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DUAL MODELS AND RELATIVISTIC QUANTUM STRINGS

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A B S T R A C T

The string picture of the dual resonance model is reviewed. It is shown that, if we assume for a relativistic string propagating through space time an action proportional to the area of the surface of evolution, the spectrum of states of the corresponding quantum system coincides with the spectrum of states of the conventional dual resonance model and contains only transverse excitations if suitable consistency conditions are met.

It is shown also how an interaction among strings can be introduced so as to obtain the dual amplitudes as quantum mechanical transition amplitudes.

The main properties of the spectrum of the closed string (Pomeron sector of the conventional dual model) and a generalization to include spin degrees of freedom are discussed.

Throughout the article, the results obtained in the theory of the string are compared with the results obtained in the conventional approach to dual models.

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1. INTRODUCTION

Since the first appearance of the dual resonance amplitudes in the theory of strong interactions, numerous investigations have been made to reach a better understanding of their mathematical properties and physical meaning.

Among the many results that have been obtained, a particularly relevant one consists in the fact that the dual models can be given an interpretation in terms of the quantum mechanics of one-dimensional extended relativistic objects.

To be more precise, it has been understood that

- a) it is possible to define a one-dimensional system moving in ordinary space-time, and
- b) it is possible to introduce a dynamics for this system by defining an action functional, in such a way that:
- c) when the system is quantized, the states of excitation coincide with the states in the spectrum of the dual resonance model,
- d) the gauge structure, which is at the origin of the no-ghost theorem of the dual resonance model, emerges as a consequence of the assumed action,
- e) an interaction can be introduced so that the quantum mechanical transition amplitudes coincide with the dual amplitudes.

The one-dimensional system is specified by assigning its spatial coordinates as continuous functions of one parameter, varying in a definite closed interval on the real axis. It has become customary to call such a system a "string", and I shall retain this name in this article. When the string is entirely defined by the position of its points, it can be used to reproduce the physics of the conventional dual resonance model. But it is possible to endow the string with a continuous spin structure, and then one can give a string interpretation of the dual models of Neveu-Schwarz and Ramond.

This article illustrates the string picture of the dual resonance models. The points (a) to (e) will be expanded in such a way as to reproduce all of the main features of the dual models in terms of the theory of the quantum string. At the end of the article it should be manifest

that the physics of the dual models is the physics of one-dimensional relativistic extended objects.

This equivalence between dual models and the quantum mechanics of strings may raise some perplexities, and so let me make a few considerations on the appearance of one-dimensional objects in the theory of strong interactions. Let me draw attention to the following three experimental facts or generally-accepted postulates of hadron physics:

- i) there is a strong cut-off in the distributions of the transverse momenta of the particles produced in hadronic collisions;
- ii) the spectrum of hadrons exhibits an exponential increase in the mass of the states, and there is a limiting temperature¹⁾;
- iii) the resonances are arranged on approximatively linearly rising Regge trajectories; the amplitudes for strong interactions obey the super-convergence relations²⁾ and the finite energy sum rules³⁾.

If we try to solve, exactly or approximately, the problem of strong interactions in terms of resonance saturation, then the points (i) to (iii) strongly suggest that the spectrum of resonances should correspond to the excitations of a one-dimensional continuum.

In a scheme of dominant resonances, a hadronic collision goes through the production of one or more resonances. The possibility of preserving a cut-off in transverse momentum [point (1)] requires a very anisotropic decay of the resonances. Degenerate resonances of different spins must be produced coherently: this is the mechanism by which the direction of the incident particles may be "remembered", and the longitudinal direction may be favoured for the decay.

We see then that, if the resonances are to be interpreted as quantum excitations of some hadronic system, this system must have enough degrees of freedom to allow for high degeneracy in the spectrum and coherence effects that single out a privileged direction. The simplest way to satisfy these requirements is to approximate the hadronic system with a string: a natural distinction between a longitudinal direction and transverse directions emerges, and the one-dimensional continuum of degrees of freedom generates a rapid increase of the density of states of the spectrum.

A hadronic system with a one-dimensional continuum of degrees of freedom seems to be implied also by point (ii). Let us accept from the thermodynamical bootstrap that the density of states of the spectrum should increase exponentially in the invariant mass. It follows that the underlying physical system should have an infinite number of degrees of freedom, because finite numbers of degrees of freedom (e.g. a finite number of harmonic oscillators) give at most a polynomial increase of the density of states. Once we accept the idea of an infinite number of degrees of freedom, the argument is rather vague to discriminate between the various possible structures of the dynamical system. Still, we notice that a one-dimensional continuum of harmonic oscillators leads exactly to the density of states predicted by the thermodynamical bootstrap^{4,5}), whereas continuous systems with higher dimensionality give origin to densities of states increasing faster⁶).

Point (iii) focuses on the existence of linear Regge trajectories, and on the validity of superconvergence relations and finite energy sum rules. A four-dimensional harmonic oscillator can give a good account of the linearly rising trajectories; but the number of states is too limited to impose consistently superconvergence relations and finite energy sum rules. Indeed, the attempt to satisfy the superconvergence relations and finite energy sum rules in a framework of linear trajectories and resonance saturation led to the dual models^{7,8}), which exhibit spectra characteristic of the excitations of a one-dimensional continuum of oscillators. It seems then that the string is the minimal system required to satisfy superconvergence relations and finite energy sum rules with states arranged on linear trajectories.

I have presented this discussion, based on the points (i) - (iii), without any pretence of rigour. I did not mean to prove that the solution to the theory of strong interactions is to be looked for in terms of strings, but I wanted to make it more plausible that we should investigate the properties of one-dimensional systems in our study of strong interactions. If we do not keep in mind this possible connection between the theory of the string and the theory of strong interactions, the content of this article may appear as an academic exercise of relativistic quantum mechanics.

As I mentioned at the beginning of this Introduction, all of the features of the dual models (spectrum, gauges and interaction) can be reproduced, step by step, in terms of the theory of the string. This gives me the possibility of making this review entirely self-contained: no prior knowledge of the dual models should be required to read it.

However, I believe that the physical significance of the theory of the string is in the interpretation it provides for the dual resonance models, and that it can be really appreciated only if a constant comparison is made between the results it allows one to obtain and the results of the theory of dual models.

For this reason I reproduce in the Appendix some basic features of the theory of dual resonance models, and I recommend that the reader, who is not yet acquainted with it, should consult some of the many review articles and lecture notes published on the subject (see, for example, Refs. 9-17). In particular, I would like to single out Ref. 15 for a very general and comprehensive introduction to the subject of dual models, and Ref. 16 for the most recent developments of the theory.

2. THE CONVENTIONAL DUAL RESONANCE MODEL AND THE RELATIVISTIC QUANTUM STRING

2.1 Historical note*)

The first interpretation of the spectrum of states of the conventional dual resonance model¹⁸⁻²²⁾ as the spectrum of excitations of a one-dimensional system (a string) is due to Nambu²³⁾, Nielsen²⁴⁾, Susskind²⁵⁾ and Takabayashi²⁶⁾. Let me recall (see Appendix) that the spectrum of the conventional dual model is conveniently described by the introduction of the Fock space spanned by an infinite number of creation operators $a_n^{\mu\dagger}$ ($n = 1, 2, \dots$)²⁷⁾.

*) Some sections are preceded by a short historical note. The purpose of these notes is to give a rapid account of how the subject dealt with in the section developed from the theory of dual models. All of the historical notes are kept very concise, and, with the exception of Section 5, the content of the rest of the section is largely independent of them.

The invariant mass M of the state is related to the occupation numbers of the Fock space by the spectral equation.

$$\alpha' M^2 = \sum_n n a_{\mu n}^\dagger a_{\mu n}^{}. \quad (2.1)$$

The crucial point in establishing the analogy with a quantum string is to notice that each oscillator contributes to the mass squared of the state in a way proportional to the mode number n . If one considers a vibrating string with free ends, one finds an analogous relation between the mode number of the oscillators and their contribution to the energy. The idea of Refs. 23-26 was to introduce a string as a one-dimensional curve in space time

$$x^\mu = x^\mu(\sigma), \quad (2.2)$$

where for definiteness we assume $0 \leq \sigma \leq \pi$, and to postulate a quadratic Lagrangian of the form

$$L = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left[\left(\frac{\partial x(\sigma, \tau)}{\partial \tau} \right)^2 - \left(\frac{\partial x(\sigma, \tau)}{\partial \sigma} \right)^2 \right] \quad (2.3)$$

to describe the evolution of the string. τ is an internal evolution parameter: in the course of its motion the string spans a bidimensional surface in space time, of parametric equation

$$x^\mu = x^\mu(\sigma, \tau). \quad (2.4)$$

α' is a constant with the dimensions of a length squared necessary to make the action $S = \int L d\tau$ dimensionless (in this article we use units where $\hbar = c = 1$). The Lagrangian (2.3) gives an equation of evolution

$$\frac{\partial^2}{\partial \tau^2} x^\mu(\sigma, \tau) - \frac{\partial^2}{\partial \sigma^2} x^\mu(\sigma, \tau) = 0 \quad (2.5)$$

and boundary conditions

$$\frac{\partial}{\partial \sigma} x^\mu(\sigma, \tau) = 0 \quad \text{at } \sigma=0, \pi. \quad (2.6)$$

Equations (2.5) and (2.6) suggest that we expand $x^\mu(\sigma, \tau)$ in normal modes:

$$x^\mu(\sigma, \tau) = x_0^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{\cos n\sigma}{\sqrt{n}} (a_n^\mu e^{-in\tau} + a_n^{\mu*} e^{in\tau}) \quad (2.7)$$

p^μ in Eq. (2.7) represents the total momentum of the string, and is canonically conjugate to the barycentric coordinate x_0^μ .

The variables a_n^μ and $a_n^{\mu\dagger}$ are also canonically conjugate variables, and the Hamiltonian that generates the τ evolution is given by

$$H = \alpha' p^2 + \sum_n n a_n^{\mu*} a_n^\mu. \quad (2.8)$$

The quantum string is defined by replacing p^μ , x_0^μ , a_n^μ and $a_n^{\mu\dagger}$ with operators obeying

$$[x_0^\mu, p^\nu] = i g^{\mu\nu}, \quad (2.9)$$

$$[a_m^\mu, a_n^{\nu\dagger}] = g^{\mu\nu} \delta_{mn}, \quad (2.10)$$

all other commutators vanishing, and $H_{\text{classical}}$ with

$$H_{\text{quantum}} = \alpha' p^2 + \sum_n n a_n^{\mu\dagger} a_n^\mu. \quad (2.11)$$

The auxiliary condition

$$H |\psi\rangle = \alpha_0 |\psi\rangle, \quad (2.12)$$

to be imposed on the quantum excitations of the string, reproduces the spectral condition of the conventional dual model.

The difficulty with this interpretation is that it does not imply the existence of the Virasoro subsidiary conditions²⁸⁾ (see Appendix)

$$L_n |\psi\rangle = 0 \quad (2.13)$$

on the physical states of the spectrum. The Virasoro conditions are compatible with the spectral equation (2.12) and they must in fact be added as auxiliary requirements to the physical system outlined above to make the theory consistent.

A few years after the Lagrangian of Eq. (2.3) was introduced in the theory of the dual resonance model, Nambu²⁹⁾ (see also Refs. 30-34) proposed an alternative Lagrangian to describe the motion of the string, which is equivalent to the quadratic Lagrangian for the spectral equation, but has the additional advantage of implying the Virasoro conditions.

In the rest of this section, we shall study the new Nambu Lagrangian, following the treatment of Ref. 35.

2.2 The kinematics of the string and Nambu's action principle

The string is defined by assigning the spacial coordinates as continuous functions of one parameter σ :

$$x^i = x^i(\sigma) \quad i = 1, 2, 3. \quad (2.14)$$

Any range of variation can be taken for σ , and we fix $0 \leq \sigma \leq \pi$.

The time evolution of the string is described by the following equations

$$\begin{aligned} x^i &= x^i(\sigma, \tau) \\ x^0 &= \tau \end{aligned} \quad (2.15)$$

Equations (2.15) are the parametric equations of a bidimensional surface in space time: the surface of evolution spanned by the string during its motion.

A more general parametrization can be introduced to describe the same surface:

$$x^\mu = x^\mu(\sigma, \tau) \quad (2.16)$$

The only constraint placed on Eqs. (2.16) is that they represent a time-like surface, i.e. that any point of the surface must possess at least one time-like or light-like tangential vector. This is a necessary condition to interpret the surface as the surface of temporal evolution of a one-dimensional space-like object.

We shall always take τ as an evolution parameter, σ as a kinematical parameter, and shall impose that the surface of evolution be time-like by demanding

$$\left(\frac{\partial x}{\partial \tau}\right)^2 \leq 0, \quad \left(\frac{\partial x}{\partial \sigma}\right)^2 > 0. \quad (2.17)$$

We use a metric $g^{ii} = 1 = -g^{00}$.

Notice that if we define the evolution of the string by Eqs. (2.15), we obtain the more general parametrization of Eqs. (2.16) and (2.17) after a Lorentz transformation, which gives

$$x^0(\tau) \rightarrow x^{0'}(\sigma, \tau) = \bigwedge_{\mu}^0 x^{\mu}(\sigma, \tau). \quad (2.18)$$

Vice versa, from Eqs. (2.16), subject to the constraint Eqs. (2.17), we can recover the temporal evolution of the string in any Lorentz frame by intersecting the surface of evolution with the family of planes

$$\bigwedge_{\mu}^0 x^{\mu}(\sigma, \tau) = x^{0'}. \quad (2.19)$$

Having defined the motion of the string in terms of its surface of evolution, we introduce a dynamic by assigning to each possible motion an action functional.

The action should depend only on the shape of the surface of evolution (because this surface gives a complete description of the motion of the string). Following Nambu²⁹⁾ we take an action proportional to the area of the surface itself *) **).

*) An action of the same form, in different physical contexts, has been previously introduced in Refs. 36 and 37.

**) It is amusing to notice that a free point particle, in the course of its temporal evolution, describes a time-like curve, and that the equations of motion are obtained by assuming an action proportional to the length of the curve. We may regard the Nambu action as a quite natural generalization of the action for a point particle to a one-dimensional extended object.

We define then an action

$$S = -\frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\pi d\sigma \sqrt{-\left(\frac{\partial x}{\partial \tau}\right)^2 \left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau} \frac{\partial x}{\partial \sigma}\right)^2}. \quad (2.20)$$

The expression

$$\sqrt{-\left(\frac{\partial x}{\partial \tau}\right)^2 \left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau} \frac{\partial x}{\partial \sigma}\right)^2} d\tau d\sigma \quad (2.21)$$

is the area element between two infinitesimal displacements $(\partial x^\mu / \partial \tau) d\tau$ and $(\partial x^\mu / \partial \sigma) d\sigma$. A dimensional constant α' with the dimensions of a length squared must be introduced to make S dimensionless. In the following we shall often use the abbreviated notation

$$\frac{\partial \mathcal{L}}{\partial \tau} = \dot{\mathcal{L}} \quad \frac{\partial \mathcal{L}}{\partial \sigma} = \mathcal{L}'. \quad (2.22)$$

The motion of the string is determined by Hamilton's principle

$$\delta S = 0. \quad (2.23)$$

Let us define a Lagrangian density

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \sqrt{-\dot{x}^2 x'^2 + (\dot{x} x')^2}. \quad (2.24)$$

Equations (2.20) and (2.23) imply

$$\begin{aligned} \delta S = & \int_{\tau_0}^{\tau_1} d\tau \int_0^\pi d\sigma \left\{ -\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x'^\mu} \right\} \delta x^\mu + \\ & + \int_{\tau_0}^{\tau_1} d\tau \left\{ \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta x^\mu \right\} \bigg|_{\sigma=0}^{\pi} = 0. \end{aligned} \quad (2.25)$$

Equation (2.25) is obtained in the standard way by an integration by parts. We drop the boundary terms at $\tau = \tau_0$, $\tau = \tau_1$, because $\delta x^\mu = 0$ there, but must retain those at $\sigma = 0, \pi$.

Since δx^μ is arbitrary, Eq. (2.25) is satisfied if

$$\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x'^\mu} = 0 \quad (2.26)$$

and

$$\frac{\partial \mathcal{L}}{\partial x'^\mu} = 0 \quad \text{at} \quad \sigma = 0, \pi. \quad (2.29)$$

Equations (2.26) and (2.27) are the Euler-Lagrange equations for the classical motion of the string.

Notice that the Lagrangian density is invariant under the transformations of the Poincaré group: this implies the existence of locally conserved quantities associated with the generators of the group.

In particular, by performing an infinitesimal translation, we find that the energy momentum flowing across an infinitesimal line element $(d\sigma, d\tau)$ is given by

$$dp^\mu = \mathcal{P}^\mu d\sigma + \Pi^\mu d\tau \quad (2.28)$$

where we set

$$\mathcal{P}^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = \frac{1}{2\pi\alpha'} \frac{\dot{x}^\mu x'^2 - x'^\mu (\dot{x} x')}{\sqrt{-\dot{x}^2 x'^2 + (\dot{x} x')^2}}, \quad (2.29)$$

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial x'_\mu} = \frac{1}{2\pi\alpha'} \frac{x'^\mu \dot{x}^2 - \dot{x}^\mu (\dot{x} x')}{\sqrt{-\dot{x}^2 x'^2 + (\dot{x} x')^2}}. \quad (2.30)$$

We see that the equations of motion (2.26) and (2.27) are nothing other than the statement of the conservation of the energy momentum of the string: Eq. (2.26) says that the flow of energy momentum is locally conserved inside the string, Eq. (2.27) that no energy momentum flows in or out of the ends of the string.

The conserved total energy momentum of the string is given by

$$p^\mu = \int_c (\mathcal{P}^\mu d\sigma + \pi^\mu d\tau) = \int_{\tau \text{ fixed}} \mathcal{P}^\mu d\sigma, \quad (2.31)$$

where c is any curve that cuts the surface of evolution.

The relativistic conserved angular momentum of the string is given by

$$\begin{aligned} m^{\mu\nu} &= \int_c \{ (\mathcal{P}^\mu x^\nu - \mathcal{P}^\nu x^\mu) d\sigma + (\pi^\mu x^\nu - \pi^\nu x^\mu) d\tau \} = \\ &= \int_{\tau \text{ fixed}} (\mathcal{P}^\mu x^\nu - \mathcal{P}^\nu x^\mu) d\sigma. \end{aligned} \quad (2.32)$$

Notice that by squaring Eq. (2.30) we obtain

$$2\pi\alpha' \dot{\pi}^2 + \frac{\dot{x}^2}{2\pi\alpha'} = 0 \quad (2.33)$$

The boundary condition

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial x_\mu'} = 0 \quad \text{at } \sigma = 0, \pi$$

implies then that $\dot{x}^2 = 0$ at $\sigma = 0, \pi$, i.e., that the end points of the string move with the speed of light.

The condition $\dot{x}^2 = 0$ could not be implemented if the space had a Euclidean metric: in a Euclidean space a surface of minimal area can be found only if it has a fixed closed boundary.

In the following subsections we shall study in detail the solutions of the equations of motion of the string, but it is instructive to anticipate here a very simple example of a possible classic motion. This is given by a rigidly rotating string, which can be parametrized by

$$\begin{aligned} x^0 &= \tau \\ x^1 &= A (\sigma - \pi/2) \cos \omega \tau \\ x^2 &= A (\sigma - \pi/2) \sin \omega \tau \\ x^3 &= 0. \end{aligned} \quad (2.34)$$

From Eqs. (2.29) and (2.30) we find

$$\mathcal{P}^\mu = \frac{A}{2\pi\alpha'} \frac{1}{\sqrt{1 - \omega^2 A^2 (\sigma - \pi/2)^2}} \times$$

$$\times (1, -\omega A(\sigma - \pi/2) \sin \omega \tau, \omega A(\sigma - \pi/2) \cos \omega \tau, 0) \quad (2.35)$$

$$\Pi^\mu = -\frac{1}{2\pi\alpha'} \frac{\sqrt{1 - \omega^2 A^2 (\sigma - \pi/2)^2}}{A} \times$$

$$\times (0, A \cos \omega \tau, A \sin \omega \tau, 0) \quad (2.36)$$

With these expressions for $\mathcal{P}^\mu$ and Π^μ , the equation $\dot{\mathcal{P}} + \Pi' = 0$ is always satisfied, whereas the condition $\Pi = 0$ at $\sigma = 0, \pi$ is satisfied if $(\pi/2)\omega A = 1$.

From Eqs. (2.31) and (2.32), we compute energy and angular momentum of the string. We find

$$E = M (\text{because } \vec{p} = 0) = \frac{A \pi}{4\alpha'}$$

$$J = \frac{A^2 \pi^2}{16\alpha'}$$

It follows that $J = \alpha' M^2$. We see that angular momentum and mass squared of the string are proportional, in this particular mode of motion (linear Regge trajectory), and that the constant α' sets the scale between the two quantities. We shall find again these two features (linear Regge trajectories with slope α') in the quantum spectrum of the string.

2.3 The gauge group of the string. Hamiltonian formulation of the equations of motion.

The action S of Eq. (2.20) expresses a geometrical property of the surface of evolution and does not depend on the particular choice of parameters used to describe it. As a consequence, S is left invariant by the transformations

$$x^\mu(\sigma, \tau) \Rightarrow \tilde{x}^\mu(\sigma, \tau) = x^\mu(\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau)), \quad (2.37)$$

induced by a reparametrization of the surface:

$$\begin{aligned} \tilde{\sigma} &\Rightarrow \sigma = \sigma(\tilde{\sigma}, \tilde{\tau}) \\ \tilde{\tau} &\Rightarrow \tau = \tau(\tilde{\sigma}, \tilde{\tau}). \end{aligned} \quad (2.38)$$

The transformations (2.37) form a continuous group, which we call the gauge group of the string. Infinitesimal generators of the group are the transformations

$$x^\mu(\sigma, \tau) \Rightarrow x^\mu(\sigma, \tau) - \epsilon f(\sigma, \tau) \dot{x}^\mu(\sigma, \tau) - \epsilon g(\sigma, \tau) \dot{x}^\mu(\sigma, \tau), \quad (2.39)$$

which correspond to the infinitesimal change of parameters

$$\begin{aligned} \sigma &\Rightarrow \sigma + \epsilon f(\sigma, \tau) \\ \tau &\Rightarrow \tau + \epsilon g(\sigma, \tau). \end{aligned} \quad (2.40)$$

We assume $f(0, \tau) = f(\pi, \tau) = 0$ so as to leave the range $0 \leq \sigma \leq \pi$ unchanged. The invariance of S under this gauge group has some important consequences.

Our final goal is to quantize the equations of motion of the string, and we shall proceed by establishing first a canonical formalism. Once a classical Hamiltonian is obtained, the standard replacement of dynamical variables with operators defines the quantum system.

However, in the presence of a group of invariance of the action the introduction of a canonical formalism is not straight forward.

A general consequence of the invariance of the action under some group of transformations is that the equations that give the momenta in terms of the velocities cannot be inverted and that there exist dependences

among momenta and coordinates. Dirac calls these dependences primary constraints³⁸⁾.

In our case the momentum conjugate to $x^\mu(\sigma)$ is

$$\mathcal{P}^\mu(\sigma) = \frac{\delta}{\delta \dot{x}^\mu(\sigma)} \int d\sigma \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}(\sigma), \quad (2.41)$$

It can immediately be checked from Eq. (2.29) that the following relations hold

$$x'_\mu \mathcal{P}^\mu = 0, \quad (2.42)$$

$$2\alpha' (\pi \mathcal{P})^2 + \frac{x'^2}{2\alpha'} = 0. \quad (2.43)$$

When there are primary constraints in trying to set up a canonical formalism one faces two problems. First of all, the constraint equations are in general incompatible with canonical Poisson brackets or commutation relations where all x 's and p 's are treated as independent.

Secondly, in the presence of constraint equations of the form $h_j(p,q) = 0$, the effective Hamiltonian $H(p,q)$ that generates the evolution of the system may contain a linear combination of the constraint functions³⁸⁾. In general, we shall have

$$\begin{aligned} H(p,q) &= \sum_i p_i \dot{q}_i - L + \sum_j c_j h_j(p,q) = \\ &= H_0(p,q) + \sum_j c_j h_j(p,q). \end{aligned} \quad (2.44)$$

There are, then, two ways to proceed. One is to reduce the number of degrees of freedom of the system, by eliminating redundant variables. The dynamical variables which are left must form a complete set of independent variables; for these alone one assumes canonical Poisson brackets and commutation relations. This method of quantization will be studied in the next section.

The other possibility is to deal with all of the coordinates and momenta as if they were independent quantities, disregarding the existence of constraints when establishing Poisson brackets (or later commutation relations). The constraints are imposed as initial conditions on the classical variables (on the wave function). The Hamiltonian is then chosen in the form (2.43) and must be such that the constraint equations are preserved in the course of the motion. [This will automatically happen if the Poisson brackets (the commutators) of H and the constraint equations form a closed algebra. Otherwise, new constraints must be added to the system, but this does not occur in our case].

In this section we follow this second method of quantization. We define canonical Poisson brackets

$$\{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0, \quad (2.45)$$

$$\{x^\mu(\sigma), p^\nu(\sigma')\} = g^{\mu\nu} \delta(\sigma - \sigma'). \quad (2.46)$$

Notice the appearance of $g^{\mu\nu}$ in Eq. (2.46), which is necessary to preserve Lorentz covariance.

In Eqs. (2.45) and (2.46), the coordinates $x^\mu(\sigma)$ and the momenta $p^\mu(\sigma)$ are considered as independent dynamical variables.

Equations (2.45) and (2.46) are not compatible with the constraint equations (2.42) and (2.43). We impose these equations as initial conditions on the system.

It is convenient to introduce arbitrary linear combinations of the constraint functions in the following way.

We extend first the range of definition of $x'^\mu(\sigma)$ and $p^\mu(\sigma)$ to $-\pi \leq \sigma \leq \pi$ by the equations

$$\begin{aligned} x'^\mu(-\sigma) &= -x'^\mu(\sigma), \\ p^\mu(-\sigma) &= p^\mu(\sigma). \end{aligned} \quad (2.47)$$

We define then constraint functionals

$$L_f = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma f(\sigma) \left(\sqrt{2\alpha'} \pi \mathcal{P}(\sigma) + \frac{x'(\sigma)}{\sqrt{2\alpha'}} \right)^2. \quad (2.48)$$

Clearly, the constraint equations (2.42) and (2.43) are equivalent to

$$L_f = 0 \quad (2.49)$$

for any function $f(\sigma)$.

Notice that the Poisson brackets of the constraint functionals form a closed algebra. We have

$$\{L_f, L_g\} = L_{f \otimes g}, \quad (2.50)$$

with

$$f \otimes g = f g' - f' g. \quad (2.51)$$

If we compute the Hamiltonian

$$H_0 = \int d\sigma \mathcal{P}(\sigma) \dot{x}(\sigma) - \int d\sigma \mathcal{L}, \quad (2.52)$$

we find

$$H_0 \equiv 0. \quad (2.53)$$

It follows that the effective Hamiltonian that generates the motion will be of the form

$$H = L_h \quad (2.54)$$

for some appropriate choice of the function h .

There is some arbitrariness in the choice of this function; indeed, any specification of a definite form of H defines a gauge for the system. To see this, let us recall that the gauge invariance manifests itself in the impossibility of inverting the equations $\mathcal{P}^\mu = \mathcal{P}^\mu(x', \dot{x})$, to obtain $\dot{x}^\mu = x^\mu(\mathcal{P}, x')$.

However, once the Hamiltonian is defined, we can express the velocities in terms of coordinates and momenta through Hamilton's equations

$$\dot{x}^\mu = \{x^\mu, H\}. \quad (2.55)$$

A choice of H is then equivalent to a gauge specification.

