Regular and Irregular Boundary Conditions in the AdS/CFT Correspondence

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June 21, 1999

Abstract

We expand on Klebanov and Witten’s recent proposal for formulating the AdS/CFT correspondence using irregular boundary conditions. The proposal is shown to be correct to any order in perturbation theory.

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1 Introduction

The celebrated AdS/CFT correspondence relates field theories on anti-de Sitter (AdS) space with conformal field theories (CFTs) living on the AdS horizon. The main prediction of this duality is that CFT correlation functions of conformal operators can be calculated by evaluating the AdS action on-shell as a functional of prescribed boundary values.

For example, using a scalar field theory on AdS space, CFT correlators of conformal fields of scaling dimensions $\Delta \geq d/2$ have been calculated [1, 2, 3, 4, 5, 6, and references therein]. Until recently, no prescription was known to include operators with scaling dimension $\Delta$, $d/2 - 1 < \Delta < d/2$. Here, $d/2 - 1$ is the unitary bound on the conformal dimension of scalar operators. Recently, Klebanov and Witten [7] proposed a method to do just that. They used the fact that a scalar field on AdS space can obey two types of boundary conditions [8]. The regular one, which can always be imposed, leads to the CFT correlators with $\Delta \geq d/2$, whereas the irregular one would lead to $d/2 - 1 < \Delta \leq d/2$. They realized that the respective boundary fields are conjugate to each other and proposed to use a Legendre transform of the action, expressed as a functional of the irregular boundary value, as the generating functional. They also demonstrated the correctness of this proposal for CFT two point functions.

In this article, we would like to expand on their proposal and demonstrate its correctness to all orders in perturbation theory. A second result of our analysis is that a different Green’s function must be used for internal lines in second or higher order graphs.

The outline of the article shall be as follows. In the remainder of this section motivating arguments about the origin of the irregular boundary conditions will be given. In section 2 we will for completeness repeat the formalism using regular boundary conditions. Then, in section 3 Klebanov and Witten’s proposal to include irregular boundary conditions shall be analyzed and shown to be correct to any order in perturbation theory.

To start, consider an interacting scalar field, whose action is given by

$$I = \frac{1}{2} \int_{\Omega} \mathit{d}x \left(D_{\mu} \phi D^{\mu} \phi + m^2 \phi^2 \right) + I_{\text{int}},$$  (1)

where $I_{\text{int}}$ denotes the interaction terms and $\mathit{d}x = d^{d+1}x \sqrt{g(x)}$ is the invariant volume integral measure. The equation of motion following from the action (1) is given by

$$\left(D_{\mu} D^{\mu} - m^2\right) \phi(x) = B(x),$$  (2)

where

$$B(x) = \frac{\delta I_{\text{int}}}{\delta \phi(x)}.$$  

Using as AdS representation the conventional upper half space $x \in \mathbb{R}^d$, $x_0 > 0$ with the metric

$$ds^2 = (x_0)^{-2} dx^\mu dx^\mu,$$  (3)

the solution to equation (2) can be written in the form

$$\phi(x) = \int d^4y \left[ \frac{x_0}{(x - y)^2} \right]^{\frac{d}{2} \pm \alpha} f_\pm(y) + \int_{\Omega} \mathit{d}y G(x, y) B(y),$$  (4)

1The functional variation is done covariantly, cf. [9].
where \( \alpha = \sqrt{d^2/4 + m^2} \) and \( G(x, y) \) is a standard Green’s function satisfying

\[
(D_\mu D^\mu - m^2) G(x, y) = \frac{\delta(x - y)}{\sqrt{x}}.
\]

The free field solution with the lower sign exists classically for \( \alpha < d/2 \), but the unitary bound restricts it further to \( \alpha < 1 \) [8]. The functions \( f_- \) and \( f_+ \) are called regular and irregular boundary values and are conformal fields of scaling dimensions \( d/2 - \alpha \) and \( d/2 + \alpha \), respectively.

In the AdS/CFT correspondence the fields obeying regular boundary conditions give rise to CFT correlation functions of operators with conformal dimensions \( \Delta \geq d/2 \). Hence, the use of irregular boundary conditions enables one to obtain correlation functions for operators with scaling dimensions \( d/2 - 1 < \Delta < d/2 \).

