We use the present observed number density of large X-ray clusters to constrain the amplitude of matter density perturbations induced by cosmic strings on the scale of $8h^{-1}\text{Mpc}$ ($\sigma_8$), in both open cosmologies and flat models with a non-zero cosmological constant. We find a slightly lower value of $\sigma_8$ than that obtained in the context of primordial Gaussian fluctuations generated during inflation. This lower normalization of $\sigma_8$ results from the mild non-Gaussianity on cluster scales, where the one point probability distribution function is well approximated by a $\chi^2$ distribution. We use our estimate of $\sigma_8$ to constrain the string linear energy density $\mu$ and show that it is consistent with the COBE normalization.

A. Introduction

Current theories for structure formation can be divided into two broad categories: inflation and cosmic defects. While inflation predicts primordial and Gaussian fluctuations (in the simplest inflationary models), topological defects [1] induce active and non-Gaussian perturbations.

One of the most important constraints on models of structure formation is the observed abundance of galaxy clusters. Although the cluster abundance has been widely used to constrain cosmological models with primordial Gaussian fluctuations (e.g. [2–6]) there have been few studies in the context of non-Gaussian perturbations such as those generated by topological defects. This is due to the difficulty of making robust predictions in topological defect scenarios, owing to their non-linear effects which are difficult to model and require large-scale numerical simulations (e.g. [7,8]).

The present work relies on high resolution numerical simulations of cosmic string seeded structure formation [9–12], which are first used to estimate the power spectrum and one point probability distribution functions (PDF) of the induced density perturbations (see also [13–16] for different approaches). We then employ a simple generalization of the Press-Schechter formalism [17], which is suitable for non-Gaussian perturbations with a general one point PDF [18], in order to obtain the expected number density of collapsed objects with a mass greater that a given threshold. This generalized Press-Schechter formalism has been used to constrain the Gaussianity of the density fluctuations in the Universe [19,20], and has been verified for a particular set of non-Gaussian structure formation models including a simplified flat space cosmic string model [21]. We finally estimate the amplitude of matter density perturbations induced by cosmic strings on the standard scale of $8h^{-1}\text{Mpc}$, using the presently observed number density of large X-ray clusters. We do this for cosmic string models in open universes without a cosmological constant (SOCDM), and also in flat universes with a non-zero cosmological constant (SACDM). We use our estimate of $\sigma_8$ to constrain the string linear energy density $\mu$ and show that it is consistent with the COBE normalization.

B. Power spectrum and PDF

In ref. [12] we described the results of high-resolution numerical simulations of string-induced structure formation. The power spectrum of CDM perturbations induced by long strings can be approximately described by

$$S_{\text{CDM}}(q) \equiv 4\pi k^3P(k) \propto (0.7q)^{p(q)},$$  

where

$$p(q) = 3.9 - \frac{2.7}{1 + (2.8q)^{-0.44}},$$

$q = k/\Gamma$ and the shape parameter $\Gamma$ is defined as

$$\Gamma = \Omega_0^0 h \exp(-2\Omega_0^0 - \Omega_0^0 / \Omega_0^m).$$

This fit to our numerical results has an accuracy of better than 10% in the range $k = 0.01–100h\text{Mpc}^{-1}$, provided that the baryon to dark matter ratio is relatively small ($2\Omega_0^0 + \Omega_0^B / \Omega_0^m \lesssim 0.3$ for 10% accuracy in $P(k)$). Note, however, that there are specified uncertainties in the underlying numerical results which are significantly larger [12]. This result also does not include the contribution from small loops which significantly enhance the overall power while leaving the overall shape unchanged [11] (as we will discuss later). A $\chi^2$ analysis using the observational power spectrum reconstructed by Peacock and Dodds [22] gives a best fit to long string power spectrum (1) with $\Gamma = 0.074 \times 10^{+0.1}$ at the 95 per cent confidence level. For a baryon energy density $\Omega_B = 0.05$, this implies string models provide a consistent match to observations in the acceptable cosmological parameter range $\Omega_0^m h = 0.14 \times 10^{+0.1}$.