Let us compute*)

$$\begin{aligned} \{L_h, x^\mu(\sigma)\} &= -\frac{h(\sigma)}{2} (2\pi\alpha' \mathcal{P}^\mu(\sigma) + x'^\mu(\sigma)) + (\sigma \leftrightarrow -\sigma) = \\ &= -\frac{h(\sigma) + h(-\sigma)}{2} 2\pi\alpha' \mathcal{P}^\mu(\sigma) - \frac{h(\sigma) - h(-\sigma)}{2} x'^\mu(\sigma). \end{aligned} \quad (2.56)$$

We see from this equation that the even part of L_h transforms x into $\mathcal{T}$, and thus generates a dynamical evolution of the system. The odd part corresponds instead to a reparametrization of the points of the string, at fixed τ . Taking h odd we find

$$\delta x^\mu = \epsilon \{L_h, x^\mu(\sigma)\} = -\epsilon h(\sigma) x'^\mu(\sigma), \quad (2.57)$$

which is the change of parametrization induced by $\sigma \rightarrow \sigma + \epsilon h(\sigma)$ [see Eqs. (2.36) and (2.37)].

We shall fix a gauge for the string by choosing $h = \text{const} = 1$.

With

$$\begin{aligned} H &= \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma \left(\sqrt{2\alpha'} \pi \mathcal{P}(\sigma) + \frac{x'(\sigma)}{\sqrt{2\alpha'}} \right)^2 = \\ &= \pi \int_0^\pi d\sigma \left(\alpha' \mathcal{P}^2(\sigma) + \frac{x'^2(\sigma)}{4\pi^2\alpha'} \right), \end{aligned} \quad (2.58)$$

*) Equation (2.56), as well as Eq. (2.51) can be derived only under some assumptions of regularity at $\sigma = 0, \pi$ and $-\pi$. In particular, we must have $x'(0) = x'(\pi) = 0$. We shall see later that this boundary condition does indeed arise from the equations of motion; but it is important to keep in mind that a correct dynamical treatment must include proper care of the boundary conditions.

we find Hamilton's equation of motion

$$\dot{x} = 2\pi\alpha' \mathcal{P} , \quad (2.59)$$

$$\dot{\mathcal{P}} = \frac{1}{2\pi\alpha'} x'' . \quad (2.60)$$

Eliminating $\mathcal{P}^\mu$ we obtain

$$\ddot{x} - x'' = 0 . \quad (2.61)$$

By inserting Eq. (2.59) in the definition of Π^μ we obtain

$$\Pi^\mu = - \frac{x'^\mu}{2\pi\alpha'} . \quad (2.62)$$

The boundary condition then becomes

$$x'^\mu(0) = x'^\mu(\pi) = 0 . \quad (2.63)$$

We see that the non-linear equations of motion (2.26) and (2.27) simplify remarkably and become linear in the gauge we have chosen. There is a geometrical reason for this.

It is a known result from the theory of Euclidean surfaces³⁹⁾ that the equations for a surface of minimal area linearize and become Laplace equations if one requires that the net of coordinates form a locally orthonormal system.

This result can be extended to a time-like surface. The orthonormality conditions become in this case

$$\dot{x} x' = 0 \quad (2.64)$$

and

$$x'^2 - |\dot{x}|^2 = x'^2 + \dot{x}^2 = 0 , \quad (2.65)$$

and the equations for a minimal surface become D'Alembert equations.

Substituting Eq. (2.59) in the constraint equations (2.42) and (2.43), these become precisely Eqs. (2.64) and (2.65). We see that a simplification of the equations of motion occur in our gauge, because the coordinate lines on the surface of evolution are locally orthonormal. Because of this feature, we shall call this gauge the "orthonormal gauge" of the string.

2.4 The quantum system

Once we have established a canonical formalism, we can quantize the string by the replacement

$$i \{ \text{Poisson bracket} \} \rightarrow [\text{commutator}].$$

It is convenient, however, to expand first the coordinates and momenta into normal modes, so as to deal with a discrete set of dynamical variables.

We expand $x(\sigma)$ and $\mathcal{P}(\sigma)$ in $\cos n\sigma$, as required by $x'(0) = x'(\pi) = 0$ and by $\mathcal{P} = (1/2\pi\alpha')\dot{x}$:

$$\begin{aligned} x^\mu(\sigma) &= x_0^\mu + \sum_n x_n^\mu \cos n\sigma, \\ \mathcal{P}^\mu(\sigma) &= \frac{p^\mu}{\pi} + \sum_n \mathcal{P}_n^\mu \cos n\sigma. \end{aligned} \quad (2.66)$$

Inserting into Eq. (2.58) we obtain

$$H = \alpha' p^2 + \sum_n \left(\frac{\alpha' \pi^2}{2} \mathcal{P}_n^2 + \frac{n^2 x_n^2}{8\alpha'} \right). \quad (2.67)$$

We diagonalize H by introducing

$$a_n^\mu = \frac{i\pi}{2} \sqrt{\frac{2\alpha'}{n}} \mathcal{P}_n^\mu + \frac{1}{2} \sqrt{\frac{n}{2\alpha'}} x_n^\mu. \quad (2.68)$$

Solving for $\mathcal{P}_n^\mu$ and x_n^μ we find

$$\begin{aligned} x_n^\mu &= \sqrt{\frac{2\alpha'}{n}} (a_n^\mu + a_n^{\mu*}), \\ \mathcal{P}_n^\mu &= \frac{1}{\pi} \sqrt{\frac{n}{2\alpha'}} (i a_n^{\mu*} - i a_n^\mu). \end{aligned} \quad (2.69)$$

Substituting in Eq. (2.66) we obtain

$$H = \alpha' p^2 + \sum_n n a_{n\mu}^* a_n^\mu. \quad (2.70)$$

The Poisson brackets of a_n^μ and $a_n^{\mu*}$ are

$$\{a_m^\mu, a_n^{\nu*}\} = -i g^{\mu\nu} \delta_{mn}, \quad (2.71)$$

all other Poisson brackets vanishing.

From Eqs. (2.70) and (2.71) we derive a τ evolution:

$$a_n^\mu(\tau) = e^{-in\tau} a_n^\mu(0). \quad (2.72)$$

We have then the following expansions

$$x^\mu(\sigma, \tau) = x_0^\mu + 2\alpha' p^\mu \tau + \sum_{n=1}^{\infty} \sqrt{\frac{2\alpha'}{n}} \cos n\sigma (a_n^\mu e^{-in\tau} + a_n^{\mu*} e^{in\tau}), \quad (2.73)$$

$$\mathcal{P}^\mu(\sigma, \tau) = \frac{p^\mu}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \sqrt{\frac{n}{2\alpha'}} \cos n\sigma (-i a_n^\mu e^{-in\tau} + i a_n^{\mu*} e^{in\tau}). \quad (2.74)$$

In the theory of dual resonance models one often encounters the fields (introduced by Fubini and Veneziano^{40,41}) and Gervais⁴²):

$$Q^\mu(\tau) = x_0^\mu + 2\alpha' p^\mu \tau + \sum_{n=1}^{\infty} \sqrt{\frac{2\alpha'}{n}} (a_n^\mu e^{-in\tau} + a_n^{\mu*} e^{in\tau}),$$

$$P^\mu(\tau) = \frac{d}{d\tau} Q^\mu(\tau) = 2\alpha' p^\mu + \sum_{n=1}^{\infty} \sqrt{2n\alpha'} (-ia_n^\mu e^{-in\tau} + ia_n^{\mu*} e^{in\tau}). \quad (2.75)$$

(frequently Q^μ and P^μ are expressed as functions of $z = e^{i\tau}$).

The meaning of P and Q is clear: we have

$$Q^\mu(\tau) = x^\mu(0, \tau),$$

$$P^\mu(\tau) = \dot{x}^\mu(0, \tau) = 2\pi\alpha' \mathcal{P}^\mu(0, \tau). \quad (2.76)$$

We can also write

$$P^\mu(\sigma) = 2\pi\alpha' \mathcal{P}^\mu(\sigma) + \dot{x}^\mu(\sigma), \quad (2.77)$$

where the reflection principle [Eq. (2.49)] has been used to extend the range of σ .

Summarizing, we have the following dynamical description of the classical string. In the orthonormal gauge, the coordinates and momenta of the string are a superposition of normal modes. The normal mode variables a_n^μ and $a_n^{\mu*}$ evolve according to the Hamiltonian

$$H = \alpha' p^2 + \sum_n n a_n^{\mu*} a_n^\mu. \quad (2.70)$$

Their initial values are not arbitrary, but subject to the constraints

$$L_h = 0.$$

We transform the constraint equations into a discrete set by choosing a basis made of the functionals

$$L_n \equiv L_h e^{in\tau} \quad (2.78)$$

Expanding into normal modes we find

$$\begin{aligned} L_n = & -i\sqrt{2\alpha'} p a_n + \sum_{m \geq 0} \sqrt{(m+n)m} a_m^* a_{m+n} \\ & - \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{(n-m)m} a_m a_{n-m} \quad (n > 0), \end{aligned} \quad (2.79)$$

$$L_{-n} = L_n^*,$$

and

$$L_0 \equiv H. \quad (2.80)$$

The initial conditions are then

$$L_n = 0 \quad n = 0, \pm 1, \dots \quad (2.81)$$

Notice that $L_0 = H = 0$ is a spectral equation for the classical system. It gives indeed*)

$$-\alpha' p^2 = \sum_n n |a_n|^2. \quad (2.82)$$

It is straightforward now to quantize the system. We replace the c-numbers a_n^μ , $a_n^{\mu*}$, x_0^μ and p^μ with operators satisfying

$$[a_m^\mu, a_n^{\nu\dagger}] = g^{\mu\nu} \delta_{mn} \quad (2.83)$$

and

$$[x_0^\mu, p^\nu] = i g^{\mu\nu}, \quad (2.84)$$

all other commutators vanishing. These operators are represented in the Fock space spanned by the $a_n^{\mu\dagger}$ and $e^{ik \cdot x_0}$ operators acting on a ground state vector $|0\rangle$ satisfying

$$a_m^\mu |0\rangle = p^\mu |0\rangle = 0. \quad (2.85)$$

Notice that we have chosen a ground state vector carrying zero momentum. An eigenstate of momentum, with eigenvalue k^μ , is obtained by acting on $|0\rangle$ with the operator $e^{ik \cdot x_0}$. One has

*) The particular solution to the classical equations of motion studied at the end of Subsection 2.2 is obtained by choosing

$$a_1^1 = - \frac{\pi A}{4\sqrt{2\alpha'}} e^{-in\tau}$$

$$a_1^2 = - \frac{i\pi A}{4\sqrt{2\alpha'}} e^{-in\tau}$$

$$p^0 = \frac{\pi A}{4\alpha'}$$

all other a 's, x 's and p 's vanishing. This Ansatz clearly satisfies the constraint equations (2.81).

$$p^\mu |k^\mu\rangle = p^\mu e^{i k_\mu x_0^\mu} |0\rangle = k^\mu |k^\mu\rangle. \quad (2.86)$$

The normalization of the eigenstates of momentum is

$$\langle k | k' \rangle = (2\pi)^4 \delta^{(4)}(k - k'). \quad (2.87)$$

A basis of the Fock space is given by the vectors

$$|\lambda, k\rangle = a_1^{\mu_1 \dagger} \dots a_1^{\mu_{\lambda_1} \dagger} a_2^{\mu_2 \dagger} \dots a_2^{\mu_{\lambda_2} \dagger} \dots e^{i k x_0} |0\rangle. \quad (2.88)$$

The quantum Hamiltonian $H \equiv L_0$, the constraint operators L_n and the expansions of the quantum fields $x^\mu(\sigma, \tau)$, $\mathcal{P}^\mu(\sigma, \tau)$, $Q^\mu(\tau)$, $P^\mu(\tau)$ are obtained by making the appropriate replacements in Eqs. (2.70), (2.79), (2.80), (2.73), (2.74) and (2.75). We do not rewrite these operators here.

It is important to notice that in the definition of the quantum L_0 one encounters an ambiguity from the ordering of the operators^{*)}. We fix this ambiguity by defining

$$L_0 = : L_0 : = \alpha' p^2 + \sum_n n a_n^\dagger a_n. \quad (2.89)$$

We must allow, however, for the presence of an undetermined additive constant in the quantum equation corresponding to $L_0 = 0$.

The excitations of the quantum system are not arbitrary. The wavefunctions $|\phi\rangle$ of the physical states must satisfy the conditions

$$\langle \phi | L_n - \alpha_0 \delta_{n0} | \phi \rangle = 0.$$

*) No such ambiguities occur in the definition of the L_n with $n \neq 0$.

We separate this equation into a creation and an annihilation part, and obtain

$$L_0 |\varphi\rangle = \alpha_0 |\varphi\rangle, \quad (2.90)$$

$$\begin{aligned} L_n |\varphi\rangle &= 0 \quad (n > 0), \\ \langle \varphi | L_n &= \langle \varphi | L_{-n}^\dagger = 0 \quad (n < 0). \end{aligned} \quad (2.91)$$

These conditions replace the operator equations $L_n = 0$, which are incompatible with the commutation relations (2.82) and (2.83).

Notice that the Fock space with basis vectors $|\lambda, k\rangle$ of Eq. (2.88) is precisely the space used to factorize the conventional dual resonance model, and that Eqs. (2.90) and (2.91) are the identities satisfied by the on-mass shell physical states (see Appendix).

Equation (2.91) is a spectral condition. It gives

$$(-\alpha' p^2 + \alpha_0) |\varphi\rangle = (\alpha' s + \alpha_0) |\varphi\rangle = \alpha(s) |\varphi\rangle = \sum_n n a_n^\dagger a_n |\varphi\rangle$$

and assigns the quantum excitations to Regge trajectories with slope α' and intercept α_0 . We shall see in the next section that consistency requires $\alpha_0 = 1$.

We conclude this section with an important remark on the commutators of the L_n operators. From Eqs. (2.78), (2.53) and (2.54) we might expect

$$[L_m, L_n] = (m-n) L_{m+n}$$

However, this commutator is affected by an anomaly, which results from the normal ordering in the definition of L_0 . By explicit computation one finds⁴³⁾

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{12} (m^3 - m) \delta_{m,-n}, \quad (2.92)$$

where d is the number of dimensions of space time.

The number d occurs in the anomaly, because the a^μ and $a^{\mu\dagger}$ operators with different Lorentz indices give independent contributions.

The presence of a c-number term in the commutator of L_m and $L_m^\dagger = L_{-m}$ will be of extreme importance when we study the internal consistency of our method of quantization.

3. QUANTIZATION IN THE TRANSVERSE GAUGE

3.1 Historical note

An important problem in the theory of dual resonance models has been whether states of negative norm (ghost states) are present in the spectrum of physical states. These states are not acceptable for a consistent interpretation of the theory; however their presence cannot be *a priori* ruled out, because the Fock space introduced to describe the excited states does not have a positive definite norm. States of negative norm occur in this space because of the minus sign in the commutators

$$[a_m^\circ, a_m^{\circ\dagger}] = g^{\circ\circ} = -1, \quad (3.1)$$

which is unavoidable if one wants to have explicit Lorentz covariance.

Of course, the crucial point is that not all the vectors of the Fock space represent physical states of excitations, but only those satisfying the conditions

$$L_0 |\psi\rangle = \alpha_0 |\psi\rangle, \quad (2.90)$$

$$L_n |\psi\rangle = 0 \quad (n > 0). \quad (2.91)$$

The question is then whether Eqs. (2.90) and (2.91) are sufficient to rule out the occurrence of ghosts in the physical spectrum.

This question remained for a long time an open problem, until an affirmative answer was established by Brower⁴⁴⁾ and Goddard and Thorn⁴⁵⁾.

An extremely important intermediate step toward the proof of the no-ghost theorem was made by Del Giudice, Di Vecchia and Fubini⁴⁶⁾. These authors constructed a set of "transverse excitation operators" $A_n^{i\dagger}(k)$, with the following fundamental properties:

$$[L_m, A_n^{i\dagger}(k)] = 0, \quad (3.2)$$

$$[A_m^i(k), A_n^{j\dagger}(k)] = m \delta^{ij} \delta_{mn}. \quad (3.3)$$

Equation (3.2) implies that the transverse excitation operators transform physical states into physical states; in particular, that they span a subspace $\mathcal{G}$ of the space of physical states when applied to the ground state vector. Equation (3.2) implies then that $\mathcal{G}$ is a space with positive definite norm.

The operators $A_n^{i\dagger}(k)$ can be defined only in a non-covariant manner. One must choose a definite Lorentz frame and specify in it a longitudinal direction. Then the index i labels the transverse directions, whereas k is a light-like vector with a purely longitudinal space part: $k^\mu = (k, 0, 0, -k)$. The magnitude of k is fixed by the requirement that, if $A_n^{i\dagger}(k)$ acts on a state with momentum p^μ , one must have $2\alpha' p \cdot k = 1$. n is an index, running from 1 to ∞ , analogous to the index n of $a_n^{\mu\dagger}$.

The relevance of the space $\mathcal{G}$ is that, while in general it is only a subspace of the space Ψ of physical states, it becomes equivalent to it if the intercept parameter $\alpha_0 = 1$ and one enlarges the dimensionality d of space time to accommodate 24 transverse dimensions.

Equivalent means that any physical state can be represented as the sum of a state of $\mathcal{G}$ with a state of zero norm, decoupled from all physical states.

The central part of the no-ghost theorem of Refs. 44 and 45 is the proof that the two spaces $\mathcal{G}$ and ψ are equivalent for $\alpha_0 = 1$ and $d = 26$. Since $\mathcal{G}$ is a space with positive definite norm, the no-ghost theorem follows for these critical values of α_0 and d . The theorem can then be extended also to $d < 26$. The most important concept, which we extract from this brief discussion, is that relevant physical information can be obtained on the spectrum of the dual resonance model starting from a non-covariant approach.

The problem of the possible occurrence of ghost arises because we quantized the system in an explicitly covariant way. We shall see in this section that, abdicating explicit Lorentz covariance, we can quantize the string in a space of positive definite norm. The problem becomes then whether this space represents indeed all possible excitations of the system, or, equivalently, how can Lorentz covariance be reinstated.

3.2 Quantization in the transverse gauge

In general, the presence of constraints among coordinates and momenta indicates that there is a redundancy in the set of dynamical variables introduced to describe the system.

An alternative method of quantization is then to eliminate all superfluous degrees of freedom, until one is left with a set of independent coordinates and momenta, which are quantized canonically.

When the redundancy derives from the invariance of the action under some gauge group of transformations, it is eliminated by choosing a gauge for the system.

In our case, this requires that we fix a definite parametrization for the surface of evolution of the string, which is most conveniently done in the following way.

We choose the τ parameter proportional to the time coordinate in some given Lorentz frame:

$$\tau = n_\mu x^\mu / (2\alpha' n_\mu p^\mu), \quad (3.4)$$

with $n_\mu n^\mu \leq 0$. We allow for $n^2 = 0$ because later on we shall be interested precisely in the limit where n_μ becomes light-like. A factor $1/(2\alpha' n_\mu p^\mu)$ has been introduced in Eq. (3.4) for normalization purposes (p^μ is the total momentum of the string).

Once the τ parameter has been fixed, we have a freedom in the choice of the σ parameter, that labels the points of the string at fixed "time" τ .

Notice that there is no rest mass distribution to which we may attach a definition of τ . We use therefore the energy content of the string to label its points. Precisely, we take σ proportional to the total flow of energy between one boundary of the string and the point considered:

$$\sigma = \pi \frac{(n_\mu p^\mu)^{\sigma \rightarrow \sigma'}}{(n_\mu p^\mu)^{\text{total}}} = \pi \int_0^\sigma \frac{n_\mu \mathcal{P}^\mu(\sigma') d\sigma'}{n_\mu p^\mu} . \quad (3.5)$$

An equivalent statement is that we label the points so that the density of the momentum in the n_μ direction is a constant, precisely

$$n_\mu \mathcal{P}^\mu(\sigma) = \text{const} = \frac{n_\mu p^\mu}{\pi} . \quad (3.6)$$

Equations (3.4) and (3.5) constitute a unique gauge specification: a net of coordinates is set on the surface of evolution by first intersecting it with the planes $n_\mu x^\mu = 2\alpha' (n_\mu p^\mu) \tau$, (This defines the τ coordinate lines), and then using as σ coordinate lines the lines of flux of the n_μ component of the momentum density.

We can now use Eqs. (3.4) and (3.5), together with the constraint equations (2.42) and (2.43) to reduce the number of independent variables. This reduction is most easily performed when n_μ is light-like. We shall refer to the gauge we have defined above, with $n^2 = 0$, as "the transverse gauge" of the string.

It is convenient to introduce for any vector u^μ the light-like components

$$u^\pm = \frac{u^0 \pm u^{d-1}}{\sqrt{2}} , \quad u_\perp = \{u^i\} = \{u^1 \dots u^{d-2}\} . \quad (3.7)$$

In Eq. (3.7) d is the number of dimensions of space time.

For definiteness we choose $n^- = 1$, $n^+ = n^i = 0$. Equations (3.4) and (3.6) give us then

$$\frac{d}{d\sigma} x^+(\sigma) = 0 , \quad (3.8)$$

$$\mathcal{P}^+(\sigma) = \frac{p^+}{\pi} . \quad (3.9)$$

Inserting these equations in the constraint equations (2.42) and (2.43), we find

$$x^-(\sigma) = \frac{\pi}{p^+} x_\perp'(\sigma) \mathcal{P}_\perp(\sigma), \quad (3.10)$$

$$\mathcal{P}^-(\sigma) = \frac{1}{4\alpha' \pi p^+} \left(2\alpha' (\pi \mathcal{P}_\perp(\sigma))^2 + \frac{x_\perp'(\sigma)^2}{2\alpha'} \right). \quad (3.11)$$

We see that the "-" components of coordinates and momenta are dependent variables. Independent dynamical variables in the transverse gauge are $x_\perp(\sigma)$, $\mathcal{P}_\perp(\sigma)$, p^+ and x_0^- . The need to add to the list of independent variables a barycentric coordinate

$$x_0^- = \frac{1}{\pi} \int_0^\pi d\sigma x^-(\sigma) d\sigma \quad (3.12)$$

comes from the fact that the constraint equations contain only $(\partial/\partial\sigma)x^-(\sigma)$.

For the independent variables alone we assume canonical Poisson brackets

$$\{x^i(\sigma), \mathcal{P}^j(\sigma')\} = \delta^{ij} \delta(\sigma - \sigma'), \quad (3.13)$$

$$\{x_0^-, p^+\} = -1. \quad (3.14)$$

The Hamiltonian that generates the motion is proportional to the p^- total momentum, because the evolution variable τ is proportional to x^+ , and p^- is the generator of infinitesimal displacements in the x^+ direction.

By comparing the equations

$$\frac{d}{dx^+} f = \{f, p^-\}$$

and

$$\frac{d}{d\tau} f = 2\alpha' p^+ \frac{d}{dx^+} f = \{f, H\},$$

we find immediately:

$$\begin{aligned} H &= 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi \mathcal{P}^-(\sigma) d\sigma = \\ &= \frac{1}{2\pi} \int_0^\pi d\sigma \left[2\alpha' (\pi \mathcal{P}_\perp)^2 + \frac{x_\perp'^2}{2\alpha'} \right]. \end{aligned} \quad (3.15)$$

Hamilton's equations of motion follow:

$$\dot{x}_\perp = 2\pi\alpha' \mathcal{P}_\perp, \quad (3.16)$$

$$\dot{\mathcal{P}}_\perp = \frac{1}{2\pi\alpha'} x_\perp''. \quad (3.17)$$

We also have, from the definition of the transverse gauge:

$$\dot{x}^+ = 2\pi\alpha' \mathcal{P}^+, \quad (3.18)$$

$$\dot{\mathcal{P}}^+ = x^{+'''} = 0. \quad (3.19)$$

Substituting Eqs. (3.16) and (3.19) in Eqs. (3.10) and (3.11), we find

$$\dot{x}^- = 2\pi\alpha' \mathcal{P}^- \quad (3.20)$$

$$\dot{\mathcal{P}}^- = \frac{1}{2\pi\alpha'} x^{-''}. \quad (3.21)$$

Notice that Eqs. (3.16) to (3.21) are identical to Eqs. (2.59) and (2.60), which are valid in the orthonormal gauge, introduced in Section 2. That gauge could be equivalently defined by the choice of the Hamiltonian $H = L_0$, by the requirements $\dot{x}^2 + x'^2 = \dot{x}x' = 0$ or by Eqs. (2.59) and (2.60).

It follows that the transverse gauge is a further gauge specification of the orthonormal gauge. Indeed, the gauge equations (2.61), $\dot{x}^2 + x'^2 = 0$ and $\dot{x}x' = 0$, do not specify a unique gauge. Within the orthonormal gauge one can perform changes of parametrization

$$\begin{aligned}\sigma &\rightarrow \hat{\sigma} = \hat{\sigma}(\sigma, \tau), \\ \tau &\rightarrow \hat{\tau} = \hat{\tau}(\sigma, \tau),\end{aligned}$$

provided that $\dot{\hat{\sigma}}' = \dot{\hat{\tau}}$ and $\dot{\hat{\sigma}} = \dot{\hat{\tau}}'$. These last two equations in turn imply

$$\ddot{\hat{\tau}} - \hat{\tau}'' = 0,$$

i.e. the new τ parameter must satisfy the D'Alembert equation as a function of the old ones.

Since the coordinates $x^\mu(\sigma, \tau)$ do satisfy the D'Alembert equation, it follows that the linear combination $n_\mu x^\mu$ can be taken as proportional to a new τ parameter, without spoiling the orthonormal gauge.

Having established Eqs. (3.10), (3.11), (3.13), (3.14) and (3.15), it is possible to proceed immediately to the quantization of the string in the transverse gauge. It is convenient, as we did for the orthonormal gauge, to expand first the variables into normal coordinates. Since the expansions are the same as those of Eqs. (2.73) and (2.74) we do not reproduce them here. We only notice that, of course, the operators a_n^+ and a_n^{++} are not defined in the transverse gauge, where x^+ is given by

$$x^+ = 2\alpha' p^+ \tau. \quad (3.22)$$

The operators a_{1n} , a_{1n}^+ , p^+ and x_0^- are independent operators, with commutation relations

$$[a_m^i, a_n^{j\dagger}] = \delta^{ij} \delta_{mn}, \quad (3.23)$$

and

$$[x_0^-, p^+] = -i. \quad (3.24)$$

The Fock space of transverse excitations spanned by the operators $a_n^{i\dagger}$ is clearly a space with positive definite norm. The "-" variables are dependent ones. The expressions for the a^- modes are easily found to be [from Eqs. (2.73), (2.74), (3.10) and (3.11)]

$$a_n^- = \frac{i}{(\sqrt{2\alpha' n}) p^+} L_n^{tr}. \quad (3.25)$$

By L_n^{tr} we denote the L_n operators, where only the transverse modes have been kept. We also have

$$p^- = \frac{1}{2\alpha' p^+} (L_0^{tr} - \alpha_0), \quad (3.26)$$

where a constant α_0 has been introduced because of the ordering ambiguity in the definition of L_0 . Equation (3.26), which can be rewritten

$$2\alpha' p^+ p^- = \alpha' p_\perp^2 + \sum_n n a_{\perp n}^\dagger a_{\perp n} - \alpha_0, \quad (3.27)$$

plays the role of the spectral equation in the quantization in the transverse gauge.

3.3 Proof of Lorentz covariance

The method of quantization described in the previous subsection is clearly non-covariant. The theory will be therefore self-consistent only if we can show that there exists a representation of Lorentz transformations.

