Large-scale structure, the cosmic microwave background, and primordial non-gaussianity

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ABSTRACT

Since cosmic-microwave-background (CMB) and large-scale-structure (LSS) data will shortly improve dramatically with the Microwave Anisotropy Probe (MAP) and Planck Surveyor, and the Anglo-Australian 2-Degree Field (2dF) and Sloan Digital Sky Survey (SDSS), respectively, it is timely to ask which of the CMB or LSS will provide a better probe of primordial non-gaussianity. In this paper we consider this question, using the bispectrum as a discriminating statistic. We consider several non-gaussian models and find that in each case the CMB will provide a better probe of primordial non-gaussianity. Since the bispectrum is the lowest-order statistic expected to arise in a generic non-gaussian model, our results suggest that if CMB maps appear gaussian, then apparent deviations from gaussian initial conditions in galaxy surveys can be attributed with confidence to the effects of biasing. We demonstrate this precisely for the spatial bispectrum induced by local non-linear biasing.

1 INTRODUCTION

It is widely accepted that the large-scale structures we observe in the Universe today originated from gravitational evolution of small primordial fluctuations in the matter density. Information about the physical processes that generated these primordial fluctuations can be gleaned by testing whether their statistical distribution is well approximated by a gaussian random field. In particular, the simplest versions of inflation predict gaussian initial fluctuations (e.g., Guth & Pi 1982; Hawking 1982; Starobinsky 1982; Bardeen, Steinhardt & Turner 1983), but there are other models of inflation (Allen, Grinstein & Wise 1987; Kofman & Pogosian 1988;Salopek, Bond & Bardeen 1989) and models where structure is seeded by topological defects (Vilenkin 1985; Vachaspati 1986; Hill, Schramm & Fry 1989; Turok 1989; Albrecht & Stebbins 1992) that generate non-gaussian fluctuations. By looking at cosmic-microwave-background (CMB) anisotropies we can probe cosmic fluctuations at a time when their statistical distribution should have been close to their original form. At present, the limited signal-to-noise or sky coverage of existing experiments is not sufficient to provide conclusive evidence either for or against non-gaussianity (e.g., Heavens 1999; Ferreira, Maguejo & Gorski 1998; Kamionkowski & Jaffe 1998; Pando, Valls-Gabaud & Fan 1998; Bromley & Tegmark 1999). An alternative approach is to analyze the present-day statistics of density or velocity fields of large scale structure (LSS). In principle, this is a more complicated approach, since gravitational instability and bias can introduce non-gaussian features in an initially gaussian field, and these may mask the signal we desire to measure. Since the CMB and LSS data will shortly improve dramatically with the Microwave Anisotropy Probe (MAP) and Planck Surveyor satellites, and the Anglo-Australian 2-Degree Field (2dF) and Sloan Digital Sky Survey (SDSS), respectively, it is timely to ask which of the CMB or LSS is the better place to look to detect a primordial non-gaussian signal.

In this paper, we use the skewness and bispectrum to determine which of the CMB or LSS provides a better probe of non-gaussianity. To do so, we consider several models with a primordial non-gaussianity whose amplitude can be dialed from zero (the gaussian limit). We then calculate the smallest non-gaussian amplitude that can be detected with a plausible CMB map and with a plausible galaxy survey. In each case, we find that the smallest non-gaussian amplitude detectable with a CMB map is smaller than that from a galaxy survey, even if we neglect the complicating effects of biasing.

Of course, there is an infinitude of possible deviations from gaussianity and we cannot address them all. However, physical mechanisms that produce non-gaussianity generically produce a non-vanishing bispectrum. Since this is the lowest-order non-gaussian statistic, it is usually the most easily detectable. Although this argument is not fully general, we show that it holds in several non-gaussian models that have been considered in the recent literature. Our results suggest that if the CMB maps provided by MAP and Planck are consistent with gaussian initial conditions, then any signatures of non-gaussianity found in galaxy surveys (apart from those from non-linear clustering) can be attributed to biasing.

