Two Outcomes for Two Old (Super-)Problems

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Abstract: I briefly review the outcomes of two very different old questions – where global SUSY improves on ordinary QFT – when they are in turn posed in the local, SUGRA, context. The first concerns the unexpectedly powerful role of the “Dirac square root” graded algebra relation $E = Q^2$, originally found in $D=4$ SUSY by Gol'fand and Likhtman, in proving positive energy theorems both in SUGRA and in ordinary gravity theories, where $E$ and $Q$ have very different – gauge generator – definitions. The second seeks SUGRA counterparts of the boson/fermion loop ultraviolet cancellations in SUSY models. Here the result is negative: local supersymmetry cannot overcome the infinities associated with the dimensional gravitational $\kappa^2$ even, as recently shown, in maximal $D=11$ SUGRA.

1. Introduction

We are here commemorating the birth of $D=4$ supersymmetry, (SUSY), as well as the memory of the late Yuri Abramovich Gol’fand, who with Evgeny Pinkhusovich Likhtman, discovered it [1] in one of those Soviet-era works often overlooked in the West – and in this case, also in the East!

So universal has the SUSY gospel become in so short a time that few of us still envisage particle theory without (broken) SUSY or gravity without supergravity (SUGRA), even in the absence of any experimental evidence. This is a strong testimonial to the naturalness of SUSY, since it differs from ordinary QFT in a “discrete” way. By this I mean that a generalization, such as Einstein’s of Newtonian gravity, is usually continuously related to it by a natural group contraction, there $c \to \infty$. Here there is no such parameter (except the far too strong $\hbar \to 0$) to rely on. Nor do we have a unique natural SUSY-breaking mechanism. While awaiting explicit verification of uniquely SUSY predictions, its theoretical persuasiveness rests on tangible formal successes. In this essay, I will concentrate on two problems where SUSY undeniably improved on “ordinary” QFT, but take them one step further, translated to the local SUSY, namely SUGRA, context. I chose the two for balance: one succeeded beyond expectation, the other failed according to expectation.

The first problem stems from perhaps the deepest aspect of the graded Poincaré group, the supercharge anticommutator given in [1], whose relevant part for us is that the square of two supercharges is the energy. I will recall the adaptation of this relation to the proof of positive energy of supergravity and of Einstein gravity, where both supercharge and energy have very different meanings. My other example involves heavier machinery, and in its latest and final incarnation the result has only recently been obtained: what is the impact of supersymmetry on ultraviolet

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1Dedicated to two founders of $D=4$ SUSY on its (nearly) 30th birthday.
divergences in SUGRA? Are any supergravity models renormalizable, or more exactly, finite? Since this is neither a report of new work nor a real review, all details (and many references) are left to the references.

2. Positive Energy

If one looks at the graded algebra of SUSY, it contains (besides the existence of a constant supercharge) one really novel relation, that the square of $Q$ is the total energy. This is the true “improvement” on the Poincaré algebra, imposing the huge (and desirable) restriction on supersymmetrizable models, that more generally, their 4-momentum is future timelike. I should caution that it does not exclude systems having a trivial kind of negative energy. Consider for example, two copies of a free multiplet, the second of whose actions has relative ghost sign; each is separately supersymmetric and yet the total energy need not be positive. Indeed, the more refined arena of SUGRA models with actions quadratic in curvature is essentially of the above type, and provides some instructive examples [2] of the role of the Hilbert space metric in interpreting “$E = Q^2$”. Still, this caveat is more of a curiosity; every bose/fermi pair of excitations must share purely positive (or purely negative) energy, and that is really the heart of SUSY’s “Dirac square root” character, as was already exploited (for global SUSY) in [3].