To find out what this representation may be, let us compare the transverse gauge descriptions of the motion of the string in two different Lorentz frames. It is convenient to consider two frames related by an infinitesimal Lorentz transformation

$$\Lambda_\nu^\mu = \delta_\nu^\mu + \epsilon m_\nu^\mu. \quad (3.28)$$

We shall denote by a symbol $\sim$ all the quantities defined in the frame obtained after the transformation (3.28).

In the two frames we define transverse gauges by means of Eqs. (3.4) and (3.5). Notice that the parametrizations defined in the two frames in general will not coincide. We can easily find the relation between the parameters $\tilde{\sigma}, \tilde{\tau}$ and σ, τ . Equation (3.4) gives us

$$\begin{aligned} \delta\tau &= \tilde{\tau} - \tau = \frac{\tilde{x}^+}{2\alpha' \tilde{p}^+} - \frac{x^+}{2\alpha' p^+} = \\ &= \frac{x^+ + \epsilon m_\nu^+ x^\nu}{2\alpha' (p^+ + \epsilon m_\nu^+ p^\nu)} - \frac{x^+}{2\alpha' p^+} = \\ &= \frac{\epsilon m_\nu^+}{p^+} \left(\frac{x^\nu}{2\alpha'} - p^\nu \tau \right), \end{aligned} \quad (3.29)$$

whereas from Eq. (3.5) we obtain

$$\begin{aligned} \delta\sigma &= \tilde{\sigma} - \sigma = \frac{\pi \int_0^\sigma \tilde{\mathcal{P}}^+(\sigma', \tau) d\sigma'}{p^+} - \sigma = \\ &= \frac{\epsilon m_\nu^+}{p^+} \left(\pi \int_0^\sigma \mathcal{P}^\nu(\sigma', \tau) d\sigma' - \frac{\sigma p^\nu}{p^+} \right). \end{aligned} \quad (3.30)$$

We have, of course,

$$\tilde{x}^\mu(\tilde{\sigma}, \tilde{\tau}) = \bigwedge_\nu^\mu x^\nu(\sigma, \tau) = x^\mu(\sigma, \tau) + \epsilon m_\nu^\mu x^\nu(\sigma, \tau). \quad (3.31)$$

From Eqs. (3.29) to (3.31) we deduce a variation in form of the functions $x^\mu(\sigma, \tau)$:

$$\begin{aligned} \delta x^\mu(\sigma, \tau) &= \hat{x}^\mu(\sigma, \tau) - x^\mu(\sigma, \tau) = \hat{x}^\mu(\tilde{\sigma} - \delta\sigma, \tilde{\tau} - \delta\tau) - x^\mu(\sigma, \tau) = \\ &= \epsilon m_\nu^\mu x^\nu - \epsilon \frac{m_\nu^+}{p^+} x^\nu \\ &\quad \times \left\{ x^\mu \left(\pi \int_0^\sigma d\sigma' \mathcal{P}^\nu(\sigma', \tau) - \sigma p^\nu \right) + \dot{x}^\mu \left(\frac{x^\nu}{2\alpha'} - \tau p^\nu \right) \right\}. \end{aligned} \quad (3.32)$$

Equations (3.32) tell us how the transverse gauge description of the motion of the string is modified by an (infinitesimal) change of frame.

The same equation, together with the corresponding equation for $\delta \mathcal{P}_\mu$, gives a non-linear realization of Lorentz transformations in the space of the transverse variables. This realization consists of the linear Lorentz transformation $\delta x^\mu = \epsilon m_\nu^\mu x^\nu$ followed by a change of gauge.

It can be verified that the transformation (3.32) is generated (through Poisson brackets) by the quantity

$$\epsilon m_{\mu\nu} \mathcal{M}^{\mu\nu} = \epsilon m_{\mu\nu} \int d\sigma (\mathcal{P}^\mu x^\nu - \mathcal{P}^\nu x^\mu),$$

provided we substitute for x^+ , x^- , $\mathcal{P}^+$ and $\mathcal{P}^-$ their expressions in terms of x_0^- , p^+ , $x_\perp(\sigma)$ and $\mathcal{P}_\perp(\sigma)$. One can check that

$$\delta x^\mu = \epsilon m_{\nu\rho} \{ \mathcal{M}^{\nu\rho}, x^\mu \},$$

and also that the Poisson brackets of the quantities $\mathcal{M}^{\mu\nu}$ [expressed as functions of x_0^- , p^+ , $x_\perp(\sigma)$, $\mathcal{P}_\perp(\sigma)$] close correctly into the algebra of Lorentz generators. This proves the covariance of the classical system. We may expect at this point that the covariance of the quantization in the transverse gauge should result from the existence of a representation of the Lorentz group in the Hilbert space of transverse excitations, generated by the quantum operators $\mathcal{M}^{\mu\nu}$. We must be careful, however, because we know that the normal ordering of operators can produce differences between Poisson brackets and commutators. Therefore, we check explicitly whether the commutators of the operators $\mathcal{M}^{\mu\nu}$ close into the algebra of the Lorentz group.

For simplicity, we consider only the generators of the little group of a string with momentum^{*)}

$$p^\mu = (p^+, p^-, p_\perp = 0).$$

These are

$$m^{ij} = i \sum_{n \neq 0} (a_n^{it} a_n^j - (i \leftrightarrow j)) \quad (3.33)$$

and

$$m^{Li} = \sum_{n \neq 0} \left(\frac{L_n^t a_n^i}{\sqrt{n}} + \frac{a_n^{it} L_n}{\sqrt{n}} \right). \quad (3.34)$$

Notice that no ordering ambiguity is present in the definition of $\mathcal{M}^{ij}$ and $\mathcal{M}^{Li}$.

*) The results that we shall obtain for the little group can easily be extended to the algebra of the whole Poincaré group.

From the algebra of Lorentz generators we expect

$$[m^{Li}, m^{Lj}] = -4i\alpha' p^+ p^- m^{ij}. \quad (3.35)$$

Computing the commutator we find⁽⁴⁷⁾ *)

$$\begin{aligned} [m^{Li}, m^{Lj}] = & -2i \left(L_0^{tr} - \frac{d^{tr}}{24} \right) m^{ij} - \\ & -i \left(\frac{d^{tr}}{24} - 1 \right) \sum_{n \neq 0} n^2 (a_n^{i+} a_n^j - (i \leftrightarrow j)), \end{aligned} \quad (3.36)$$

where d^{tr} is the number of transverse indices. We see that the commutators of the operators $\mathcal{M}^{\mu\nu}$ can reproduce the algebra of the Lorentz group, but only if we enlarge the dimensionality of space time to allow for 24 transverse dimensions. When $d^{tr} = 24$, comparing Eqs. (3.35) and (3.36), we find

$$2\alpha' p^+ p^- = \alpha' M^2 = L_0^{tr} - 1, \quad (3.37)$$

which fixes the intercept parameter $\alpha_0 = 1$.

* To derive Eq. (3.36) one must normal order the various terms obtained in the commutation of $\mathcal{M}^{Li}$ and $\mathcal{M}^{Lj}$, being careful to avoid formal manipulations involving infinite shifts in summation indices. The c-number term in the commutator [see Eq. (2.92)]

$$[L_n^{tr}, L_n^{tr}] = 2n L_0^{tr} + \frac{d^{tr}}{12} (n^3 - n)$$

plays a very important role: it is through this c-number term that the dimension dependent factors are introduced in Eq. (3.36).

Summarizing, we have found that the quantization of the string in the transverse gauge leads to a covariant quantum system only when $d = 26$ and $\alpha_0 = 1$.

We can understand the physical reason for $\alpha_0 = 1$ (but $d = 26$ seems to have no intuitive explanation). In the transverse gauge, the whole Lorentz group must be represented in the space of transverse excitations. At the first excited level we have $d^{tr} = d - 2$ spin states: $a_1^{i\dagger} |0\rangle$, which is short of one state to allow for a representation of the little group of a massive particle, i.e. $O(d - 1)$. We must therefore have a massless particle.

It is interesting to notice that at the second level the states $a_1^{i\dagger} a_1^{j\dagger} |0\rangle$ [$(d - 2)(d - 1)/2$ states] and $a_2^{i\dagger} |0\rangle$ ($d - 1$ states) combine into a single irreducible representation of $O(d - 1)$ [of spin "2", with $d(d - 1)/2 - 1$ states].

3.4 Relation between the quantization in the transverse gauge and the covariant method of quantization

The quantization of the string in the transverse gauge makes fundamental use of the equations [equivalent to Eqs. (3.4) and (3.5)]

$$x^{+'}(\sigma) = \mathcal{P}^{+'}(\sigma) = 0, \quad (3.38)$$

which can always be imposed on the classical system.

If we quantize the system covariantly, it is not obvious that a gauge can be found where Eqs. (3.38) are valid. Indeed, we shall see that Eqs. (3.38) can be imposed in the covariant method of quantization only when the consistency requirements $\alpha_0 = 1$ and $d = 26$ are satisfied: the two methods of quantization become then equivalent.

Equations (3.38) cannot be imposed as operator identities in the covariant formalism. We require that they are satisfied in a weak form, when matrix elements are taken between physical states

$$\langle \psi | x^{+'}(\sigma) | \psi \rangle = \langle \psi | \mathcal{P}^{+'}(\sigma) | \psi \rangle = 0. \quad (3.39)$$

Separating the creation and annihilation parts of Eqs. (3.39), we obtain the conditions

$$a_n^+ |\psi\rangle = 0, \quad n > 0, \quad (3.40)$$

that the states should satisfy, besides the on-mass-shell requirement

$$L_0 |\psi\rangle = \alpha_0 |\psi\rangle \quad (2.91)$$

and the Virasoro conditions

$$L_n |\psi\rangle = 0, \quad n > 0. \quad (2.92)$$

We define transverse states, and denote by $|\psi\rangle_{tr}$, the simultaneous solutions of Eqs. (2.91), (2.92) and (3.40). The space $\mathcal{G}$ of all transverse states will be called the transverse space.

Notice that, by counting arguments or explicit construction, it can be shown that the transverse space is always a proper subspace of the space Ψ of physical states. So, when we demand the validity of Eqs. (3.39), we do not require that they should be satisfied by all physical states, but we do demand that, by a gauge transformation, any physical state can be replaced by a state obeying that equation.

In other words, we demand that the transverse states form a complete set of physical states, in the sense that every physical state can be expressed as a sum of a transverse state and a state of zero norm, decoupled from all physical states:

$$|\psi\rangle = |\psi\rangle_{tr} + |\text{decoupled state}\rangle. \quad (3.41)$$

The transverse states can be constructed explicitly, by acting on a ground state vector

$$|\psi_0\rangle = e^{ipx_0} |0\rangle \quad (\alpha' p^2 = \alpha_0), \quad (3.42)$$

with a suitable set of transverse excitation operators $A_n^{i\dagger}(k)$ ⁴⁶.

These operators are given by

$$A_n^{i\dagger}(k) = \frac{1}{2\pi i \sqrt{2\alpha'}} \oint_0 \frac{dz}{z} P^i(z) e^{-in k Q^+(z)}, \quad n=1,2,\dots, \quad (3.43)$$

where the fields $Q^\mu(z = e^{i\tau})$ and $P^\mu(z)$ are those defined in Section 2, Eq. (2.75), and k is a parameter, which must satisfy $2\alpha' k p^+ = 1$, if $A_n^{i\dagger}(k)$ acts on a state with momentum p^μ *).

It can be shown that the transverse excitation operators satisfy the following fundamental equations **)

$$[L_m, A_n^{i\dagger}(k)] = 0, \quad (3.44)$$

$$[a_m^+, A_n^{i\dagger}(k)] = 0, \quad (3.45)$$

$$[A_m^i(k), A_n^{j\dagger}(k)] = n \delta_{mn} \delta^{ij}. \quad (3.46)$$

*) The exponential $e^{-in k Q^+(z)}$, when acting on a state of momentum p^μ , generates a factor

$$e^{-2n\alpha' p^+ k \log z} = z^{-2n\alpha' p^+ k},$$

which is single valued for all n only when $2\alpha' p^+ k = 1$.

**) Equations (3.44) to (3.46) will be proved in Section 4, where also the physical arguments that led to the introduction of the $A_n^{i\dagger}(k)$ operators will be illustrated.

Equations (3.44) and (3.45) imply that all the states generated by the action of the $A_n^{i\dagger}(k)$ operators on the ground state vector $|\psi_0\rangle$ are transverse states, and, by counting arguments, it can be shown that the whole transverse space $\mathcal{G}$ is spanned by the $A_n^{i\dagger}(k)$ operators.

Equation (3.46) then proves that $\mathcal{G}$ is a space with positive definite norm. In general, Eq. (3.41) is not valid. But it can be proved^{44,45)} that it becomes true when $\alpha_0 = 1$, $d = 26$. In this case the transverse excitation operators span a complete set of physical states.

Notice that the space generated by the $A_n^{i\dagger}(k)$ operators is isomorphic to the space generated by the $a_n^{i\dagger}$ operators. Thus we see that, when $\alpha_0 = 1$, $d = 26$, the two alternative methods of quantization lead to equivalent descriptions of the quantum string.

The completeness of the transverse states for the critical values of α_0 and d can be also shown by constructing the projection operator τ over the transverse space $\mathcal{G}$.

This projection operator has been first obtained by Brink and Olive⁴⁸⁾. The main idea in Ref. 48 is to compare the excitation number operators

$$R = \sum_n n a_n^\dagger a_n \quad (3.47)$$

and*)

$$R^{tr} = \sum_n A_{\perp n}^\dagger A_{\perp n} \quad (3.48)$$

We notice a few algebraic properties of these operators. We have

$$R = L_0 - \alpha' p^2 \quad (3.49)$$

Since $A_n^{i\dagger}(k)$ increases the "-" component of momentum by nk , we have

*) The factor n is missing in Eq. (3.48) because the A operators are normalized to $[A_n^i, A_n^{i\dagger}] = n$.

$$\begin{aligned}
 [-\alpha' p^2, A_n^{i\dagger}(k)] &= 2\alpha' p^+ [p^-, A_n^{i\dagger}(k)] = \\
 &= 2\alpha' p^+ n k A_n^{i\dagger}(k) = n A_n^{i\dagger}(k).
 \end{aligned} \tag{3.50}$$

From Eqs. (3.44), (3.49) and (3.50) it follows that

$$[R, A_n^{i\dagger}(k)] = n A_n^{i\dagger}(k). \tag{3.51a}$$

We also have

$$[R^{tr}, A_n^{i\dagger}(k)] = n A_n^{i\dagger}(k), \tag{3.51b}$$

$$[R, L_n^{\dagger}] = [L_0, L_n^{\dagger}] = n L_n^{\dagger}, \tag{3.52a}$$

$$[R^{tr}, L_n^{\dagger}] = 0, \tag{3.52b}$$

$$[R, a_n^{++}] = n a_n^{++}, \tag{3.53a}$$

$$[R^{tr}, a_n^{++}] = 0. \tag{3.53b}$$

Equations (3.51) to (3.53) imply that

$$[R - R^{tr}, A_n^{i\dagger}(k)] = 0, \tag{3.54}$$

$$[R - R^{tr}, a_n^{++}] = n a_n^{++}, \tag{3.55}$$

$$[R - R^{tr}, L_n^{\dagger}] = n L_n^{\dagger}. \tag{3.56}$$

Since all the states of the Fock space spanned by the $a_n^{\mu\dagger}$ operators are also generated by the operators $A_n^{i\dagger}$, a_n^{++} and $L_n^{\dagger}$, Eqs. (3.54) to (3.56) imply that the operator $R - R^{tr}$ has only non-negative integer eigenvalues, and, in particular, that the transverse states (and only these) are eigenstates with zero eigenvalue.

It follows that the operator

$$\tau = \frac{1}{2\pi i} \oint_0 \frac{dz}{z} z^{R-R^{\text{tr}}} \quad (3.57)$$

acts as the identity operator on $\mathcal{G}$, and annihilates the orthogonal subspace. τ is then the required projection operator.

A very important result, established in Ref. 48, is that τ can be re-expressed in a form which involves the gauge operators L_n . With some algebraic manipulations, for which I refer to their original article, Brink and Olive were able to prove that

$$\begin{aligned} R^{\text{tr}} - R = & (D_0 - 1) \left(L_0 - \frac{d^{\text{tr}}}{24} \right) + \\ & + \sum_{n \geq 0} (D_n^\dagger L_n + L_n^\dagger D_n) + \\ & + \left(1 - \frac{d^{\text{tr}}}{24} \right) \oint_0 \frac{dz}{2\pi i z} \frac{1}{k P^+(z)} \left[z \frac{d}{dz} \log k P^+(z) \right]^2, \end{aligned} \quad (3.58)$$

where

$$D_n = \frac{1}{2\pi i} \oint \frac{dz}{z} z^n k P^+(z).$$

Notice that, for $d^{\text{tr}} = 24$, Eq. (3.58) reduces to *)

$$R^{\text{tr}} - R = (D_0 - 1)(L_0 - 1) + \sum_{n \geq 0} (D_n^\dagger L_n + L_n^\dagger D_n). \quad (3.59)$$

*) For an interesting independent derivation of Eq. (3.59) see Ref. 49.

If the on-mass-shell condition is $(L_0 - 1)|\psi\rangle = 0$, i.e. if $\alpha_0 = 1$, the matrix elements of $R^{tr} - R$ between any on-mass-shell physical states vanish. It follows then that

$$\langle 4 | 4 \rangle = \langle 4 | T | 4 \rangle = t_1 \langle 4 | 4 \rangle_{t_1} \quad (3.60)$$

with

$$|4\rangle_{tr} = T |4\rangle.$$

Equation (3.60) is another statement of the completeness of the transverse states for the critical values of the intercept parameter and dimensionality of space time.

4. STRINGS IN INTERACTION - I

4.1 Strings in an external field

The quantization of the string leads to the same spectrum of excited states as is found in the factorization of the conventional dual resonance model. It is then an obvious question whether it is possible to introduce an interaction among strings that gives origin to the dual amplitudes such as quantum mechanical transition amplitudes.

The answer is affirmative. Indeed, the dual amplitudes can be obtained following two rather different methods: one makes use of an operatorial formalism, and will be described in this section; in the other one, which will be described in Section 5, the dual amplitudes are obtained as the result of a functional integration.

The clue to the first approach comes from the multiperipheral factorization of the dual amplitudes. It is known (see Appendix) that the dual amplitude $A(k_1 \dots k_N)$ has poles in all planar channels, and that it can be fully factorized, so as to exhibit the maximum number of compatible resonant poles. A very useful factorization is the multiperipheral one (see Fig. 4.1), where one singles out the poles in the subenergies $-(k_1 + \dots + k_i)^2$ ($i=2, \dots, N-2$).

Fig. 4.1

Correspondingly to this factorization, the dual amplitude is written

$$A(k_1 \dots k_N) = \langle k_N | V(k_{N-1}) D V(k_{N-2}) \dots D V(k_2) | k_1 \rangle, \quad (4.1)$$

where $|k_1\rangle$ and $|k_N\rangle$ are vectors that represent the initial and final particles of the multiperipheral chain, $V(k_i)$ is the vertex for the emission of a particle with momentum k_i , and

$$D = \int_0^1 dz z^{L_0-2} = \frac{1}{L_0-1} \quad (4.2)$$

is the propagator of a free string (with $\alpha_0 = 1$).

It is important to notice that the initial and final particles do not have to be in the ground state. The same expression (4.1) can be used to describe amplitudes where the particles 1 and N are in arbitrary states of excitation λ_1 and λ_N , provided the appropriate vectors $|k_1\lambda_1\rangle$ and $|k_N\lambda_N\rangle$, satisfying the Virasoro conditions Eqs. (2.91) are substituted for $|k_1\rangle$ and $|k_N\rangle$. The dual amplitude emerges then as the amplitude for the transition between the states $|k_1\lambda_1\rangle$ and $|k_N\lambda_N\rangle$, induced by the emission of quanta of some external field with momenta $k_2 \dots k_{N-1}$.

At this point, it is rather natural to identify the states $|k_1\lambda_1\rangle$ and $|k_N\lambda_N\rangle$ as states of the string, and the problem becomes to introduce an interaction between the string and the external field that reproduces the dual amplitude. This problem has been solved by the authors of Ref. 50, whose derivation of the dual amplitude is outlined in this subsection.

The amplitude $A_{\lambda_1 \lambda_N}(k_1 \dots k_N)$ of Eq. (4.1) describes the emission of ground state particles. For reasons which will become clear later, it is more convenient to consider the amplitude $A_{\lambda_1 \lambda_N}^{(1)}$ for emission of particles in the first state of excitation. These are massless vector particles, having the same properties as the photons.

$$A_{\lambda_1 \lambda_N}^{(1)} = A_{\lambda_1 \lambda_N}^{(1)}(k_1; \epsilon_2, k_2, \dots, \epsilon_{N-1}, k_{N-1}; k_N)$$

is obtained by substituting for $V(k)$ in Eq. (4.1) the "photon" emission vertex $V(k, \epsilon) = \epsilon_\mu V^\mu(k)$, where ϵ_μ is the polarization vector of the emitted particle.

To obtain $A_{\lambda_1 \lambda_N}^{(1)}$ as transition amplitude for a string, we must consider the interaction of the string with an external field $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, obeying the same equations as the electromagnetic field. We couple the string to the field $F_{\mu\nu}(x)$ by adding an interaction term S_I to the free action S_0 of Eq. (2.20). S_I must be invariant under reparametrizations of the surface of evolution $x^\mu(\sigma, \tau)$ and must depend on $\dot{x}^\mu$, x'^μ , $F_{\mu\nu}$, or on A_μ in a gauge-invariant way.

The possible choices for S_I are:

$$S_I^a = - \frac{e}{\pi} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma F_{\mu\nu}(x) \dot{x}^\mu x'^\nu \quad (4.3)$$

and

$$\begin{aligned} S_I^b = & - \frac{g_0}{\pi} \int_{\tau_i}^{\tau_f} d\tau A_\mu(x(\tau=0)) \dot{x}^\mu(\tau=0) - \\ & - \frac{g_\pi}{\pi} \int_{\tau_i}^{\tau_f} d\tau A_\mu(x(\tau=\pi)) \dot{x}^\mu(\tau=\pi). \end{aligned} \quad (4.4)$$

However, S_I^a gives origin to the same equations of motion as S_I^b with $g_0 = g_\pi = e$. Indeed, we have

$$\begin{aligned} F_{\mu\nu}(x) \dot{x}^\mu x'^\nu &= x'^\nu (\dot{x}^\mu \partial_\mu) A_\nu(x) - \dot{x}^\nu (x'^\mu \partial_\mu) A_\nu(x) = \\ &= x'^\nu \frac{d}{d\tau} A_\nu(x) - \dot{x}^\nu \frac{d}{d\sigma} A_\nu(x). \end{aligned} \quad (4.5)$$

Integrating by parts, we find $S_I^a = S_I^b + 2$ terms at τ_i and τ_f , which do not contribute to δS . We consider therefore only S_I^b .

The gauge invariance of S_I^b is proved with an integration by parts. If we add to A_μ the quantity $\partial_\mu X$, the extra term which is introduced in S_I^b can be integrated away to $\tau = \tau_i$ and $\tau = \tau_f$, where it does not contribute to the equations of motion.

Adding $S_I^b = S_I$ to the free action is equivalent to adding a term

$$\mathcal{L}_I = \dot{x}^\mu A_\mu(x) \Delta(\sigma), \quad (4.6)$$

$$\Delta(\sigma) = -\frac{g_0}{\pi} \delta(\sigma) - \frac{g_\pi}{\pi} \delta(\pi - \sigma), \quad (4.7)$$

to the free Lagrangian density $\mathcal{L}$ of Eq. (2.24). The Euler-Lagrange Equations (2.26) remain unchanged, but the boundary conditions [Eq. (2.27)] $\partial \mathcal{L} / \partial x'^\mu = 0$ at $\sigma = 0, \pi$ are modified and become

$$\begin{aligned} & -\frac{g_0}{\pi} \left(\frac{d}{d\tau} A_\mu - \dot{x}^\nu \frac{\partial}{\partial x'^\mu} A_\nu \right) = \\ & = -\frac{g_0}{\pi} \dot{x}^\nu F_{\nu\mu}(x) = -\frac{\partial \mathcal{L}}{\partial x'^\mu} \quad \text{at } \sigma=0, \end{aligned} \quad (4.8a)$$

$$\begin{aligned}
 & - \frac{g\pi}{\pi} \left(\frac{d}{d\tau} A_\mu - \dot{x}^\nu \frac{\partial}{\partial x^\mu} A_\nu \right) = \\
 & = - \frac{g\pi}{\pi} \dot{x}^\nu F_{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial x'^\mu} \quad \text{at } \sigma = \pi. \quad (4.8b)
 \end{aligned}$$

Eqs. (4.8) are non-linear in the variable x^μ , because x^μ appears as an argument of the external field $A_\mu(x)$. However, if we consider a plane wave

$$A_\mu(x) = \epsilon_\mu e^{-i k_\mu x^\mu} \quad (4.9)$$

with

$$k^2 = k^\mu \epsilon_\mu = 0 \quad (4.10)$$

then the Eqs. (2.26) and (4.8) can be solved exactly. Indeed, in this case we can separate

$$X^\mu(\sigma, \tau) = X_0^\mu(\sigma, \tau) + X_I^\mu(\sigma, \tau), \quad (4.11)$$

where x_0^μ satisfies the equations of motion of a free string in the orthonormal gauge (in particular $x_0'^\mu = 0$ at $\sigma = 0, \pi$) and x_I^μ consists of two terms, proportional to k^μ and ϵ^μ :

$$X_I^\mu(\sigma, \tau) = \epsilon^\mu a(\sigma, \tau) + k^\mu b(\sigma, \tau). \quad (4.12)$$

Because of Eqs. (4.9) and (4.10), only the free part x_0^μ is present in the argument of the field $A_\mu(x) = A_\mu(x_0)$, and this allows one to solve the equations of motion in a compact form.

It is convenient (although not necessary) to consider circularly-polarized waves so that $\epsilon^2 = 0$. We still require that the complete solution of the equations of motion satisfies the orthonormality conditions

$$\dot{x}^2 + x'^2 = 0$$

and

$$\dot{x}x' = 0.$$

(4.13)

Then the equations of motion become

$$\ddot{x}^\mu - x''^\mu = 0$$

(4.14)

and the boundary conditions become

$$x'^\mu = 2i\alpha' g_0 (\epsilon^\mu k_\nu - \epsilon_\nu k^\mu) \dot{x}^\nu e^{-ikx} \text{ at } \sigma=0, \quad (4.15a)$$

$$-x'^\mu = 2i\alpha' g_\pi (\epsilon^\mu k_\nu - \epsilon_\nu k^\mu) \dot{x}^\nu e^{-ikx} \text{ at } \sigma=\pi. \quad (4.15b)$$

Equations (4.13), (4.14) and (4.15) have the solution (which we express in terms of $\dot{x}_\mu$ and x'_μ):

$$\begin{aligned} \dot{x}_\mu(\sigma, \tau) = & \dot{x}_{0\mu}(\sigma, \tau) + 2\alpha' (\epsilon_\mu k_\nu - \epsilon_\nu k_\mu) \times \\ & \times [R_+ \dot{x}_0^\nu(0, \tau-\sigma) e^{-ikx_0(0, \tau-\sigma)} + R_- \dot{x}_0^\nu(0, \tau+\sigma) e^{-ikx_0(0, \tau+\sigma)}] \end{aligned} \quad (4.16a)$$

$$\begin{aligned} x'_\mu(\sigma, \tau) = & x_{0\mu}'(\sigma, \tau) - 2\alpha' (\epsilon_\mu k_\nu - \epsilon_\nu k_\mu) \times \\ & \times [R_+ \dot{x}_0^\nu(0, \tau-\sigma) e^{-ikx_0(0, \tau-\sigma)} - R_- \dot{x}_0^\nu(0, \tau+\sigma) e^{-ikx_0(0, \tau+\sigma)}] \end{aligned} \quad (4.16b)$$

where $x_0^\mu(\sigma, \tau)$ represents any solution of the free equations of motion,

$$R_{\pm} = \frac{g_{\pi} + g_0 e^{\pm 2i\alpha'(p \cdot k)\pi}}{2 \sin 2\alpha'(p \cdot k)\pi} \quad (4.17)$$

and p^μ is the free momentum:

$$p^\mu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \dot{x}^\mu(\sigma, \tau).$$

The constants $R_{\pm}$ are determined by the boundary conditions [Eq. (4.15)].