2 THE SKEWNESS

In order to determine whether a field is gaussian, we need a discriminating statistic. We shall principally be concerned with the bispectrum, as it is the lowest-order non-gaussian statistic that generically arises in physical mechanisms that produce non-gaussianity, and it is able in principle to distinguish between various sources of non-gaussianity (e.g. primordial, non-linear growth, bias). However, we
begin by discussing the skewness in LSS and the CMB, as this is simpler, and illustrates some of the effects.

2.1 Skewness in Large scale structure

The statistical properties of the fluctuations in the cosmological mass density field \( \delta(x) = \rho(x) - \bar{\rho} / \bar{\rho} \) can be characterized by the \( n \)-point moments, \( \langle \delta^n \rangle \). By definition, \( \langle \delta(x) \rangle = 0 \). If the fluctuation field is gaussian, then the probability distribution for \( \delta \) is

\[
p(\delta) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{\delta^2}{2\sigma^2} \right],
\]

from which the moments \((n = 0, 1, \ldots)\) can be calculated to be

\[
\langle \delta^{2n} \rangle = (2n - 1)!! \langle \delta^2 \rangle^n = (2n - 1)!! \sigma^{2n},
\]

where \( \sigma^2 \equiv \langle \delta^2 \rangle \). The odd moments are of course zero. To linear order in perturbation theory, \( \delta \) grows by an overall normalization factor, \( \delta(x, t) = D(t) \delta(x, t_0) \), so an initially gaussian distribution will remain gaussian as long as linear perturbation theory holds.

To higher order in perturbation theory, gravitational instability will induce departures from gaussianity. To describe the evolution of non-linear fluctuations in perturbation theory we expand the fluctuation field in a gaussian, where the subscript \( D \) in the RHS of equation (5) are time-independent apart from \( n = 3 \). In the weakly non-linear regime, gaussian initial conditions give rise to a non-vanishing skewness (e.g. Peebles 1980),

\[
S_3 \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^{3/2}} = \frac{3}{7},
\]

in second-order perturbation theory (2OPT).

For generic non-gaussian initial conditions, Fry & Scherrer (1994) found the skewness in second-order perturbation theory (2OPT) to be

\[
S_3 = S_{3,0} + \frac{26}{21} \frac{\langle \delta^2 \rangle^2}{\langle \delta^4 \rangle} - \frac{8}{7} \frac{\langle \xi_{\delta\delta\delta} \rangle}{\langle \delta^2 \rangle^3} + \frac{10}{7} \frac{\langle \xi_{\delta\delta} \rangle}{\langle \delta^2 \rangle} + \frac{6}{7} \frac{\langle \xi_{\delta} \rangle}{\langle \delta \rangle} - \frac{34}{7} \frac{\langle \delta \rangle}{\langle \delta^2 \rangle}.
\]

where the subscript 0 denotes the quantity linearly evolved from the initial density field [e.g., in an Einstein-de Sitter Universe, \( \delta_0 = \delta(z_0)(1+z_0) \)] to the present epoch. The quantity \( I[\xi_{\delta\delta\delta}](n = 3, 4) \) is an integral which depends on the specific linearly-evolved connected (or irreducible) three- and four-point functions, \( \xi_{\delta\delta\delta} \) and \( \xi_{\delta\delta\delta\delta} \), respectively. For gaussian initial conditions, \( \xi_{\delta\delta\delta\delta}(0) = 0 \) for \( n \geq 3 \). Fry & Scherrer (1994) find that \( I[\xi_{\delta\delta\delta}](n = 3, 4) \) for \( n = 3, 4 \) for several non-gaussian models they explore. All terms in the RHS of equation (5) are time-independent apart from \( S_{3,0} \) which scales like \( S_{3,0}(z) \propto S_{3,0}(z = 0)(1+z) \) in an Einstein-de Sitter Universe.

It is therefore useful to define the time-independent quantities \( p_3 = \xi_{\delta\delta\delta}/\sigma^3 \) (normalized skewness) and \( p_4 = \xi_{\delta\delta\delta\delta}/\sigma^4 \) (normalized kurtosis). When written in terms of the relevant quantities at decoupling (i.e., at \( z \approx 1100 \)),

\[
S_3 \approx \frac{34}{7} + \frac{p_3}{1100 \sigma(z = 1100)} + d_1 p_3^2 + d_2 p_4,
\]

where \( d_1 \approx d_2 \approx 2 \).