We have used equation (1) to obtain a numerical fit for the standard deviation of matter density perturbations at
where 

\[ g(R) = r^{-\gamma(r)} \quad \text{and} \quad \gamma(r) = 0.5 + \frac{1.48}{1 + (6/r)^{0.7}}. \]  

The red-shift dependence of \( \sigma \) for arbitrary \( \Omega_m, \Omega_\Lambda \) is described accurately by

\[ \sigma(R, z) = \sigma(R) \frac{g(\Omega_m \cdot \Omega_\Lambda)}{(1+z)g(\Omega_m, \Omega_\Lambda)}, \]

where

\[ g(\Omega_m, \Omega_\Lambda) = \frac{2.5 \Omega_m}{\left[ \sqrt{\Omega_m^{0.7} - \Omega_\Lambda + (1 + \Omega_m/2)(1 + \Omega_\Lambda/70)} \right]^7}. \]

Here, \( g(\Omega_m, \Omega_\Lambda) \) gives the suppression of the growth of density perturbations relative to that of a critical-density-universe [12,23,24], and we can describe the evolution of the \( \Omega_m \) with red-shift as

\[ \Omega_m \equiv \Omega_m(z) = \frac{\Omega_m^0 (1+z)^3}{(1+z)^2 (1 + \Omega_m^0 z - \Omega_\Lambda^0) + \Omega_\Lambda^0}. \]

Using the simulation results for string-induced matter perturbations, we also found a reasonable fit to the positive side of one point PDF as follows:

\[ \chi^2(n) = \frac{n - \chi^2(\sigma)}{\sqrt{2n}}, \]

where \( \chi^2(\sigma) \) is the standard \( \chi^2 \) variant with the number of degrees of freedom \( n \), given as

\[ n = 0.4 (R \Gamma + 2)^{3.6}. \]

Here \( R \) is again the smoothing radius in units of \( h^{-1}\text{Mpc} \) and the fit is accurate within 10% error for all scales.

C. Modified Press Schechter formalism

The Press-Schechter formalism [17] relates the mass fraction of collapsed objects whose masses are larger than some given threshold \( M \), with the fraction of space in which the evolved linear density field exceeds some threshold \( \delta_c \). It has been extensively tested with great success against N-body simulations in the context of primordial Gaussian fluctuations, allowing for the computation of the number density of clusters within a given background cosmology. Here we used an extension of the Press-Schechter formalism for non-Gaussian perturbations proposed by Chiu, Ostriker and Strauss [18] and verified for a particular set of non-Gaussian structure formation models in ref. [21]. The fraction of the total mass within collapsed objects larger than a given mass \( M \) is given by:

\[ \frac{\Omega_m > M}{\Omega_m} = f_R \times \int_{\delta_c}^{\infty} P_R(\nu) d\nu, \]

where \( M \) is the cluster mass defined by \( M = 4\pi R^3 \rho_0 / 3 \), \( f_R = 1/\int_{\delta_c}^{\infty} P_R(\nu) d\nu \), \( P_R(\nu) \) is the one point PDF for a given smoothing radius \( R \), \( \nu \) is the number of standard deviations from mean density, \( \delta_c = \delta_c/\sigma(R, z) \) and \( \delta_c = 1.7 \pm 0.2 \) (95% confidence interval) assuming spherical collapse [25].

To obtain the number density of clusters in a mass interval \( dM \) about \( M \) at a redshift \( z \), we differentiate the Press-Schechter formula (11) to obtain (c.f. ref. [4])

\[ n(M, z) dM \approx -f_R \frac{\Omega_m^0 (\delta_c/\sigma(R, z))^3}{\sigma(R, z) dM} P_R \left[ \frac{\delta_c}{\sigma(R, z)} \right] dM. \]

In the derivation of equation (12) we have ignored other terms resulting from the \( R \) dependence of \( P_R \) and \( f_R \). This is a mathematically motivated approximation in the regime of mild non-Gaussianity, and we have verified in our case that it gives rise to at most an extra 2% error in the final estimate of \( \sigma_s \). Substituting (6) into (12) gives

\[ n(M, z) dM = f_R \frac{\Omega_m^0 (\delta_c/\sigma(R, z))^3}{\sigma(R, z) dM} P_R \left[ \frac{\delta_c}{\sigma(R, z)} \right] \times \left\{ \left[ \frac{w(z)}{\sqrt{2}} \right]^3 \log(\Gamma R) - \chi^2(\sigma) + \frac{7}{3} \right\} dM. \]