## 2 Regular Boundary Conditions

Let us start by rewriting the expression (4) as

\[
\phi(x) = \phi^{(0)}(x) + \int \Omega dG(x, y)B(y),
\]

where the Green’s function \( G(x) \) is given by [10, 3]

\[
G(x, y) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot (x-y)} \left\{ \frac{I_\alpha(kx_0)K_\alpha(ky_0)}{k^d} \right. \quad \text{for } x_0 < y_0,
\]

\[
= -\frac{c_\alpha}{2} \xi^{d/2 + \alpha} F\left(d/2, d/2 + \alpha; 1 + \alpha; \xi^{-2}\right),
\]

where \( F \) is the hypergeometric function,

\[
\xi = \frac{1}{2x_0 y_0} \left\{ \frac{1}{2} \left[ (x - y)^2 + (x - y^*)^2 \right] + \sqrt{(x - y^2)^2 + (x - y)^2} \right\}
\]

\((y^* \text{ denotes the vector } (-y_0, y))\), and

\[
c_\alpha = \frac{\Gamma(d/2 + \alpha)}{\pi^{d/2} \Gamma(1 + \alpha)}.
\]

Moreover, the free field solution \( \phi^{(0)} \) shall be written as

\[
\phi^{(0)}(x) = \int d^d y K_\alpha(x, y)\phi^{(0)}(y) = \int d^d y K_{-\alpha}(x, y)\phi^{(0)}(y).
\]

The bulk-boundary propagators occuring in equation (10) are given by

\[
K_{\pm \alpha}(x, y) = \pm \alpha c_{\pm \alpha} \left[ \frac{x_0}{(x - y)^2} \right]^{\frac{d}{2} + \alpha},
\]

where \( c_{\pm \alpha} \) is given by equation (9), and their Fourier transforms read

\[
K_{\pm \alpha}(x, k) = \frac{\pm 2\alpha}{\Gamma(1 + \alpha)} e^{ik \cdot x} \left( \frac{k}{2} \right)^{\alpha} x_0^{\frac{d}{2} - \alpha} K_\alpha(kx_0).
\]
Equations (12) and (10) imply that the boundary functions $\phi_0^+$ and $\phi_0^-$ are related by

$$
\phi_0^+(k) = -\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \left( \frac{k}{2} \right)^{2\alpha} \phi_0^-(k).
$$

(13)

Obviously, the free field $\phi_0$ can be written as a sum of two series, whose leading powers are \(x_0^{d/2-\alpha}\) and \(x_0^{d/2+\alpha}\), respectively. Thus, one finds by direct comparison with equations (10) and (12) that the small $x_0$ behaviour of $\phi_0$ is

$$
\phi_0(x) \approx \frac{x_0^{d-\alpha}}{x_0^{d+\alpha}} \phi_0^-(x) + x_0^{d+\alpha} \phi_0^+(x),
$$

(14)

where subleading terms have been dropped. Moreover, the Green’s function (8) goes like

$$
G(x, y) \approx -\frac{1}{2\alpha} \int d^d y K_\alpha(x, y).
$$

(15)

Hence, the interaction contributes only to the $\phi_+$ part of the asymptotic boundary behaviour, i.e. one can write

$$
\phi(x) \approx \frac{x_0^{d-\alpha}}{x_0^{d+\alpha}} \phi_-(x) + x_0^{d+\alpha} \phi_+(x),
$$

(16)

where

$$
\phi_-(x) = \phi_0^-(x), \quad \phi_+(x) = \phi_0^+(x) - \frac{1}{2\alpha} \int d^d y K_\alpha(x, y) B(y).
$$

(17)

(18)

Identical relations hold for the Fourier transformed expressions.

Now consider the on-shell action, treated as a functional of the regular boundary values $\phi_-$. Integrating equation (1) by parts yields

$$
I = \frac{1}{2} \int d^dx \int d^d x_0 \int d^d \phi \partial_\mu \phi - \frac{1}{2} \int \Omega d \phi(x) B(x) + I_{int}.
$$

The first term must be regularized, which is done by writing

$$
x_0^{-d} n^\mu \partial_\mu \phi = -x_0^{-d} \phi \left[ \left( \frac{d}{2} - \alpha \right) \frac{x_0^{\frac{d}{2} - \alpha}}{x_0^{\frac{d}{2} + \alpha}} \phi_- + \left( \frac{d}{2} + \alpha \right) x_0^{\frac{d}{2} + \alpha} \phi_+ + \cdots \right]
$$

$$
= -x_0^{-d} \left( \frac{d}{2} - \alpha \right) \phi^2 - 2\alpha \phi_- \phi_+ + \cdots,
$$

where the ellipses indicate contributions from subleading terms and other terms which vanish for $x_0 = 0$. The first term in the last line is cancelled by a covariant counterterm. Hence, the renormalized on-shell action is

$$
I[\phi_] = -\alpha \int \frac{d^dk}{(2\pi)^d} \phi_-(k) \phi_+(-k) - \frac{1}{2} \int \Omega d \phi(x) B(x) + I_{int}
$$

$$
= I^{(0)}[\phi_+] - \frac{1}{2} \int \Omega d \phi(x) B(x) G(x, y) B(y) + I_{int},
$$