Now suppose we have a survey of \( N \) independent volumes in which the rms fractional density contrast is \( \sigma \). Then, from equations (2) and (4), it follows that in the mildly non-linear regime, the standard error due to cosmic variance with which the skewness can be recovered is

\[
\Delta S_3 = \frac{1}{\sqrt{N}} \sqrt{\frac{15}{\sigma^2} + 17S_3^2},
\]

where we have used \( \Delta(\langle \delta^3 \rangle) = \sqrt{15\sigma^6 + 10(\delta^3)^2} \) and \( \Delta(\langle \delta^2 \rangle) \sim \sigma_2 \). Of course, since the mass will be traced by discrete objects (i.e., galaxies), the shot noise may increase the error estimate in equation (7).

So now let us consider the 2dF and/or SDSS. The present-day skewness in volumes of side \( 10 h^{-1} \) Mpc could be measured with a standard error at least \( \Delta S_3 \sim 20 N^{-1/2} \sim 10^{-3} \), where \( N \) is the number of such cubes in the survey volume. Thus, equation (6) tells us that a primordial normalized skewness on the \( 10 h^{-1} \) Mpc scale could be identified in a statistically-significant manner in 2dF and/or SDSS only if \( p_3 \) exceeded about \( 10^{-2} \).

2.2 Skewness in the CMB

For simplicity, consider an Einstein-de Sitter Universe. Then a region of comoving size \( 10 h^{-1} \) Mpc \((h \equiv H_0/100 \text{ km sec}^{-1} \text{ Mpc}^{-1})\) subtends an angle of \( 0.1 \). Now suppose that a full-sky cosmic-variance-limited 0.1-degree-resolution CMB map (close to the Planck Surveyor’s specifications) finds that the distribution of temperature fluctuations \( \Delta T/T \) is consistent with gaussian with a variance \( \sigma^2 = \langle (\delta\rho/\rho)^2 \rangle \sim 0.1(\Delta T/T)^2 \sim (10^{-3})^2 \) (where \( \delta\rho/\rho \) is the fractional density perturbation at a redshift \( z \approx 1100 \), when the CMB decouples), the largest primordial normalized skewness \( p_3 \) (\( \xi_{\delta\delta\delta}/\sigma^3 \)) that would be consistent with such a map would be \( \sqrt{15/N_{\text{pix}} \sim 10^{-3}} \), where \( N_{\text{pix}} \sim 10^7 \) is the number of \( 0.1 \times 0.1 \) degree pixels. We have neglected instrumental noise and assumed systematic effects will be under control. Still, a sensitivity to a value as small as \( p_3 \sim 10^{-3} \), or perhaps an order of magnitude larger, is a realistic expectation of CMB maps.

If we compare this with the nominal smallest normalized skewness \( p_3 \sim 10^{-2} \) accessible with LSS, it appears that the CMB has, perhaps an extra order of magnitude in sensitivity. On the other hand, there may be systematic effects in both the CMB and LSS measurements that may affect both estimates, and our argument was only qualitative. Thus, we conclude from this exercise that the CMB and LSS should provide roughly comparable sensitivity to a primordial skewness on \( 10 h^{-1} \) Mpc scales, with perhaps a slight edge to the CMB. Since these heuristic arguments are inconclusive to orders of magnitude as to which of the CMB and LSS provides a better probe of primordial non-gaussianity, we now proceed to consider realistic models more carefully.

