THE QUESTION OF INVARIANT RENORMALIZABILITY OF THE MASSLESS

YANG-MILLS THEORY IN A MANIFEST COVARIANT APPROACH

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A B S T R A C T

By introducing a classical field the generating functional of the Green's functions for the Yang-Mills theory is written as a sum of gauge invariant functionals depending on the classical field. An invariant regularization of the functionals corresponding to the class of one-loop diagrams is given so that the finite part fulfils the Ward identities automatically.

1. INTRODUCTION

Recently much attention has been paid to a unified theory of the weak and electromagnetic interaction of leptons proposed by Salam ^{1),2)} and Weinberg ³⁾. The intermediate bosons and the photon are united into a multiplet of gauge fields, a mass splitting for these gauge fields is achieved by an $SU(2) \times U(1)$ spontaneous breakdown mechanism, firstly described by Higgs ⁴⁾ and Kibble ⁵⁾. The Lagrangian for the gauge field is that of a Yang-Mills theory, and the question of convergence of this unified theory is connected with the renormalizability of the Yang-Mills Lagrangian. There has been much work done on the Yang-Mills Lagrangian, in deriving the correct Feynman rules ^{6),7)} and in studying the renormalizability of the massless and the massive case ⁷⁾⁻⁹⁾. A simple mass term for all vector-mesons leads to a non-renormalizable theory ⁷⁾, whereas a model in which the charged vector mesons acquire a mass by the Higgs-Kibble mechanism, remains to be renormalizable, while the infra-red problem of the pure massless Yang-Mills theory is avoided.

However, the question in these theories is not only whether there exists a renormalization procedure, but also, whether this renormalization procedure is gauge invariant so that the renormalized theory is also invariant under local gauge transformations. To obtain a gauge invariant renormalized theory one may exploit - in renormalizing - the Ward identities which are an expression for the invariance under the group.

There exists another method by which the Ward identities are fulfilled automatically because manifest covariance is preserved at each stage of the calculation. This method is constructed in the spirit of the gauge invariant renormalization in Q.E.D., which Schwinger ¹⁰⁾ formulated in his famous article from 1951. The electromagnetic field can be treated in his problem as an external one, and to get in an arbitrary theory an analogue of the external field one has to introduce properly defined classical fields. The generating functional of the Green's functions then can be written as a sum of functionals which depend on the classical fields and which are invariant under the transformations of the classical fields. Hence, the divergent parts of these functionals turn out to be invariants under the group, and an invariant regularization prescription is possible. The purpose of this paper is to apply this method to the Yang-Mills theory in its massless version.

For the class of the one-loop diagrams it is rigorously shown that the renormalization can be done in a gauge invariant manner. The proof for an arbitrary number of loops is briefly indicated. It will be carried out in connection with a subsequent discussion of a massive version of the Yang-Mills theory.

In Section 2 we write down the invariant perturbation theory, in Section 3 we treat the one-loop functional in momentum space as well as in co-ordinate space to give the gauge invariant regularization prescription. The relation to De Witt's approach ¹¹⁾ of "variation of the background field" is indicated in the Conclusions.

2. THE GAUGE INVARIANT PERTURBATION THEORY

The Lagrangian for the massless Yang-Mills fields reads

$$L_{y.m.} = -\frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - ig \epsilon^{abc} V_\mu^b V_\nu^c \quad (2)$$

The Lagrangian (1) is invariant under the transformation

$$V_\mu' = e^{i\alpha} V_\mu e^{-i\alpha} - \frac{1}{ig} e^{i\alpha} \partial_\mu e^{-i\alpha} \quad (3)$$

where

$$V_\mu = V_\mu^a t_a \quad ; \quad \alpha = \alpha_a t_a$$

$\alpha^a(x)$ are the group parameters, t_a ($a=1,2,3$) are 3×3 matrices, $(t_a)_{bc} = \epsilon_{bac}$; $[t_a, t_b] = \epsilon_{abc} t_c$.

In order to construct a perturbation theory we begin by writing provisorily for the generating functional of the Green's function

$$Z(j_\mu) = N^{-1} \int \prod_a \prod_\mu d\varphi_\mu^a e^{i \int d^4x L_{Y.M.}(\varphi_\mu) + i \int d^4x j_\mu^a(x) \varphi_\mu^a(x)} \quad (4)$$

N is a normalization factor so that $Z(0) = 1$, j_μ^a an external source. The intention is to use the method described in ¹²⁾, namely, to expand the exponential in (4) at some classical fields $\varphi_{\mu,cl}$, which are solutions of a specific classical equation of motions and then to reduce the integral in (4) to be a sum of functional integrals of Gaussian type. Each of them, after evaluation, may be interpreted as a sum of diagrams with a definite number of loops and a definite topological structure.

The naïve field equations would read $[\bar{S} = \int d^4x L_{Y.M.}(\varphi_\mu(x))]$

$$\frac{\delta \bar{S}}{\delta \varphi_\mu^a(x)} + j_\mu^a(x) = 0 \quad (5a)$$

or

$$-(D_\nu)_{ac} \varphi_{\mu\nu}^c + j_\mu^a(x) = 0 \quad (5b)$$

where

$$(D_\mu)_{ac} = \partial_\mu \delta_{ac} - ig (\varphi_\mu)_{ac} \quad (6)$$

$(D_\mu)_{ab}$ is the covariant derivative. We have

$$[D_\mu, D_\nu] = -ig \varphi_{\mu\nu} \quad (7)$$

where D_μ is the matrix with components $(D_\mu)_{ab}$ and $(\varphi_{\mu\nu})_{ab} = \varphi_{\mu\nu}^c (t_c)_{ab}$

One easily deduces

$$(D_\mu)_{ac} (D_\nu)_{cb} \varphi_{\mu\nu}^c = 0 \quad (8)$$

for every J_μ , hence from (5b) we infer

$$(\mathcal{D}_\mu)_{ac} \partial_\mu^c \equiv 0 \quad (9)$$

contrary to the assumption that $J_\mu^b(x)$ is an arbitrary external source which does not depend on $J_\mu^a(x)$.

