hbar}\right)^3 \frac{9}{4\pi} \frac{Z^{2/3}\alpha^5}{10} \frac{1}{r_e^4}. \quad (13) \]
In this model, the total pressure is then given by
\[ P = P_e + P_C. \quad (14) \]

Thus, since \(P_C\) is negative, the equation of state would lead to negative pressures at low density (Salpeter 1961).

In integrating the equations of hydrostatic equilibrium, we follow the classical procedure of Hamada & Salpeter (1961). Figures 1, 3, and 5 show the mass-radius relation of \(^4\text{He}, ^{12}\text{C}, \) and \(^{56}\text{Fe} \) white dwarfs for a given magnetic field strength. Figures 2, 4, and 6 show the relation between mass and central density of these white dwarfs for a given magnetic field strength. Here it can be seen that our results approach the Hamada & Salpeter (1961) results as the magnetic field strength decreases. For high central field strengths \(\gamma \approx 0.01 - 1 [B \approx 4.4 \times (10^{11} - 10^{13}) \text{G}],\) both the mass and radius of magnetic white dwarfs increase compared to non-magnetic white dwarfs of the same central density. For instance, for \(\gamma \approx 0.8\) carbon white dwarfs, the radius increases by about 30% for \(M \approx 1 M_\odot\). Similarly for \(R \approx 0.01 R_\odot\) the mass also increases by about 25%. These results are approximately consistent with Ostriker & Hartwick (1968), i.e. the radius increases while the central density \(\rho_c\) decreases as the magnetic field increases for fixed \(M.\) As expected, for \(B \leq 10^{10} \text{G},\) internal magnetic fields do not affect the white-dwarf mass-radius relation.
Mass-Radius Relation for Magnetic White Dwarfs

Fig. 1 Relation between the mass $M$ and radius $R$ of a $^4$He magnetic white dwarf for the indicated magnetic-field strengths. The solid line denotes the Hamada & Salpeter model for non-magnetic white dwarfs ($\gamma = 0$). The dashed lines are magnetic white dwarfs.

Fig. 2 Relation between the central density $\rho_c$ (in g/cm$^3$) and mass $M$ for $^4$He magnetic white dwarfs. The solid line denotes the Hamada-Salpeter model for non-magnetic white dwarfs ($\gamma = 0$). The dashed lines are for magnetic white dwarfs.

A striking feature of these results is that white dwarfs with strong interior magnetic fields should be massive. This is simply because a star becomes unbound if the magnetic plus matter pressure force exceeds the gravitational force. For example, there is no stable solution with $M \lesssim 0.5 M_\odot$ for $\gamma = 0.3$ carbon white dwarfs. This is consistent with recent observations that on average magnetic white dwarfs have a higher mass than typical non-magnetic white dwarfs (see Table 1). Note, however, that for a field strength of $B \lesssim 1 \times 10^{13}$ G, magnetic fields only give a relatively small change on the mass-radius relation, that is, we can not distinguish between magnetic and non-magnetic white dwarfs for $\gamma \lesssim 0.2$.

4. DISCUSSION

In this work, we have calculated the equation of state for an electron gas in a magnetic field at zero temperature. For simplicity, we have assumed a uniform composition to obtain the relation between mass and radius for magnetic white dwarfs. For high internal magnetic fields $B \simeq 4.4 \times (10^{11} - 10^{13})$ G, ($\gamma \simeq 0.01 - 1$), the mass-radius relation is modified. Our results not only confirm the Ostriker & Hartwick (1968) result of increasing radius and decreasing central density $\rho_c$ with increasing field, but also are consistent with the suggestion (Liebert 1988) that observed magnetic white dwarfs have masses which are on average larger than non-magnetic white dwarfs, implying more massive and younger progenitors.

The question remains, however, as to whether it is reasonable to consider such high internal field strengths for magnetic white dwarfs. First, one must assume that the magnetic fields are well hidden beneath the surface, while the surface fields are several orders of magnitude less. Second, assuming that flux is conserved during the collapse to a white dwarf, the progenitor of the white dwarfs must have had sufficiently large fields to produce the required white dwarf internal field strengths. Flux conservation implies that the central field strength of the progenitors is of order $\sim 10^8$ G, assuming $R \approx 1 R_\odot$. This is reasonable. During star formation, the collapse of a typical interstellar cloud with radius $\sim 0.1$ pc, mass $\sim 1 M_\odot$, and protostellar magnetic field of magnitude $\sim 3 \times 10^{-6}$ G would result in a field strength of $\sim 3 \times 10^4$ G in a solar type star formed from this material (Spitzer 1978). Although there is no evidence for main sequence stars with such field strengths, they are not ruled out by observations either (Shapiro & Teukolsky 1983).