It is important to remember that [see Eq. (2.73)]

$$\begin{aligned} x_0^\mu(\sigma, \tau) = & \text{const} + 2\alpha' p^\mu \tau + \\ & + f^\mu(\sigma + \tau) + g^\mu(\sigma - \tau), \end{aligned} \quad (4.18)$$

where f^μ and g^μ are periodic functions with period 2π . It is then straightforward to check that $\dot{x}^\mu$ and x'^μ as given in Eqs. (4.16) satisfy Eqs. (4.14) and (4.15). The validity of Eqs. (4.13) is best checked by rewriting them in the equivalent form $(\dot{x} \pm x')^2 = 0$. Because of Eq. (4.18), we have

$$(\dot{x}_0 \pm x'_0)(\sigma, \tau) = \dot{x}_0(0, \tau \pm \sigma) \quad (4.19)$$

Equations (4.13) then give

$$\begin{aligned} & (\dot{x}_0^\mu(0, \tau \pm \sigma) + 4\alpha'(\epsilon^\mu{}_\nu k^\nu - \epsilon^\nu{}_\mu k^\mu) \times \\ & \times R_{\mp} \dot{x}_{0\nu}(0, \tau \pm \sigma) + i k x_0(0, \tau \pm \sigma))^2 = 0, \end{aligned} \quad (4.20)$$

which in turn reduce to

$$\dot{X}_0^2(0, \tau \pm \sigma) = (\dot{X}_0 \pm \dot{X}_0')^2 = 0, \quad (4.21)$$

i.e., to the orthonormality conditions for the free coordinates x_0^μ .

Notice that even at the classical level the interaction of the string with the external field $A_\mu(x)$ gives origin to a resonant phenomenon. The amplitude becomes infinite whenever $2\alpha' p \cdot k = n(\text{integer})$. This condition is equivalent to $(p - k)^2 - p^2 = n/\alpha'$. But n/α' is the characteristic spacing between the mass squared of the n^{th} excited level and the mass squared of the ground state, which appears now as a condition for a resonance even before quantization.

Once a classical solution to the equations of motion is found, it is easy to quantize the system. Normal coordinates are introduced by expanding the free coordinates and momenta, $x_0^\mu(\sigma, \tau)$ and $\mathcal{P}_0^\mu(\sigma, \tau)$ in a cosine series, [see Eqs. (2.65) and (2.69)]. The variables x_0^μ , p^μ , a_m^μ and $a_m^{\mu*}$ are quantized according to Eqs. (2.82) and (2.83).

The choice of a Hamiltonian is again equivalent to a choice of a gauge. In the orthonormal gauge H is given by

$$H = H_0 - \int_0^\pi d\sigma \mathcal{L}_I = H_0 + H_I, \quad (4.22)$$

with

$$\begin{aligned} H_I(\tau) = & \frac{g_0}{\pi} \epsilon_\mu \dot{X}_0^\mu(\sigma=0, \tau) e^{-i k_\mu x_0^\mu(\sigma=0, \tau)} + \\ & + \frac{g_\pi}{\pi} \epsilon_\mu \dot{X}_0^\mu(\sigma=\pi, \tau) e^{-i k_\mu x_0^\mu(\sigma=\pi, \tau)} \end{aligned} \quad (4.23)$$

With the interaction hamiltonian of Eq. (4.23) we can compute the amplitude for the emission of different photons in successive laps of τ . We obtain

$$A_{\lambda, \lambda_N}^{(1)} \propto \int_{-\infty}^{\infty} d\tau_{N-1} \int_{-\infty}^{\tau_{N-1}} d\tau_{N-2} \dots \int_{-\infty}^{\tau_3} d\tau_2$$

$$\langle k_N \lambda_N | H_I(\epsilon_{N-1}, k_{N-1}; \tau_{N-1}) \dots$$

$$H_I(\epsilon_2, k_2; \tau_2) | k_1 \lambda_1 \rangle.$$

We use now

$$H_I(\tau) = e^{i\tau L_0} H_I(0) e^{-i\tau L_0} \quad (4.25)$$

which follows from

$$[x_0^\mu, H_0] = i \frac{d}{d\tau} x_0^\mu, \quad [p_0^\mu, H_0] = i \frac{d}{d\tau} p_0^\mu \text{ and } H_0 \equiv L_0,$$

to cast $A_{\lambda_1 \lambda_N}^{(1)}$ in the form^{*)}

$$A_{\lambda, \lambda_N}^{(1)} = \langle k_N \lambda_N | H_I(\epsilon_{N-1}, k_{N-1}; 0) \frac{1}{L_0 - 1 - i\epsilon} \dots$$

$$\frac{1}{L_0 - 1 - i\epsilon} H_I(\epsilon_2, k_2; 0) | k_1 \lambda_1 \rangle, \quad (4.26)$$

*) To obtain Eq. (4.26) remember that the on-mass-shell physical states $|k_1 \lambda_1\rangle$ and $|k_N \lambda_N\rangle$ satisfy the equation $L_0 |k\lambda\rangle = |k\lambda\rangle$.

[a δ -function due to invariance under τ translation has been factored out in the passage from Eq. (4.24) to Eq. (4.26)].

If we set $g_\pi = 0$, the amplitude of Eq. (4.26) coincides with the planar dual amplitude for emission of particles of the first excited level. The "photon emission" vertex of the dual theory is indeed proportional to

$$V(k, \epsilon) = \epsilon_\mu P^\mu(0) e^{-i k_\mu Q^\mu(0)} \quad (4.27)$$

[$P^\mu(\tau)$ and $Q^\mu(\tau)$ are defined in Eq. (2.75)] which equals $H_I(\epsilon, k; 0)$ with $g_0 = \pi$ and $g_\pi = 0$.

More generally, if we allow for emission with equal strength at $\sigma = 0$ and $\sigma = \pi$, we obtain the amplitudes associated with the generalized multiperipheral diagrams of Fig. 4.2, with emission from the two sides of the line.

Fig. 4.2

If we let the τ_i cover all the range $(-\infty, +\infty)$, dropping the restrictions $\tau_i < \tau_{i+1}$ (but of course τ -ordering the product of the interaction Hamiltonians) we obtain the full dual amplitude, sum of all the non planar contributions.

4.2 The transverse excitation operators

Using the photon emission vertex we can generate physical states⁴⁶). We consider the transition induced by the emission of the first photon (of polarization vector ϵ_2^μ and momentum k_2^μ) in Eq. (4.24). The initial state $|k_1 \lambda_1\rangle$ is transformed into a state

$$\begin{aligned}
 |\varphi\rangle &= \int_{-\infty}^{\tau_3} d\tau_2 H_I(\epsilon_2, k_2; \tau_2) |k, \lambda_1\rangle = \\
 &= \int_{-\infty}^{\tau_3} d\tau_2 e^{i(L_0 - 1 - i\epsilon)\tau_2} H_I(\epsilon_2, k_2; 0) |k, \lambda_1\rangle = \\
 &= \frac{e^{iL_0\tau_3}}{L_0 - 1 - i\epsilon} H_I(\epsilon_2, k_2; 0) |k, \lambda_1\rangle.
 \end{aligned} \tag{4.28}$$

The vector $|\varphi\rangle$ is a function of the momenta k_1 and k_2 ; as such, it has poles when the condition for a resonance $-\alpha'(k_1 - k_2)^2 = n$ is satisfied. These poles correspond to transitions to new on-mass-shell physical states, represented by the residue of the poles themselves.

Notice that when $-\alpha'(k_1 - k_2)^2$ is an integer, the exponential $e^{iL_0\tau_2}$ is a periodic function of τ_2 with period 2π . It follows that the residuum at the pole can be expressed as

$$\begin{aligned}
 \text{Res } |\varphi\rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\tau e^{i(L_0 - 1)\tau} H_I(\epsilon_2, k_2; 0) |k, \lambda_1\rangle = \\
 &= \frac{1}{2\pi} \int_0^{2\pi} d\tau H_I(\epsilon_2, k_2; \tau) |k, \lambda_1\rangle.
 \end{aligned} \tag{4.29}$$

We see that the operator

$$A = \frac{1}{2\pi} \int_0^{2\pi} d\tau H_I(\epsilon_2, k_2, \tau) \quad (4.30)$$

converts the on-mass-shell physical state $|k_1 \lambda_1\rangle$ into a new on-mass-shell physical state $|\phi'\rangle$. We modify now the notation to conform to the notation used in Subsection 3.4. We denote by $p^\mu = (p^+, p^-, p_\perp)$ the momentum of the initial physical state $|\lambda_1\rangle$ and assume, for definiteness, the momentum of the emitted photon to be given by $k_2^\mu = n(0, k, 0_\perp)$. The on-mass-shell condition $-\alpha'(p - k_2)^2 = \text{integer}$ becomes then $2\alpha'p^+k = 1$.

We assume ϵ^μ to be transverse, $\epsilon^\mu = (0, 0, \epsilon_i)$ ($\epsilon^+ = 0$, because of $k_\mu \epsilon^\mu = 0$, and a component $\epsilon^- \neq 0$ can always be eliminated by a gauge transformation); we introduce a variable $z = e^{i\tau}$, and define a normalization by taking $g_0 = \pi/\sqrt{2\alpha'}$, $g_\pi = 0$. With these conventions we have

$$H_I(\epsilon_2, k_2; \tau) = \frac{\epsilon_i}{\pi\sqrt{2\alpha'}} P^i(z) e^{in k Q^+(z)} \quad (4.31)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} d\tau = \frac{1}{2\pi i} \oint_0 \frac{dz}{z}$$

The operators that generate a transition between the two physical states $|p \lambda_1\rangle$ and $|\phi'\rangle$ are then given by

$$A_n^i(k) = \frac{1}{2\pi i \sqrt{2\alpha'}} \oint_0 \frac{dz}{z} P^i(z) e^{in k Q^+(z)} \quad (4.32)$$

We recognize in this expression the transverse excitation operators of Eq. (3.43)*). We also see how the theory of interacting strings gives a motivation for the introduction of these operators, which may have seemed rather *ad hoc* in Section 3.

*) We obtain in Eq. (4.32) the annihilation type operator A , instead of $A^\dagger$ of Eq. (3.43), because the emission of a "photon" lowers the excitation number of the state.

We prove now Eqs. (3.44) and (3.46)^{*)}. The commutation relations of the operators L_m with the fields $Q(z)$ and $P(z)$ are given by^{**)}

$$[L_m, Q^\mu(z)] = z^{m+1} \frac{d}{dz} Q^\mu(z), \quad (4.33)$$

$$[L_m, P^\mu(z)] = z \frac{d}{dz} z^m P^\mu(z). \quad (4.34)$$

It follows that

$$\begin{aligned} [L_m, A_n^i(k)] &= \frac{1}{2\pi i \sqrt{2\alpha'}} \oint_0 \frac{dz}{z} \\ &\quad \left\{ \left(z \frac{d}{dz} z^m P^i \right) e^{in k Q^+} + P^i z^{m+1} \frac{d}{dz} e^{in k Q^+} \right\} = \\ &= \frac{1}{2\pi i \sqrt{2\alpha'}} \oint_0 dz \frac{d}{dz} \left\{ z^m P^i(z) e^{in k Q^+(z)} \right\} = 0, \end{aligned} \quad (4.35)$$

which proves Eq. (2.47).

*) Equation (3.45) is obvious, since the right-hand side of Eq. (3.43) contains only operators that commute with a_n^+ and $a_n^{+\dagger}$.

***) Equation (4.34) can either be checked directly or derived from Eqs. (2.46), (2.48), (2.76) and (2.78).

To demonstrate Eq. (2.49) we use the equation:

$$\begin{aligned}
 & P^i(z_1) P^j(z_2) \text{ (defined only for } |z_1| > |z_2| \text{)} = \\
 & = \left\{ 2\alpha' p^i + \sum_{n=1}^{\infty} \sqrt{2\alpha' n} (-i a_n^i z_1^{-n} + i a_n^{i\dagger} z_1^n) \right\} \\
 & \quad \left\{ 2\alpha' p^j + \sum_{n=1}^{\infty} \sqrt{2\alpha' n} (-i a_n^j z_2^{-n} + i a_n^{j\dagger} z_2^n) \right\} = \quad (4.36) \\
 & = : P^i(z_1) P^j(z_2) : + 2\alpha' \delta^{ij} \sum_{n=1}^{\infty} n \left(\frac{z_2}{z_1} \right)^n = \\
 & = : P^i(z_1) P^j(z_2) : + 2\alpha' \delta^{ij} \frac{z_1 z_2}{(z_1 - z_2)^2}.
 \end{aligned}$$

We have

$$\begin{aligned}
 A_m^i A_n^{j\dagger} &= \frac{1}{(2\pi i)^2} \frac{1}{2\alpha'} \oint_0 \oint_0 \frac{dz_1}{z_1} \frac{dz_2}{z_2} \quad (4.37) \\
 & \quad |z_1| > |z_2| \\
 & \left\{ : P^i(z_1) P^j(z_2) : + \frac{2\alpha' \delta^{ij} z_1 z_2}{(z_1 - z_2)^2} \right\} e^{ik(n_1 Q(z_1) - n_2 Q^*(z_1))}.
 \end{aligned}$$

The other term of the commutator of A_m^i and $A_n^{j\dagger}$ differs from the right-hand side of Eq. (4.37) only in that the contours are subject to the opposite prescription $|z_2| > |z_1|$. Distorting the contours in Eq. (4.37) to $|z_2| > |z_1|$, we pick up the singularity due to the double pole $1/(z_1 - z_2)^2$ [$: P^i(z_1) P^j(z_2) :$ is regular for $z_1 = z_2$].

We thus obtain

$$\begin{aligned}
 [A_m^i, A_n^{j\dagger}] &= \frac{\delta^{ij}}{(2\pi i)^2} \oint_0 dz_1 \oint_{z_2} \frac{dz_2}{(z_1 - z_2)^2} e^{ik(n_1 Q(z_1) - n_2 Q(z_2))} \\
 &= \frac{\delta^{ij}}{2\pi i} \oint_0 dz_2 (i n_1 k) \left[\frac{d}{dz_2} Q(z_2) \right] e^{ik(n_1 - n_2) Q(z_2)} \\
 &= \begin{cases} \frac{\delta^{ij}}{2\pi i} \oint_0 dz_2 \frac{n_1}{n_1 - n_2} \frac{d}{dz_2} e^{ik(n_1 - n_2) Q(z_2)} & \text{for } n_1 \neq n_2, \\ \delta^{ij} \frac{n}{2\pi} \oint_0 dz_2 \frac{d}{dz_2} k Q(z_2) & \text{for } n_1 = n_2 = n. \end{cases} \quad (4.38)
 \end{aligned}$$

But $kQ^+(z_2)$, acting on a state of momentum p^μ , contains terms regular for $z = 0$ plus a term, $-2\alpha' ikp^- \log z_2$, that increases by $2\pi(2\alpha' kp^-) = 2\pi$ in a complete tour around the origin. It follows that

$$[A_m^i, A_n^{j\dagger}] = \begin{cases} 0 & \text{for } n_1 \neq n_2 \\ n & \text{for } n_1 = n_2 \end{cases} \quad (4.39)$$

as we wanted to prove.

4.3 Some remarks on the general vertex operators

The reason why we restricted our attention to the vertex operator for the emission of the massless particles of the theory is that only when $k^2 = 0$ the passage from the classical expression $e^{ikx(\sigma=0, \tau)}$ to the corresponding quantum operator is not ambiguous.

We expect that every time we consider the interaction of strings with some external field, the wave-function of the external field $e^{ikx(\sigma, \tau)}$ will appear in the interaction Hamiltonian. If, as in the case of the electromagnetic field, the interaction occurs at the boundary, H_I will contain the expressions $e^{ikx(0, \tau)}$ and $e^{ikx(\pi, \tau)}$. But the operator $e^{ikx(\sigma, \tau)}$ is not defined for $k^2 \neq 0$, and one has to use the normal ordered

expression

$$: e^{i k x(\sigma, \tau)} : \left(= e^{i k x(\sigma, \tau)} e^{\frac{1}{2} k^2 \log \infty} \right).$$

The passage from the classical system to the quantum system involves then some non-trivial problems of ordering, and the derivation of the vertex from the theory of the string becomes rather involved.

The problem has been studied in Ref. 50, and, for what concerns the emission of ground state particles, also in Ref. 51. One finds that it is possible to reproduce the vertices of the dual model, and that the interaction occurs always at the boundary, but we shall not deal here with this problem, also because the vertex for the coupling of strings in arbitrary states of excitation can be found more easily by using the functional integration formalism, explained in Section 5.

Let us consider here a feature that the general vertex must have, and which is shared by the photon emission vertex. When we factorize any dual amplitude $A(k_1 \dots k_N)$, we should find only physical states (satisfying the Virasoro conditions) as on-mass-shell intermediate states. This means in particular that the residues at the poles of the amplitude

$$\langle k_N, \lambda_N | V \frac{1}{L_0 - 1} V \dots \frac{1}{L_0 - 1} V | k_1, \lambda_1 \rangle$$

should contain only physical states, i.e., that the vector

$$|\phi\rangle = L_n \frac{1}{L_0 - 1} V \dots \frac{1}{L_0 - 1} V | k_1, \lambda_1 \rangle \quad (4.40)$$

should not have the poles corresponding to the first propagator from the left.

We know that

$$L_n | k_1, \lambda_1 \rangle = 0 \quad (n > 0).$$

The absence of the poles in $|\phi\rangle$ means that somehow the gauge operators L_n "propagate" through the vertices and propagators in Eq. (4.40). This propagation will occur only if the vertices possess definite gauge transformation properties.

It is interesting to see which is the identity satisfied by the photon emission vertex that guarantees the correct propagation of the gauge operators. This is expressed by the equation:

$$\begin{aligned} [L_n, \epsilon_\mu V^\mu(k, z = e^{i\tau})] &= \\ &= \left(z \frac{d}{dz}\right) z^n \epsilon_\mu V^\mu(k, z), \end{aligned} \quad (4.41)$$

which follows immediately from Eqs. (4.33) and (4.34) and

$$\epsilon_\mu V^\mu(k, z) = \epsilon_\mu P^\mu(z) e^{-i k Q(z)} \quad \begin{array}{l} \text{(with } \epsilon \cdot k = \\ (= k^2 = 0). \end{array}$$

When a vertex $V(k, z)$ obeys to the equation

$$[L_n, V(k, z)] = z \frac{d}{dz} z^n V(k, z) \quad (4.42)$$

it can be shown (see Appendix, Eqs. (A46) and following ones) that the vector

$$|\varphi\rangle = \frac{1}{L_0 - 1} V(k, 1) \frac{1}{L_0 - 1} \dots V(k, 1) |k, \lambda_1\rangle \quad (4.43)$$

satisfies the equation $(L_n - L_0 + 1) |\phi\rangle = 0$, so that $|\phi'\rangle = \text{Res}_{L_0=1} |\phi\rangle$ is annihilated by the gauge operators L_n .

In the theory of dual resonance models, Eq. (4.42) is taken as the defining equation for the general vertex operators (of the conventional model) (see Refs. 52 and 53).

5. STRINGS IN INTERACTION - II

5.1 Splitting and joining of strings

The introduction of an interaction at the boundary, between the string and some external fields, leads quite naturally, as we saw in Section 4, to the multiperipheral form of the dual resonance amplitude. However, it is not clear in this approach why the resulting amplitudes should obey the constraints of duality and be symmetric for cyclic or anticyclic permutations of the external lines.

The formalism of Section 4 treats in a non-symmetrical way the states of the string and the quanta of the external fields, even though at the end the symmetry between the two is recovered. One would think that the duality properties of the amplitude should become more manifest in a formalism, where all of the external particles are treated as states of excitations of interacting strings. The problem, then, is to find an interaction among strings that may lead to the dual amplitudes.

The results obtained in Section 4 give an indication that the strings should always interact at their ends. However, with two strings in interaction, an interaction at the ends of both strings may occur only if the end points occupy a common position, i.e. if the two strings merge into a single one. Thus we are led to the following picture: the two strings join together, merge into a single one and finally separate (Fig. 5.1). The interaction occurs while the two strings propagate as a single one.

Fig. 5.1

At this point it is very tempting to assume that no particular interaction should occur while the two strings propagate together, and that the whole interaction is represented by the fact that two strings can merge into one or that one string can split into two.

In this section we shall see that this very simple mechanism of interaction gives origin to the dual resonance amplitudes. The problem, of course, is how to transform the physical assumption that the interaction is just given by the merging and splitting of strings into quantum mechanical transition amplitudes. It is very convenient, for this purpose, to follow Feynman's formulation⁵⁴⁾ of quantum mechanics, where the transition amplitudes are obtained by means of an average of a phase factor proportional to the action over all possible evolutions of the classical system.

If one or more strings merge and later split, they span a surface in space time which has one or more ramifications (Fig. 5.2). For every surface of evolution an action S is defined as $1/2\pi\alpha'$ times the area of the surface. The dual amplitude is then evaluated through an integral over all possible evolutions of the system (path integral) of the exponential of iS .

Fig. 5.2

Notice that this approach gives us a clear clue on why the transition amplitudes do satisfy planar duality: indeed, the surfaces of evolution that describe a definite process $A: a_1 + a_2 + \dots + a_i \rightarrow a_{i+1} + \dots + a_n$, are topologically equivalent to the surfaces of evolution

for the processes obtained from A by cyclic permutations of the external lines, e.g. A': $a_2 + a_3 + \dots + a_i + a_{i+1} \rightarrow a_{i+2} + \dots + a_n + a_1$ *). Then, a single amplitude describes all processes related by planar crossing of external legs.

The path integral formulation of dual amplitudes is antecedent to the theory of the string, and may be traced to the analogue model of Fairlie and Nielsen⁵⁵⁾ and to further developments by Gervais, Hsue, Nielsen, Olesen, Sakita and Virasoro⁵⁶⁻⁵⁹⁾. In the next subsection I shall give a very short account of the main ideas underlying the works of Refs. 55-59. This historical note is introductory to the arguments presented in the rest of the section.

5.2 Historical development

The origin of the path integral formulation of the dual amplitudes may be found in the analogue model⁵⁵⁾. Let us recall that the momentum-dependent part of the integrand of the dual amplitude has the form (see Appendix)

$$J = \prod_{i < j} |\rho_i - \rho_j|^{2\alpha' p_i \cdot p_j}, \quad (5.1)$$

where the integration variables ρ_i lie on the real axis.

Equation (5.2) can be rewritten as

$$J = \exp \frac{1}{2} \sum_{i \neq j} 2\alpha' p_i \cdot p_j \log |\rho_i - \rho_j|. \quad (5.2)$$

It is convenient to introduce a quantity

$$J^{\text{extended}} = N^{-1} \exp \frac{1}{4} \int_{\text{Im } z \geq 0} d^2 z \int_{\text{Im } z' \geq 0} d^2 z' \quad (5.3)$$

$$(\sum_i \sqrt{2\alpha'} p_i^\mu j_i(z)) (\sum_i \sqrt{2\alpha'} p_i^\mu j_i(z')) G(z, z'),$$

*) The surfaces A and A' cannot be transformed into each other if we want to preserve the character (time-like or space-like) of the separation of the points on the surfaces. We assume that the character of the surface can be modified in the process of crossing.

where

$$G(z, z') = \log |z - z'| + \log |z - z'^*|, \quad (5.4)$$

the sources $j_i(z)$ are normalized to 1 $[\int_{\text{Im } z \geq 0} d^2z \rho_i(z) = 1]$ and the normalization factor N is given by

$$N = \exp \frac{1}{4} \iint_{\text{Im } z, z' \geq 0} d^2z d^2z' \sum_i 2\alpha' p_i^2 j_i(z) j_i(z') G(z, z'). \quad (5.5)$$

J is the limiting value taken by (J^{extended}) when the sources $j_i(z)$ tend to a Dirac δ function localized at $z = \rho_i$.

Notice that $G(z, z')$ is the Neumann function for the upper complex plane defined by the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G(z, z') = 2\pi \delta^2(z - z') \quad (x = \text{Re } z, y = \text{Im } z), \quad (5.6)$$

$$\frac{\partial}{\partial y} G(z, z') = 0 \quad \text{for } y = 0. \quad (5.7)$$

This gives the possibility of interpreting NJ^{extended} as the exponential of the energy of interaction of a flow of momentum determined by the source distribution $J^\mu(z) = \sum_i \sqrt{2\alpha'} p_i^\mu j_i(z)$ (analogue model, Ref. 55). For the purpose we assume that the components of the flow of momentum

$$\vec{P}^\mu = (P_x^\mu, P_y^\mu)$$

are given in terms of a potential function by

$$\vec{P}^\mu(x, y) = \vec{\nabla} X^\mu(x, y) \quad (5.8)$$

and that the energy of interaction is given by

$$S = \frac{1}{4\pi\alpha'} \int_{\text{Im } z \geq 0} d^2z (\vec{\nabla} X(z))^2 + \int_{\text{Im } z \geq 0} d^2z \frac{X^\mu(z)}{\sqrt{2\alpha'}} J_\mu(z). \quad (5.9)$$

The condition that S be minimal implies then

$$\nabla^2 \chi^\mu(z) = \pi \sqrt{2\alpha'} J^\mu(z) \quad (5.10)$$

and

$$\frac{\partial}{\partial y} \chi^\mu(z) = 0 \quad \text{at } y=0. \quad (5.11)$$

It follows that

$$\chi^\mu(z) = \frac{1}{2} \int_{\text{Im } z' \geq 0} d^2 z' G(z, z') \sqrt{2\alpha'} J^\mu(z'), \quad (5.12)$$

$$\begin{aligned} S &= \frac{1}{2} \int_{\text{Im } z \geq 0} d^2 z \chi^\mu(z) J_\mu(z) / \sqrt{2\alpha'} = \\ &= 2\alpha' \iint_{\text{Im } z, z' \geq 0} d^2 z d^2 z' \left(\xi_i p_i^\mu j_i(z) \right) \left(\xi_i p_{i\mu} j_i(z') \right) G(z, z'), \end{aligned} \quad (5.13)$$

and

$$J^{\text{extended}} = N^{-1} S. \quad (5.14)$$

The normalization factor N^{-1} subtracts from S the self energy of the sources, which becomes infinite in the limit $j_i(z) \rightarrow \delta^2(z - \rho_i)$.

An alternative interpretation of the integrand of the conventional dual amplitude can be given in the following way⁵⁶⁻⁵⁹). The Neumann function $G(z, z')$ is the propagator of a quantum field $\chi^\mu(z)$ defined over the upper complex plane with an action

$$A = \frac{1}{2\pi} \int_{\text{Im } z \geq 0} d^2 z \left[\left(\frac{\partial \chi}{\partial x} \right)^2 + \left(\frac{\partial \chi}{\partial y} \right)^2 \right], \quad (5.15)$$

It should then be possible to express the quantity J^{extended} as a functional average of $\exp \left[i/\sqrt{2\alpha'} \int_{\text{Im } z \geq 0} d^2 z \chi^\mu(z) J_\mu(z) \right]$ with the weight factor $e^{-A/2\alpha'}$.