This is a well-known difficulty ¹¹⁾ with non-Abelian gauge theories. In the following we will therefore use instead of Eq. (5) the equation

$$-(\mathcal{D}_\nu)_{ac} J_{\mu\nu}^c + \partial_{\mu,a} - (\mathcal{D}_\mu \bar{G} \mathcal{D}_\sigma \partial_\sigma)_a = 0 \quad (10a)$$

where $\bar{G}_{bc}(x, x', J_\mu)$ is defined by

$$(\mathcal{D}_\tau \mathcal{D}^\tau)_{ac} \bar{G}_{\mu c}(x, x', J_\mu) = \delta_{ac} \delta(x, x') \quad (11)$$

and $(\mathcal{D}_\mu \bar{G} \mathcal{D}_\sigma J_\sigma)^a$ does mean:

$$\int d^4x' (\mathcal{D}_\mu)_{ac} \bar{G}_{\mu c}(x, x', J) (\mathcal{D}_\sigma')_{ca} \partial_\sigma^c(x')$$

Equation (10) may be written as

$$\partial_\nu (\partial_\mu J_\nu^a - \partial_\nu J_\mu^a) = \partial_\mu^a - \partial_\mu G_0 \partial_\sigma \partial_\sigma + O(g)$$

hence

$$J_\mu^a(x) = \int d^4x' G_{\mu\nu}^F(x, x') \left\{ \partial_\nu^a(x') - \partial_\nu'(G_0 \partial_\sigma \partial_\sigma)(x') + O(g) \right\} \quad (10b)$$

where

$$G_{\mu\nu}^F(x, x') = (2\pi)^{-4} \int d^4k e^{ik(x-x')} \frac{1}{k^2 - i\epsilon} (g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2})$$

α arbitrary.

By iteration of Eq. (10b) one obtains an expansion of φ_μ^a in terms of J_μ^a . Graphically, each term of this series corresponds to a tree graph¹³⁾.

At the very end of the calculation of some Green's function we may insert for $J_\mu^a(x)$, e.g., $j_\mu^a(x) \square$, where $j_\mu^a(x)$ is a field being on-mass shell identical to $\varphi_\mu^{a,in}$, which obeys the equation

$$\square \varphi_\mu^{a,in} = 0 \quad ; \quad \partial_\mu \varphi_\mu^{a,in} = 0$$

This prescription will make the term $\int d^4x' G_{\mu\nu}^F(x, x') J_\nu^a(x')$ for an external leg to just $\varphi_\mu^{a,in}(x)$, whereas all other expressions coupled to the source will be reduced to zero. In this way we are led to an on-mass shell amplitude.

The solution of Eq. (10) will be denoted by $\varphi_{\mu,cl}^a$. We write

$$\varphi_\mu^a = \varphi_{\mu,cl}^a + A_\mu^a$$

and $\varphi_{\mu,cl}$ transforms exactly as φ_μ [see Eq. (3)] while for A_μ we have

$$\tilde{A}_\mu^i = e^{i\alpha} \tilde{A}_\mu^i e^{-i\alpha} \quad (12)$$

The expansion of the exponent in (4) at the classical field $\varphi_{\mu,cl}$ now reads:

$$\begin{aligned} \mathcal{J}(\varphi_\mu) + \int d^4x \partial_\mu^a \varphi_\mu^a &= \mathcal{J}(\varphi_{\mu,cl}) + \int d^4x \partial_\mu^a \varphi_{\mu,cl}^a \\ &+ (\varphi_{\mu,cl}^a + \partial_\mu^a) A_a^\mu + \frac{1}{2!} \int_{\mu\nu}^{ab}(\varphi_{cl}) A_a^\mu A_b^\nu \\ &+ \frac{1}{3!} \int_{\mu\nu\sigma}^{abc}(\varphi_{cl}) A_a^\mu A_b^\nu A_c^\sigma + \frac{1}{4!} \int_{\mu\nu\sigma\tau}^{abcd}(\varphi_{cl}) A_a^\mu A_b^\nu A_c^\sigma A_d^\tau \end{aligned}$$

where in the last four terms we have used the following abbreviations.
An index, e.g., like μ does not only stand for itself but also for the co-ordinate variable x , and the summation convention now means a summation over μ and an integration over x . Thus, e.g.,

$$\int_{\mu\nu}^{ab} A_a^\mu A_b^\nu = \int d^4x d^4x' \sum_{\mu\nu} \sum_{a,b} \int_{\mu\nu}^{ab}(x,x',\xi_\mu^a) A_a^\mu(x) A_b^\nu(x')$$

In particular we get

$$\begin{aligned} \int_{\mu\nu}^{ab} &= \frac{\delta^2 \mathcal{L}}{\delta \xi_\nu^b(x) \delta \xi_\mu^a(x)} = \\ &= ((D_\tau D^\tau)_{ab} g_{\mu\nu} - (D_\mu D_\nu)_{ab} - 2ig (\rho_{\mu\nu})_{ab}) \delta(x,x') \quad (14a) \end{aligned}$$

$$\begin{aligned} \int_{\mu\nu\sigma}^{abc} &= -ig g_{\mu\nu} [\epsilon_{acd} (D_\sigma \delta(x,x'))_{ab} \delta(x,x'') + \\ &+ \epsilon_{acd} (D_\sigma \delta(x,x''))_{ab} \delta(x,x') + \epsilon_{bad} (D_\sigma \delta(x,x'))_{ac} \delta(x,x'')] \\ &- ig g_{\mu\sigma} [- \epsilon_{acd} (D_\nu \delta(x,x'))_{ab} \delta(x,x'') - 2\epsilon_{bad} (D_\nu \delta(x,x''))_{ac} \delta(x,x')] \\ &- ig g_{\nu\sigma} [2 \epsilon_{bad} (D_\mu \delta(x,x''))_{ac} \delta(x,x') - \\ &- \epsilon_{acd} (D_\mu \delta(x,x''))_{ab} \delta(x,x') - \epsilon_{bad} (D_\mu \delta(x,x'))_{ac} \delta(x,x'')] \quad (14b) \end{aligned}$$

$$\begin{aligned} \int_{\mu\nu\sigma\tau}^{abcd} &= g^2 \delta(x,x') \delta(x,x'') \delta(x,x''') [\epsilon_{acm} \epsilon_{mcd} (g_{\mu\tau} g_{\nu\sigma} - \\ &- g_{\nu\tau} g_{\mu\sigma}) + \epsilon_{acm} \epsilon_{mcb} (g_{\mu\nu} g_{\sigma\tau} - g_{\mu\tau} g_{\sigma\nu}) + \\ &+ \epsilon_{adm} \epsilon_{mcb} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\nu} g_{\sigma\tau})] \quad (14c) \end{aligned}$$

The generating functional (4) may now be written as

$$Z(\mathcal{J}_\mu) = \bar{N}^1 e^{i\mathcal{J}(\varphi_c) + i \int d^4x \mathcal{J}_\mu^a \varphi_{a,c}^\mu} \int \prod_x \prod_a \prod_\mu dA_\mu^a(x) e^{i(\mathcal{J}_\mu^a + \mathcal{J}_\mu^a) A_\mu^a + i/2 \int_{\mu\nu}^{ab} A_\mu^a A_\nu^b (1+\dots)} \quad (15)$$

where the exponential function has been expanded with the exception of the terms linear and bilinear in $A_\mu^a(x)$. If one changes the gauge of $\mathcal{J}_{\mu,cl}(x)$, one can absorb this with a change of A_μ^a like (12) *) because the functional derivatives of the action transform covariantly. Hence, the integral (15) is independent of the gauge of the classical field.