3 CMB AND LSS BISPECTRA

The skewness has the advantage of being far easier to calculate than the full three-point correlation function, but it does not contain as much information. We therefore prefer to investigate the bispectrum (the three-point function in Fourier space) for reasons which have been rehearsed before (e.g., Matarrese, Verde & Heavens 1997). In addition, various theoretical models for structure formation yield directly Fourier-space quantities, so the bispectrum allows a more straightforward relation between measurable quantities and theoretical predictions.
3.1 LSS Bispectrum

We define the Fourier transform of the fractional overdensity perturbation by $b_{k} = \int \frac{d^{3}x}{(2\pi)^{3}} \delta(x) \exp(-ix \cdot k)$. The spatial bispectrum $B(k_{1}, k_{2}, k_{3})$ is defined by

$$\langle \delta_{k_{1}} \delta_{k_{2}} \delta_{k_{3}} \rangle = B(k_{1}, k_{2}, k_{3}) \delta^{2}(k_{1} + k_{2} + k_{3})$$

(8)

where the angle brackets denotes an ensemble average or, by the ergodic theorem, the average over a large volume (fair sample), and $\delta^{2}$ is the Dirac delta function.

To second-order in perturbation theory, the bispectrum may be written as a sum of a primordial part and a part induced by gravitational instability:

$$B(k_{1}, k_{2}, k_{3}) \approx B_{0}(k_{1}, k_{2}, k_{3}) + [2(J_{2}(k_{1}, k_{2})P_{0}(k_{1})P_{0}(k_{2}) + \text{cyc.})$$

$$+ \int d^{3}k_{4}(J_{3}(k_{4}, k_{3} - k_{4}, k_{1} - k_{4}, k_{2}) + \text{cyc.})] \delta^{2}(k_{1} + k_{2} + k_{3})$$

(9)

where $J_{2}$ and $J_{3}$ is a function almost independent of the non-relativistic-matter density $\Omega_{0}$ and cosmological constant $\Lambda$ (Bouchet et al. 1992; Bouchet, Hivon & Juszkiewicz 1995; Bernardeau 1994; Scoccimarro et al. 1998; Kamionkowski & Burchalter 1998). Its detailed form need not concern us here. Here, $B_{0}$ is the primordial bispectrum (which we wish to probe) linearly evolved to redshift $z$, $P_{0}$ is the power spectrum and $T^{\nu}$ denotes the connected trispectrum, which is the Fourier transform of the connected 4-point correlation function. The last two terms arise from non-linear gravitational instability; this depends (see e.g. Catelan & Moscardini 1994) on the 4-point function, which has a disconnected part (present in gaussian fields), and a connected part, $T^{\nu}$ (which is zero for gaussian fields). In principle, this last term $(\equiv B_{T})$ may be important, as it grows as fast as the usual disconnected part. We will show later that this term is very small for a range of proposed models.

As already seen for the skewness, in an Einstein-de Sitter Universe, $B_{0}(z) \propto (1 + z)^{-3}$ and $[P_{0}(z)]^{2} \propto (1 + z)^{-4}$, so the primordial bispectrum redshifts away with comparison that arises from non-linear gravitational instabilities.

If we suppose that the galaxies (subscript $g$) are locally biased with respect to the mass, then the density of galaxies $\delta_{g}$ can be Taylor expanded in the mass density to second order, $\delta_{g} = b_{0} + b_{1} \delta + \frac{1}{2} b_{2} \delta^{2}$, for some biasing coefficients $b_{i}$. The quantity $b_{0}$ affects only $c = 0$, so it can be ignored. The bispectrum for the galaxies is then

$$B_{g}(k_{1}, k_{2}, k_{3}) \approx b_{1}^{2} B_{0}(k_{1}, k_{2}, k_{3}) + \left\{ P_{0}(k_{1})P_{0}(k_{2}) \left[ b_{1}^{2} 2J_{2}(k_{1}, k_{2}) + b_{2} b_{1}^{2} \right] + \text{cyc.} + b_{1}^{3} B_{T} \right\}.$$  

(10)

So, assuming that the initial (or the linear) power spectrum is precisely known, we can do a likelihood analysis only if we also have a model for the initial (or linear) bispectrum and trispectrum as a function of the three wave numbers. We found previously, the primordial bispectrum redshifts away, and we can realistically only detect non-gaussianity if $B_{T}$ is significant. In section 4.1 we will quantify further this argument by showing that for all the models considered $B_{T} \ll B_{0}$. Let us parameterize the observed bispectrum as

$$B_{g}(k_{1}, k_{2}, k_{3}) = P_{0}(k_{1})P_{0}(k_{2})[c_{2}J_{2}(k_{1}, k_{2}) + c_{3}T_{3}(k_{1}, k_{2}, k_{3})] + \text{cyc.} \quad (11)$$