In 1976, when SUGRA was found, the problem of positive energy in Einstein gravity (GR) had more or less exhausted those of us who had worked on it over the previous 15 years, ever since gravitational energy first became sufficiently understood to ask the question. While there were no (correct) counter-examples, there were also only “physicists’ proofs” [4]. It was only for special initial value configurations that explicit positivity could be exhibited, and no underlying “square of a square root” structure of the energy was ever observed in the process. It was agreed (among some of the experts at least) that it was both necessary for physical consistency and probably true, that not only did GR have positive energy but that $E = 0$ implied the vacuum, i.e., only flat space had vanishing energy (for some history, see [5]). Actually, just about that time the first positivity proof did emerge [6]; while rigorous, it was not very intuitive. The obstacle to obtaining a direct physical proof of positivity of GR in the sense that the energy of say (even nonabelian) vector gauge theory is manifestly positive, $E = \frac{1}{2} \int d^3r (E^2 + B^2)$, is that not only is GR a gauge theory with energy as the “charge”, but an infinitely self-interacting one. While its (linearized) free massless spin 2 excitations consist of a positive harmonic-oscillator frequency sum once the gauge constraints are imposed, the higher terms are not easily bounded: the full Einstein theory’s action is an infinite series $I_E \sim \int d^4x \sum_{n=1}^{\infty} h^n (\partial h)^2$, where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the deviation from flatness of the covariant metric, on top of which the ten $h_{\mu\nu}$ consist of four pairs of unphysical gauge and constraint variables (akin to the longitudinal vector potential and electric field in electrodynamics), the latter are themselves spatially non-local functionals of the two remaining physical components (as well as of the matter stress-tensor).

Aside from the above technical difficulties of the pure GR problem, interest now moved to the more complicated – because including a “matter” source and being intrinsically quantized – SUGRA, seemingly an even more difficult arena. Yet it had that magic $E = Q^2$ formula; however, when we turned it to advantage [7], we were met with scepticism because of the well-known special status as surface integrals of $E$ already in GR, and $Q$ for the spin 3/2 field. Besides, how could an ancient unsolved problem in the simpler classical setting be trivially solved in this much more complicated one? This is not the place to provide the details of our results, but only to indicate the beauty of the theory and the robustness of these “charges” upon transition to the SUGRA context.
Recall that in GR – with or without sources such as the spin 3/2 field \( \psi_\mu \) – the total energy \( E \) (defined for asymptotically flat solutions) being a gauge charge, could be expressed as a surface integral of a purely gravitational quantity at spatial infinity. To oversimplify enormously what took a long time to understand, the equivalent of the Gauss constraint on the Coulomb potential, with charge density as source \( (\nabla^2 \varphi = \rho) \) is here the \( G^0_0 = T^0_0 \) Einstein constraint, which can be rewritten to resemble linear Gauss form for a certain component \( h^T \) of the metric, \( \nabla^2 h^T = t^0_0 \). Here, however the energy density \( t^0_0 \) of gravity plus matter depends in a complicated way on \( h^T \): energy self-interacts! Still, the value \( E \) can be read off by elementary Poisson equation considerations as the coefficient of \( 1/r \) in \( h^T \) at spatial infinity (whatever the complications). But whether its sign is always positive remains hidden in the full interior “counting”, \( E = \int d^3 x t^0_0(h) \), of the local contributions, just as the total charge’s functional form, in contrast with its surface value, as the coefficient of \( 1/r \) in \( \varphi \), is only obtainable by “counting” all the interior electrons. Fortunately, if it can be shown generically that \( E \) is itself always the square of a Hermitian operator, then this suffices to prove its positivity by functional form without needing recourse to the volume integral. This is where the supercharge \( Q \) of SUGRA comes to the rescue. Everything that has just been said in our child’s description of gravitational energy holds for the spin 3/2 field’s supercharge. It is also simultaneously both a surface integral as well as one over the interior volume. What we were then able to show was that indeed the Gol’fand–Likhtman relation \( E = Q^2 \geq 0 \) continues to hold as an operator statement in SUGRA, where \( E \) is now the total (non-vanishing!) Hamiltonian operator of SUGRA, including both graviton and gravitino contributions. [This is an essential point, since the \((E,Q)\) seemingly vanish by the respective constraint equations, and a delicate analysis is required.] Furthermore, because the relevant Hilbert space (after gauge fixing) has positive metric, \( E \) can vanish if and only if \( Q \) vanishes on all states; but \( Q| \geq 0 \) in turn implies that there are no excitations: spacetime is then flat and there are no gravitinos. This result even remains valid in presence of supermatter (and hence necessarily of positive energy) sources, since \((E,Q)\) remain the total generators, and obey the same algebra independent of any sources. Those details are simply hidden in the constraints. Likewise, for \( N > 1 \) SUGRAs, the \( E-Q \) relation still holds as a sum over the various charges, \( E = \sum_{i=1}^{N} Q_i^2 \geq 0 \).