Because of the invariance of $L_{Y.M.}$ there will not only be one classical solution but a whole manifold of solutions because with $\mathcal{J}_{\mu,cl}$ also $\mathcal{J}'_{\mu,cl}(x)$ fulfils Eq. (10) up to terms on the right-hand side of (10) which vanish on the mass shell.

In order to avoid this degeneracy one usually adds to the invariant Lagrangian a non-invariant one, L' , which breaks all local gauge symmetries and fixes the gauge ¹⁴⁾. It is constructed in such a way that the S matrix on-mass shell does not depend on the general structure of L' , i.e., on the gauge.

If, e.g., we choose the gauge condition

$$\partial_\mu \varphi_a^\mu - C_a = 0 \quad (16)$$

this would mean in this approach

$$\partial_\mu A_a^\mu - C_a' = 0 \quad (17)$$

For fixed $\mathcal{J}_{\mu,cl}$ the transformation $\mathcal{J}_\mu \rightarrow \mathcal{J}'_\mu$, can be achieved with

*) Up to the term $\int_{\mu\nu}^{ab} A_\mu^a A_\nu^b$, but this corresponds to the gauge dependence off-mass shell.

$$\vec{A}_\mu \rightarrow \vec{A}_\mu^{[\alpha]} = \vec{A}_\mu + \frac{1}{g} (\vec{D}_\mu \alpha) \quad (18)$$

where

$$\vec{A}_\mu^\alpha = \vec{A}_\mu + i (\vec{\alpha} \times \vec{A}_\mu)$$

and $\vec{D}_\mu$, now and in the following, is constructed with the classical field:

$$(\vec{D}_\mu)_{ab} = \partial_\mu \delta_{ab} - ig f_{abc}^c \vec{A}_\mu^c \epsilon_{abc}$$

Hence, to take into account the constraint (17), one should insert into (4)

$$\delta(\partial_\mu A_\mu^a - C_a') \Delta(A_\mu, g_{cl}) \quad (19)$$

where $\Delta(A_\mu, g_{cl})$ has to be chosen so that ¹⁴⁾

$$\Delta(A_\mu, g_{cl}) \int d\alpha \delta(\partial_\mu A_\mu^{a[\alpha]} - C_a') = 1$$

This, obviously, leads to the term

$$\Delta(A_\mu, g_{cl}) = \text{Det} \left(\frac{1}{g} \partial_\mu (\partial_\mu - ig f_{\mu\nu}^c) \right) \quad (20)$$

given in the literature ^{6),7),8)}.

By construction, the expression thus obtained for (19) is not covariant with respect to the transformations $g_{\mu,cl} \rightarrow g'_{\mu,cl}, A_\mu \rightarrow A'_\mu$. This, of course, is just required to fix the gauge.

In this approach, in which one has introduced the classical field $g_{\mu,cl}$, one may preserve the degeneracy, and one may require in a covariant fashion

$$\vec{D}_\mu A_\mu - \vec{C} = 0 \quad (21)$$

The corresponding $\bar{\Delta}$ will be calculated from the condition

$$\bar{\Delta}(A_\mu, f_{\mu,ce}) \int dx \delta(D_\mu A^\mu(x) - \bar{C}) = 1 \quad (22)$$

This leads to

$$\bar{\Delta} = \text{Det}(M_{ac}) \quad (23a)$$

where

$$M_{ac} = \frac{1}{g} (D_\tau D^\tau)_{ac} - i(D_\mu)_{ac} \epsilon_{\mu dc} A^\mu_d \quad (23b)$$

Hence, the modified generating functional now reads

$$\begin{aligned} Z(J) = & \bar{N}^{-1} e^{i(\int f_{ce}) + \int d^4x J_\mu^a f_{a,\mu,ce}} \\ & \int \prod_x \prod_\mu \prod_a dA_\mu^a(x) \delta(D_\mu A^\mu - \bar{C}) \bar{\Delta}(A_\mu, f_{ce}) \\ & e^{-i(D_\tau D^\tau)_a \bar{G}_{ac} (D_\mu A^\mu)_c} e^{i/2 \tilde{f}_{\mu\nu}^{ab} A_a^\mu A_b^\nu} (1 + \dots) \end{aligned} \quad (24)$$

$Z(J)$ is independent of $\bar{C}_a$ by construction. To get rid of the exponent linear in $A_\mu(x)$, we will choose in the following $\bar{C} \equiv 0$.

Then the exponent bilinear in A_μ^a will reduce to

$$i/2 \tilde{f}_{\mu\nu}^{ab} A_a^\mu A_b^\nu$$

where

$$\tilde{f}_{\mu\nu}^{ab}(x, x', f_{\mu,ce}) = ((D_\tau D^\tau)_{ab} g_{\mu\nu} - 2ig(f_{\mu\nu,ce})_{ab}) \delta(x, x') \quad (25)$$

The inverse $\tilde{G}_{\mu\nu}^{ab}(x, x', \gamma_{cl})$ of this differential operator is defined by

$$\int d^4x'' \tilde{f}_{\mu\tau}^{ac}(x, x'') \tilde{G}_{\tau\nu}^{cb}(x'', x') = g_{\mu\nu} \delta^{ab} \delta(x, x') \quad (26)$$

Up to the term $\bar{\Delta}$ in (24) all functional integrals are already of Gaussian type. We write

$$\begin{aligned} \bar{\Delta} &= \text{Det} \left(\frac{1}{g} (D_\tau D^\tau)_{ac} \right) \text{Det} \left(\delta(x, x') \delta_{ac} - \right. \\ &\quad \left. -ig (D_\mu)_{an} \epsilon_{nac} A_n^\mu \bar{G}_{cb}(x, x'; \gamma_{cl}) \right) \\ &= \text{Det} \left(\frac{1}{g} (D_\tau D^\tau)_{ac} \right) \exp \left\{ -ig \int d^4x (D_\mu)_{an} \epsilon_{nac} A_n^\mu(x) \right. \\ &\quad \left. \bar{G}_{ca}(x, x', \gamma_{cl}) \right|_{x'=x} + \\ &\quad + \frac{g^2}{2} \int d^4x d^4x' \left((D_\mu)_{ab} \epsilon_{bac} A_a^\mu \bar{G}_{cn}(x, x') \right) \left((D_\tau)_{nm} \epsilon_{mnc} A_n^\tau(x') \bar{G}_{ca}(x', x) \right) \\ &\quad \left. + \dots \right\} \quad (27) \end{aligned}$$