5. COMPARISON WITH OBSERVATIONS

Figure 7 shows a comparison between our calculations and white dwarfs with known masses and radii from the HIPPARCOS survey (Vauclair, Schmidt, & Koester 1997; Provencal et al. 1998). Plotted error bars are the quoted $\pm 1\sigma$ observational uncertainties. Strong hidden interior magnetic fields would be expected to manifest themselves by a preponderance of stars with large masses and radii. Most of the data, however, are within $2\sigma$ of the non-magnetic theoretical curves. A puzzling feature however (Provencal et al. 1998) is that some of the best determined data points (e.g., EG 50 and Procyon B) can not be fit without postulating an iron composition, something which seems unlikely from a stellar evolution standpoint. Furthermore, although the evidence is not compelling, there are at least two well determined stars [i.e., GD 140 and Sirius B (Provencal et al. 1998)] as well as some field stars with $M \gtrsim 0.6 M_\odot$ and $R \gtrsim 0.012 R_\odot$ which may be better fit if strong internal magnetic fields are assumed. However, Sirius B and some of the field stars can also be accommodated by atmospheric models (Wood 1990).
Fig. 3 Same as Fig. 1, but for $^{12}\text{C}$.

Fig. 4 Same as Fig. 2, but for $^{12}\text{C}$.

Fig. 5 Same as Fig. 1, but for $^{56}\text{Fe}$.

Fig. 6 Same as Fig. 2, but for $^{56}\text{Fe}$.
Some typical magnetic white dwarfs which have reported masses $M$ and surface magnetic fields $B_s$ are summarized in Table 1. This table shows that magnetic white dwarfs are typically more massive than non-magnetic white dwarfs (on average $\sim 0.6M_\odot$). Among them RE J0317-858 ($B_s > 10^8$ G) is approaching the Chandrasekhar mass limit (Barstow et al. 1995).

Not listed in Table 1, however, are the two known magnetic white dwarfs with the strongest surface magnetic field, GD 229 ($B_s \gtrsim 10^{10}$ G) and PG 1031+234 ($B_s \approx (0.5-1) \times 10^9$ G). Unfortunately, there are no reported masses or radii for those magnetic white dwarfs. If their masses and radii could be measured, then our magnetic field model might be tested.

Perhaps the most interesting object in Table 1 is LB 11146 (PG 0945+245). This is an unresolved binary system consisting of two degenerate stars (Liebert et al. 1993): one component is a normal DA white dwarf ($M = 0.91 \pm 0.07M_\odot$) with no detectable magnetic field; the other has a strong magnetic field ($B_s > 3 \times 10^8$ G) and a mass in the range $0.76 \leq M/M_\odot \lesssim 1.0$. Thus, assuming for illustration a stronger interior field of $\gamma \sim 0.5$, and that this is a typical equal mass system with $M = 0.9M_\odot$ each, then the radius of the magnetic component would be larger by about 10% than the normal star. Alternatively, if they have similar radii of $R \sim 0.01R_\odot$, the magnetic star would have a heavier mass by about 12% than the normal one for $\gamma \sim 0.5$.

Regarding the radius of magnetic white dwarfs, Greenstein & Oke (1982) have reported radii of $R \sim 0.0066R_\odot$ for Grw +70°8247 ($B_s \approx 3.2 \times 10^8$ G) and $R \sim 0.01R_\odot$ for Feige 7 ($B_s \approx 3.5 \times 10^7$ G) from the interpretation of their spectra. Assuming an interior field which is about $10^5$ times stronger than the surface field, then we would deduce $\gamma \approx 0.8$ and $M \sim 1.1-1.2M_\odot$ for Grw +70°8247, consistent with the expectation of high masses for magnetic white dwarfs. Similarly, we would deduce $\gamma \approx 0.075$ and $M \approx 0.8M_\odot$ for Feige 7. This star however would be nearly indistinguishable from a non-magnetic white dwarf.

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**Table 1**

**Mass and Surface Magnetic Field Strength in Some Typical Magnetic White Dwarfs**

| Object          | Mass \(M_\odot\) | 
|-----------------|-----------------|-----------------|-----------------|-----------------|
| PG 2329+267     | ~ 0.9           | 2.3             | Moran et al. 1998 |
| LB 11146B       | 0.76 - 1.0      | > 300           | Liebert et al. 1993 |
| Grw +70°8247    | > 1.0           | 320             | Greenstein & Oke 1990 |
| IRXS J0823.6-2525 | 1.2             | ~ 3             | Ferrario et al. 1998 |
| PG 0136+251     | 1.28            | 1.3 ?           | Vennes et al. 1997 |
| PG 1658+441     | 1.31            | 3.5 ?           | Schmidt et al. 1992 |
| RE J0317-858    | 1.35            | 660             | Barstow et al. 1995 |