The positive energy proof was perhaps the first formal benefit that SUGRA brought to the gravity world, but at first it seemed both too much and too little. The “too much” was that it was a quantum result; what was its implication to the “smaller” world of classical and non-SUSY gravity, with sources such as stars or photons? The “too little” was that formal results of SUGRA could not be trusted, since its likely nonrenormalizability (see Sec. 3) would render all quantities infinite and the relations between them suspect. Given that the magic \( E = Q^2 \) mantra yielded so much for so little effort compared to the frontal assaults, both unsuccessful and successful, in the classical theory, we cautiously claimed only the formal SUGRA victory in [7]. It was soon pointed out, however, that at this same formal level our proof remained valid in the classical and non-fermionic limit [8]: as \( \hbar \rightarrow 0 \), all internal, closed loop, contributions from either the graviton or from \( \psi_\mu \) disappear, while considering matrix elements of the \( E = Q^2 \) operator relation between purely bosonic states then reduces everything, including the result, to ordinary tree-level/classical GR.

The next step was to find a purely classical proof that utilized only the “smile” of SUGRA, i.e., the necessary positiveness of pure GR that underlies its supersymmetrizability. The “Dirac square” character of the classical GR constraint equations that define the four-momentum for asymptotically flat space was exhibited, in a celebrated work [9], in terms of a classical spinor...
parameter that is the sole (gauge function) remnant of SUGRA; this was refined and spelled out further in many subsequent papers (e.g., [10]). So the SUGRA argument really did teach us a lot about the physical constraints imposed on a classical theory, in this case GR, for it to be part of a SUSY one, thereby also providing yet another lesson in the liberating effects of (Grassmann) extending number systems. There have been many subsequent examples of SUSY constraints, from the restrictions on potentials allowed in normal SUSY models to the simple helicity-conservation rules dictated for graviton-graviton scattering in GR due to its SUGRA embeddability. [There is undoubtedly still progress to be made in this connection at higher dimensions, by finding the right D>4 helicity generalizations and thence their SUSY constraints.] Let me also mention that the currently important world of ADS gravity, with negative cosmological constant, also benefits from the above energy results. For, not only can SUGRA be extended to encompass anti-deSitter sign cosmological terms (except for the maximal, D=11, case [11]) but so can the notion of energy – or at least the nearest that the asymptotically ADS (rather than flat) boundary conditions allow there [12].

3. Nonrenormalizability of SUGRAs

My second, very different, topic has been studied almost as long as SUGRA itself; its renormalizability (or more precisely, finiteness; the dimensional nature of the gravitational constant precludes counterterms of the same derivative order as the original action). Although not discussed directly in [1], it was soon realized, in the realm of ordinary SUSY theories, that having equal numbers of fermionic and bosonic excitations leads to all sorts of marvellous cancellations of ultraviolet loop divergences, such as that of zero-point energy [3]. It was no wonder then, that one of the hopes following the discovery of D=4 SUGRA was that, despite describing gravity plus (spin 3/2) “matter” (which was already known to be one-loop divergent generically), it would be more convergent by virtue of its extended symmetries. Indeed, there was both one- and two-loop improvement: The one-loop counterterms, precisely as in pure gravity (rather than as in gravity coupled to “ordinary” matter), were “field-redefinable-away” arrays proportional to the field equations of supergravity, e.g., \( \Delta I_1 = \int d^4x (G_{\mu\nu} - \kappa^2 T_{\mu\nu}(\psi)) X^{\mu\nu} \), while the two-loop term (where GR first fails [13]) was absent altogether due to supersymmetry, there being no SUSY companion to \( \kappa^2 R_{\mu\nu\alpha\beta}^3 \). Alas, [14], three-loop invariants did exist both for the original N=1 and higher \( N \) models. Their structure follows the lines of the simpler global SUSY case, where the system’s stress tensor \( T_{\mu\nu} \), super-current \( J_{\mu} \) and chiral current \( C_{\mu} \) form a supermultiplet from which a quadratic SUSY invariant \( \sim \int d^4x \left[ T^{2}_{\mu\nu} - iJ_{\mu} \frac{\partial}{\partial x} J^{\nu} + \frac{3}{2} C_{\mu} \Box C^{\nu} \right] \) can be constructed. As we know, there is no tensorial stress-tensor for the gravitational field itself, the nearest, higher derivative, analog being the Bel–Robinson tensor \( B_{\mu\nu\alpha\beta} \) quadratic in the curvature. On-shell (\( R_{\mu\nu} = 0 \)) \( B \) is totally symmetric, (covariantly) conserved and traceless and there is again a quadratic invariant \( \sim \int d^4x [B^2 - iJ \frac{\partial}{\partial x} J + \frac{3}{2} C \Box C] \) constructed from \( B \) and corresponding (also higher derivative) super- and chiral currents. Surprisingly, it was even possible to learn something about the actual coefficients of such counterterms: For the non-maximal \( 1 \leq N < 8 \) models, where covariant superspace quantization is possible, it was concluded [15] that they are uniformly non-vanishing. For the special maximal \( N=8 \) case, very beautiful recent work [16] suggests that, while the three-loop coefficient may vanish, its 5-loop coefficient probably does not. Similarly, for dimensions, \( 4 < D < 11 \), it is possible to construct SUSY counterterms explicitly where a suitable superspace formulation exists and the construction and negative conclusions of [16] apply.