Expanding also the exponential of (27) we may evaluate all functional integrals with the help of the formula

$$\begin{aligned} &\int \prod_x \prod_\mu \prod_a dA_\mu^a(x) \delta(D_\mu A^\mu) e^{i/2 \int_{\mu\nu} \tilde{f}_{\mu\nu}^{ab} A_a^\mu A_b^\nu} A_{\sigma_1}^{c_1} \dots A_{\sigma_{2n}}^{c_{2n}} \\ &= \begin{cases} 0 & \text{if } 2n \text{ odd} \\ \text{const.} \times (\text{Det } \tilde{f}_{\mu\nu}^{ab})^{-1/2} \text{Det} (D_\mu \tilde{G}_{\mu\nu} D_\nu)^{-1/2} \\ \quad i^{-3n} \left(\hat{G}_{\sigma_1 \sigma_2}^{c_1 c_2} \dots \hat{G}_{\sigma_{2n-1} \sigma_{2n}}^{c_{2n-1} c_{2n}} + \text{all permutations of } \sigma_i \right) & \text{if } 2n \text{ even} \end{cases} \quad (28) \end{aligned}$$

where

$$\hat{G}_{\sigma_1 \sigma_2}^{c_1 c_2} = \tilde{G}_{\sigma_1 \sigma_2}^{c_1 c_2} - \tilde{G}_{\sigma_1 \mu}^{c_1 d_1} (\tilde{D}_{\mu})^{d_1 d_2} F^{d_2 d_3} (\tilde{D}_{\lambda})^{d_3 d_4} \tilde{G}_{\lambda \sigma_2}^{d_4 c_2} \quad (29a)$$

and

$$\tilde{F} = (\tilde{D}_{\mu} \tilde{G}_{\mu\nu} \tilde{D}_{\nu})^{-1} \quad (29b)$$

Note that for $\hat{G}$ we have

$$\tilde{D}_{\sigma_1} \hat{G}_{\sigma_1 \sigma_2} = \tilde{D}_{\sigma_1} \tilde{G}_{\sigma_1 \sigma_2} - \tilde{D}_{\lambda} \tilde{G}_{\lambda \sigma_2} \equiv 0$$

as well as

$$\hat{G}_{\sigma_1 \sigma_2} \tilde{D}_{\sigma_2} \equiv 0$$

Formula (28) has been obtained by writing

$$\delta(\tilde{D}_{\mu} A^{\mu}) = \int \prod_x \prod_a d\beta^a(x) e^{i \int d^4x \beta^a(x) (\tilde{D}_{\mu} A^{\mu})_a}$$

and substituting

$$A_{\mu} = A'_{\mu} + \tilde{G}_{\mu\nu} D_{\nu} B$$

Hence, we finally get

$$Z(\beta) = \tilde{N}^{-1} e^{i \left(\int (\beta_{\mu}) + \int d^4x \beta_{\mu}^a \tilde{J}_{\mu}^a \right)}$$

$$(\text{Det } \tilde{J}_{\mu\nu}^{ab})^{-1/2} (\text{Det } (D_{\tau} D^{\tau})_{ac}) (\text{Det } \tilde{F})^{1/2}$$

$$\left(1 + \frac{i}{2!3!3!} \tilde{J}_{\mu\nu\sigma}^{abc}(\beta_{\mu}) \tilde{J}_{\tau\lambda\kappa}^{def} (\hat{G}_{\mu\nu}^{ab}(\beta_{\mu}) \hat{G}_{\tau\lambda}^{de} \hat{G}_{\sigma\kappa}^{ef} + \dots \right)$$

$$\begin{aligned}
 & + \hat{G}_{\mu\tau}^{ad} \hat{G}_{\nu\lambda}^{bc} \hat{G}_{\sigma\kappa}^{cf} + \dots) \\
 & + \frac{(-i)}{4!} \int_{\mu\nu\sigma\tau}^{abcd} (y_{ce}) (\hat{G}_{\mu\nu}^{ae} (1_{ce}) \hat{G}_{\sigma\tau}^{cd} + \dots) + \dots \\
 & - i \frac{g^2}{2} \int d^4x d^4x' (D_\mu)_{ac} \epsilon_{bad} \bar{G}_{ca}(x, x'') (D'_\tau)_{nm} \epsilon_{mek} \bar{G}_{kn}(x', x''') \\
 & \hat{G}_{\mu\tau}^{ae} (x, x'))_{x''=x, x'''=x'} \\
 & - \frac{g}{3!} \int d^4x d^4x' d^4x'' d^4y \int_{\mu\nu\sigma}^{abc} (x, x', x'') \hat{G}_{\nu\sigma}^{bc}(x', x'') (D_\tau^\eta)_{en} \epsilon_{nac} \bar{G}_{ce}(y, y') \\
 & \hat{G}_{\mu\tau}^{ad} (x, y))_{y'=y} \\
 & + i \frac{g^2}{2} \int d^4x d^4x' (D_\mu)_{ac} \epsilon_{bad} \bar{G}_{cn}(x, x') (D'_\tau)_{nm} \epsilon_{mek} \\
 & \bar{G}_{ka}(x', x) \hat{G}_{\mu\tau}^{ae} (x, x') + \dots) \tag{30}
 \end{aligned}$$

There exists a definite meaning of all terms in this series.

The first term

$$e^{i \left(\int (y_{ce}) + \int d^4x j_\mu^a(x) p_{a,ce}^\mu(x) \right)}$$

corresponds to the tree diagrams, connected and disconnected ones.

The term

$$\Sigma = \left(\text{Det } \tilde{f}_{\mu\nu}^{ab} \right)^{-1/2} \text{Det } (D_\tau^\tau D_\tau^\tau)_{ac} (\text{Det } \tilde{F})^{1/2}$$

corresponds to the sum of the one-loop diagrams with an arbitrary number of external $\int_{\mu, cl}$ legs.

The expression for the connected one-loop diagrams may be written as

$$-i \log \Sigma = -\frac{i}{2} \text{Tr} \log \tilde{G}_{\mu\nu}^{ab} + i \text{Tr} \log \bar{G}_{ab} - \frac{i}{2} \text{Tr} \log F$$

The third term $\text{Tr} \log (D_\mu (\tilde{G}_{\mu\nu} D_\nu))$ is equal to zero on the mass shell, as will be shown in a moment. Graphically, the first two terms may be represented as depicted in Fig. 1, where a wavy line symbolizes the propagator $\tilde{G}$, a dashed line the propagator $\bar{G}$. The external $\not{p}_{\mu,cl}$ legs here and in the following graphs will not be indicated.