Indeed, let us consider the expression

$$I = \frac{\int D X^\mu(z) e^{-\frac{A}{2\alpha'}} e^{-i \int_{\text{Im } z \geq 0} d^2 z \frac{X^\mu(z) J_\mu(z)}{\sqrt{2\alpha'}}}}{\int D X^\mu(z) e^{-\frac{A}{2\alpha'}}} \quad (5.16)$$

We can perform the functional integration at the numerator of I by a change of variable. Let us define

$$X^\mu(z) = \tilde{X}^\mu(z) + \frac{i}{2} \int_{\text{Im } z' \geq 0} d^2 z' G(z, z') \sqrt{2\alpha'} J^\mu(z'). \quad (5.17)$$

We have

$$\begin{aligned} & -\frac{1}{4\pi\alpha'} \int_{\text{Im } z \geq 0} d^2 z (\vec{\nabla} X)^2 - i \int_{\text{Im } z \geq 0} d^2 z \frac{X^\mu(z) J_\mu(z)}{\sqrt{2\alpha'}} = \\ & = -\frac{1}{4\pi\alpha'} \int d^2 z (\vec{\nabla} \tilde{X})^2 - \frac{i}{4\pi\alpha'} \int d^2 z d^2 z' \vec{\nabla} \tilde{X}^\mu(z) \times \\ & \times \vec{\nabla} G(z, z') \sqrt{2\alpha'} J_\mu(z') + \frac{1}{16\pi\alpha'} \int d^2 z d^2 z' d^2 z'' (\vec{\nabla} G(z, z') \times \\ & \times \sqrt{2\alpha'} J^\mu(z')) (\vec{\nabla} G(z, z'') \sqrt{2\alpha'} J_\mu(z'')) - i \int d^2 z \frac{\tilde{X}^\mu(z)}{\sqrt{2\alpha'}} J_\mu(z) \\ & + \frac{1}{2} \int d^2 z d^2 z' J^\mu(z) G(z, z') J_\mu(z') \quad (\text{Im } z, z', z'' \geq 0) = \\ & \stackrel{\text{integrating by parts}}{=} -\frac{1}{4\pi\alpha'} \int_{\text{Im } z \geq 0} d^2 z (\vec{\nabla} X)^2 + \frac{1}{4} \int_{\text{Im } z, z' \geq 0} d^2 z d^2 z' J^\mu(z) G(z, z') J_\mu(z'). \end{aligned} \quad (5.18)$$

It follows then that

$$I = e^{\frac{1}{4} \int_{\text{Im } z, z' \geq 0} d^2 z d^2 z' J^\mu(z) G(z, z') J_\mu(z')} = \mathcal{N} \mathcal{J}^{\text{extended}} \quad (5.19)$$

We deduce from this result the following considerations:

- a) It is possible to obtain the integrand of the conventional dual amplitude through a functional (path) integral.
- b) The functional integration is performed over all of the possible values of a function $\chi^\mu(x,y)$ defined over the upper half plane.
- c) The functional integral uses as a weight factor the exponential of the negative of the quadratic action

$$A = \frac{1}{4\pi\alpha'} \int_{y \geq 0} dx dy \left[\left(\frac{\partial \chi}{\partial x} \right)^2 + \left(\frac{\partial \chi}{\partial y} \right)^2 \right].$$

- d) The sources are proportional to the external momenta and are localized at the boundary.

Notice that we used for convenience the upper complex plane as a domain of definition of the "field" $\chi^\mu(x,y)$; however, all of the equations that we have introduced are invariant under conformal transformations, and we could have employed a different domain for the variables x,y . For instance, we could have used as domain the one illustrated in Fig. 5.3.

Fig. 5.3

The strip of Fig. 5.3 is cut by lines parallel to the boundary of the strip, departing from the points A,B,C and D. The whole cut strip can be mapped conformally onto the complex plane, and the points at ∞ (P_1 - P_6) of the six branches cut into the strip are mapped into points on the real axis ρ_1 - ρ_6 .

A choice of the domain of definition of the "field" $\chi^\mu(x,y)$ like the one of Fig. 5.3 helps us to give a more intuitive interpretation of the functional integral. We may consider indeed the function $\chi^\mu(x,y)$ as providing a parametric representation of a surface of evolution, where

two or more strings come from asymptotic points, join [points of junction are $\chi^\mu(A)$, $\chi^\mu(B)$], separate at $\chi^\mu(C)$, $\chi^\mu(D)$, and proceed to some other asymptotic points. The sum over the functions $\chi^\mu(x,y)$ represents a sum over all possible classical evolutions of the strings; the transition amplitude is obtained by performing the functional average of the exponential of the action associated to the motion of the string; the exponentials of the source functions that contain the external p^μ are the wave-functions of the incoming or outgoing strings.

However, even if it is suggestive, this interpretation is not entirely satisfactory. The following points should be kept in mind:

- i) In arriving at Eq. (5.19) we have used Euclidean and not time-like surfaces.
- ii) The action we used is a quadratic action and not the square root action that gives the area of the surface of evolution.
- iii) The functional integration is over more variables than are actually needed to parametrize a surface of evolution.
- iv) We have not justified the further integration over the variables $\rho_1 - \rho_n$ which is needed to obtain the dual amplitude.

In the next subsection we shall see that it is possible to modify the path integral formulation of the dual amplitudes, so as to avoid the objections implicit in the points (i) - (iv).

5.3 Path integral formulation of the dual amplitudes in the transverse gauge

We consider a set of one or more strings that join at the end points and subsequently split into another set of strings. In the course of its time development this system of strings spans a surface in space time, which in general will have many branches, extending from the initial configurations of the strings to the final one (Figs. 5.1 and 5.2). In this section we do not consider the possibility that a string splits into two, which subsequently merge again (Fig. 5.4).

Fig. 5.4

Let us denote by $S^{(i)}$ and $S^{(f)}$ the initial and final configurations of the strings. To every possible surface Σ of evolution from the system of strings $S^{(i)}$ to the system of strings $S^{(f)}$ we associate an action $S = (1/2\pi\alpha')A$ where A is the area of Σ . The path integral formulation of quantum mechanics⁵⁴⁾ leads us to consider the functional

$$I(S^{(i)}, S^{(f)}) = \int D\Sigma e^{iS} \quad (5.20)$$

where the symbol $\int D\Sigma$ indicates a summation (to be properly defined) over all possible surfaces of evolution. The transition amplitude is extracted from $I(S^{(i)}, S^{(f)})$ by multiplying I with the wave-functions of the free strings in some definite state of excitation, and letting then the initial and final configurations of the strings become asymptotic.

$$T = \lim_{\tau_i \rightarrow -\infty, \tau_f \rightarrow \infty} \int \pi_h D S_h^{(i)} \pi_k D S_k^{(f)} \quad (5.21)$$

$$\pi_h \Psi_{\alpha_h}(S_h^{(i)}, \tau_i) \pi_k \Psi_{\alpha_k}^*(S_k^{(f)}, \tau_f) I(S_h^{(i)}, \tau_i; S_k^{(f)}, \tau_f).$$

The surface of evolution Σ can be parametrized by assigning the coordinates of its points as functions of two parameters

$$X^\mu = X^\mu(\sigma, \tau). \quad (5.22)$$

σ and τ are assumed to vary in some domain D of the plane: it does not have to be the strip $0 \leq \sigma \leq \pi$, $\tau_i \leq \tau \leq \tau_f$ which was convenient to

parametrize the evolution of the free strings. The path integral $\int D\Sigma$ is then expressed as a functional integral over functions $\chi^\mu(\sigma, \tau)$ defined over D .

Notice that the task of performing a functional integral

$$J = \int D\chi^\mu(\sigma, \tau) e^{iS(\dot{\chi}^\mu, \chi^\mu)} \quad (5.23)$$

is almost hopeless, because of the occurrence of a square root expression exponentiated in the integrand. At the same time, however, an integration over all functions $\chi^\mu(\sigma, \tau)$ would be redundant, because we know that many different functions, related by a change of parametrization, can represent the same surface of evolution. We must impose a constraint on the functional integral, in the form of a gauge requirement that specifies a parametrization^{60, 61}). By choosing a suitable gauge, we can reduce the square-root action to a quadratic one, and then perform exactly the functional integration. This line of approach to the dual amplitudes has been followed by Gervais and Sakita⁶²) and Mandelstam⁶³).

The authors of Ref. 62 focus on the problem of reducing the square root action to a quadratic one, both for the case of a free and of an interacting string. The redundant variables of integration are eliminated from the right-hand side of Eq. (5.23) by inserting in the integrand the (continuous) δ functions:

$$\delta[(\dot{\chi} - \chi')^2] \delta[x^+ - f(\sigma, \tau)].$$

The constraints $(\dot{\chi} - \chi')^2 = 0$, $x^+ - f'(\sigma, \tau) = 0$ specify uniquely the parametrization. When $(\dot{\chi} - \chi')^2 = 0$ the square root action can be rewritten as a quadratic one. Indeed from

$$(\dot{\chi} - \chi')^2 = \dot{\chi}^2 + \chi'^2 - 2\dot{\chi}\chi' = 0 \quad (5.24)$$

we deduce

$$(\dot{\chi}\chi')^2 = \frac{(\dot{\chi}^2)^2 + (\chi'^2)^2}{4} - \frac{\dot{\chi}^2 \chi'^2}{2} \quad (5.25)$$

$$\sqrt{-\dot{x}^2 x'^2 + (\dot{x} x')^2} = \frac{1}{2} (x'^2 - \dot{x}^2). \quad (5.26)$$

Two two δ functions reduce the functional integral to an integral over the transverse coordinates $x^i(\sigma, \tau)$ alone. By appropriately choosing the function $f(\sigma, \tau)$ one can perform the integral and one obtains the integrand of the conventional dual model.

In this section I shall present in more detail the treatment of Ref. 63. The main idea behind this work is to evaluate the functional integral directly in the transverse gauge, and to check afterwards that the answer is covariant.

Let us recall from Section 3 that any surface of evolution can be parametrized in the transverse gauge by identifying the parameter τ with the x^+ coordinate, and the parameter σ with the total amount of p^+ momentum flowing between the boundary of the string and the point considered. The independent dynamical variables are then $x^i(\sigma, \tau)$, $\mathcal{P}^i(\sigma, \tau)$, $p^+(\tau)$ and $x^-(\tau)$, while the "-" variables are obtained from the equations of constraint

$$\begin{aligned} x^- &= \frac{1}{\mathcal{P}^+} (\mathcal{P}_\perp x'_\perp), \\ \mathcal{P}^- &= \frac{1}{2\mathcal{P}^+} \left(\mathcal{P}_\perp^2 + \frac{x'^2_\perp}{(2\pi\alpha')^2} \right). \end{aligned} \quad (5.27)$$

In this section we shall change the normalization factors used in Section 3, and set

$$x^+ = \sqrt{\alpha'} \tau \quad (5.28)$$

$$\mathcal{P}^+(\sigma) = \frac{1}{2\pi\sqrt{\alpha'}} = \text{const.} \quad (5.29)$$

With the different normalization we now use, the range of the σ parameter becomes

$$0 \leq \sigma \leq 2\pi \sqrt{\alpha'} p^+ \quad (5.30)$$

as follows from the requirement

$$p^+ = \int_0^{\tau_{\max}} d\tau \mathcal{P}^+(\sigma) = \frac{\sigma_{\max}}{2\pi \sqrt{\alpha'}}.$$

As a matter of fact, it will be convenient to assume a range

$$\sigma_0 \leq \sigma \leq \sigma_0 + 2\pi \sqrt{\alpha'} p^+ \quad (5.31)$$

where the value of σ_0 will be specified later.

Any surface of evolution can be parametrized by Eqs. (5.27) to (5.29). For a general surface of evolution (not satisfying the classical equations of motion) $p^+(\tau)$ will not be a constant. In the quantum-mechanical sum over paths (5.20) we should include all possible surfaces of evolution. But we know that the (quantum) variable $p^+(\tau)$ will finally be conserved. We assume therefore that we can restrict the summation over paths to those surfaces of evolution that conserve p^+ , so that the path-integral is a functional integration over the transverse variables $x^i(\sigma, \tau)$ only.

It is convenient to parametrize the whole surface of evolution using a common domain

$$\begin{aligned} \tau_i \leq \tau \leq \tau_f \\ 0 \leq \sigma \leq 2\pi \sqrt{\alpha'} \sum p_{\text{incoming}}^+ = 2\pi \sqrt{\alpha'} \sum p_{\text{outgoing}}^+. \end{aligned} \quad (5.32)$$

The initial strings are parametrized with σ 's varying in the ranges

$$0 \leq \sigma \leq 2\pi \sqrt{\alpha'} p_{1,\text{in}}^+, \quad 2\pi \sqrt{\alpha'} p_{1,\text{in}}^+ \leq \sigma \leq 2\pi \sqrt{\alpha'} (p_{1,\text{in}}^+ + p_{2,\text{in}}^+)$$

etc. (see Fig. 5.5). The x^+ time $\sqrt{\alpha'} \tau_1$ represents in Fig. 5.5 the time when the strings $s_1^{(i)}$ and $s_2^{(i)}$ merge into a single one. Because of p^+ conservation, the new string will be parametrized with $0 \leq \sigma \leq 2\pi \sqrt{\alpha'} (p_{1,\text{in}}^+ + p_{2,\text{in}}^+)$.

Fig. 5.5

The surface of evolution is of course discontinuous across the cuts extending from τ_i or τ_f to the interaction times τ_n .

In the transverse gauge the action is quadratic, and the transition amplitude is given by

$$\begin{aligned}
 T = & \lim_{\tau_i \rightarrow -\infty, \tau_f \rightarrow \infty} \int \Pi_n d\tau_n V(\tau_1 \dots \tau_n, \tau_i, \tau_f) \times \\
 & \times \int D x^i(\sigma, \tau) \Pi_h \psi_{\alpha_h}(x_{\perp}^{(h)}(\sigma, \tau_i)) \Pi_k \psi_{\alpha_k}(x_{\perp}^{(k)}(\sigma, \tau_f)) \times \\
 & \times e^{\frac{-i}{4\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{2\pi} d\sigma (\dot{x}_{\perp}^2(\sigma, \tau) - x_{\perp}'^2(\sigma, \tau))} \quad (5.33)
 \end{aligned}$$

In Eq. (5.33) $V(\tau_1 \dots \tau_n, \tau_i, \tau_f)$ is a volume element that must be determined in such a way that T satisfies the correct factorization properties and $\psi_{\alpha_h}, \psi_{\alpha_k}$ are the free wave-functions of the initial and final strings. In the momentum representation we have

$$\psi(x_\perp(\sigma, \tau)) = \int \Pi_{i,n} \mathcal{P}_n^i \hat{\psi}(\mathcal{P}_n^i) \times \\ \times e^{-i \int_{\sigma_0}^{\sigma_0 + 2\pi\sqrt{\alpha'} p^+} d\sigma \frac{1}{2\sqrt{\alpha'} p^+} \mathcal{P}^i\left(\frac{\sigma - \sigma_0}{2\sqrt{\alpha'} p^+}\right) x_i(\sigma, \tau) + i\sqrt{\alpha'} \tau p^-} \quad (5.34)$$

where $\mathcal{P}_n^i$ represents the Fourier components of the transverse momentum $\mathcal{P}^i(\sigma)$ and $\mathcal{P}^i$ appears in the form $(1/2\sqrt{\alpha'} p^+) \mathcal{P}^i(\sigma - \sigma_0 / 2\sqrt{\alpha'} p^+)$ because of the change of the range of variation of σ .

It is convenient to perform a Wick rotation in Eq. (5.33), replacing τ with $i\tau$. Then the functional integral takes the form ^{*)}

$$I = \int D x_\perp(\sigma, \tau) e^{-\frac{1}{4\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_{max}} d\sigma (\dot{x}_\perp^2(\sigma, \tau) + x_\perp'^2(\sigma, \tau)) - i \int_{\tau_i, \tau_f} d\sigma x^i(\sigma, \tau) J^i(\sigma, \tau)} \quad (5.35)$$

The source function J^i stands for the momentum distributions $(1/2\sqrt{\alpha'} p^+) \mathcal{P}^i(\sigma - \sigma_0 / 2\sqrt{\alpha'} p^+)$ of Eq. (5.34).

Notice that the expression (5.35) is invariant under conformal transformations of the domain of definitions of the variables σ and τ . We introduce a complex variable

$$z = \tau + i\sigma \quad (5.36)$$

and factors α_ℓ proportional to the σ length of the ℓ^{th} string:

$$|\alpha_\ell| = 2\sqrt{\alpha'} p_\ell^+ \quad (5.37)$$

We define α_ℓ to be negative for outgoing strings, positive otherwise. It is very useful to consider the mapping

*) After the Wick rotation the explicit τ dependence of the external wave-functions involves increasing exponentials ($\exp -\sqrt{\alpha'} p^- \tau_i$ for $\tau_i \rightarrow -\infty$, $\exp \sqrt{\alpha'} p^- \tau_f$ for $\tau_f \rightarrow \infty$) so that factors in I with decreasing exponential behaviour must be kept in the limiting procedure $\tau_i \rightarrow -\infty$, $\tau_f \rightarrow \infty$.

$$z = \sum_{n=1}^{N-1} \alpha_n \log(\tilde{z} - z_n) \quad (z_n \text{ real}; z_1 > z_2 > \dots). \quad (5.38)$$

This mapping transforms the upper complex $\tilde{z}$ plane into the strip $-\infty \leq \tau \leq \infty$, $0 \leq \sigma \leq 2\pi\sqrt{\alpha'} \sum p_{in}^+$ (N is the total number of strings). If we let $\tilde{z}$ vary on the real axis from 0 to Z_1 , z varies from $+\infty$ to $-\infty$ (lower boundary of the strip) with $\text{Im } z = 0$; $\text{Im } z$ suddenly jumps to $\pi\alpha_1 = 2\pi\sqrt{\alpha'} p_1^+$ when $\tilde{z}$ goes through Z_1 ; then z follows the line of separation between the first and second branches of the evolution surface, while $\tilde{z}$ decreases approaching Z_2 , and so on. In the mapping, the points at $\tau_i = -\infty$ and $\tau_f = \infty$ of the strips representing the strings are mapped onto the points Z_ℓ on the real axis, the point at infinity of the last outgoing string onto the point of infinity of the real axis. The position of the Z_ℓ variables, together with the value of the α_ℓ parameters, determine the value of the interaction times τ_n . The finite strip $\tau_i \leq \text{Re } z \leq \tau_f$ is mapped into a domain D , given by the upper complex plane with small regions around the points Z_ℓ taken off. These regions are delimited by arcs γ_ℓ , which form the image under the mapping of the lines $\text{Re } z = \tau_i$ or τ_f , $\text{Im } z$ between $\sigma_0 + \pi\alpha_\ell$. In the limit $\tau_i \rightarrow -\infty$, $\tau_f \rightarrow \infty$ the arcs γ_ℓ shrink to the points Z_ℓ .

To evaluate the functional integral in Eq. (5.35) we proceed in a way analogous to the one followed in Subsection 5.2. We shift the variables of integration by setting

$$x^i(z) = \hat{x}^i(z) + i\alpha' \int_{\tau_i, \tau_f} d\sigma' G(z, z') J^i(z') \quad (5.39)$$

where $G(z, z')$ is a Neumann function for the strip, defined by

$$\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \tau^2} \right) G(z, z') = 2\pi \delta^2(z - z') \quad (5.40a)$$

inside the strip, and

$$\frac{\partial}{\partial \eta} G(z, z') = f(z) \quad (5.40b)$$

(independent of z') at the boundary of the strip. $\partial/\partial \eta$ in Eq. (5.41) stands for the normal derivative at the boundary; the boundary includes the cuts extending from τ_i and τ_f to the interaction times τ_n . After the change of variables of Eq. (5.39), we obtain

$$\begin{aligned} I &= e^{\frac{\alpha'}{2}} \int_{\tau=\tau_i, \tau_f} d\sigma \int_{\tau'=\tau_i, \tau_f} d\sigma' J^i(z) J^i(z') G(z, z') \\ &\quad \times \int D\tilde{x}_\perp(\sigma, \tau) e^{-\frac{1}{4\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\tau_{max}} d\sigma (\dot{x}_\perp^2(\sigma, \tau) + x_\perp'^2(\sigma, \tau))} \\ &= e^{\frac{\alpha'}{2}} \int_{\tau=\tau_i, \tau_f} d\sigma \int_{\tau'=\tau_i, \tau_f} d\sigma' J^i(z) J^i(z') G(z, z') \\ &\quad \times N(\tau_n, \tau_i, \tau_f; \alpha_e), \end{aligned} \quad (5.41)$$

which solves the problem of computing I .

To derive the transition amplitude T from Eq. (5.41), we must further specify the explicit form of $G(z, z')$, and of the volume element in Eq. (5.33). Notice that the Neumann functions are invariant under conformal transformations; we can therefore determine $G(z, z')$ if we know the Neumann function for the domain D :

$$G(z, z') = G_D(\tilde{z}(z), \tilde{z}(z')) \quad (5.42)$$

where the relation between z and $\tilde{z}$ is given by Eq. (5.38). In particular, in the limit $\tau_i \rightarrow -\infty$, $\tau_f \rightarrow \infty$

$$G(z, z') \rightarrow \log |\tilde{z}(z) - \tilde{z}(z')| + \log |\tilde{z}(z) - \tilde{z}(z')^*| \quad (5.43)$$

More generally it can be shown (see Ref. 63) that, when z and z' lie on the initial or final boundary of the strip, the Neumann function has the expansion

$$\begin{aligned}
 G(z, z') = & -\delta_{ee'} \sum_{n=1}^{\infty} \frac{4}{n} \cos n \eta_e \cos n \eta_{e'} + \\
 & + \sum_{m,n=1}^{\infty} 8 N_{mn,ee'} e^{m \xi_e + n \xi_{e'}} \cos m \eta_e \cos n \eta_{e'} + \\
 & + \sum_{m=1}^{\infty} 4 N_{m0,ee'} e^{m \xi_e} + \sum_{n=1}^{\infty} 4 N_{0n,ee'} e^{n \xi_{e'}} + \\
 & + 2(\delta_{ee'} \xi_e - \delta_{eN} \xi_e - \delta_{e'N} \xi_{e'} + b_{ee'}).
 \end{aligned} \tag{5.44}$$

In this last equation, ℓ , ℓ' and N are indices labelling the strings,

$$\eta_e = \frac{\sigma - \sigma_{e,0}}{\alpha_e} \tag{5.45}$$

is defined so as to vary between 0 and π when the parameter σ describes the ℓ^{th} string,

$$\begin{aligned}
 \xi_e &= \frac{\tau_i}{2\alpha_e} && \text{for incoming strings} \\
 \text{and} \\
 \xi_e &= \frac{\tau_f}{2\alpha_e} && \text{for outgoing ones.}
 \end{aligned} \tag{5.46}$$

Notice that $\xi_\ell \rightarrow -\infty$ for $\tau_i \rightarrow -\infty$ or $\tau_f \rightarrow \infty$, so that the exponentials in Eq. (5.44) are always decreasing exponentials. The explicit form of the coefficients $N_{mn,\ell\ell'}$ and $b_{\ell\ell'}$ depends on the values of the interaction times τ_n (or alternatively of the variables Z_ℓ) and of the momenta p_ℓ^+ .

We can substitute Eq. (5.44) in Eq. (5.41), remembering Eq. (5.34), to obtain

$$\begin{aligned}
 I = & N(\tau_n, \tau_i, \tau_f; \alpha_e) \exp \sum_e \left(-\frac{\pi \alpha'}{2} \sum_{n=1}^{\infty} \frac{p_{n,e}^2}{n} \right) \times \\
 & \times \exp \sum_{e,e'} -\pi^2 \alpha' \left\{ \sum_{m,n=1}^{\infty} N_{mn,ee'} e^{m \xi_e + n \xi_{e'}} p_{me}^i p_{ne'}^i \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{m=1}^{\infty} N_{m0,e} e^{m \xi_e} \frac{P_{m,e}^i}{\pi} + \sum_{n=1}^{\infty} N_{0n,e} e^{n \xi_{e'}} \frac{P_{n,e'}^i}{\pi} \\
 & - \alpha' \sum_{e,e'} p_e^i p_{e'}^i (\delta_{ee'} - \delta_{eN} \xi_e - \delta_{e'N'} \xi_{e'} + b_{ee'}) \} = \\
 & = N(\tau_n, \tau_i, \tau_f; \alpha_e) \exp \sum_e \left(- \frac{\pi \alpha'}{2} \sum_{n=1}^{\infty} \frac{P_{n,e}^2}{n} \right) \times \\
 & \times \exp A(P, p, N, b, \xi).
 \end{aligned} \tag{5.47}$$

This expression is to be multiplied with the wave-functions of the external particles in the momentum representation and integrated over all $\mathcal{P}_{l,n}^\perp$. Notice that

$$\exp - \sum_e \left(\frac{\pi \alpha'}{2} \sum_{n=1}^{\infty} \frac{P_{n,e}^2}{n} \right)$$

is the wave-function of the ground state. This means that we can go to the occupation number representation simply by replacing that exponential factor with a vacuum bra

$$\langle 0 | = \Pi_e \langle 0 |_e \tag{5.48}$$

and contracting the resulting expression with the kets representing the external particles:

$$|\alpha_1 \dots \alpha_e\rangle = \Pi_e \Psi_{\alpha_e}(a_{e,n}^{\perp+}) |0\rangle. \tag{5.49}$$

We deduce then the following expression for the transition amplitude

$$\begin{aligned}
 T = & \lim_{\tau_i \rightarrow -\infty, \tau_f \rightarrow \infty} \int \prod_n d\tau_n W(\tau_n, \tau_i, \tau_f, \alpha_e) \times \\
 & \times \exp \left(- \sum_{e_{in}} \sqrt{\alpha'} \tau_i p_e^- + \sum_{e_{out}} \sqrt{\alpha'} \tau_f p_e^- \right) \times \\
 & \times \langle 0 | \exp A(P, p, N, b; \xi) \Pi_e \Psi_{\alpha_e}(a_{e,n}^{\perp+}) |0\rangle,
 \end{aligned} \tag{5.50}$$

where we defined

$$W(\tau, \alpha) = V(\tau) N(\tau, \alpha).$$

In the matrix A the coefficients $\mathcal{P}_{\ell,n}^{\perp}$ must be expressed in terms of $a_{\ell,n}^{\perp}$ and $a_{\ell,n}^{\perp\dagger}$:

$$\mathcal{P}_{\ell,n}^{\perp} = \frac{2}{\pi} \sqrt{\frac{n}{2\alpha'}} (-i a_{\ell,n}^{\perp} + i a_{\ell,n}^{\perp\dagger}). \quad (5.51)$$

We can normal order the exponential; this introduces factors

$$\exp(-\text{const} \cdot e^{n\xi_{\ell}}),$$

which tend to 1 when $\xi_{\ell} \rightarrow -\infty$. Therefore, we may retain only the annihilation operators in A. We finally obtain

$$\begin{aligned} T = & \lim_{\tau_i \rightarrow -\infty, \tau_f \rightarrow \infty} \int \prod_n d\tau_n W(\tau, \alpha) \times \\ & \times \exp(-\sum_{e \text{ in}} \sqrt{\alpha'} \tau_i p_e^- + \sum_{e \text{ out}} \sqrt{\alpha'} \tau_f p_e^-) \times \\ & \times \langle 0 | \exp A\left(\frac{2}{\pi} \sqrt{\frac{n}{2\alpha'}} a_{\ell,n}^{\perp}; p, N, b, \xi\right) \prod_e \Psi_{\alpha_e}(a_{\ell,n}^{\perp\dagger}) | 0 \rangle \end{aligned} \quad (5.52)$$

Equation (5.52) gives a remarkably compact expression for the transition amplitudes among excited states, when the coefficients N and b are known^{*)}. Notice that the $a_{\ell,n}^{\perp}$ operators in the wave-functions of the external particles bring down $e^{n\xi_{\ell,n}}$ factors, and these exactly compensate for the increasing exponentials

$$e^{-\sum_{e \text{ in}} \sqrt{\alpha'} \tau_i p_e^-}$$

and

$$e^{\sum_{e \text{ out}} \sqrt{\alpha'} \tau_f p_e^-}$$

We turn now to the formula for the ground state transition amplitude. If we put the vacuum state at the right of the matrix element in Eq. (5.52),

*) Equation (5.52) can be used to obtain the vertex among three transverse states. One recovers the result obtained in the operator formalism by Ademollo, Del Giudice, di Vecchia and Fubini⁶⁴⁾.

only the zero modes of the expansion of the momentum distributions survive. We can replace then the source functions in Eq. (5.41) with the piecewise constant expression

$$J^i(z) = \sum_e \theta(\sigma - \sigma_{0,e}) \theta(\sigma_{0,e+1} - \sigma) \frac{p_e^i}{2\pi\sqrt{\alpha'}} p_e^+ . \quad (5.53)$$

We get

$$I = N(\tau, \alpha) e^{\frac{\alpha'}{2} \sum_{ee'} p_e^i p_{e'}^i \int_0^\pi d\eta_e \int_0^\pi d\eta_{e'} G(z_e, z_{e'})} \quad (5.54)$$

[for the definition of η_ℓ see Eq. (5.45)].