This left only D=11 SUGRA [17] as the remaining candidate to which the earlier arguments did not directly apply. After a long period of neglect because it fell beyond string the-
ory’s D=10 regime, it has revived in the wake of its M-theoretical connections. It represents the highest-dimensional SUSY model allowed by kinematic consistency (a single graviton, no higher spins etc.). It is unique in many other respects, including being only N=1, i.e., simultaneously “minimal/maximal” and matter-free because there is no lower spin SUSY system at D=11. Unfortunately, it also lacks any formalism that would enable one to test the SUSY of, let alone construct, candidate invariant counterterms. It is this construction and its consequences that I now describe; details are in [18]. The celebrated D=11 SUGRA action has, in its bosonic sector, in addition to the graviton, a 3-form potential $A_{\mu\nu\rho}$ with associated field strength $F_{\mu\nu\rho\beta\gamma} \equiv \partial_{[\beta}A_{\mu\nu\rho\gamma]}$. It contains besides their kinetic terms, a cubic metric-independent Chern–Simons (CS) term in which the 11 indices of the epsilon symbol are saturated by two $F$’s and one $A$. It is clear, because dimension is odd (and $\kappa^2 \sim L^{+9}$ since the Einstein action here is $I_E = \kappa^{-2} \int d^{11}\sqrt{-g} R$), that no 1-loop $\sim \kappa^0 \int d^{11}x$ candidate $\Delta I_1$ exist – one cannot make gravitational scalars with odd numbers of derivatives. At 2-loop, lowest possible, order then, $\Delta I_2 \sim \kappa^2 \int d^{11}x \Delta L_{20}$ where $\Delta L_{20}$ has dimension 20. The dimensional regularization we use here uniformly has logarithmic cutoff so all $\Delta L$ are $\Delta L_{20}$’s. Now since a curvature is dimensionally equivalent to two covariant derivatives $D_{\mu
u}$, candidate terms are schematically of the form $\Delta L_{20} \sim \sum_{n=4}^{10} R^n D^{(10-n)}$. The simplest and first interesting level is $n=4$ because clearly the lower $n$’s are either (like $R^3$) not parts of super-invariants or are leading order trivial (like $R^2$ which can be removed by field redefinitions). Thus, our lowest possible choice is $\Delta I_2 \sim \kappa^2 \int d^{11}x [R^4D^{12} + \ldots]$ where the ellipsis represents the SUSY completion, if any. How does one construct a suitable $\Delta I_2$ in absence of any guiding super-calculus? Our procedure was the following. As was also recognized in [16], there is certainly one on-shell nonvanishing lowest order SUSY invariant that starts out quartic in $h_{\mu\nu}$: the tree-level 4-point scattering amplitude $M$ generated by the D=11 action itself. It has the enormous advantage that, since there are no loops, and SUSY transformations are linear at this level, the purely bosonic terms are guaranteed to be part of the overall SUSY invariant that is the total 4-point amplitude. However, its usefulness in providing a counterterm faces two immediate obstacles: We want a local invariant and indeed one with twelve explicit derivatives, whereas the amplitude has a denominator, from virtual particle exchanges; can we extract such a local but still SUSY residue $L$ from $M$? The answer is yes, because each term in the amplitude is in fact proportional to the product $(1/stu)$ of the Mandelstam variables and we can insert the additional $D^{12}$, without losing SUSY or having everything vanish on-shell, by further multiplication with $(stu)^2$ or $(s^6 + t^6 + u^6)$, after the initial $stu$ one.