The other terms written out in (30) correspond to the two-loop diagrams shown in Fig. 2 with the propagator $\hat{G}$ symbolized by a solid line. The functional determinant $\bar{\Delta}$ in (25) produces all diagrams with the dashed line (or the propagator $\bar{G}_{ab}$) which correspond in the usual approach to the graphs with a fictitious scalar particle ^{6),7),8)}.

An iterative solution of the equation for $\not{p}_{\mu,cl}$ like (10a) will exhibit $\not{p}_{\mu,cl}$ as a sum of tree diagrams with the source J_μ^a at the end of each leg. Hence, one may attach any tree graph structure to the loop diagrams using this expansion of the classical field in terms of J_μ^a .

Now let us look at the term $\text{Tr} \log (D_\mu \tilde{G}_{\mu\nu} D_\nu)$. With the help of Eq. (7) we derive

$$(\tilde{D}_{\mu\nu} D_\nu)_{ab} = (D_\nu)_{ac} (D_\tau D_\tau^\dagger)_{cb} - ig (D_\tau p_{\tau\nu})_a (t_a)_{cb}$$

Hence, using the classical field equation (10):

$$D_\nu \bar{G} = \tilde{G}_{\nu\sigma} D_\sigma + ig \tilde{G}_{\nu\sigma} (\partial_\sigma^\alpha - (D_\sigma \bar{G} D_\sigma)^\alpha) t_\alpha \bar{G}$$

Both source terms of the right-hand side will not contribute on the mass shell, so that up to terms which vanish on the mass shell we have

$$D_\nu \bar{G} = \tilde{G}_{\nu\sigma} D_\sigma$$

Then

$$(D_\nu \tilde{G}_{\nu\sigma} D_\sigma)_{ab} = (D_\tau D_\tau^\dagger)_{ac} \bar{G}_{cb}(x,x') = \delta_{ab} \delta(x,x')$$

and the term $\text{Det}(D_\mu \tilde{G}_{\mu\nu} D_\nu)$ is just equal to one on the mass shell.

Till now we have chosen the special constraint $D_\mu A^\mu \equiv 0$ to get rid of the exponent in Eq. (24) linear in A_μ . However, the term $\bar{G} D_\mu J^\mu$ does not contribute on the mass shell. Leaving $\bar{C}$ in Eq. (24) arbitrary, one may integrate the right-hand side over $\bar{C}$ with the weight

$$e^{-i/2 \int d^4x \bar{C}_a(x) \bar{C}^a(x)}$$

leading to

$$Z(\bar{J}_\mu) = \bar{N}^{-1} e^{i \left(\int d^4x \bar{J}_\mu^a A_\mu^a + \int d^4x \bar{J}_\mu^a \hat{P}_\mu^a \right)}$$

$$\int \prod_{x, \mu, a} dA_\mu^a(x) \bar{\Delta}(A_\mu, \bar{J}_\mu) e^{i/2 \int d^4x \tilde{F}_{\mu\nu}^{ac} A_\mu^a A_\nu^c} (1 + \dots)$$

Evaluating this integral in the same fashion as above we obtain a series which is identical to that of Eq. (30) with the exception that one always has to use instead of the propagator $\hat{G}$ the propagator $\tilde{G}$ in the functionals which describe the diagrams with two and more loops. The first terms in the expansions of $\hat{G}$ and $\tilde{G}$ in terms of the classical field correspond to a free propagator in the Landau gauge and in the Feynman gauge respectively. Hence, the independence on $\bar{C}$ of the generating functional on the mass shell tells us that we may use the propagator $\hat{G}$ or, equally, $\tilde{G}$, or more generally, that we may take the free propagator in any gauge. The expression for the one-loop functional is always the same on the mass shell. For the expression of the higher diagrams this can be shown also. We will postpone a detailed discussion of these points to a treatment of a massive version of the Yang-Mills theory.

3. THE ONE-LOOP FUNCTIONAL

In this Section we will calculate the first terms of the expansion of the one-loop functional in terms of the classical field.

All divergence problems which will arise in this massless Yang-Mills theory may already be seen at those terms.

Let us first evaluate

$$\log \text{Det}((D_\mu D^\mu)_{ab} G_0(x, x')) = \text{Tr} \log \left(\delta(x, x') \delta_{ab} - (2ig \hat{F}_{\mu\nu}^{ac} \partial^\mu + \right.$$

$$\left. ig (\partial_\mu \hat{F}_{\nu\alpha}^{ac} + g^2 (\hat{F}_{\mu\alpha}^{ac} \hat{F}_{\nu\alpha}^{ac})) G_0(x, x') \right)$$

where $G_0(x, x')$ is the free propagator with zero mass

$$G_0(x, x') = (2\pi)^{-4} \int d^4p \, e^{ip(x-x')} \frac{1}{-p^2 + i0} \quad (31)$$

Expanding the logarithm we obtain

$$\begin{aligned} & - \int d^4x \left[(2ig \, \tilde{f}_{\mu, ce}^{ab}(x) \partial_\mu + ig \, (\partial_\mu \tilde{f}_{ce}^{\mu, ab})^{ab}) \delta_{ab} G_0(x, x) + \right. \\ & \quad \left. + g^2 \, (\tilde{f}_{ce}^{\mu, ab} \tilde{f}_{\mu, ce})_{aa} G_0(x, x) \right] \\ & - \frac{1}{2} \int d^4x \, d^4x' \left(2ig \, \tilde{f}_{\mu, ce}^{ab}(x) \partial_\mu G_0(x, x') + ig \, \partial_\mu \tilde{f}_{ce}^{\mu, ab}(x) G_0(x, x') + O(g^2) \right) \cdot \\ & \quad \cdot \left(2ig \, \tilde{f}_{\nu, ce}^{ba}(x') \partial'_\nu G_0(x', x) + ig \, \partial'_\nu \tilde{f}_{ce}^{\nu, ba}(x') G_0(x', x) + O(g^2) \right) \\ & - \frac{1}{3} \dots \end{aligned} \quad (32)$$

$$\begin{aligned} & = -g^2 \int d^4x \, \tilde{f}_{\mu, ab}^{ce}(x) \, \tilde{f}_{\mu, ba}^{ce}(x) G_0(x, x) \\ & - \frac{1}{2} (2\pi)^{-4} \int d^4p \, d^4k \, \frac{1}{p^2 + i0} \frac{1}{(p+k)^2 + i0} \left(g(2p_\mu + k_\mu) \tilde{f}_{ab, ce}^\mu(k) + O(g^2) \right) \\ & \quad \left(g(2p_\nu + k_\nu) \tilde{f}_{ba, ce}^\nu(-k) + O(g^2) \right) \\ & - \frac{1}{3} \dots \end{aligned} \quad (33)$$

where we have introduced the Fourier-transformed by

$$\tilde{f}_\mu^{ab}(x) = (2\pi)^{-2} \int d^4k \, \tilde{f}_\mu^{ab}(k) e^{ikx}$$