In order to assess the precision with which the bias could be measured, Matarrese, Verde & Heavens (1997) evaluated the precision with which the parameters $c_{2}$ and $c_{3}$ would be recovered from a likelihood analysis of the 2dF and/or SDSS. From their Figure 7, we conclude that if the positions of all of the galaxies in the survey volume are known, then (a) $c_{2}$ can be determined with an error of $\sim 6 \times 10^{-3}$ if $c_{3}$ is fixed; (b) $c_{3}$ can be determined with an error of $\sim 1 \times 10^{-2}$ if $c_{3}$ is fixed; and (c) the joint determination of the two parameters allows $c_{1}$ and $c_{2}$ to be recovered with errors of $1 \times 10^{-2}$ and $4 \times 10^{-2}$, respectively. The analysis of Matarrese, Verde & Heavens (1997) further shows that the bispectrum signal comes primarily from $k \approx 0.1 - 1$ h Mpc$^{-1}$. (All of these estimates would be reduced only by about a factor 20 if we could map the mass throughout the entire Hubble volume rather than just in the survey volume.) We will use these results to assess the smallest detectable primordial non-gaussianity.

3.2 CMB Bispectrum

A CMB map of the temperature $T(\hat{n})$ as a function of position $\hat{n}$ on the sky can be decomposed into spherical harmonics,

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}),$$

(12)

where the multipole coefficients are given by the inverse transformation,

$$a_{lm} = \int d\hat{n} Y_{lm}^{\ast}(\hat{n}) \frac{\Delta T(\hat{n})}{T}.$$  

(13)

The CMB bispectrum $B_{1,2,3}$ is defined by

$$\langle a_{1m_{1}} a_{2m_{2}} a_{3m_{3}} \rangle = B_{1,2,3} \left( \frac{l_{1}}{m_{1}}, \frac{l_{2}}{m_{2}}, \frac{l_{3}}{m_{3}} \right).$$  

(14)

where the factorization ensures statistical isotropy, and the last term is the Wigner 3 J symbol.

If we parametrize the bispectrum $X$ and some fixed function—as in equation (14)—of $l_{1}, l_{2}, l_{3}$, then the error on $X$ is given by (see, e.g., Matarrese, Verde & Heavens 1997)

$$\sigma_{X}^{2} \equiv \left( \frac{\partial^{2} \ln C}{\partial X^{2}} \right) \approx \sum_{l_{1} \leq l_{2} \leq l_{3}} \frac{(l_{1}l_{2}l_{3})^{2}}{C_{l_{1}l_{2}l_{3}}} \left[ \left( \frac{m_{1} l_{1}}{m_{2} l_{2}} \right)^{2} \left( \frac{m_{1} l_{1}}{m_{3} l_{3}} \right)^{2} \times \sum_{m_{1}, m_{2}, m_{3}} N_{N(m_{1}, l_{1})}^{m_{1}} \right],$$

(15)

where $C_{l}$ denotes the likelihood function and $C_{T} \equiv \langle |a_{lm}|^{2} \rangle$, the power spectrum of the sky fluctuation. Here we assumed that the departures from gaussianity are small and therefore the covariance matrix can be approximated by the covariance of a gaussian field that has the same power spectrum (Jungman et al. 1996; Heavens 1998), and we ignore the mixing which arises from partial sky coverage.