After expanding the Einstein plus form field actions to lowest relevant order in $h_{\mu\nu}$, one computes the various 4-part amplitudes. These consist of four different types of scatterings: a) graviton-graviton ($\sim R^4$), b) graviton-form ($\sim R^2F^2$), c) form-form ($\sim F^4$) and d) form-graviton bremsstrahlung ($\sim F^3R$). It is a straightforward, if index-intensive, procedure to perform the explicit calculations. The worst, graviton-graviton scattering, is fortunately already known [19] and provides a very useful check. It is proportional to the famous lowest string correction to the Einstein action in D=10: $L_g = stu M_g \sim t_s t_{s_R} RRR$ where $t_s$ is a constant 8-index tensor. The three remaining, form-dependent, types of amplitudes can also be calculated. [The resulting total bosonic component of the full SUSY invariant is interesting in its own right as the correction to the D=11 SUGRA action from M-theory, of which we mostly know that it contains SUGRA as the local limit.] Let me summarize the various “localized” on-shell 4-point amplitudes a–d). Schematically (see [18] for details), with $B$ a Bel–Robinson-like curvature quadratic and $F$ always appearing with a gradient, $\Delta I_2^B(g,F) = \kappa^2 \int d^{11}x [B^2 + (\partial F)^4 + B(\partial F)^2 + R(\partial F)^3]$, where we can actually write each part as squares of (themselves quadratic) currents. These terms are at this stage only accurate to lowest, fourth, order in the combined $R$’s and $F$’s, and of course are to
be supplemented by fermion-dependent terms that we do not write down. Nevertheless linearized 
SUSY is guaranteed by our construction, and the existence of a full on-shell invariant (however 
horrible) is also secure.

Of course, a counterterm is only dangerous if its coefficient is non-zero. A powerful tool 
for answering this question was provided by the amazing correspondence between SUGRA and 
Super-Yang–Mills models established and exploited in [16] to obtain many otherwise “impossible” 
results, such as mentioned above for N=8 D=4 SUGRA. While SYM is only defined for D\leq 10 one 
can argue, quite convincingly, that the results as provided really do not depend directly on D, and 
extend analytically also to D=11. This extension may also be possible to verify intrinsically; thus, 
already at leading 2-loop level, the strong odds are against finiteness of even this maximal theory.

Summary

In this celebration of the birth of D=4 SUSY, we have discussed some consequences of the 
supersymmetry algebra and of its invariants within the local, SUGRA, extension. The core of SUSY 
grading is $E = Q^2 \geq 0$ and requires candidate SUSY models to have positive energy, up to some 
(noted) fine print. This seemingly banal requirement on supersymmetrizable QFT had an enormous 
impact on SUGRA and thence on pure GR because the supersymmetrizability-Dirac square root 
relation yielded an immediate physical understanding of positive gravitational (as well as SUGRA) 
energy. The second effect of SUSY, to soften ultraviolet divergences by producing cancellations in 
fermion/fermion loop contributions, could not, however, overcome the dimensionality of the Einstein 
constant even in “maximal” D=11 SUGRA: There exist invariants that can provide counterterms 
and their coefficients do not vanish. This negative result points to the need for string-like (but 
still SUSY) nonlocality to provide a finite description of gravity; a fitting conclusion, given the still 
more elemental, D=2, source of supersymmetry!

Acknowledgements: This work was supported by the National Science Foundation under grant 
PHY99-73935. It is a pleasure to have collaborated respectively with C. Teitelboim and D. Seminara 
on the joint work reported here.

References


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