Lacey and Cole constructed a merging history for dark matter halos based on the excursion set approach and obtained an analytical expression for the probability that a galaxy cluster with present virial mass \( M \) would have formed at some redshift \( z \) [26]. The probability that a galaxy cluster with present virial mass \( M \) would have formed at a given redshift \( z \) is given by:

\[ p(z) = p(w(z)) \frac{dw(z)}{dz}, \]

where

\[ p(w(z)) = 2w(z)(F^{-1} - 1) \text{erfc} \left( \frac{w(z)}{\sqrt{2}} \right) - \sqrt{\frac{2}{\pi}} (F^{-1} - 2) \exp \left( -\frac{w^2(z)}{2} \right), \]

\[ w(z) = \frac{\delta_c/\sigma(M, z) - 1}{\sqrt{\sigma^2(F M, z) - \sigma^2(M, z)}}, \]

and \( F = 0.75 \pm 0.15 \) (95% confidence interval) is the fraction of the cluster mass assembled by a red-shift \( z \) [27]. We note that this result was derived in the context of primordial Gaussian fluctuations and must ultimately
be verified using N-body simulations. Although it may not be valid for generic non-Gaussian models, we still expect this to be a good approximation in the context of the cosmic string scenario for structure formation as the departures from Gaussianity on clusters scales are relatively small [10].

D. The Mass-temperature relation

In order to use the generalized Press-Schechter formalism to determine the abundance of X-ray clusters with a given temperature, we need to relate the X-ray temperature of a cluster with its virial mass. Here, we use the results of ref. [4] for the normalized virial mass-temperature relation (modified from [4]):

\[
M_v = (1.23 \pm 0.33) \times 10^{15} \left( \frac{\Omega^{\Lambda}(z_t)}{\Omega_{m}} \right)^{1/2} \left[ \frac{1.67 \times 10^{-5} T_{\text{B}}}{\sqrt{15-36y+24y^2-7y^3+y^4}} \right]^{1/2} h^{-1} M_{\odot},
\]

where \(z_t\) is the turnaround redshift, \(\Omega_{m} \equiv \Omega_m(z_t)\), and

\[
\eta \equiv \eta(z_t) = \frac{32}{9\pi^2} \Omega_{m}^{0.5} \Omega_{m}^{0.5}(1 + z_t)^3,
\]

\[
b(\Omega) = \begin{cases} 
0.76 - 0.25\Omega & (\text{LCDM}), \\
0.73 - 0.23\Omega & (\text{ΛCDM}).
\end{cases}
\]

For a given redshift of cluster collapse \(z_c\), the turnaround redshift \(z_t\) is easily obtained using the fact that \(2t(z_c) = t(z_t)\).

Hence we can now estimate the present comoving number density of galaxy clusters per temperature interval \(d(k_{\text{B}}T)\) with a mean X-ray temperature \(k_{\text{B}}T\) which were formed at a given redshift \(z\) as:

\[
n_T(k_{\text{B}}T, z) = \frac{3M}{2k_{\text{B}}T} n(M, 0)p(z) d(k_{\text{B}}T) dz.
\]

The present abundance of X-ray clusters with a temperature \(k_{\text{B}}T\) greater than 6.2keV can be estimated by integrating equation (20) from \(z = z_c = 0\) to infinity. A comparison between the observed cluster abundance and its theoretical prediction will give an estimate of \(\sigma_s\).

We will use the observation for the number density of galaxy clusters at \(z = 0.05\) with X-ray temperature exceeding 6.2keV, given by Viana and Liddle [4], based on the dataset presented by Henry and Arnaud [28], and updated by Henry [29]:

\[
N(> 6.2\text{keV}, 0.05) = 1.53 \times 10^{-7} \pm 0.16 \ h^3 \text{Mpc}^{-3}.
\]

The uncertainty in (21) is the 1-sigma interval, and these results have taken into account the effect of temperature measurement errors. The reasons for concentrating on galaxy clusters with temperature larger than 6.2keV has been extensively discussed by Viana and Liddle [4].

E. Results and discussion

By comparing (21) with the result integrated from (20), we obtain the observationally constrained \(\sigma_s\) as plotted in figure 1. The overall error in the value of \(\sigma_s\) was estimated by Monte Carlo simulations over \(10^4\) realizations, treating the intrinsic uncertainties in \(\Gamma\), \(N(> 6.2\text{keV}, 0.05)\) as lognormal, and those in \(\delta_c\), \(M_v\), \(F\) as Gaussian. An accurate numerical fit to this result is

\[
\sigma_s = \begin{cases} 
0.44 \Omega_{m}^{0.5} - 0.45 \Omega_{m}^{0.5} & (\text{SOCDM}), \\
0.44 \Omega_{m}^{0.5} - 0.6 + 0.35 \Omega_{m}^{0.5} & (\text{SACDM}).
\end{cases}
\]