If one considers only the real part of $\langle a_{lm}^{\ast} a_{lm}^{\ast} a_{lm}^{\ast} \rangle$ (see Matarrese, Verde & Heavens 1997) then $n = 1/2$. The quantity $N_{N(m_{1}, l_{1})}$ is related to the number of non-zero terms like $C_{l_{1}l_{2}l_{3}}$ in the covariance and ranges from 1 to 30. Equation (15) is valid as long as the noise does not dominate the signal. If $N = 1$, then the sum over the Wigner 3J symbols is unity; the fact that $N$ is not equal to one reduces the sum by a only few percent, so
We note that such non-gaussianity can arise in standard slow-roll inflation: a Gaussian random field (in conformal Newtonian gauge) is a linear combination of a \( \Phi \) and \( \alpha \) models and introduce some non-gaussianity in several different ways. Specifically, we consider models in which the gravitational potential contains a part that is the square of a gaussian random field and models in which the density contains a part that is the square of a gaussian random field. We note that such non-gaussianity may arise in slow-roll and/or nonstandard (e.g., two-field) inflation models (Luo 1994; Falk, Rangarajan & Srednicki 1993; Gangui et al. 1994; Fan & Bardeen 1992). Moreover both of these models may be considered as Taylor expansions of more general fields, and are thus a fairly generic form of non-gaussianity. We also consider \( O(N) - \sigma \) models, as these will approximate the non-gaussianity expected in topological-defect models.

4 SOME NON-GAUSSIAN MODELS

We now proceed to consider several classes of physically-motivated models with primordial non-gaussianity in order to investigate more precisely the relative sensitivities of the CMB and LSS. Since current measurements of CMB and LSS power spectra are roughly compatible with cold-dark-matter (CDM) models for structure formation, we first consider CDM-like models and introduce some non-gaussianity in several different ways. Specifically, we consider models in which the gravitational potential contains a part that is the square of a gaussian random field and models in which the density contains a part that is the square of a gaussian random field. We note that such non-gaussianity may arise in slow-roll and/or nonstandard (e.g., two-field) inflation models (Luo 1994; Falk, Rangarajan & Srednicki 1993; Gangui et al. 1994; Fan & Bardeen 1992). Moreover both of these models may be considered as Taylor expansions of more general fields, and are thus a fairly generic form of non-gaussianity. We also consider \( O(N) - \sigma \) models, as these will approximate the non-gaussianity expected in topological-defect models.

4.1 Quadratic model for the potential

We start by considering a model in which the gravitational potential \( \Phi \) (in conformal Newtonian gauge) is a linear combination of a gaussian random field \( \phi \) and a term proportional to the square of the same random field,

\[
\Phi = \phi + \alpha (\phi^2 - \langle \phi^2 \rangle),
\]

where \( \alpha \) parametrizes the non-gaussianity; in the limit \( \alpha \to 0 \), the model becomes gaussian.† These models contain a primordial bispectrum for the gravitational potential,

\[
B_\Phi(k_1, k_2, k_3) \approx 2\alpha [P_B(k_1)P_B(k_2) + \text{cyc.}].
\]

The leading terms in the connected trispectrum for the \( \Phi \) field will be

\[
T_\Phi(k_1, \cdots, k_4) \simeq 4\alpha^2 P_B(k_1)P_B(k_2)[P_B(|k_1 + k_3|) + P_B(|k_1 + k_4|)] + \text{cyc.}
\]

We use a scale-invariant primordial spectrum, \( P_B = A_H k^{-3} \), where the amplitude is fixed by COBE to be, in our Fourier transform conventions, \( A_H \simeq 10^{-10} \).

The relative contribution to the bispectrum of \( B_0 \) and \( B_T \), \( R \equiv B_0/B_T \), can be obtained from equation (9) and (19). By evaluating the integral we find:

\[
R \simeq \frac{1}{\alpha A_H}.
\]

† We note that such non-gaussianity can arise in standard slow-roll inflation, and the parameter \( \alpha \) can be related to inflaton-potential parameters (e.g., Falk, Rangarajan & Srednicki 1993; Gangui et al. 1994; Wang & Kamionkowski 1999).

Thus, \( B_T \ll B_0 \) for \( \alpha \ll 10^6 \), so we can safely neglect the contribution from the trispectrum. The \( B_T \) contribution to the bispectrum will therefore be negligible compared with \( B_0 \) if \( \alpha \ll (A_H)^{-1} \).