(19)
where equations (18), (17), (13), (6) and (10) have been used. The term $I^{(0)}$ in equation (19) is given by

$$I^{(0)}[\phi_-] = \alpha \Gamma(1 - \alpha) \int \frac{d^d k}{(2\pi)^d} \left( \frac{k}{2} \right)^{2\alpha} \phi_-(k) \phi_-(\mathbf{-k})$$

$$= - \alpha^2 c_\alpha \int d^d x \ d^d y \frac{\phi_-(x) \phi_-(y)}{|x-y|^{d+2\alpha}}$$

(20)

and thus yields the correct two point function of scalar operators of conformal dimension $\Delta = \frac{d}{2} + \alpha$, if one uses the AdS/CFT correspondence formula

$$e^{-I[\phi_-]} = \left\langle \exp \left[ \alpha \int d^d x \mathcal{O}(x) \phi_-(x) \right] \right\rangle.$$ 

(21)

The other two terms have to be expressed as a perturbative series in terms of $\phi^{(0)}$. However, by virtue of equations (10) and (17) this naturally yields a perturbative series in terms of the boundary function $\phi_-.

### 3 Irregular Boundary Conditions

The treatment of irregular boundary conditions follows an idea by Klebanov and Witten [7]. Consider the expression

$$\frac{\delta I[\phi_-]}{\delta \phi_-(\mathbf{k})} = 2\alpha \Gamma(1 - \alpha) \int \frac{d^d k}{(2\pi)^d} \left( \frac{k}{2} \right)^{2\alpha} \phi_-(\mathbf{-k}) + \int \Omega \, dx \, \frac{\delta \phi(x)}{\delta \phi_-(\mathbf{0})}$$

$$- \int \Omega \, dy \, dz \frac{\delta^2 I_{int}}{\delta \phi(y) \delta \phi(z)} G(x, y) B(y)$$

Using equation (13) and the formula

$$\frac{\delta \phi(x)}{\delta \phi_-(\mathbf{k})} = K_\alpha(x, -\mathbf{k}) + \int \Omega \, dy \, dz \frac{\delta^2 I_{int}}{\delta \phi(y) \delta \phi(z)}$$

one finds

$$\frac{\delta I[\phi_-]}{\delta \phi_-(\mathbf{k})} = -2\alpha \phi^{(0)}_+(\mathbf{-k}) + \int \Omega \, dx \, K_\alpha(x, -\mathbf{k}) B(x) = -2\alpha \phi_+(-\mathbf{k}),$$

or, after an inverse Fourier transformation,

$$\frac{\delta I[\phi_-]}{\delta \phi_-(\mathbf{x})} = -2\alpha \phi_+(\mathbf{x})$$

(22)

This expression holds to any order in perturbation theory. This fact was obtained in [7] using graph arguments. Furthermore, it shows first that $\phi_+$ can be regarded as the conjugate field of $\phi_-$ and secondly that the functional

$$J[\phi_-, \phi_+] = I[\phi_-] + 2\alpha \int d^d x \phi_-(\mathbf{x}) \phi_+(\mathbf{x})$$

(23)

has a minimum with respect to a variation of $\phi_-$. 

5
Klebanov and Witten’s idea [7] is to formulate the AdS/CFT correspondence by the formula

\[ e^{-J[\phi_+]} = \left\langle \exp \left[ \alpha \int d^d x \mathcal{O}(x) \phi_+(x) \right] \right\rangle. \]  

(24)

Here, the functional \( J[\phi_+] \) is a Legendre transform of the action \( I \), i.e. it is the minimum value of the expression (23), expressed in terms of \( \phi_+ \).

In the following Klebanov and Witten’s result about the correctness of the two point function [7] shall be confirmed and interactions included. The minimum of \( J \) is easiest found from equations (19) and (23), giving

\[
J[\phi_+] = \alpha \int \frac{d^d k}{(2\pi)^d} \phi_-(k) \phi_+(-k) - \frac{1}{2} \int d x \phi(x) B(x) + I_{int}
\]

\[
= -\alpha \frac{\Gamma(1 + \alpha)}{\Gamma(1 - \alpha)} \int \frac{d^d k}{(2\pi)^d} \left( \frac{k}{2} \right)^{-2\alpha} \phi_+(k) \phi_+(-k) - \frac{1}{2} \int d x \phi(x) B(x) + I_{int}
\]

\[
+ \frac{1}{2} \int d^d x \int d y K_{\alpha}(y, x) \phi_+(x) B(y)
\]

\[
= -\alpha \frac{\Gamma(1 + \alpha)}{\Gamma(1 - \alpha)} \int \frac{d^d k}{(2\pi)^d} \left( \frac{k}{2} \right)^{-2\alpha} \phi_+(k) \phi_+(-k) - \frac{1}{2} \int d x \int d y \ B(x) G(x, y) B(y)
\]

\[
+ I_{int} - \frac{1}{4\alpha} \int d^d z \int d x d y K_{\alpha}(x, z) K_{\alpha}(y, z) B(x) B(y).
\]  