It is convenient to rewrite the factor

$$K = e^{-\tau_i \sum_{e \text{ in}} \sqrt{\alpha'} p_e^- + \tau_f \sum_{e \text{ out}} \sqrt{\alpha'} p_e^-} \quad (5.55)$$

in a form analogous to the right-hand side of Eq. (5.54). For the purpose we notice that τ is a solution of the Laplace equation with normal derivative -1 for $\tau = \tau_i$, 1 for $\tau = \tau_f$ and 0 at the ends of the string. It follows that

$$\tau = \frac{1}{2\pi} \int d\sigma' N(\sigma, \tau; z'), \quad (5.56)$$

where the integral is to be taken over the strings at $\tau = \tau_i$ and $\tau = \tau_f$. With the aid of Eq. (5.56) we can express K as

$$K = e^{-\alpha' \sum_{ee'} p_e^+ p_{e'}^- \int_0^\pi d\eta_e \int_0^\pi d\eta_{e'} G(z_e, z_{e'})} \quad (5.57)$$

We combine the two factors in a covariant expression

$$KI = N(\tau, \alpha) e^{-\frac{\alpha'}{2} \sum_{ee'} p_{\mu e} p_{\nu e'}^{\mu\nu} \int_0^\pi d\eta_e \int_0^\pi d\eta_{e'} G(z_e, z_{e'})} \quad (5.58)$$

In the evaluation of the right-hand side of this equation we need to retain only the limiting form

$$G(z, z') = 2 \log |z_e - z_{e'}| \quad (5.59)$$

when the points z and z' are on different strings [see Eq. (5.43) and recall that $\tilde{z} \rightarrow z_\ell$ when the string becomes asymptotic], whereas we use the expression (see Ref. 62)

$$G(z, z') = \frac{2 \operatorname{Re} z}{\alpha_e} + 2 \log |2(\cos \eta_e - \cos \eta_{e'})| \\ - \sum_{e' \neq e} \frac{\alpha_{e'}}{\alpha_e} \log |z_{e'} - z_e|, \quad (5.60)$$

when the points z and z' are both on the same string. Inserting Eq. (5.59) and (5.60) in Eq. (5.58), the factor $\ln |2(\cos \eta_\ell - \cos \eta_{\ell'})|$ disappears (its integral vanishes) and we obtain

$$IK = N(\tau, \alpha) \exp \left\{ \sum_e \frac{1}{\alpha_e} \left(\tau_e - \sum_{e' \neq e} \alpha_{e'} \log |z_{e'} - z_e| \right) \right\} \times \\ \times \exp - \alpha' \sum_{e \neq e'} p_{\mu e} p_{\mu e'}^* \log |z_e - z_{e'}| \quad (5.61)$$

(we have used $\alpha' p_\ell^2 = -1$).

We first exponential tends to 0 when $\tau_i \rightarrow -\infty$, $\tau_f \rightarrow \infty$, but it can be absorbed in the volume element. We finally obtain

$$T = \int \pi_n d\tau_n U(\tau_n, \alpha_e) e^{-\alpha' \sum_{e \neq e'} p_e p_{e'}^* \log |z_e - z_{e'}|} \quad (5.62)$$

where

$$U(\tau, \alpha) = V(\tau, \alpha) N(\tau, \alpha) e^{\sum_e \frac{\tau_e - \sum_{e' \neq e} \alpha_{e'} \log |z_{e'} - z_e|}{\alpha_e}} \quad (5.63)$$

Equation (5.62) gives the familiar form for the conventional dual resonance amplitude if we set

$$\int \pi_n d\tau_n U(\tau_n, \alpha_e) \dots = \int \prod_{i=1}^{N-3} d\tau_i \dots \quad (5.64)$$

It would take too long here to illustrate the further results obtained in Ref. 63, so let me briefly comment on some of the points discussed by Mandelstam.

The transition amplitude among excited states, Eq. (5.50), has been obtained in a non-covariant way, as well as the ground state transition amplitude (5.62). One should check whether the result is indeed covariant. We notice that Eq. (5.62) becomes covariant if we assume the form (5.64) for the volume element, but it is not *a priori* obvious that Eq. (5.64) is consistent with factorization.

In Ref. 63 it is shown that the expression (5.50) is covariant when the number d of dimensions of space time is 26. The proof goes through the use of the representation of the Lorentz group in the space of transverse excitations introduced in Section 3. By using the covariance of the amplitude one can show that the volume element must have the form (5.64) in 26 dimensions. The form of the volume element can be established also from the requirement of factorization. By analysing the four-point amplitude, Mandelstam shows that the volume element assumes the form (5.64) only for $d = 26$.

Summarizing, in the transverse gauge we can define transition amplitudes consistent with factorization for any number of space time dimensions; but they are covariant and reproduce the dual amplitudes only for $d = 26$.

I think it is appropriate, at this point, to make a few considerations on the results obtained in this section.

A quantum mechanical interaction among strings can be introduced in a quite natural way. It is not necessary to introduce an explicit coupling or energy of interaction. The system of one or more strings is always governed by the same action. Only the boundary conditions change when the strings interact. All this is quite different from what we are

used to, in the standard formulations of field theory, where the boundary conditions are fixed, and there is an interaction part in the Lagrangian. Here the boundary conditions change in the course of the interaction, and the transition amplitude is a sort of overlap function between the systems defined with different boundary conditions. Beyond the dynamics of the string, this new picture of the interaction may be relevant for the physics of composite systems.

6. CLOSED STRINGS

6.1 Dual loops

In Section 5 we considered surfaces of evolution without any holes. When we sum the classical action over these surfaces we obtain the tree graphs of the dual resonance theory. We can consider also virtual processes where a string splits into two strings, which later on merge again. We obtain then surfaces of evolution with one or more holes, which correspond to the dual loop diagrams^{*)}.

In the formalism of Section 4 a one-loop diagram can be interpreted as a vacuum polarization process, where the surface of evolution of the string closes into itself in space-time, with one or more external lines coupled at the boundary. (See Fig. 6.1.)

Fig. 6.1

The corresponding amplitude can be written as

$$A(p_1, \dots, p_m, q_1, \dots, q_n) = T_2 (V D V \dots D P) \quad (6.1)$$

where $p_1 \dots p_m$ and $q_1 \dots q_n$ are the momenta of the lines coupled to the two different boundaries of the surface of evolution which (after Fig. 6.1) we will call external and internal boundary, D is the propagator of a free string, V is the vertex for emission of the external particle at the appropriate boundary, and P is a projection operator that ensures that only physical states propagate in the loop.

*) For an account of dual loops see Ref. 10, and references contained therein.

Notice that P must be inserted in Eq. (6.1). Indeed, as we saw in Section 4, the vertices V allow for a propagation of the gauges, so that physical states are converted into physical states by V . But spurious states can also propagate through the vertices V if they are present in the residues of any of the propagators in the loop. In 26 dimensions the transverse states form a complete set of physical states, and we can insert for P the transverse projection operator τ of Brink and Olive (see Subsection 3.4). In spite of the apparent non-covariance of τ , the loop amplitude $A(p_1 \dots p_m, q_1 \dots q_n)$ is Lorentz-covariant⁶⁵⁾.

When one computes explicitly $A(p_1 \dots p_m, q_1 \dots q_n)$, one finds singularities in the energy variable $-(p_1 + \dots p_m)^2 = -(q_1 + \dots q_n)^2$ ^{56,66,67)}. In general, these singularities are unphysical, but they become simple poles when $d = 26$ ^{*)}. The poles lie then on a Regge trajectory with slope $\frac{1}{2} \alpha'$ and intercept $\alpha_0 = 2$.

We can give an intuitive explanation for the appearance of these new poles in the loop amplitude. The surface of evolution of Fig. 6.1 is spanned by an open string making a closed loop through space time. But we can think of it also as generated by the evolution of a closed string that moves from the outer boundary to the inner boundary of the surface. The poles in the variable $-(p_1 + \dots p_m)^2 = -(q_1 + \dots q_n)^2$ correspond then to the states of the spectrum of a free quantized closed string.

6.2 The spectrum of the closed string

The theory of a free closed string can be obtained with only few modifications from the theory of the open string, presented in Sections 2 and 3.

It is convenient to take a σ variable ranging from 0 to 2π . The canonically conjugate variables are always $x^\mu(\sigma)$ and $\dot{x}^\mu(\sigma)$, and are subject to the constraint equations (2.42) and (2.43):

$$x'_\mu \dot{x}^\mu = 0 \quad \text{and} \quad 2\alpha'(\pi \dot{x})^2 + \frac{\dot{x}^2}{2\alpha'} = 0.$$

*) It is precisely through the requirement that the singularities in $-(p_1 + \dots p_m)^2$ were poles, and not cuts violating unitarity, that the magic number $d = 26$ made its appearance in the theory of dual resonance models [see Lovelace⁶⁸⁾].

The equation of motion (2.26), $\dot{p}^\mu + \pi'^\mu = 0$, is unchanged, but the boundary condition, Eq. (2.27), is replaced by the requirements of continuity:

$$\begin{aligned} x^\mu(2\pi) &= x^\mu(0), \\ p^\mu(2\pi) &= p^\mu(0). \end{aligned} \quad (6.2)$$

We define the orthonormal gauge by choosing a Hamiltonian [cf. Eq. (2.58)]

$$H = \pi \int_0^{2\pi} d\sigma \left(\alpha' \dot{p}^2(\sigma) + \frac{x'(\sigma)^2}{4\pi\alpha'} \right). \quad (6.3)$$

In this gauge coordinates and momenta satisfy the equation of D'Alembert. We expand therefore into normal modes:

$$\begin{aligned} x^\mu(\tau) &= x_0^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (c_n^\mu e^{-in\tau} + \\ &+ \bar{c}_n^\mu e^{in\tau} + c_n^{\mu\dagger} e^{in\tau} + \bar{c}_n^{\mu\dagger} e^{-in\tau}) \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} p^\mu(\sigma) &= \frac{p_0^\mu}{2\pi} + \frac{1}{2\pi\sqrt{2\alpha'}} \sum_{n=1}^{\infty} \sqrt{n} (-i c_n^\mu e^{-in\tau} - \\ &- i \bar{c}_n^\mu e^{in\tau} + i c_n^{\mu\dagger} e^{in\tau} + i \bar{c}_n^{\mu\dagger} e^{-in\tau}). \end{aligned} \quad (6.5)$$

Notice that because of the different boundary conditions the expansion into normal modes involves a double set of oscillators, with respect to the case of the string with free ends. We denote the corresponding creation and annihilation operators by the symbols $c_n^{\mu\dagger}$, $\bar{c}_n^{\mu\dagger}$, c_n^μ and $\bar{c}_n^\mu$. The non-vanishing commutators are

$$\begin{aligned} [c_m^\mu, c_n^{\nu\dagger}] &= \delta_{mn} g^{\mu\nu}, \\ [\bar{c}_m^\mu, \bar{c}_n^{\nu\dagger}] &= \delta_{mn} g^{\mu\nu}. \end{aligned} \quad (6.6)$$

In terms of the normal coordinates H becomes

$$H = \alpha' p^2 + \sum_n n (c_n^\dagger c_n + \bar{c}_n^\dagger \bar{c}_n), \quad (6.7)$$

so that the dynamical variables evolve according to the equations

$$\begin{aligned} x_0^\mu(\tau) &= x_0^\mu(0) + 2\alpha' p^\mu \tau, \\ c_m^\mu(\tau) &= c_m^\mu(0) e^{-im\tau}, \\ \bar{c}_m^\mu(\tau) &= \bar{c}_m^\mu(0) e^{-im\tau}, \\ c_m^{\mu\dagger}(\tau) &= c_m^{\mu\dagger}(0) e^{im\tau}, \\ \bar{c}_m^{\mu\dagger}(\tau) &= \bar{c}_m^{\mu\dagger}(0) e^{im\tau}. \end{aligned} \quad (6.8)$$

To analyse the constraints imposed on the physical states by the conditions $x'^2 = 0$ and $2\alpha'(\pi\mathcal{P})^2 + (x'^2/2\alpha') = 0$, we introduce the functionals [cf. Eq. (2.48)]

$$L_f = \frac{1}{4\pi} \int_0^{2\pi} d\sigma f(\sigma) \left(\sqrt{2\alpha'} \pi \dot{P} + \frac{x'}{\sqrt{2\alpha'}} \right)^2 \quad (6.9)$$

and

$$\bar{L}_f = \frac{1}{4\pi} \int_0^{2\pi} d\sigma f(\sigma) \left(\sqrt{2\alpha'} \pi \dot{P} - \frac{x'}{\sqrt{2\alpha'}} \right)^2.$$

The constraint equations are of course equivalent to $L_f = \bar{L}_f = 0$ for any f . From Eqs. (6.4) and (6.5) it is easily seen that L_f contains only the operators p^μ , c_m^μ and $c_m^{\mu\dagger}$; $\bar{L}_f$ only the operators p^μ , $\bar{c}_m^\mu$ and $\bar{c}_m^{\mu\dagger}$. The explicit form of the L_f and $\bar{L}_f$ operators is obtained from the form of the L_f operators of Section 2 by making the replacements $a_m^\mu \rightarrow c_m^\mu$ (or $\bar{c}_m^\mu$) and $p^\mu \rightarrow p^\mu/2$. In particular we have now

$$L_0 = \frac{\alpha' p^2}{4} + \sum_n n c_n^\dagger c_n. \quad (6.10)$$

(Notice the coefficient $\alpha'/4$ of p^2).

Repeating the arguments presented in Section 2, we find that the states of the closed string obey the constraint equations

$$L_n |\psi\rangle = \bar{L}_n |\psi\rangle = 0, \quad (6.11)$$

$$(L_0 - \bar{L}_0) |\psi\rangle = 0 \quad (6.12)$$

and

$$(L_0 + \bar{L}_0) |\psi\rangle = \alpha_0 |\psi\rangle. \quad (6.13)$$

The last equation is the only one affected by an ordering ambiguity, and we introduce therefore an intercept parameter α_0 , which is undetermined for the moment. Equation (6.13):

$$\left[\frac{\alpha'}{2} p^2 + \sum_n n (c_n^\dagger c_n + \bar{c}_n^\dagger \bar{c}_n) \right] |\psi\rangle = \alpha_0 |\psi\rangle$$

is a spectral equation for the system. The states are arranged on linear Regge trajectories with slope $\alpha'/2$ (one half the slope of the trajectories of the string with free ends).

The closed string can also be quantized in the transverse gauge, defined by

$$X^{+'}(\sigma) = P^{+'}(\sigma) = 0.$$

We shall not enter here into the details of the non-covariant method of quantization, but only remark that the consistency conditions turn out to be $d = \text{number of dimensions of space time} = 26$, again, and, for what concerns the intercept parameter, it is fixed by $\alpha_0 = 2$. The spectrum of the closed string has then a massless excited state of spin 2, with all the properties of a (strong) graviton.

The spectrum of excitations of the closed string is identical to the spectrum of intermediate physical states which is found in the non-planar dual model of Shapiro-Virasoro⁶⁹⁻⁷²). We can say then that the closed string constitutes the underlying dynamical system of the non-planar dual model, in the same sense as the open string provides an interpretation for the conventional dual model.

The interaction of a system of closed strings has not been studied yet in as much detail as the interaction of a system of open strings. But in Ref. 50 it has been shown that it is possible to derive the dual vertex for emission of "strong gravitons" (the massless excitations of spin 2) in the framework of the string theory. Following a method analogous to the one used to derive the "photon emission vertex" (see Section 4), the authors of Ref. 50 consider the interaction of a closed string with an external gravitational field. In the case of a plane monochromatic gravitational wave, the classical equations of motion can be

solved exactly, and this enables one to quantize the system. The quantum-mechanical transition amplitudes coincide then with the dual amplitudes of the non-planar model (with massless particles of spin 2 on the external legs).

6.3 The Reggeon-Pomeron transition vertex

The states of the closed string which are found in the loop amplitude of Eq. (6.1) (and Fig. 6.1) are often called states of the "Pomeron sector" of the dual resonance model. The reason for the name is that a common way to introduce internal symmetry variables is to multiply the dual amplitudes by factors, that represent the coupling of some "quarks" or "antiquarks", running at the boundary of the surface evolution, with the external field. These factors have the form of a trace of τ or λ matrices, and give origin to a definite SU(2) or SU(3) structure of the various channels of the amplitudes⁷³⁻⁷⁵). In particular, all of the channels where an open string propagates have the quantum numbers of a quark-antiquark pair, whereas those where a closed string propagates have the (internal symmetry) quantum numbers of the vacuum. Hence the names of Reggeon sector and Pomeron sector given to the corresponding channels of the dual amplitudes.

It is possible to express the loop amplitude $A(p_1 \dots p_m, q_1 \dots q_n)$ of Eq. (6.1) by means of a Reggeon-Pomeron transition vertex. In Refs. 76 and 77 it has been shown that

$$\begin{aligned} A(p_1 \dots p_m, q_1 \dots q_n) = \\ = \langle q_1 \dots q_n | V_0^\dagger D_0 P V_0 | p_1 \dots p_m \rangle, \end{aligned} \quad (6.14)$$

where

$$|p_1 \dots p_m \rangle = D V(p_m) \dots V(p_1) |0 \rangle \quad (6.15)$$

and $|q_1 \dots q_n\rangle$ represent states of the Reggeon sector, V_0 is a transition vertex between the Reggeon sector and the Pomeron sector, D_0 is the propagator of a closed string, and P is a projection operator that ensures that only physical states propagate in the Pomeron sector.

Equation (6.14) has been derived within the conventional approach to the theory of dual models, but we can get an intuitive idea of the meaning of that equation and of the vertex V_0 in the theory of the string. For the purpose, let us think of deforming the surface of evolution of Fig. (6.1) to the topologically equivalent shape represented in Fig. 6.2.

Fig. 6.2

In Fig. 6.2 the external lines of momenta $p_1 \dots p_m$ are coupled at the boundary of a surface of evolution of an open string, which at some point closes, propagates as a closed string, then opens up again and is coupled at the boundary to the external lines of momenta $q_1 \dots q_n$. Correspondingly, we have the representation of Eq. (6.14) for the transition amplitude: in particular, the Reggeon-Pomeron transition vertex V_0 represents the quantum mechanical transition amplitude from a state of open string to a state of closed string^{*)}.

*) This interpretation of V_0 has not been mathematically proved. As stated before, V_0 has been obtained by a factorization of the loop amplitude and not from the theory of the string. It seems plausible, however, that one may derive the expression of V_0 , or an equivalent expression, starting directly from the string picture of the amplitude.

The explicit form of V_0 is given by (after Ref. 78):

$$V_0 = \left(\frac{\pi}{2}\right)^{p^2} \pi^{-\alpha(p^2)} \times \\ \times \sum_a \langle 0 | \exp[\theta_{ac} + \theta_{a\bar{c}} + \theta_{c\bar{c}}] | 0 \rangle_{c\bar{c}} e^{\tilde{\Delta}}, \quad (6.16)$$

where p is the momentum running through the vertex, $|0\rangle_a$ and $|0\rangle_{c\bar{c}}$ represent the ground states of the Reggeon and the Pomeron sector, and θ_{ac} , $\theta_{a\bar{c}}$, $\theta_{c\bar{c}}$ and $\tilde{\Delta}$ are bilinear forms in the annihilation operators a_n^μ of the Reggeon sector and in the creation operators $c_n^{\mu\dagger}$ and $\bar{c}_n^{\mu\dagger}$ of the Pomeron sector. These bilinear forms can be expressed as contour integrals. Defining

$$P_a^\mu(z) = \frac{1}{\sqrt{2\alpha'}} P^\mu(z)$$

$[P^\mu(z)$ as defined in Eq. (2.75)],

$$P_c^\mu(z) = \sqrt{2\alpha'} \pi \mathcal{P}^\mu(\sigma = -i \log z) + \frac{x^1}{\sqrt{2\alpha'}} (\sigma = -i \log z),$$

$$P_{\bar{c}}^\mu(z) = \sqrt{2\alpha'} \pi \mathcal{P}^\mu(\sigma = -i \log z) - \frac{x^1}{\sqrt{2\alpha'}} (\sigma = -i \log z),$$

$[g^\mu$ and x^μ as defined in Eqs. (6.4) and (6.5)], we have

$$\tilde{\Delta} = \frac{1}{2(2\pi i)^2} \oint \frac{dx dy}{xy} : P_a(x) P_a(y) : \log \frac{\sin \pi(x-y)}{\pi(x-y)}, \quad (6.17)$$

$$\Theta_{ac} = \frac{-1}{(2\pi i)^2} \oint \frac{dx dz}{xz} P_a(x) P_c(z^{-1}) \log(1 - ze^{2i\pi x}), \quad (6.18)$$

$$\Theta_{a\bar{c}} = \frac{-1}{(2\pi i)^2} \oint \frac{dx dz}{xz} P_a(x) P_{\bar{c}}(z^{-1}) \log(1 - z\bar{e}^{2i\pi x}), \quad (6.19)$$

$$\Theta_{c\bar{c}} = \frac{1}{(2\pi i)^2} \oint \frac{dx dy}{xy} P_c(x^{-1}) P_{\bar{c}}(y^{-1}) \log(1 - xy). \quad (6.20)$$

The contours in Eqs. (6.17) to (6.20) include the origin and exclude the singularities of the logarithms present in the integrand.

Of particular interest are the gauge properties satisfied by the vertex V_0 . These are analysed in detail in Refs. 78 and 79.

One finds that definite linear combinations of gauge operators $L_n(a)$ in the Reggeon sector are transformed by V_0 into linear combinations of gauge operators $L_n(c)$ and $\bar{L}_n(\bar{c})$ in the Pomeron sector plus c-number terms proportional to $d - 26$, where d is the number of dimensions of space time. (Since in general these linear combinations involve both the annihilation gauges L_n and the creation gauges $L_{-n} = L_n^\dagger$, one speaks of "transmission" and partial "reflection" of the gauges at the vertex). It is shown in Refs. 78 and 79 that these gauge identities guarantee that only physical states are present as on-mass-shell intermediate states both in the Pomeron sector and in the Reggeon sector of the loop amplitude, provided that the consistency conditions $d = 26$, $\alpha_0 \text{ Pomeron} = 2$ and $\alpha_0 \text{ Reggeon} = 1$ are satisfied.

7. THE DUAL MODELS OF NEVEU-SCHWARZ AND RAMOND

7.1 Strings with a continuum spin degree of freedom

The conventional dual model remained for some time the only known example of a solution to the basic requirements of duality, in the narrow resonance approximation. Extremely important progress was made when two new dual models, the models of Neveu and Schwarz, and Ramond, with an entirely different spectrum of intermediate states, were discovered⁸⁰⁻⁸⁴). In these two models one introduces, besides the operators a_m^μ and $a_m^{\mu\dagger}$, an infinite set of operators, b_r^μ ($r = \pm\frac{1}{2}, \pm\frac{3}{2} \dots$) for the model of Neveu and Schwarz, Γ_m^μ ($m = 0, \pm 1, \pm 2 \dots$) for the model of Ramond, commuting with the a_m^μ operators, and satisfying the anticommutation relations^{*)}

$$\{b_r^\mu, b_{-s}^{\nu\dagger}\} = g^{\mu\nu} \delta_{rs} \quad (7.1)$$

$$\{\Gamma_m^\mu, \Gamma_{-n}^{\nu\dagger}\} = -2g^{\mu\nu} \delta_{mn} \quad (7.2)$$

The $b_r^{\mu\dagger}$ and $\Gamma_m^{\mu\dagger}$ operators, together with the $a_m^{\mu\dagger}$ operators, span a Fock space, which is used to represent the intermediate resonant states of the two models.

The states of the spectrum of the model of Neveu and Schwarz are bosons, whereas those of the model of Ramond are fermions. In spite of this difference, the two models are very similar, and can be thought of as describing the states of excitations of a common underlying dynamical system. This system, loosely speaking, is a string on which one has superimposed a continuum spin structure^{85,86}). At each point of the string, labelled by the parameter σ , a set of local γ^μ matrices, $\Gamma^\mu(\sigma)$, is defined. The operators b_r^μ , or Γ_m^μ , represent the coefficients of the expansion of the field $\Gamma^\mu(\sigma)$ in a Fourier series, so that the creation operators $b_r^{\mu\dagger}$ and $\Gamma_m^{\mu\dagger}$ excite a sort of coherent spin wave on the string.

*) The anticommutation relations (7.1) and (7.2) are consistent with the equations

$$b_r^{\mu\dagger} = b_{-r}^\mu \quad \text{and} \quad \Gamma_m^{\mu\dagger} = \Gamma_0^0 \Gamma_{-m}^\mu \Gamma_0^0,$$

which will define for us the operators $b_r^{\mu\dagger}$ and $\Gamma_m^{\mu\dagger}$.

More precisely, to give a string picture of the models of Neveu-Schwarz and Ramond one must define, as done in Section 2, a coordinate function $x^\mu(\sigma)$ ($0 \leq \sigma \leq \pi$), and a momentum distribution over the string $\mathcal{P}^\mu(\sigma)$, and two γ fields $\Gamma_1^\mu(\sigma)$ and $\Gamma_2^\mu(\sigma)$.

It is convenient to extend the range of definition of σ from $-\pi$ to π , and to group together the quantities $\mathcal{P}^\mu(\sigma)$ and $x'^\mu(\sigma)$, $\Gamma_1^\mu(\sigma)$ and $\Gamma_2^\mu(\sigma)$ defining the fields^{*})

$$P^\mu(\sigma) = \pi\sqrt{2\alpha'} \mathcal{P}^\mu(\sigma) + \frac{x'^\mu(\sigma)}{\sqrt{2\alpha'}}$$

$$(-\pi \leq \sigma \leq \pi, \mathcal{P}^\mu(-\sigma) = \mathcal{P}^\mu(\sigma), x'^\mu(-\sigma) = -x'^\mu(\sigma)) \quad (7.3)$$

and

$$\begin{aligned} \Gamma^\mu(\sigma) &= \Gamma_1^\mu(\sigma) & 0 \leq \sigma \leq \pi \\ &\Gamma_2^\mu(\sigma) & -\pi \leq \sigma \leq 0 \end{aligned} \quad (7.4)$$

(In what follows we shall sometimes express P^μ and Γ^μ as functions of $z = e^{i\sigma}$).

^{*}) The definition of $P^\mu(\sigma)$ is analogous to the definition given by Eq. (2.75), apart from a factor $1/\sqrt{2\alpha'}$ introduced in Eq. (7.3) to make $P^\mu(\sigma)$ dimensionless.