The Fourier coefficients \( \delta(k, z) \) for the gravitational potential remain constant to linear order in perturbation theory in an Einstein-de-Sitter Universe. The Fourier coefficients \( \delta(k, z) \) of the density field evolve with time and are related to those of the gravitational potential by the Poisson equation, which can be written

\[
\delta(k, z) = M_k(z) \Phi(k), \quad M_k(z) = \frac{2k^2 T(k)(1 + z)}{3H_0^2}.
\]

The inverse of the function \( M_k(z) \) is plotted as a function of wavenumber \( k \) in Fig. 1. Thus, the linearly-evolved power spectrum for the mass, \( P_0(k, z) \) at redshift \( z \) is related to that for the gravitational potential (which remains constant in and Einstein-de-Sitter Universe) by

\[
P_0(k, z) = [M_k(z)]^2 P_B(k),
\]

where the transfer function is (e.g., Bardeen et al. 1986)

\[
T(k) \simeq \frac{1}{(1 + Bk + Ck^3/2 + Dk^2)},
\]

and \( B = 1.7Mpc(\Omega h^2)^{-1}, C = 0Mpc^3/2(\Omega h^2)^{-3/2} \) and \( D = 1Mpc^2(\Omega h^2)^{-2} \). The bispectrum for the mass for this model is thus

\[
B(k_1, k_2, k_3) \simeq \left\{ P(k_1)P(k_2) \left[ \frac{2\alpha M_k^2}{M^2_i M^2_j} + 2f(k_1, k_2) \right] \right\} + \text{cyc.}
\]

Comparing equation (24) with equation (11), we see that this particular form of primordial non-gaussianity leads to a present-day bispectrum that looks like a scale-dependent non-linear bias. In particular, for equilateral configurations, equation (24) becomes identical to equation (11) if we identify \( c_2 = 2\alpha/M_k \). We can therefore use the results of Matarrese, Verde & Heavens (1997) to determine the smallest \( \alpha \) that would be detectable by 2dF/SDSS, under the assumption that there were no non-linear biasing. In the range where most of the signal comes from, \( M_k^{-1} \simeq 10^{-5} - 10^{-6} \).

Thus, we conclude that the smallest \( \alpha \) that would give rise to an observable signal in the 2dF/SDSS bispectrum is \( \alpha \sim 10^3 - 10^4 \). The primordial gravitational-potential bispectrum in equation (18) will lead to a non-zero bispectrum in the CMB via the Sachs-Wolfe effect, and this bispectrum can be calculated to be (Luo...
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\[ B_{123} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \times \frac{2\alpha}{A_{SW}} \left[ C_{12} + C_{13} + C_{23} \right], \]

where \( A_{SW} \approx 1/3 \) is the Sach-Wolfe coefficient. Plugging this bispectrum (and a scale-invariant set of \( C_i \)) into equation (16), we learn that the smallest \( \alpha \) that could be detectable with a CMB map (using only \( l \leq 100 \)) is \( \sim 20 \). (The dashed curve in Fig. 2 shows the smallest \( \alpha \) detectable with the CMB as a function of the largest multipole moment \( l \) used in the analysis.) Thus, we conclude that the CMB will be at least two orders of magnitude more sensitive to a non-zero value of \( \alpha \) than LSS.

### 4.2 Quadratic model for the density

We now consider an alternative model in which the density field (rather than the gravitational potential) contains a term that is the square of a gaussian random field,

\[ \delta = \phi + \alpha \left( \delta^2 - \langle \delta^2 \rangle \right), \]

where now \( \phi \) is some other gaussian random field. Such a model has been considered in some two-field inflation models (e.g. Luo & Schramm 1993).

Since the density perturbation evolves in linear theory with time (unlike the gravitational potential), we must specify the epoch at which the density perturbation \( \delta \) is related to \( \phi \) through equation (26). We choose this epoch to be the current epoch, \( z = 0 \). Doing so, the spatial mass bispectrum today due to primordial non-gaussianity is

\[ B_\theta(k_1, k_2, k_3) \approx 2\alpha P(k_1)P(k_2) + \text{cyc.}, \]

and Luo & Schramm (1993) have shown that the contribution to the present-day bispectrum from the primordial trispectrum is negligible compared with this (\( B \phi \ll B_\theta \)), as long as 20PT holds. By comparing with equation (11), we see that this form of primordial non-gaussianity gives rise to a present-day skewness that mimics precisely that due to a scale-independent non-linear bias. Again, from the results of Matarrese, Verde & Heavens (1997), the smallest detectable \( \alpha \) in this model is \( \sim 0.01 \).