Shifting $p \rightarrow p - \frac{1}{2}k$ we encounter the integral

$$\int d^4p \ 4 p_\mu p_\nu \frac{1}{(p - \frac{1}{2}k)^2} \frac{1}{(p + \frac{1}{2}k)^2}$$

which, following Schwinger ¹⁰⁾ may be written as

$$\lim_{s_0 \rightarrow 0} \left\{ 2\pi^2 \int_{s_0}^{\infty} ds \ \bar{s}^{-2} g_{\mu\nu} + i\pi^2/2 \int_{s_0}^{\infty} ds \ \bar{s}^{-1} \int_{-1}^{+1} dv v^2 (k_\mu k_\nu - g_{\mu\nu} k^2) e^{-i\frac{k^2}{4}(1-v^2)s} \right\} \quad (34)$$

The first term on the right-hand side of (34) just cancels the first term (33), because we may write

$$\begin{aligned} G_0(x, x) &= (2\pi)^{-4} \int d^4p \frac{1}{-p^2 + i0} = -(2\pi)^{-4} \int d^4p \ i \int_{s_0 \rightarrow 0}^{\infty} ds \ e^{-ip^2 s} \\ &= - \int_{s_0 \rightarrow 0}^{\infty} ds \ \frac{\pi^2}{(2\pi)^4} \bar{s}^{-2} \end{aligned} \quad (35)$$

For the second divergent integral in (34) we may write

$$\int_{s_0 \rightarrow 0}^{\infty} ds \ \bar{s}^{-1} e^{-i\frac{k^2}{4}(1-v^2)s} = C - \log(-\frac{k^2}{4}(1-v^2)) \quad (36)$$

where C means a logarithmic divergent constant. Hence, to the order g^2 we obtain

$$\begin{aligned} \log \text{Det}((D_\tau D^\tau)_{ab} G_0(x, x)) &= \\ &= g^2 \frac{i\pi^2}{(2\pi)^4} \frac{2}{3} \frac{1}{4} \int d^4k \ \tilde{p}_{\mu,ce}^{ab}(k) \tilde{p}_{\nu,ce}^{ba}(-k) (k_\mu k_\nu - g_{\mu\nu} k^2) (\log(-k^2/a^2) + C) \\ &= \frac{i\pi^2}{(2\pi)^4} g^2 \left(-\frac{1}{12}\right) \int d^4k (\partial_\mu \tilde{p}_{\nu,ce}^{ab} - \partial_\nu \tilde{p}_{\mu,ce}^{ab})(k) (\partial_\mu \tilde{p}_{\nu,ce}^{ba} - \partial_\nu \tilde{p}_{\mu,ce}^{ba})(-k) \\ &\quad (\log(-k^2/a^2) + C) \end{aligned} \quad (37)$$

where a is some reference momentum.

With some more effort one may complete the expressions $\partial_\mu \tilde{p}^{ab} - \partial_\nu \tilde{p}^{ab}$ in (37) to $\tilde{p}_{\mu\nu}^{ab}(k)$.

Next we have to calculate

$$\begin{aligned} \frac{1}{2} \log \text{Det}(\tilde{G}_{\mu\nu}^{ab}) &= -\frac{1}{2} \log \text{Det}((D_\tau D^\tau)^{ab} g_{\mu\nu}) - \\ &- \frac{1}{2} \log \text{Det}(d(x, x') \delta^{ab} g_{\mu\nu} - 2ig \tilde{p}_{\mu\nu, ce}^{ac}(x) \bar{G}_{ce}(x, x', s_{ce})) \\ &= -2 \log \text{Det}((D_\tau D^\tau)_{ab}) - g^2 \int d^4x d^4x' \tilde{p}_{\mu\nu, ce}^{ac}(x) \bar{G}_{ce}(x, x') \\ &\quad \tilde{p}_{\nu\mu, ce}^{ba}(x') \bar{G}_{ba}(x', x) + \dots \end{aligned} \quad (38)$$

Up to the order g^2 we may use the second term of (38) $\delta_{cb} G_0(x, x')$ for $\bar{G}_{cb}(x, x'; \tilde{p}_{\mu, ce})$ leading to

$$- g^2 (2\pi)^4 \int d^4k \tilde{p}_{\mu\nu, ce}^{ac}(k) \tilde{p}_{\nu\mu, ce}^{ba}(-k) (i\pi^2 \log(-k^2/a^2) + c)$$

so that we effectively get to order g^2

$$g^2 \frac{i\pi^2}{(2\pi)^4} \int d^4k \tilde{p}_{\mu\nu, ce}^{ac}(k) \tilde{p}_{\nu\mu, ce}^{ba}(-k) \left(-\frac{11}{12} \log(-k^2/a^2) + c \right)$$

The coefficient in front of the divergent constant c is just proportional to

$$\int d^4x \tilde{p}_{\mu\nu, ce}^{ac}(x) \tilde{p}_{\nu\mu, ce}^{ba}(x)$$

so that this term may be added to $S(\tilde{p}_{\mu, ce})$ to give a wave function renormalization and a renormalization of g . Because, if one writes

$$Z \tilde{p}_{\mu, ce} = \tilde{p}_{\mu, ce}^{\text{ren.}}$$

then, one has to write

$$g/Z = g^{\text{ren.}}$$

to get

$$Z \tilde{p}_{\mu\nu, ce} = \tilde{p}_{\mu\nu, ce}^{\text{ren.}}$$

This situation, that the equality of the renormalization constants for the wave function and the coupling constant comes out automatically, is very similar to the Q.E.D. ¹⁰⁾.

The finite part looks also very similar to the Q.E.D., for massless electrons. There, we get for the self-energy graph ¹⁰⁾

$$\int d^4k F_{\mu\nu}(k) F^{\mu\nu}(-k) \log(-k^2/a^2)$$

But under gauge transformations we have

$$F_{\mu\nu}(k) \rightarrow F_{\mu\nu}(k)$$

The gauge group is Abelian and the finite part is gauge invariant. But in the Yang-Mills case we have under gauge transformations

$$\delta \vec{F}_{\mu\nu}(x) = \vec{A}(x) \times \vec{F}_{\mu\nu}(x)$$

in momentum space this means a convolution and the finite part of this self-energy graph alone cannot be gauge invariant.