(25)

Here equations (17), (13), (18), (12) and (6) have been used. The first term in equation (25) can be inversely Fourier transformed, which yields

\[ J^{(0)} = -\alpha^2 c_{-\alpha} \int d^d x d^d y \frac{\phi_+(x) \phi_+(y)}{|x - y|^{d-2\alpha}}. \]  

(26)

According to the correspondence formula (24), this yields the correct two point function.

Then, the second and fourth term in equation (25) can be combined by defining the Green’s function

\[ \tilde{G}(x, y) = G(x, y) + \frac{1}{2\alpha} \int d^d z K_{\alpha}(x, z) K_{\alpha}(y, z). \]  

(27)

This modified Green’s function \( \tilde{G} \) also satisfies equation (5), because the second term in equation (27) does not contribute to the discontinuity. Moreover, using equations (7) and (12) one finds

\[
\tilde{G}(x, y) = -(x_0 y_0)^\frac{d}{2} \int \frac{d^d k}{(2\pi)^d} e^{-i k (x - y)} \times \left[ \frac{2K_{\alpha}(k x_0) K_{\alpha}(k y_0)}{\Gamma(1 - \alpha)} + \left\{ \begin{array}{ll}
K_{\alpha}(k y_0) I_{\alpha}(k x_0) & \text{for } x_0 < y_0, \\
K_{\alpha}(k x_0) I_{\alpha}(k y_0) & \text{for } x_0 > y_0,
\end{array} \right. \right]
\]

\[
= -(x_0 y_0)^\frac{d}{2} \int \frac{d^d k}{(2\pi)^d} e^{-i k (x - y)} \left\{ \begin{array}{ll}
K_{\alpha}(k y_0) I_{-\alpha}(k x_0) & \text{for } x_0 < y_0, \\
K_{\alpha}(k x_0) I_{-\alpha}(k y_0) & \text{for } x_0 > y_0,
\end{array} \right.
\]

which differs from equation (7) only by interchanging \( \alpha \) and \( -\alpha \). Hence, the result (8) can be taken over, yielding

\[ \tilde{G}(x, y) = -\frac{c_{-\alpha}}{2} \xi^{-(\frac{d}{2}-\alpha)} F \left( d/2, d/2 - \alpha; 1 - \alpha; \xi^{-2} \right). \]  

(28)
Thus, inserting equation (27) into equation (25) yields

\[
J[\phi_+] = J^{(0)}[\phi_+] - \frac{1}{2} \int_{\Omega} dx \, dy \, B(x) \tilde{G}(x, y) B(y) + I_{\text{int}}.
\] (29)

Moreover, one can see from equation (28) that for small \( x_0 \) \( \tilde{G} \) behaves as

\[
\tilde{G}(x, y) \approx x_{\alpha}^{-d} \frac{1}{2\alpha} x_0^{-\alpha} K_{-\alpha}(x, y).
\] (30)

Hence, writing

\[
\phi(x) = \int d^d y \, K_{-\alpha}(x, y) \phi_+(y) + \int_{\Omega} dy \, \tilde{G}(x, y) B(y),
\] (31)

the interaction contributes only to \( \phi_- \). This in turn means that, expressing \( I_{\text{int}} \) and \( B \) as a perturbative series and using equation (31), the functional \( J \) is naturally expressed in terms of the irregular boundary value \( \phi_+ \). Moreover, it has the expected form, in that it is obtained from equation (19) by replacing \( \alpha \) with \(-\alpha\) and \( \phi_- \) with \( \phi_+ \). An important point is that the Green’s function \( \tilde{G} \) must be used for the calculation of internal lines.

Finally, by a calculation similar to that of the derivation of equation (22) one finds

\[
\frac{\delta J[\phi_+]}{\delta \phi_+(x)} = 2\alpha \phi_-(x).
\] (32)

This is a final confirmation of the fact that the fields \( \phi_- \) and \( \phi_+ \) are conjugate to each other.

In conclusion, we have expanded on Klebanov and Witten’s recent idea for formulating the AdS/CFT correspondence using irregular boundary conditions, showing it to give the expected answers to any order in perturbation theory.

Acknowledgments

This work was supported in part by a grant from NSERC. W. M. is very grateful to Simon Fraser University for financial support.

References