One assumes commutation and anticommutation relations

$$[P^\mu(\sigma), P^\nu(\sigma')] = 2\pi i g^{\mu\nu} \delta'(\sigma - \sigma'), \quad (7.5)$$

$$[P^\mu(\sigma), \Gamma^\nu(\sigma')] = 0 \quad (7.6)$$

and

$$\{\Gamma^\mu(\sigma), \Gamma^\nu(\sigma')\} = -4\pi g^{\mu\nu} \delta(\sigma - \sigma'). \quad (7.7)$$

The $P^\mu(\sigma)$ field, according to the standard expansion of $x^\mu(\sigma)$ and $\dot{x}^\mu(\sigma)$ in $\cos n\sigma$ [Eqs. (2.71) and (2.72)] is expanded into $\exp in\sigma$, with n integer

$$P^\mu(\sigma) = \sqrt{2\alpha'} p^\mu + \sum_{n=1}^{\infty} \sqrt{n} (-i\alpha_n^\mu e^{-in\sigma} + i\alpha_n^\mu e^{in\sigma}) \quad (7.8)$$

For the $\Gamma^\mu(\sigma)$ field, we can impose two different types of boundary conditions:

$$a) \quad \Gamma^\mu(-\pi) = -\Gamma^\mu(\pi)$$

$$b) \quad \Gamma^\mu(-\pi) = \Gamma^\mu(\pi)$$

Corresponding to the choices (a) and (b) we expand $\Gamma^\mu(\sigma)$ into $\exp ir\sigma$, with r half-integer, or $\exp im\sigma$ with m integer. We obtain:

$$\text{case (a)} \quad \Gamma^\mu(\sigma) = i\sqrt{2} \sum_r b_r^\mu e^{ir\sigma}, \quad r = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \quad (7.9)$$

case (b)
$$\Gamma^\mu(\tau) = \sum_m \Gamma_m^\mu e^{im\tau}, \quad m=0, \pm 1, \pm 2, \dots \quad (7.10)$$

The anticommutation relations Eqs. (7.1) and (7.2) follow.

We interpret the operators b_r^μ and Γ_m^μ with a positive index as annihilation operators, the operators $b_{-r}^{\mu\dagger} = b_r^{\mu\dagger}$ and $\Gamma_{-m}^{\mu\dagger} = \Gamma_0^\mu \Gamma_m^{\mu\dagger} \Gamma_0^\mu$ with a negative index as creation operators. In case (a) we can assume the existence of a single ground state vector $|0\rangle$. The Fock space of excitations H_a is obtained by acting with $a_m^{\mu\dagger}$ and $b_r^{\mu\dagger}$ on $|0\rangle$:

$$H_a : \{ b_{r_1}^{\mu_1\dagger} b_{r_2}^{\mu_2\dagger} \dots a_{m_1}^{\nu_1\dagger} \dots |0\rangle \} \quad (7.11)$$

In case (b) the ground state must allow for a representation of the zero mode γ matrices Γ_0^μ . The ground state must be degenerate, and will be represented by vectors $|u\rangle$, where u is an ordinary Dirac spinor. The action of the operators Γ_0^μ on $|u\rangle$ is given by

$$\Gamma_0^\mu |u\rangle = i\gamma^\mu u, \quad (7.12)$$

and the Fock space of excitations H_b is spanned by the vectors

$$\Gamma_{m_1}^{\mu_1\dagger} \Gamma_{m_2}^{\mu_2\dagger} \dots a_{m_1}^{\nu_1\dagger} \dots |u\rangle. \quad (7.13)$$

We see that H_a describes a spectrum consisting of an infinite number of bosons, H_b a spectrum consisting of an infinite number of fermions. The two cases (a) and (b) correspond to the models of Neveu and Schwarz and Ramond, respectively. The dynamic of the system is defined by a set of constraint equations (analogous to Eqs. (2.41) and (2.42) $[\mathcal{P}x' = 0$ and $2\alpha'(\pi\mathcal{P})^2 + x'^2/2\alpha' = 0$ for the ordinary string] and by the choice of a Hamiltonian function. The equations of constraint are the following ones:

$$P^2(\sigma) - \frac{i}{2} \Gamma^\mu(\sigma) \frac{d}{d\sigma} \Gamma_\mu(\sigma) = 0, \quad (7.14)$$

$$\Gamma^\mu(\sigma) P_\mu(\sigma) = 0. \quad (7.15)$$

For the quantum system, the expressions like $P^2(z)$ must be suitably defined (by a normal ordering, for instance) and both Equations are valid up to a c-number term, in a weak sense. Notice that the left-handside of Eq. (7.15) represents a generalized Dirac operator $\Gamma^\mu(\sigma)P_\mu(\sigma)$. One can check, using the algebra of Eqs. (7.5) to (7.7), that these equations close under the operations of commutation or anticommutation.

Equations (7.14) and (7.15) are transformed into a discrete set of conditions on the physical states by defining the gauge operators

$$\begin{aligned} L_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} \left\{ :P'(\sigma): - \frac{i}{2} : \Gamma(\sigma) \frac{d}{d\sigma} \Gamma(\sigma) : \right\} = \\ &= \frac{1}{2\pi i} \oint_0 \frac{dz}{z} z^n \left\{ :P'(z): + \frac{1}{2} : \Gamma(z) z \frac{d}{dz} \Gamma(z) : \right\}, \end{aligned} \quad (7.16)$$

$$\begin{aligned} G_r &= \frac{i}{2\pi\sqrt{2}} \int_{-\pi}^{\pi} d\sigma e^{ir\sigma} \Gamma_\mu(\sigma) P^\mu(\sigma) = \\ &= \frac{1}{2\pi\sqrt{2}} \oint_0 \frac{dz}{z} z^r \Gamma_\mu(z) P^\mu(z) \end{aligned} \quad (7.17)$$

(r half-integer),

$$\begin{aligned}
 F_n &= \frac{i}{2\pi\sqrt{2}} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} \Gamma^\mu(\sigma) P_\mu(\sigma) = \\
 &= \frac{1}{2\pi\sqrt{2}} \oint_0 \frac{dz}{z} z^n \Gamma^\mu(z) P_\mu(z) \quad (7.18) \\
 &\quad (n - \text{integer}).
 \end{aligned}$$

The algebra of these gauges is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{8}(m^3-m)\delta_{m,-n}, \quad (7.19)$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \quad (7.20a)$$

$$[L_m, F_n] = \left(\frac{m}{2} - n\right) F_{m+n}, \quad (7.20b)$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{d}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \quad (7.21a)$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{d}{2}\left(m^2 - \frac{1}{4}\right)\delta_{m,-n}. \quad (7.21b)$$

Notice the different value ($d/8$ versus $d/12$ in the conventional model) of the coefficient of the c-number term in Eq. (7.19), which results from an additional contribution from the spin degrees of freedom.

The constraint equations (7.14) and (7.15) are imposed on the quantum system as matrix element equations, valid between any two physical states. Separating the equations into a creation and an annihilation part, we obtain the conditions

$$L_n |\varphi\rangle_B = G_n |\varphi\rangle_B = 0, \quad n, r > 0,$$

on the bosonic excitations, and

$$L_n |\varphi\rangle_F = F_m |\varphi\rangle_F = 0, \quad n, m > 0, \quad (7.22)$$

on the fermionic excitations. The spectral equations are given by

$$L_0 |\varphi\rangle_B = \alpha_0 |\varphi\rangle_B \quad (7.23)$$

and

$$\begin{aligned} F_0 |\varphi\rangle_F &= \left\{ i\sqrt{\alpha'} p \Gamma_0 + \sum_{m=1}^{\infty} \frac{\sqrt{m} (a_m \Gamma_m^\dagger - h.c.)}{\sqrt{2}} \right\} |\varphi\rangle_F \\ &= i\sqrt{\alpha'} M |\varphi\rangle_F. \end{aligned} \quad (7.24)$$

We shall see in the next section that consistency conditions fix the intercept parameter α_0 of the leading bosonic trajectory and the mass M of the ground state fermion to the values $\alpha_0 = \frac{1}{2}$, $M = 0$. The orthonormal gauge is fixed by the choice

$$H = L_0. \quad (7.25)$$

The variables $P^\mu(\sigma)$ and $\Gamma^\mu(\sigma)$ evolve then according to the D'Alembert equation.

The reader will have noticed that we have defined the system directly from the constraint equations satisfied by the dynamical variables. This is legitimate, provided we can show that the constraint equations allow for the elimination of the unphysical time-like oscillations, or, in other words, that, even if the system has been defined independently from the choice of any frame of reference, we can always by a suitable gauge transformation, identify the evolution variable τ with a time-like coordinate $n_\mu x^\mu$, and all other dynamical variables with physical variables defined in that frame. In the next section we shall see that this is indeed possible. One can also give a Lagrangian formulation of the equations of motion, from which the constraint equations (7.14) and (7.15) follow as primary constraints⁸⁷⁻⁸⁹). Without giving the details of the construction of the Lagrangian, which is not straightforward, let me mention here that the main problem is to define the transformation properties of the spin fields $\Gamma_1^\mu(\sigma, \tau)$ and $\Gamma_2^\mu(\sigma, \tau)$ under general reparametrizations of the surface of evolution [the invariance of the action under these transformations generates the primary constraints of Eq. (7.14)], and also to define a new set of transformations, mixing the orbital variables x^μ and $\mathcal{J}^\mu$ with the spin variables Γ_1^μ and Γ_2^μ , that also leave the action invariant and generate the constraint equation (7.15). These new supergauge transformations can be introduced (see Refs. 88 and 89), and form a Lie group with anti-commuting parameters, belonging to a Grassmann algebra. A subgroup of the general supergauge transformations is generated by the linear combinations $\sum_r \alpha_r G_r$, or $\sum_m \alpha_m F_m$, where the α 's are anticommuting c-numbers, and the G 's and F 's are the gauge operators defined by Eqs. (7.17) and (7.18). Besides the dual models, the theory of supergauge transformations finds applications in other fields of particle physics⁹⁰).

7.2 The transverse gauge

A transverse gauge can be defined for the string with spin by demanding that, before quantization, the dynamical variables are subject to the conditions

$$P^+(\tau) = \sqrt{2\alpha'} p^+ = \text{const.} \quad (7.26a)$$

$$\Gamma^+(\tau) = 0. \quad (7.26b)$$

These conditions can be imposed within the orthonormal gauge by performing a reparametrization and a supergauge transformation⁸⁷⁾.

Independent dynamical variables are then p^+ , x_0^- , $P_\perp(\sigma)$, $\Gamma_\perp(\sigma)$, whereas the variables $P^-(\sigma)$ and $\Gamma^-(\sigma)$ are dependent variables that can be expressed in terms of the independent ones by solving the constraint equations (7.14) and (7.15).

The independent dynamical variables are quantized according to the commutation and anticommutation relations (7.5) to (7.7). As in the case of the string without spin in the transverse gauge (see Section 3.2) we have

$$x^+ = 2\alpha' p^+ \tau \quad (7.27)$$

and the Hamiltonian that generates the τ evolution of the system is given by

$$\begin{aligned} H &= 2\alpha' p^+ p^- = L_0^{tr} - \alpha_0 = \\ &= \alpha' p_\perp^2 + \sum_{n=1}^{\infty} n a_{\perp n}^\dagger a_{\perp n} + \sum_{n=\frac{1}{2}}^{\infty} 2 b_{\perp n}^\dagger b_{\perp n} - \alpha_0 \\ &\quad (\text{in the bosonic sector}) \\ &= \alpha' p_\perp^2 + \sum_{n=1}^{\infty} n a_{\perp n}^\dagger a_{\perp n} + \sum_{m=1}^{\infty} m \frac{\Gamma_{\perp m}^\dagger \Gamma_{\perp m}}{2} + M^2 \quad (7.28) \\ &\quad (\text{in the fermionic sector}). \end{aligned}$$

Equation (7.28) is a spectral equation for the system.

We see that in the transverse gauge the system is quantized in a space of positive norm, but one must check whether the method of quantization is consistent with Lorentz covariance. Consistency requires that a representation of the Lorentz group can be found within the space of transverse excitations. The form of the generators can be found by

following a line of reasoning analogous to the one of Section 3.3. If we consider the little group of a particle with zero transverse momentum, the non-trivial generators should be given by^{47,87)}

$$E_{Li} = \sum_{n=1}^{\infty} \frac{a_n^{i+} L_n^{t_2}}{\sqrt{n}} + h.c. + \sum_n b_n^{i+} G_{-n}^{t_2} \quad (7.29)$$

for the bosonic sector, and by

$$E_{Li} = \sum_{n=1}^{\infty} \frac{a_n^{i+} L_n^{t_2}}{\sqrt{n}} + h.c. + \sum_n \frac{\Gamma_n^{i+} E_n^{t_2}}{i\sqrt{2}} \quad (7.30)$$

for the fermionic sector.

The algebra of the Lorentz group requires

$$[E_{Li}, E_{Lj}] = -2\alpha' M^2 E_{ij}. \quad (7.31)$$

Computing the commutators, we find instead, in both cases

$$[E_{Li}, E_{Lj}] = -2(L_0^{t_2} - \frac{d^{t_2}}{16})E_{ij} + (\frac{d^{t_2}}{8} - 1)O_{ij}. \quad (7.32)$$

We see from this equation that the quantization in the transverse gauge leads to a covariant system only when $d^{tr} = 8$. We get then, for the masses of the ground states [cf. Eqs. (7.31) and (7.32) with $d^{tr} = 8$]

$$\alpha' M_{gs}^2 = \left\langle \left| L_0^{t_2} - \frac{1}{2} \right| \right\rangle_{gs}. \quad (7.33)$$

We obtain $M_{gs}^2 = -(1/2\alpha')$ for the model of Neveu and Schwarz, where $_{gs}\langle |L_0^{tr}| \rangle_{gs} = 0$. But from Eq. (7.21b) we deduce $_{gs}\langle |L_0^{tr}| \rangle_{gs} = d^{tr}/16$ in the model of Ramond, so that $M_{gs} = 0$.

It is important to notice that in the case of the string with spin the critical values of the number of dimensions of space time and of the intercept parameters are different from the case of the spinless string. These critical values here enter into the theory as consistency requirements for the transverse quantization, but one finds the same conditions if one wants to introduce a completely consistent interaction, even in the covariant gauge. (The same happens in the conventional model, where absence of ghosts in the tree amplitudes demands $\alpha_0 = 1$, and in the Pomeron sector of the loop amplitudes demands $d = 26$).

An interesting modification of the model has been proposed by Schwarz⁹¹⁾. By redefining some of the a_m^μ , b_r^μ and Γ_m^μ operators, it is possible to interpret $2N$ of the Lorentz indices as labelling an internal degree of freedom. If one takes in particular $N = 3$, the critical number of space-time dimensions is reduced to four.

The modification is very simple in the bosonic sector; one assumes that all the states have (illustrating the case of $N = 3$) $p^4 = p^5 = \dots p^9 = 0$. One redefines then

$$\begin{aligned}\alpha_n^1 &= a_n^4 + i a_n^5, & \bar{\alpha}_n^1 &= a_n^4 - i a_n^5, \\ \alpha_n^2 &= a_n^6 + i a_n^7, & \bar{\alpha}_n^2 &= a_n^6 - i a_n^7, \\ \alpha_n^3 &= a_n^8 + i a_n^9, & \bar{\alpha}_n^3 &= a_n^8 - i a_n^9,\end{aligned}\tag{7.34}$$

and analogously defines β_r^i and $\bar{\beta}_r^i$ operators starting from $b_r^4 - b_r^9$. At this point one assumes that the operators α_n^i and β_n^i transform according to the triplet representation of $SU(3)$, $\bar{\alpha}_n^i$ and $\bar{\beta}_n^i$ according to the conjugate representation, and that the corresponding excitations are not longer orbital or spin excitations, but excitations of an intrinsic quantum number.

The changes required for the fermionic sector involve a different expansion of the fields $P^i(\sigma)$ and $\Gamma^i(\sigma)$ ($i = 4 - 9$) and will not be reproduced here.

Transverse states can be defined in the covariant formalism by the additional requirements on the physical states

$$\begin{aligned} a_n^+ |\varphi\rangle &= 0 & n > 0 \\ b_n^+ |\varphi\rangle &= 0 \text{ or } \Gamma_n^+ |\varphi\rangle = 0, \quad n > 0. \end{aligned} \quad (7.35)$$

The transverse states can be generated by acting on the ground state with suitable transverse excitation operators^{47,92,93}, and a projection operator over transverse physical states can be constructed^{48,49}.

When $d = 10$, for the critical values of the intercept parameters, the transverse states form a complete set of physical states and the no-ghost theorem for the models of Neveu-Schwarz and Ramond^{45,92,93} follows.

7.3 The main features of the interaction

In the models of Neveu-Schwarz and Ramond one can have three types of interaction vertices:

- i) the vertex for the transition between two bosonic excitations with emission of a boson;
- ii) the vertex for the transition between two fermionic excitations with emission of a boson;
- iii) the vertex for emission of a fermion from a fermionic excitation, which is converted into a bosonic excitation.

A derivation of these vertices within the string interpretation of the models, analogous for instance to the derivation of the photon emission vertex presented in Section 4, has not been given yet. Their form can be found in many articles on the theory of dual resonance models, where their properties are also discussed.

For completeness, I reproduce here the expressions of the vertices for emission of a ground state particle. The transition between two bosonic states with emission of a ground state boson is generated by the vertex⁸⁰⁻⁸²)

$$V_{BB}(k) = \frac{1}{i\sqrt{2}} k^\mu \Gamma_\mu(\tau=0) V_0(k), \quad (7.36)$$

where k^μ is the momentum of the emitted particle and

$$V_0(k) = : e^{-ikQ(\tau=0)} :$$

is the vertex for the emission of a ground state particle in the conventional model.

The vertex for a transition between two fermionic states, with emission of a ground state boson is given by^{83,84)}

$$V_{FF}(k) = \Gamma_s V_0(k), \quad (7.37)$$

where

$$\Gamma_s = \gamma_s(-)^{-\frac{1}{2}} \sum_{m=1}^{\infty} \Gamma_{\mu m}^\dagger \Gamma_m^\mu \quad (7.38)$$

The expression of the vertex for the conversion of a fermion into a boson, with emission of a ground state fermion is more complicated, because it involves a passage between two different Fock spaces. It has been found in Ref. 94 and is given by

$$V_{FB}(k) = e^{-L_1^\Gamma} W V_0(k), \quad (7.39)$$

where L_1^Γ is the part of the L_1 gauge operator containing the Γ_m^μ operators, and

$$\begin{aligned} W = & {}_B \langle 0 | \exp \left(\frac{1}{i\sqrt{2}} \sum_{\substack{m=0 \\ 2=\frac{1}{2}}}^{\infty} (-)^{m-2} \binom{m-\frac{1}{2}}{2-\frac{1}{2}} \Gamma_{-m}^\mu b_{\mu 2} \right) \times \\ & \times \exp \left(-\frac{1}{4} \sum_{\substack{2,s=\frac{1}{2}}}^{\infty} \frac{2-s}{2+s} (-)^{2+s} \binom{-\frac{1}{2}}{2-\frac{1}{2}} \binom{-\frac{1}{2}}{s-\frac{1}{2}} b_2^\mu b_{s\mu} \right) \times \\ & \times \gamma^5 |u\rangle_F . \end{aligned} \quad (7.40)$$

In Eq. (7.40) the vector $|u\rangle_F$ represents the emitted fermion, $|0\rangle_B$ is the bosonic vacuum vector. W acts to the left on the fermionic states, to the right on the bosonic states.

The amplitudes are constructed by joining the vertices and the propagators, which are $1/(L_0 - \frac{1}{2})$ for the bosons, $1/(F_0 - i\sqrt{\alpha'}M)$ for the fermions^{*)}.

The vertices of Eqs. (7.36) satisfy definite gauge identities [in particular, at the vertex V_{FB} between the bosonic sector and the fermionic sector a phenomenon of transmission and partial reflection of the gauges occurs^{95,96}), rather similar to what happens with the Reggeon-Pomeron transition vertex]. Using the gauge identities satisfied by the vertices, one can show that the amplitudes are ghost free, provided that the conditions $\alpha_{0B} = \frac{1}{2}$, $M_F = 0$ and $d = 10$ are met.

The amplitudes of interaction for a system of strings with spin can also be found by following a different method, based on a functional integration over all paths of evolution of the classical strings. This approach has been followed by Mandelstam⁹⁷), who has been able to construct the amplitudes of interaction for strings with spin in the transverse gauge.

The method is a generalization of the one illustrated in Section 5, where we considered strings without spin. The basic idea is that a classical string with spin is described in the transverse gauge by the variables $x_\perp(\sigma, \tau)$, $P_\perp(\sigma, \tau)$ and $\Gamma_\perp(\sigma, \tau)$, and that the Lagrangian is quadratic in the transverse gauge. One integrates the exponential of i x the classical action over all possible evolutions of the system, assuming the conservation of the p^+ momentum. The classical quantities $\Gamma_\perp(\sigma, \tau)$ must be considered as anticommuting c numbers (elements of a Grassmann algebra), and the functional integral over these variables must be properly defined^{**)}

*) The vertices and propagators given above are expressed in the so-called F_2 representation. There exists another representation of the amplitudes (the F_1 representation), where the vertices and propagators have a slightly different form (cf. Refs. 82-84 and 16).

**) A path integral formulation of the amplitudes of the models of Neveu-Schwarz and Ramond had been considered also by the authors of Ref. 86.

It would take too long here to give the details of the construction of the amplitudes, and I refer to the reader for this to the articles by Mandelstam⁹⁷⁾. Let me only mention that, by the functional integration method, Mandelstam was able to compute the fermion-fermion scattering amplitude, which had remained for some time an open problem in the theory. Soon after, in Refs. 98 and 99, it was shown how to construct the same amplitude in the operator approach. The results obtained by the two completely different methods agree, and this provides another brilliant example of the consistency of the theory.

8. CONCLUSION

The theory of the string provides us with a physical interpretation of the dual models. We can think of the dual amplitudes as being the amplitudes of interaction of definite systems, the strings, propagating through space time.

But the dynamics that emerges from this interpretation has many novel, unusual features. Among these, the most important in my opinion are the following two:

- i) the fundamental objects that interact are extended objects;
- ii) the interaction results out of a change in the boundary conditions.

These aspects of the theory are rather different from what we were used to from standard field theory, where the fundamental objects are point-like and the interaction originates from an explicit term added to the Lagrangian.

Today many investigations are made to see whether those features of the theory of the string can be found in some non-perturbative approach to field theory. In particular, let me mention the work of Nielsen and Olesen¹⁰⁰⁾, who interpret the string in terms of vortex line solutions of gauge theories; the proposal of Fritsch and Gell-Mann¹⁰¹⁾ that the string should be related to path-dependent generalizations of bilocal operators; and the suggestion of Amati and Testa¹⁰²⁾ that, in a conventional field theoretical approach to weak and electromagnetic interactions, the strong interactions may result from boundary conditions that prohibit single quarks from being detected.

This new vision of the structure of the fundamental objects and of the mechanisms of interaction is surely one of the major contributions that the theory of dual models and their string interpretation has brought to the theory of elementary particles.

The dual models we know today cannot account for the true spectrum of hadrons. Maybe one day the correct model will be found. But, independently from this possibility, the theory of dual models has provided us with new ideas, which may form the basis for the future theory of strong interactions.

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APPENDIX

THE CONVENTIONAL DUAL RESONANCE MODEL

1. DEFINITION OF THE CONVENTIONAL DUAL RESONANCE MODEL

The conventional dual resonance model satisfies planar duality. The transition amplitude $A(k_1 \dots k_N)$ for a process with N external lines is expressed as a sum of as many terms as there are unequivalent planar orderings, of the external momenta

$$A(k_1 \dots k_N) = \sum_{\text{orderings}} V(k_{i_1} \dots k_{i_N}). \quad (\text{A.1})$$

Two orderings are defined as unequivalent when they cannot be obtained from each other with a cyclic or anticyclic permutations of the external momenta.

The amplitudes $V(k_1 \dots k_N)$ are invariant under cyclic or anticyclic permutations of the external momenta, and have resonant poles in all planar channels associated with a given ordering of the momenta. Planar channels are those made of lines labelled by indices in succession; i.e. (123), (23), (3456) are planar channels, whereas (13), (245) are non-planar ones.

In the so-called tree approximation to the dual amplitudes, the resonances are supposed to be infinitely narrow, and are represented by poles on the real axis. $V(k_1 \dots k_N)$ is then a meromorphic function of the external momenta, with poles whenever the real trajectory functions

$$\alpha(S_{ij}) = \alpha' S_{ij} + \alpha_0 = -\alpha' (k_i + \dots k_j)^2 + \alpha_0 \quad (\text{A.2})$$

equal a non-negative integer n . We have chosen a linear trajectory function to make $\alpha(s)$ compatible with $\text{Im } \alpha(s) = 0$, and have chosen for simplicity a common intercept α_0 . More generally, the intercepts could depend on the channel, and we could have more Regge trajectories in the same channel.

The simplest example of an amplitude satisfying planar duality is the original proposal by Veneziano⁷⁾:

$$A(s, t, u) = V(s, t) + V(t, u) + V(u, s), \quad (A.3)$$

with

$$V(s, t) = \int_0^1 dx x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} = B(\alpha(s), \alpha(t)). \quad (A.4)$$

It is an amplitude for a process involving four external scalar particles. (As a matter of fact, the original proposal by Veneziano was for a process of the type $\pi\omega \rightarrow \pi\pi$, but the different structure of spin and isospin is not relevant in this context.)

It is useful to analyse the origin of the poles in $\alpha(s)$ and $\alpha(t)$ in Eq. (A.4). They result from a divergence of the integral, which is originally defined for $\alpha(s)$ and $\alpha(t)$ negative, at the end points $x = 0$ and $x = 1$. Notice that when $x \rightarrow 0$, $y = 1 - x \rightarrow 1$; as a consequence, the residuum of the n^{th} pole in $\alpha(s)$ [the pole at $\alpha(s) = n$] is a polynomial of degree n in $\alpha(t)$ and, because of Eq. (A.2), in the variable t . The residua of the poles at $\alpha(s) = n$ have to be polynomials in the transferred momenta of degree not larger than n . This is necessary if we wish to interpret the poles as due to the exchange of one or more degenerate resonances of spins ranging up to n , and has to be true for every dual amplitude.

The conventional dual resonance model is a generalization of Eqs. (A.3) and (A.4) to a set of amplitudes for any number of external scalar particles¹⁸⁻²²⁾. The amplitudes $V(k_1 \dots k_N)$ must have poles in all planar channels; moreover, the residua of the poles have to be polynomials in the transferred momenta. Given any planar channel $(i, i+1, \dots, j)$, its momentum transfer variables are the subenergies $s_{i',j'} = -(k_{i'} + \dots + k_{j'})^2$ of the overlapping channels [channels having some but not all of the lines in common with $(i, i+1, \dots, j)$]. $V(k_1 \dots k_N)$ must then have poles in all planar subenergies $s_{ij} = -(p_i + \dots + p_j)^2$ with residua which are polynomials in the overlapping subenergies $s_{i',j'} = -(p_{i'} + \dots + p_{j'})^2$.

We use Eq. (A.4) to conjecture a form for $V(k_1 \dots k_N)$. Let us rewrite Eq. (A.4) as

$$V(s, t) = \int_0^1 dx u_{12}^{-\alpha(s)-1} u_{23}^{-\alpha(t)-1} \quad (A.5)$$

with

$$u_{12} + u_{23} = 1, \quad u_{12} = x. \quad (A.6)$$

In Eq. (A.5) we associate with each planar channel $(i, i+1 \dots j)$ an internal variable u_{ij} and integrate the product ($s_{12} = s, s_{23} = t$):

$$\prod_{i < j} u_{ij}^{-\alpha(s_{ij})-1}$$

over a domain such that:

- i) the u_{ij} variables are not independent, but constrained in such a way that when $u_{ij} \rightarrow 0$, all $u_{i',j'}$ associated with overlapping channels tend to 1.
- ii) $0 \leq u_{ij} \leq 1$ within the region of integration, with the value 0 reached at the boundary.