The 95% confidence limits are \(\pm 32\%\) in the SOCDM case, and \(+35\%\) and \(-32\%\) in the SACDM case. We note that the overall shape of \(\sigma_s\) in figure 1 is more or less the same as that for inflationary models [4], while the amplitude here is about 10–20% lower than standard inflation. The emergence of the same shape from these very different scenarios is due to the fact that both the string-induced and inflationary power spectra used in the calculation of cluster abundance are constrained by the same observation [22] and have roughly the same shape within the scales of interest (though with quite different choices of \(\Gamma\)). We have also verified that the lower normalization of \(\sigma_s\) is due to a slightly larger right-hand side tail of the
FIG. 2. $B_{G\mu}$ as a function of $\Omega_m^0$ and $\Omega_\Lambda^0$. The hatched (SOCDM) and shaded (SACDM) areas are within a 95% confidence level.

PDF in the cosmic string case, i.e. a high-density region then becomes consistent with a smaller $\sigma_8$.

The value of $\sigma_8$ in the context of the cosmic string model for structure formation was also investigated by Bruck [30]. In his work he assumed the one point PDF to be independent of scale by taking the distribution at the non-Gaussian scale ($\sim 1.5(\Omega_m^0 h^2)^{-1}$Mpc [10]) to be valid up to scales relevant for the cluster abundance calculation. Although this assumption can give the right qualitative results, for small values of $\Omega_m^0 h$, the $R$ dependence of the one point PDF needs to be taken into account in order to obtain more accurate results. This improvement has been incorporated in our work, which also took into consideration the merging history of dark matter halos.

To see how cluster abundances constrain the string energy density per unit length $\mu$, we can define a string “bias” parameter,

$$B_{G\mu}(\Omega_m^0, \Omega_\Lambda^0) = \frac{\sigma_8(\Omega_m^0, \Omega_\Lambda^0)}{\sigma_8^{(S)}(\Omega_m^0, \Omega_\Lambda^0)}, \quad (23)$$

where $\sigma_8$ is the cluster result (22) and $\sigma_8^{(S)}$ is directly inferred from the long string-induced power spectrum (1), given the assumptions that $\Omega_B^0 = 0.05$, $\Omega_m^0 h = 0.14$, and $G\mu_6 = G\mu \times 10^6 = 1$ for $\Omega_m^0 = 1$ with $\Lambda = 0$ (with the open and $\Lambda$-parameter dependence for $G\mu_6$ given by $G\mu_6 \propto \Omega_m^0^{-0.3}$ for SOCMD and $G\mu_6 \propto \Omega_m^0^{-0.05}$ for SACDM [13]).

We can observe in figure 2 for the flat $\Lambda$-model that $B_{G\mu}(\Omega_m^0, \Omega_\Lambda^0)$ ranges from $2.2 \pm 0.8$ to $3.9 \pm 1.3$ (95% confidence level) for different choices of $\Omega_m^0$ and $\Omega_\Lambda^0$. These values of $B_{G\mu}$ are actually the cluster normalized values for $G\mu_6$ at $\Omega_m^0 = 1$ with $\Lambda = 0$. For small $\Omega_m^0 \approx 0.2$–0.3, the result is slightly higher than the previously COBE-normalized $G\mu_6 = 1.7$ [31], but it is certainly consistent within the uncertainties. For large $\Omega_m^0$, the result appears to be inconsistent with the COBE constraint. However, as recently discussed in [11], the inclusion of perturbations induced by cosmic string loops can boost the string-induced power spectrum by a factor of about two under reasonable assumptions. This may serve to remove even this apparent discrepancy in $G\mu_6$ between the COBE and cluster abundance constraints. We also note that the exact contribution of the background of gravitational radiation emitted by cosmic string loops remains a significant uncertainty [32].

F. Conclusion

We have constrained the amplitude of matter density perturbations induced by cosmic strings on the scale of $8h^{-1}$Mpc in both open cosmologies and flat models with a non-zero cosmological constant, using the currently observed number density of large X-ray clusters. Because string-seeded matter perturbations are mildly non-Gaussian on cluster scales, we obtained a slightly lower normalization of $\sigma_8$ than that found for cosmological models with primordial Gaussian fluctuations. We used the calculated $\sigma_8$ to constrain the string linear energy density $\mu$, which we found to be consistent with the latest COBE normalization when current uncertainties in the normalization are taken into account.

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[1] For a review see A. Vilenkin and E. P. S. Shellard, Cosmic strings and other topological defects (Cambridge University Press, 1994)
\[ \Omega_0^\Lambda = 0 \]

\[ \Omega_0^0 + \Omega_0^\Lambda = 1 \]