Now let us consider the CMB bispectrum of this model. The spatial bispectrum for the gravitational potential here is

\[ B_\phi(k_1, k_2, k_3) = \frac{2\alpha}{M_{k_1}M_{k_2}M_{k_3}} \left[ P_\phi(k_1)P_\phi(k_2) + \text{cyc.} \right]. \]

Although we have not carried out an exact calculation of the bispectrum for this model, it is easily seen (at least in the \( l \gg 1 \) limit) to be

\[ B_{\text{tris}} \approx \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \times \frac{2\alpha}{A_{SW}} \left[ \frac{2}{3} C_{12} \right]. \]

\[ \text{(29)} \]

Applying equation (16), we find that the smallest detectable \( \alpha \) in this model is \( \sim 0.01 \) (using only \( l \leq 100 \); results for other \( l_{\text{max}} \) are shown in Fig. 2), which is comparable to the LSS error. However, since this model has more power in the CMB bispectrum at larger \( l \), the smallest detectable \( \alpha \) decreases by roughly an order of magnitude even if we go out only to \( l_{\text{max}} = 200 \). Also in this case we find that CMB will provide a more precise probe of a primordial bispectrum. Moreover, as a corollary, this particular result demonstrates that if the maps provided by MAP and Planck are consistent with gaussian, then any measurement of a non-zero \( c_2 \) from the LSS bispectrum can be interpreted unambiguously as evidence for non-linear biasing, rather than some primordial non-gaussianity.

### 4.3 \( O(N) \) \( \sigma \) models

The \( O(N) \) sigma model provides an approximation to the non-gaussianity expected in topological-defect or scalar-field-alignment models (Turok & Spergel 1991; Jaffe (1994)). The \( N = 1 \) model has domain walls, \( N = 2 \) global strings, \( N = 3 \) global monopoles, \( N = 4 \) global textures, and higher \( N \) correspond to non-topological-defect models. For large \( N \) this model approaches the gaussian model, so we take \( \alpha = N^{-1/2} \) and as \( \alpha \rightarrow 0 \) the models become asymptotically gaussian. Since calculation of power spectra and higher-order statistics for the CMB and LSS are quite involved for these models, our analysis will be only approximate. As we will see below, these order-of-magnitude estimates will be sufficiently precise for our purposes.

For equilateral-triangle configurations and for values of \( \ell \) that can be probed with 2dF/SDSS, the power spectrum, bispectrum and trispectrum for LSS in the linear regime are (Jaffe 1994)

\[ P \approx 12.5 K^2 T^2(k), \quad B_0 \approx 1.6 K^3 \alpha T^3(k), \quad T \approx K^4 \frac{1}{k\alpha^2} T^4(k), \]

\[ \text{(30)} \]

\[ \text{(31)} \]
5 CONCLUSIONS

We addressed the question of which of the CMB or LSS is better poised to detect primordial non-gaussianity of several varieties. We used the bispectrum as a discriminating statistic since it is the lowest-order quantity that has zero expectation value for a gaussian field. We considered three forms of non-gaussianity: one in which the density field was the square of a gaussian field; another in which the density field was the square of another, linear bias, and the others mimicked a scale-independent non-linear bias. We showed that in all cases, the CMB is likely to provide a more precise probe of such non-gaussianity. One of these models produced a mass bispectrum that mimicked a scale-dependent non-linear bias, and the others mimicked a scale-independent non-linear bias.

Of course, our results are not fully general. In principle, it is possible to think of some other type of non-gaussianity for which our conclusions would not hold. However, plausible physical mechanisms that produce nearly scale-invariant power spectra should generically produce non-gaussian signals that have scale dependences roughly like those that we investigated. Thus we may conclude that if CMB maps turn out to be consistent with gaussian initial conditions, any non-gaussianity seen in the LSS bispectrum can be unambiguously attributed to the effects of biasing.

Details will be presented elsewhere. Our estimate differs from that of Luo (1994) for reasons that escape us. We have checked, however, that our conclusions are unaltered if we use his results.