Equations (A.5) and (A.6) can be generalized to an amplitude for N external lines by defining

$$V(k_1 \dots k_N) = \int dV \prod_{i < j} u_{ij}^{-\alpha(s_{ij})-1}, \quad (A.7)$$

where we let the product in the integrand run over all planar channels. The region of integration must be defined so as to satisfy the requirements (i) and (ii).

The constraints (i) and (ii) can be solved in a very elegant way by the following parametrization of u_{ij} ¹⁸⁾:

$$u_{ij} = \frac{(p_i - p_j)(p_{i-1} - p_{j+1})}{(p_i - p_{j+1})(p_j - p_{i-1})} = w(p_i, p_{i-1}, p_j, p_{j+1}). \quad (A.8)$$

$[w(\rho_1, \rho_2, \rho_3, \rho_4)]$ is the anharmonic ratio of the variables $\rho_1, \rho_2, \rho_3, \rho_4$. The variables ρ are taken on the real axis; the region of integration is defined by

$$\rho_i \leq \rho_j \quad \text{for } i < j.$$

Three out of the N variables ρ_i (say ρ_a, ρ_b, ρ_c) are kept fixed in the integration, and the volume element is given by ($\rho_{N+1} \equiv \rho_1$)

$$dV = \prod_i \frac{d\rho_i}{|\rho_{i+1} - \rho_i|} \bigg/ \frac{|\rho_a - \rho_b| |\rho_b - \rho_c| |\rho_c - \rho_a|}{d\rho_a d\rho_b d\rho_c}. \quad (\text{A.9})$$

We can show that this ansatz satisfies the requirements (i) and (ii). Indeed, from $\rho_{i-1} \leq \rho_i \leq \rho_j \leq \rho_{j+1}$ it follows that

$$0 \leq w(\rho_i, \rho_{i-1}, \rho_j, \rho_{j+1}) \leq 1.$$

We have $w(\rho_i, \rho_{i-1}, \rho_j, \rho_{j+1}) = 0$ for $\rho_i = \rho_j$, which is on the boundary of the integration region. Moreover, let $i \leq i' - 1 < i' \leq j$, but $j' \geq j$ so that the channel $(i'j')$ overlaps with the channel (ij) . When $\rho_i = \rho_j$, then $\rho_{i'-1} = \rho_{i'} = \rho_i = \rho_j$; it follows that $u_{i'j'} = w(\rho_{i'}, \rho_{i'}, \rho_{j'}, \rho_{j'+1}) = 1$.

Equations (A.7), (A.8) and (A.9) define the conventional dual resonance model.

The fact that three ρ 's have been taken as fixed in the definition of the integral might lead one to think that invariance under cyclic and anticyclic permutation of the external momenta is lost. However, Eqs. (A.8) and (A.9) [and therefore Eq. (A.7)] are invariant under simultaneous projective transformations of the ρ variables. This is an extremely important feature of the model. By a projective transformation we can send any three ρ variables onto any other set of three. It follows that the expression of the amplitude is in fact independent from the specific choice of ρ_a, ρ_b and ρ_c , and this proves the invariance of $V(k_1 \dots k_N)$ under cyclic or anticyclic permutations of the external momenta.

2. FACTORIZATION

With a little algebra, from Eqs. (A.7), (A.8) and (A.9) we obtain

$$V(k_1, \dots, k_N) = \int \prod_i \theta(p_{i+1} - p_i) |p_{i+1} - p_i|^{\nu_0-1} / dV_{abc} \times \quad (A.10)$$

$$\times \prod_{i < j} |p_j - p_i|^{2\alpha' k_i \cdot k_j},$$

where we set

$$dV_{abc} = \frac{dp_a dp_b dp_c}{|p_a - p_b| |p_b - p_c| |p_c - p_a|} \quad (A.11)$$

and used the fact that the on-mass-shell external momenta satisfy

$$-\alpha' k_i^2 + \alpha_0 = 0. \quad (A.12)$$

Through the use of a suitable operator formalism^{27,40-42}), Eq. (A.10) can be rewritten in a form which exhibits explicitly the resonant poles, and also the factorization properties of the residues. For the purpose, it is necessary to introduce the Fock space spanned by the infinite set of operators $a_n^{\mu\dagger}$ ($n = 1, \dots, \infty$), with commutation relations

$$[a_m^\mu, a_n^{\nu\dagger}] = g^{\mu\nu} \delta_{m,n}. \quad (A.13)$$

We introduce also a momentum operator p^μ , and a position operator x_0^μ , with commutation relations

$$[p^\mu, x_0^\nu] = -ig^{\mu\nu}. \quad (A.14)$$

We assume that the ground state vector $|0\rangle$ of the Fock space is an eigenstate of momentum

$$p^\mu |0\rangle = 0. \quad (A.15)$$

A convenient basis is then given by the vectors

$$|\lambda, k\rangle = \frac{a_{\mu_1,1}^+ a_{\mu_2,1}^+ \dots a_{\mu_{\lambda_1},1}^+}{\sqrt{\lambda_1!}} \frac{a_{\nu_1,2}^+ \dots a_{\nu_{\lambda_2},2}^+}{\sqrt{\lambda_2!}} e^{ip_\mu k^\mu} |0\rangle. \quad (A.16)$$

Notice that $|\lambda, k\rangle$ is an eigenstate of momentum with $p^\mu |\lambda, k\rangle = k^\mu |\lambda, k\rangle$.

We introduce⁴⁰⁻⁴²⁾ a $Q^\mu(\rho)$ field:

$$Q^\mu(\rho) = x_0^\mu - 2i\alpha' p^\mu \log \rho + \sum_{n=1}^{\infty} i\sqrt{2\alpha'} \frac{Q_n^\mu}{\sqrt{n}} \rho^{-n} + \\ + \sum_{n=1}^{\infty} -i\sqrt{2\alpha'} \frac{Q_n^{\mu\dagger}}{\sqrt{n}} \rho^n = x_0^\mu - 2i\alpha' p^\mu \log \rho + Q_{(-)}^\mu(\rho) + Q_{(+)}^\mu(\rho). \quad (A.17)$$

It is easy to check that

$$[Q_{(-)}^\mu(\rho_2), Q_{(+)}^\nu(\rho_1)] = -2\alpha' g^{\mu\nu} \log(1 - \frac{\rho_1}{\rho_2}), \quad (A.18)$$

It follows that^{*)}

$$(\rho_1 < \rho_2).$$

$$e^{ik_2 Q^{(-)}(\rho_2)} e^{ik_1 Q^{(+)}(\rho_1)} = e^{2\alpha' k_1 \cdot k_2 \log(1 - \frac{\rho_1}{\rho_2})} \times \\ \times e^{ik_1 Q^{(+)}(\rho_1)} e^{ik_2 Q^{(-)}(\rho_2)} = (\rho_2 - \rho_1)^{2\alpha' k_1 \cdot k_2} \frac{-2\alpha' k_1 \cdot k_2}{\rho_2} \times \\ \times e^{ik_1 Q^{(+)}(\rho_1)} e^{ik_2 Q^{(-)}(\rho_2)}.$$

*) We use $e^A e^B = e^{B+A} [A, B]$, valid whenever $[A, B]$ commutes with A and B.

Equation (A.19) is the key equation to rewriting Eq. (A.10) in a factorized form. From Eq. (A.19) it follows indeed that

$$\begin{aligned}
 & \langle 0 | \left(e^{i k_N Q^+(p_N)} e^{i k_N Q^-(p_N)} \right) \dots \\
 & \dots \left(e^{i k_1 Q^+(p_1)} e^{i k_1 Q^-(p_1)} \right) | 0 \rangle = \\
 & = \prod_{i < j} | p_j - p_i |^{2\alpha' k_i \cdot k_j} \prod_j p_j^{-2\alpha' k_j \cdot (\sum_{i < j} k_i)}
 \end{aligned} \tag{A.20}$$

Notice that the right-hand side of Eq. (A.20) contains the factor

$$\prod_{i < j} | p_j - p_i |^{2\alpha' k_i \cdot k_j},$$

which is the momentum-dependent part of the integrand of the dual amplitude. The other factor

$$\prod_j p_j^{-2\alpha' k_j \cdot (\sum_{i < j} k_i)}$$

can be disposed of by using the "zero mode" part of $Q^\mu(\rho)$, i.e. $x_0^\mu - 2i\alpha' p^\mu \log \rho$. We have, indeed^{*)},

$$\begin{aligned}
 & \langle 0 | e^{i k_N (x_0 - 2i\alpha' p \log p_N)} \dots e^{i k_1 (x_0 - 2i\alpha' p \log p_1)} | 0 \rangle = \\
 & = \prod_j p_j^{-\alpha' k_j^2} \prod_j p_j^{2\alpha' k_j \cdot \sum_{i < j} k_i},
 \end{aligned} \tag{A.21}$$

so that, introducing vertices

$$\begin{aligned}
 V(k, p) = & : e^{i k_N Q^+(p)} : = e^{i k Q^+(p)} e^{i k (x_0 - 2i\alpha' p \log p)} \\
 & \times e^{i k Q^-(p)},
 \end{aligned} \tag{A.22}$$

*) The expression $\langle 0 | \exp [i x_0 \sum_n k_n] | 0 \rangle$ is ambiguous, because $\langle 0 | x_0^\mu | 0 \rangle$ is a singular expression. A careful treatment leads to

$$\langle 0 | \exp [i x_0 \sum_n k_n] | 0 \rangle = (2\pi)^4 \delta(\sum_n k_n)$$

We omit this δ function, expressing momentum conservation, from the following equations.

we can write

$$V(k_1, \dots, k_N) = \int \prod_i \frac{d\rho_i}{\rho_i} \Theta(\rho_{i+1} - \rho_i) \left| \frac{\rho_{i+1} - \rho_i}{\sqrt{\rho_{i+1} \rho_i}} \right|^{\alpha_0 - 1} / dV_{abc} \times$$

$$\times \langle 0 | V(k_N, \rho_N) \dots V(k_1, \rho_1) | 0 \rangle. \quad (\text{A.23})$$

In Eq. (A.23) the dependence of the integrand of Eq. (A.10) on the variables ρ_i and k_i is disentangled into a product of operators. It becomes then possible to write the amplitude in a fully factorized form. $V(k_1 \dots k_N)$ can be factorized for all values of the intercept α_0 , but, since we shall be finally interested only in one definite value of the intercept, let us proceed with $\alpha_0 = 1$. Equation (A.23) becomes then

$$V(k_1, \dots, k_N) = \int \prod_i \frac{d\rho_i}{\rho_i} \Theta(\rho_{i+1} - \rho_i) / dV_{abc} \times$$

$$\times \langle 0 | V(k_N, \rho_N) \dots V(k_1, \rho_1) | 0 \rangle \quad (\text{A.24})$$

It is convenient to choose $a = 1$, $b = N - 1$ and $c = N$, setting $\rho_1 = 0$, $\rho_{N-1} = 1$, $\rho_N = \infty$ ^{*)}. We obtain

$$V(k_1, \dots, k_N) = \int \prod_{i=2}^{N-2} \frac{d\rho_i}{\rho_i} \Theta(\rho_{i+1} - \rho_i) \times$$

$$\times \langle 0 | e^{ik_N x_0} V(k_{N-1}, 1) \dots V(k_2, \rho_2) e^{ik_1 x_0} | 0 \rangle. \quad (\text{A.25})$$

Notice that Eq. (A.25) resembles closely the amplitude for a transition between states $e^{ik_1 x_0} | 0 \rangle$ and $\langle 0 | e^{ik_N x_0}$ induced by interaction vertices $V(k_i, \rho_i)$. This analogy is made even more explicit by using "time"

*) The values $\rho_1 = 0$ and $\rho_N = \infty$ must be reached through a limiting procedure, since $V(k_1, \rho_1) | 0 \rangle$ and $\langle 0 | V(k_N, \rho_N)$ behave as $\rho_1^{\alpha_0}$ and $\rho_N^{-\alpha_0}$ as $\rho_1 \rightarrow 0$ and $\rho_N \rightarrow \infty$, and these factors cancel against other factors in the integrand of Eq. (A.24).

variables $\tau_i = \log \rho_i$ ^{*)}. Equation (A.25) becomes

$$V(k_1, \dots, k_N) = \int_{-\infty}^0 \prod_{i=2}^{N-2} d\tau_i \Theta(\tau_{i+1} - \tau_i) \times$$

$$\times \langle 0 | e^{i k_N x_0} \tilde{V}(k_{N-1}, 0) \dots \tilde{V}(k_1, \tau_2) e^{i k_1 x_0} | 0 \rangle. \quad (A.26)$$

We have written $\tilde{V}(k, \tau)$ for $V(k, e^\tau = \rho)$ in this equation, but in the following we shall write simply $V(k, \tau)$ [and $Q(\tau)$].

The poles of the amplitude in the channels (1, 2 ... m) are exhibited in the standard way by introducing propagators. Notice that

$$L_0 = \alpha' p^2 + \sum_n n a_n^\dagger a_n \quad (A.27)$$

generates an infinitesimal translation of the fields $Q(\tau)$ and $V(k, \tau)$:

$$[L_0, Q(\tau)] = \frac{d}{d\tau} Q(\tau),$$

$$[L_0, V(k, \tau)] = \frac{d}{d\tau} V(k, \tau). \quad (A.28)$$

It follows that

$$V(k, \tau) = e^{\tau L_0} V(k, 0) e^{-\tau L_0} \quad (A.29)$$

We substitute in Eq. (A.26), set

$$\tau_{N-2} = \mathcal{E}_{N-2}, \quad \tau_{N-3} = \mathcal{E}_{N-2} + \mathcal{E}_{N-3}, \dots$$

notice that

$$\dots \tau_2 = \mathcal{E}_{N-2} + \dots \mathcal{E}_2,$$

$$e^{-\tau_2 L_0} e^{i k_1 x_0} | 0 \rangle = e^{-\tau_2 L_0} | 0 \rangle = e^{-\sum_{i=2}^{N-2} \mathcal{E}_i L_0} | 0 \rangle,$$

*) In the main text of this article, one often finds a "time" variable τ which is related to the argument of the vertex operators by $\tau = i \log \rho$. Here it is more convenient to use the definition $\tau = \log \rho$. We can consider the variable τ used in the Appendix as having been Wick rotated to the imaginary axis.

(since $\alpha' k_1^2 = \alpha_0 = 1$), and obtain

$$\begin{aligned}
 V(k_1 \dots k_N) &= \int_{-\infty}^0 \prod_{i=2}^{N-2} d\xi_i \\
 \langle 0 | e^{i k_N x_0} V(k_{N-1}, 0) e^{\xi_{N-2}(L_0-1)} V(k_{N-2}, 0) \dots & \quad (A.30) \\
 \dots e^{\xi_2(L_0-1)} V(k_2, 0) e^{i k_1 x_0} | 0 \rangle &= \\
 = \langle 0 | e^{i k_N x_0} V(k_{N-1}, 0) D V(k_{N-2}, 0) \dots D V(k_2, 0) e^{i k_1 x_0} | 0 \rangle, & \\
 \text{with}^*) &
 \end{aligned}$$

$$D = \int_{-\infty}^0 e^{\xi(L_0-1)} d\xi = \frac{1}{L_0-1} \quad (A.31)$$

The propagator $1/(L_0 - 1)$ has poles whenever $\alpha' p^2$ takes an integer value smaller or equal to 1 [i.e. for $\alpha(s) = -\alpha' p^2 + 1 = n$, with n a non-negative integer]. This follows from

$$L_0 - 1 = \alpha' p^2 - 1 + \sum_n n a_n^\dagger a_n$$

and the fact that $R = \sum_n n a_n^\dagger a_n$ has non-negative integer eigenvalues. The residuum at the pole for $\alpha(s) = n$ is given by

$$\begin{aligned}
 \sum_\lambda \langle 0 | e^{i k_N x_0} V(k_{N-1}, 0) D \dots V(k_{m+1}, 0) | \lambda \rangle (-)^{n_0} \times & \quad (A.32) \\
 \times \langle \lambda | V(k_m, 0) D \dots V(k_2, 0) e^{i k_1 x_0} | 0 \rangle, &
 \end{aligned}$$

*) The amplitude $V(k_1 \dots k_N)$ is originally defined for values of the external momenta such that $\langle a | L_0 | b \rangle > 1$; the integral in Eq. (A.31) is therefore convergent for $\xi \rightarrow -\infty$.

where the sum ranges over all of the vectors $|\lambda\rangle$ of the basis (A.16) for which $R|\lambda\rangle = n|\lambda\rangle$. The number of these vectors is finite ($R|\lambda\rangle = n|\lambda\rangle$ implies $\lambda_1 + 2\lambda_2 + \dots = n$), and this proves that the residues of the poles can be expressed as a sum of a finite number of factorized terms, which is necessary to interpret them as due to the exchange of a finite number of degenerate resonances.

Finally, notice the factor $(-)^{n_0}$ in Eq. (A.32), where n_0 is the total number of time-like oscillators in $|\lambda\rangle$, which is due to the negative value of $[a_n^0, a_n^{0+}] = g^{00} = -1$. The factor $(-)^{n_0}$ in Eq. (A.32) means that there are ghost states (states of negative norm) in the factorization of the dual amplitude. Their contribution, however, may be cancelled if there are identities among the vertex factors

$$\langle \lambda | V(k_m, 0) D \dots V(k_2, 0) e^{i k_1 x_0} | 0 \rangle.$$

3. CONSTRAINTS ON PHYSICAL STATES

The vertex factors in Eq. (A.32) are not all independent. To prove the existence of identities among vertex factors it is convenient to write

$$\begin{aligned} |k_m \dots k_1\rangle &= (\text{definition}) = \\ &= V(k_m, 0) D V(k_{m-1}, 0) \dots D V(k_2, 0) e^{i k_1 x_0} | 0 \rangle = \\ &= \int_{-\infty}^0 \prod_{i=2}^{m-1} d\tau_i \Theta(\tau_{i+1} - \tau_i) V(k_m, 0) V(k_{m-1}, 0) \dots V(k_2, 0) e^{i k_1 x_0} | 0 \rangle. \end{aligned} \quad (\text{A.33})$$

In Eq. (A.33) the differentials $V(k_i, \tau_i) d\tau_i$ are integrated over the domain

$$-\infty \leq \tau_2 \leq \tau_3 \dots \leq \tau_{m-1} \leq 0. \quad (\text{A.34})$$

We may change integration variables in Eq. (A.33), with the transformation

$$\tau \rightarrow \tilde{\tau} = \tilde{\tau}(\tau). \quad (\text{A.35})$$

Under this transformation

$$V(k, \tau) d\tau \Rightarrow \tilde{V}(k, \tilde{\tau}) d\tilde{\tau} = V(k, \tau(\tilde{\tau})) \frac{d\tau(\tilde{\tau})}{d\tilde{\tau}} d\tilde{\tau}. \quad (\text{A.36})$$

It is convenient to deal with infinitesimal transformations

$$\tau \xrightarrow{W_f} \tilde{\tau} = \tau - \epsilon f(\tau). \quad (\text{A.37})$$

This gives

$$\begin{aligned} V(k, \tau) d\tau \xrightarrow{W_f} \tilde{V}(k, \tau) d\tau &= V(k, \tau) d\tau + \\ &+ \epsilon \left(f(\tau) \frac{d}{d\tau} + f'(\tau) \right) V(k, \tau) d\tau. \end{aligned} \quad (\text{A.38})$$

If we consider a transformation W_f that leaves invariant the points 0 and $-\infty$ [$f(0) = f(-\infty) = 0$], the effect of the change of variables is to replace the operator $V(k_{m-1}, \tau_{m-1}) \dots V(k_2, \tau_2)$ in Eq. (A.33) with their transformed $\tilde{V}$.

Suppose now that $\tilde{V}$ could be obtained from V by a τ -independent operator transformation W_f :

$$\tilde{V} = V + \epsilon [W_f, V]. \quad (\text{A.39})$$

Then we would have the identity

$$\begin{aligned} |k_m \dots k_1\rangle &= \int_{-\infty}^0 \pi d\tau_i \theta(\tau_{i+1} - \tau_i) V(k_m, 0) \tilde{V}(k_{m-1}, \tau_{m-1}) \dots \\ &\dots e^{i k_1 x_0} |0\rangle = \\ &= |k_m \dots k_1\rangle + \epsilon W_f \int_{-\infty}^0 \pi d\tau_i \theta(\tau_{i+1} - \tau_i) \times \end{aligned} \quad (\text{A.40})$$

$$\begin{aligned}
 & \times V(k_m, 0) V(k_{m-1}, \tau_{m-1}) \dots e^{ik_1 x_0} |0\rangle = \\
 & = e^{\int_{-\infty}^0 \pi d\tau_i \theta(\tau_{i+1} - \tau_i) [W_f, V(k_m, 0)] V(k_{m-1}, \tau_{m-1}) \dots} \quad (A.40) \\
 & \dots e^{ik_1 x_0} |0\rangle = e^{\int_{-\infty}^0 \pi d\tau_i \theta(\tau_{i+1} - \tau_i) V(k_m, 0) \times} \\
 & \times V(k_{m-1}, \tau_{m-1}) \dots W_f e^{ik_1 x_0} |0\rangle.
 \end{aligned}$$

In this equation we have taken into account the fact that $V(k_m, 0)$ and $e^{ik_1 x_0} |0\rangle$ should not be transformed by the operator W_f that performs the change of variables of integration.

It would take too long to motivate here why such an operator W should exist. [The study of the physical meaning of the identities (A.36) is one of the primary objects of the whole review article.] But let us show that a W operator does exist for all possible continuous transformations of the τ variables leaving the points $\tau = 0$ and $\tau = -\infty$ unchanged.

These transformations are generated by the infinitesimal transformations

$$\tau \xrightarrow{W_n} \tilde{\tau} = \tau - \epsilon (e^{n\tau} - 1). \quad (A.41)$$

Consider the simpler transformation L_n :

$$\begin{aligned}
 \tau \xrightarrow{L_n} \tilde{\tau} &= \tau - \epsilon e^{n\tau} \\
 (W_n &= L_n - L_0)
 \end{aligned}$$

Under L_n

$$Q(\tilde{\tau}) \xrightarrow{L_n} \tilde{Q}(\tilde{\tau}) = Q(\tau(\tilde{\tau})) = Q(\tilde{\tau}) + \epsilon e^{n\tilde{\tau}} \frac{d}{d\tilde{\tau}} Q(\tilde{\tau}). \quad (A.42)$$

It can be easily checked that

$$e^{n\tau} \frac{d}{d\tau} Q(\tau) = [L_n, Q(\tau)], \quad (\text{A.43})$$

where

$$L_n = i\sqrt{2\alpha'n} p \cdot a_n + \sum_{\ell=1}^{\infty} \sqrt{\ell(n+\ell)} a_{\ell}^{\dagger} a_{n+\ell} - \frac{1}{2} \sum_{\ell=1}^{n-1} \sqrt{\ell(n-\ell)} a_{\ell} a_{n-\ell}. \quad (\text{A.44})$$

Because of the normal ordering in the definition of $V(k, \tau)$ [Eq. (A.22)], V does not transform according to Eq. (A.43), but in the following way:

$$[L_n, V(k, \tau)] = e^{n\tau} \left(\frac{d}{d\tau} + n\alpha' k^2 \right) V(k, \tau). \quad (\text{A.45})$$

With $\alpha' k^2 = 1$, we have

$$[L_n, V(k, \tau)] = \left(e^{n\tau} \frac{d}{d\tau} + n e^{n\tau} \right) V(k, \tau), \quad (\text{A.46})$$

which precisely agrees with Eq. (A.38). We can use then Eq. (A.40), with $W_f = L_n - L_0 = W_n$. We have

$$[W_n, V(k, 0)] = n V(k, 0) \quad (\text{A.47})$$

$$\begin{aligned} W_n e^{ik_1 x_0} |0\rangle &= -\alpha' k_1^2 e^{ik_1 x_0} |0\rangle = \\ &= -e^{ik_1 x_0} |0\rangle. \end{aligned} \quad (\text{A.48})$$

It follows from Eq. (A.40) that

$$\begin{aligned} (W_n - n + 1) |k_m \dots k_1\rangle &= \\ &= (L_n - L_0 - n + 1) |k_m \dots k_1\rangle \end{aligned} \quad (\text{A.49})$$

and these relations constitute the Virasoro identities²⁸⁾ satisfied by the vertices of the dual resonance model.

Equation (A.49) implies the existence of linear dependences among the vertex vectors $\langle \lambda | k_m \dots k_1 \rangle$. This is seen quite easily by saturating Eq. (A.49) with a general vector $\langle \lambda' |$ and using the expansion of the identity $I = \sum_{\lambda} |\lambda\rangle \langle \lambda|$. We have

$$\begin{aligned} \sum_{\lambda} \langle \lambda' | L_0 - L_0 - n + 1 | \lambda \rangle (-)^{n_0} \langle \lambda | k_m \dots k_1 \rangle &= \\ &= \sum_{\lambda} C_{\lambda}^{\lambda'} \langle \lambda | k_m \dots k_1 \rangle. \end{aligned} \quad (\text{A.50})$$

Notice that the operator form of the Virasoro identities, Eq. (A.49), is expressed as a constraint equation on the off-mass-shell vertex vector $|k_m \dots k_1\rangle$. It is often useful to consider the form that the same equation takes on the mass shell. To find the identities satisfied by on-mass shell states, we multiply the vertex vector $|k_m \dots k_1\rangle$ by the propagator $D = 1/(L_0 - 1)$. We obtain a vector

$$|k_m \dots k_1\rangle' = \frac{1}{L_0 - 1} |k_m \dots k_1\rangle. \quad (\text{A.51})$$

It can be checked that the following commutation relation is true:

$$[L_0, L_n] = L_0 L_n - L_n L_0 = -n L_n. \quad (\text{A.52})$$

With a little algebra, Eq. (A.52) is shown to be equivalent to the equation

$$(L_0 + n - 1)(L_n - L_0 + 1) = (L_n - L_0 - n + 1)(L_0 - 1). \quad (\text{A.53})$$

From this it follows that

$$(L_n - L_0 + 1) \frac{1}{L_0 - 1} = \frac{1}{L_0 + n - 1} (L_n - L_0 - n + 1). \quad (\text{A.54})$$

Since the vertex vectors are annihilated by the operators $L_n - L_0 - n + 1$, we deduce the following equation

$$\begin{aligned} (L_n - L_0 + 1) |k_m \dots k_1\rangle' &= \\ &= (L_n - L_0 + 1) \frac{1}{L_0 - 1} |k_m \dots k_1\rangle = \\ &= \frac{1}{L_0 + n - 1} (L_n - L_0 - n + 1) |k_m \dots k_1\rangle. \end{aligned} \quad (\text{A.55})$$

Starting from this equation we can study the identities satisfied by on-mass-shell physical states $|\phi\rangle$. Indeed, the states $|\phi\rangle$ are obtained by taking the residuum of the pole for $L_0 = 1$ of $|k_m \dots k_1\rangle'$. Since $|k_m \dots k_1\rangle'$ satisfies Eq. (A.55), so do the on-mass-shell physical states $|\phi\rangle$. We have therefore

$$(L_n - L_0 + 1) |\phi\rangle = 0 \quad (n > 0). \quad (\text{A.56})$$

But the states $|\phi\rangle$ satisfies also the on-mass-shell condition $(L_0 - 1)|\phi\rangle = 0$. Equations (A.56) then simplify to

$$L_n |\phi\rangle = 0, \quad n > 0. \quad (\text{A.57})$$

These identities can be used to prove the absence of ghosts in the spectrum of the conventional dual resonance model^{44,45}.