However, this cannot be expected because it is an off-shell matrix element. But, there is no possibility to go on the mass shell, all amplitudes will diverge. This is an expression for the infra-red problem which arises because the vector mesons ρ_μ have zero mass.

At least one may say that the sum of diagrams which corresponds to a complete functional of Eq. (30) is also off-mass shell independent of the gauge of $\rho_{\mu,cl}$.

To make this statement rigorous for the case of the one-loop functional let us calculate this functional in co-ordinate space.

First, we will look for $\bar{G}_{ab}(x, x', \rho_{cl})$ defined by

$$(\partial_\tau \partial^\tau)_{ac} \bar{G}_{cb}(x, x') = \delta_{ab} \delta(x, x')$$

Following De Witt ^{6), 11)} we make the ansatz

$$\bar{G}_{ab}(x, x', \rho_{cl}) = -i \int_0^\infty ds \langle x, 0 | x', 0 \rangle_{ab} \quad (39)$$

where the transition amplitude has to obey the equation

$$(D_\tau D^\tau)_{ac} \langle x, s | x', 0 \rangle_{cb} = -i \frac{\partial}{\partial s} \langle x, s | x', 0 \rangle_{ab} \quad (40a)$$

and the boundary condition

$$\langle x, s | x', 0 \rangle_{ab} \xrightarrow{s \rightarrow 0} \delta(x, x') \delta_{ab} \quad (40b)$$

Given $\bar{G}$ in the form of (39) we have ^{11)*)}

$$-\frac{1}{2} \text{Tr} \log \bar{G} = -\frac{1}{2} \int_0^\infty ds \int d^4x \langle x, s | x, 0 \rangle_{aa} \quad (41)$$

Of course, the last equation is only a formal one because the integral over s may not exist due to a bad s behaviour at $s \rightarrow \infty$ and at $s \rightarrow 0$.

The behaviour at $s \rightarrow \infty$ is connected with the infra-red problem, which is, anyway, not tractable in the massless Yang-Mills case. Thus, we concentrate on the behaviour of the s integrand near $s=0$.

To do this we may apply a series expansion in s by making the ansatz:

$$\langle x, s | x', 0 \rangle = -i (4\pi)^{-2} s^{-2} e^{\frac{i(x-x')^2}{4s}} \sum_{n=0}^{\infty} a_n(x, x', s) (is)^n \quad (42)$$

which is motivated by the free equation ($\not{D}_{\mu, cl} = 0$). The $a_n(x, x', \not{D}_{cl})$ are matrices like the transition amplitude.

To fulfil the condition (40b) we have to require

$$a_0(x, x')_{cb} \xrightarrow{x' \rightarrow x} \delta_{cb} \quad (43)$$

and the equation (40a) now leads to a recursion relation for a_n

$$(x-x')^\mu D_\mu a_0 = 0 \quad (44a)$$

*) From the derivation of Eq. (41) in Ref. 11) one can immediately see that this formulation breaks down, e.g., in spinor electrodynamics with γ_5 invariance.

$$(x-x')^\mu \mathcal{D}_\mu a_{n+1} + (n+1) a_{n+1} = \mathcal{D}_\tau \mathcal{D}^\tau a_n, \quad n=0,1,\dots \quad (44b)$$

Repeated differentiation of (44a) and (44b) and then going to the limit leads to

$$\lim_{x' \rightarrow x} \mathcal{D}_\mu a_0 = 0$$

$$\lim_{x' \rightarrow x} \mathcal{D}_\nu \mathcal{D}_\mu a_0 = i g/2 \, \underline{\rho}_{\mu\nu}^{ce}(x)$$

$$\lim_{x' \rightarrow x} \mathcal{D}_\tau \mathcal{D}^\tau \mathcal{D}_\nu \mathcal{D}^\nu a_0 = -g^2/2 \, \underline{\rho}_{\mu\nu}^{ce} \underline{\rho}^{\mu\nu}_{ce}(x)$$

$$\lim_{x' \rightarrow x} a_1 = 0$$

$$\lim_{x' \rightarrow x} a_2 = -g^2/12 \, \underline{\rho}_{\mu\nu}^{ce} \underline{\rho}^{\mu\nu}_{ce}(x)$$

(45)

We now define the regularized version of $\log \text{Det } \bar{G}$. We write

$$\begin{aligned} -\frac{1}{2} \log \text{Det } \bar{G} \big|_{\text{Reg.}} &= -\frac{1}{2} \text{Im} \int_0^\infty d\tau \bar{\tau}^{-1} \int d^4x \langle x, s | x, 0 \rangle_{aa} \\ &- \frac{1}{2} \int_0^\infty d\tau \bar{\tau}^{-1} \left\{ \text{Im} \int d^4x \langle x, s | x, 0 \rangle_{aa} - \frac{3}{16\pi^2} \bar{\tau}^2 - \pi \frac{1}{16\pi^2} \frac{g^2}{12} \int d^4x \underline{\rho}_{\mu\nu}^{ce}(x) \underline{\rho}^{\mu\nu}_{ce}(x) \right\} \end{aligned} \quad (46)$$

where tr means the trace with respect to the isospin indices. The crucial point is the following. Equation (40a) is a covariant one. $\langle x, s | x', 0 \rangle$ transforms under a gauge transformation of $\underline{f}_{\mu, cl}$

$$\langle x, s | x', 0 \rangle' = e^{i\alpha(x)} \langle x, s | x', 0 \rangle e^{-i\alpha(x')}$$

where on the left-hand side the transition amplitude is calculated with

$$\underline{f}_{\mu', cl}^{(x)}.$$

Hence, the integrand of the s integral is a gauge invariant finite quantity as the terms $1/s^2$ and $\text{tr} \int_{\mu\nu}^{cl} \int_{cl}^{\mu\nu}$ are which we subtract from it to make the s integral finite at $s=0$. Consequently the finite expression is also gauge invariant.

The expression

$$\log \text{Det} \left((D_\mu \bar{D}^\mu)_{ab} g_{\mu\nu} - 2ig \int_{\mu\nu}^{ce} \int_{ce}^{\mu\nu} \right)$$

can be treated similarly. For the corresponding matrices $a_1(x, x')_{\mu\nu}^{ab}$, $a_2(x, x')_{\mu\nu}^{ab}$ one has

$$\lim_{x' \rightarrow x} a_1(x, x')_{\mu\nu}^{ab} = 0$$

$$\lim_{x' \rightarrow x} a_2(x, x')_{\mu\nu}^{ab} = \frac{5}{3} g^2 + \int_{\mu\nu}^{ce} \int_{ce}^{\mu\nu}$$

CONCLUSIONS

We have shown at least in the one-loop approximation that the Yang-Mills theory can be renormalized invariantly. Because we deal always with invariant - or covariant - quantities there is no necessity of exploiting Ward identities. These are usually formulated as a constraint on the generating functional to keep track of the invariance of the Lagrangian. In this covariant approach the invariance remains manifest at each stage of the calculation. Hence, the Ward identities must be fulfilled automatically.

To establish the connection with the Ward identity approach more closely let us write down these identities in the functional form derived by Lee and Zinn-Justin ⁹⁾

$$-i \partial_\mu \frac{\delta Z}{\delta J_\mu^a(x)} + \alpha \int d^4y \partial_\lambda^c(y) D_\lambda^{\lambda c} \left(\frac{i\delta}{\delta J} \right)^{ca} \underline{G}^{ca}(y, x; \frac{i\delta}{\delta J}) Z = 0 \quad (47)$$

where $Z(J)$ is the generating functional and $\underline{G}(x, y; \rho)$ is a Green's function satisfying

$$\partial^\lambda (D_\lambda)_{ac} \underline{G}_{bc}(x, y, \rho) = \delta_{ac} \delta(x, y)$$

Because we now have

$$Z(\vartheta) = e^{i \int (\mathcal{L}_{cl}(\vartheta)) + i \int d^4x \partial_a^\mu \mathcal{P}_{\mu,cl}^a(\vartheta)} F(\mathcal{L}_{cl}(\vartheta))$$

where $F(\int_{cl}(J))$ corresponds to the loop diagrams we have

$$i \frac{\delta Z}{\delta \mathcal{P}_{\mu,cl}^a(x)} = \left(- \mathcal{P}_{\mu,cl}^a(x) + \int d^4x' \frac{\delta \mathcal{P}_{\nu,cl}^c(x')}{\delta \mathcal{P}_{\mu,cl}^a(x)} \frac{\delta}{\delta \mathcal{P}_{\nu,cl}^c(x')} \right) Z$$

Now that Eq. (47) of Lee and Zinn-Justin is fulfilled is most easily seen, if we choose a classical field $\mathcal{P}_{\mu,cl}$ with $\partial^\mu \mathcal{P}_{\mu,cl} = 0$, i.e., we make the choice $\alpha = 0$. Equation (47) then reads

$$\partial_\mu \mathcal{P}_{\mu,cl}^a(x) Z + \partial_\mu \int d^4x' \frac{\delta \mathcal{P}_{\nu,cl}^c(x')}{\delta \mathcal{P}_{\mu,cl}^a(x)} \frac{\delta}{\delta \mathcal{P}_{\nu,cl}^c(x')} Z = 0$$

which, of course, is satisfied because of

$$\partial_\mu \frac{\delta \mathcal{P}_{\nu,cl}^c(x')}{\delta \mathcal{P}_{\mu,cl}^a(x)} = 0$$

In showing the last equation we have written out the tree graph structure - expansion of $\mathcal{P}_{\mu,cl}$ in terms of J_μ . Diagrammatically, each J_μ , which contributes on the mass shell, is attached to an external leg which is associated with the Green's function

$$(2\pi)^4 \int d^4k e^{ik(x-x')} \frac{1}{k^2 - i0} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

Now, a differentiation of $\mathcal{P}_{\mu,cl}$ with respect to J_μ eliminates such a J_μ and the action of the ∂_μ operation on this Green's function leads to zero.

However, it should be stressed that in the covariant approach such a discussion of the Ward identities is superfluous.

Because of the gauge independence on mass shell one may go to a gauge in the literature called U gauge ^{2),9)} in which unitarity of the S matrix is manifest.

Of course, this is only a formal proof of the unitarity which is based on formal manipulations of functional integrals. In a forthcoming discussion of a massive version of the Yang-Mills theory a rigorous proof of unitarity will be given.

This covariant approach allows an immediate generalization to other gauge theories, e.g., to the Einstein Lagrangian theory which will come out to be non-renormalizable in the usual sense. The subtraction terms under the s integral in this case contain invariants which do not occur in the Lagrangian.

Nevertheless the crucial test for this approach will be whether the proof of invariant renormalizability in any order and of unitarity can be made in a simple and transparent manner. There is a strong indication for this. Because the free propagator of the theory is that of a renormalizable theory the only problem is the invariance of the counter-terms. But, by using a representation like (39) for the Green's function one is able to isolate the invariants which are associated with the divergences of the integral over the parameters, say, s_1, s_n . Then, one may apply a subtraction method as in the one-loop case. Invariants associated with overlapping divergences will cancel with contributions from lower order diagrams produced by counter-terms. By dimensional and invariance arguments the subtraction terms can only involve a and

$$b \int d^4x \, p_{\mu\nu,ce}^a(x) p_{a,ce}^{\mu\nu}(x)$$

where a and b are some finite constants.

A detailed discussion of this, in particular of the case with overlapping divergences will be published in a subsequent paper.

In Ref. 11) De Witt has developed a manifest covariant theory to treat in particular the Yang-Mills and the Einstein-Lagrangian. He introduces some background field which is very similar to the classical field which is used throughout this paper. In order to avoid the inconsistency connected with the equation $D_\mu J^\mu \equiv 0$ for the external source De Witt dispenses with the sources completely. His background field fulfils the homogeneous field equation $S_{,\mu} \equiv 0$.

De Witt carries out the Feynman idea ¹⁵⁾ that, e.g., the one-loop diagrams can be written as a sum of tree diagrams. Thus, in requiring unitarity at each stage he derives the correct Feynman rules and only afterwards he uses as a short-cut for the rules the functional integral formalism. Hence, the line of arguments is reversed in this paper. Using powerful though still heuristic functional integral techniques one immediately arrives at an elegant prescription for all the rules. But the unitarity then is still to be shown to legitimate the functional integral manipulations.

In theories in which no gauge-type invariance is present De Witt's formalism is identical to an approach with sources, in which the sources are not coupled to the classical part of the field but just to the quantum part. Otherwise, as in the Yang-Mills case, the situation is more subtle. However, in transforming from the classical field defined as, e.g., in this paper to the background field of De Witt one should be able to develop De Witt's approach starting from a formalism with sources.

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FIG.1

a)

b)

c)

d)

e)

f)

FIG.2