A New Formulation of a 1 + 1 Dimensional Field Theory Constrained to a Box

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Abstract

We consider a 1 + 1 dimensional field theory constrained to a finite box of length $L$. Traditionally, calculations in a box are done by replacing the integrals over the spatial momenta by discrete sums and then evaluating sums and doing analytic continuations. We show that it is also possible to do such calculations using an analogy to finite temperature field theory. We develop a formalism that is similar to the closed time path formulation of finite temperature field theory. Our technique can be used to calculate spatially retarded green functions, without evaluating sums or doing analytic continuations. We calculate the self energy in a simple scalar theory as an example.

I. INTRODUCTION

We consider a 1 + 1 dimensional field theory constrained to a finite box in the spatial dimension. We show that this system can be studied in two different ways. We can impose the boundary condition in the usual way by restricting the momentum variable to take discrete values $k = 2\pi n/L$ and replacing the momentum integral by a sum over the discrete index. We will refer to this type of calculation as a ‘box’ calculation. We will show that we can formulate this theory in a simpler way, based on an analogy with finite temperature field theory. This approach will be called the ‘CSP’ [Closed Space Path] formalism (the notation will be explained below). We study the self energy function for a simple scalar theory and compare the results of the box calculation and the CSP calculation.

The motivation is as follows. There are physical systems in which the physics is essentially 1 + 1 dimensional. One example is a micromaser, where one is interested in the situation where a beam of excited atoms enters a box, interacts with a radiation field which is trapped inside the box, and emerges from the other end. The standard non-relativistic description of this system is the Jaynes-Cummings model which describes a reservoir of two level atoms in interaction with a single mode oscillator representing a coherent electromagnetic field [1]. It has been shown that this simple non-relativistic model can be obtained as the non-relativistic limit of a relativistic quantum field theory [2]. The micromaser is studied experimentally by measuring the probability that an atom will emerge in the excited state, given that it entered
in the excited state. It is known that this probability exhibits oscillations, which are known as Rabi oscillations. Non-relativistically, it is straightforward to obtain these oscillations from a calculation of transition amplitudes using the Jaynes-Cummings Hamiltonian. To obtain the relativistic generalization of this result, we need to calculate the imaginary part of the atomic self energy. Measurements of the final atomic state are performed at a specific point in space, and thus it is the spatially retarded self energy that is relevant, and not the time retarded one. One can obtain this spatially retarded self energy in a straightforward way using the CSP formalism. The CSP formalism is constructed in analogy to the closed time path formalism (CTP) of finite temperature field theory [3–5] and has the same advantages with respect to the box calculation as the CTP formalism has over the imaginary time formalism of finite temperature field theory [6–8]. The CSP formalism allows us to obtain spatially retarded green functions without explicitly worrying about the difficulties associated with analytic continuation.

It is straightforward to see the similarity between a 1 + 1 dimensional field theory constrained to a box and finite temperature field theory. Finite temperature field theory involves expectation values of operators weighted with Boltzmann factors,

$$\langle O \rangle = \sum_n \langle \phi_n | e^{-\beta H} O | \phi_n \rangle$$

The Boltzmann factor is equivalent to an imaginary time evolution operator,

$$\langle O \rangle = \sum_n \langle \phi_n(x); t - i\beta | O | \phi_n(x); t \rangle$$

If the system obeys periodic boundary conditions such that

$$\phi_n(t - i\beta) = \phi_n(t)$$

then we have

$$\langle O \rangle = \Sigma_n \langle \phi_n(t) | O | \phi_n(t) \rangle$$

Thus, imposing a periodic boundary condition (in imaginary time) is equivalent to taking expectation values that are weighted with a distribution function $e^{-\beta H}$, which means that finite temperature field theory can be formulated as a field theory constrained to a box in imaginary time. This constraint is imposed at the level of the action by using momentum expansions of the fields with the Fourier integral over the zeroth momentum component replaced by a sum over the discrete values $k_0 = 2\pi nT$. The generating functional leads to Feynman rules which are the same as the zero temperature rules except that the zero momentum components take discrete values, and the part of the loop integral that corresponds to integrals over the zeroth momentum components are replaced by summations. In the evaluation of Feynman integrals, these sums are replaced by contour integrals that contain distribution functions which have poles at the same discrete values $k_0 = 2\pi nT$. These contour integrals can be rotated so that the integration contour encloses the real axis, running from negative infinity to positive infinity above the real axis, and backwards from positive infinity to negative infinity below the real axis. This contour is called the closed time path contour.
We want to consider a 1 + 1 dimensional field theory constrained to a box in real space, instead of a box in imaginary time. To achieve this, we will define a field theory on a contour that encloses the real \( x \)-coordinate axis. This closed space path (CSP) formalism will be mathematically very similar to the CTP formalism of finite temperature field theory. Consider what is involved in reversing the roles of space and time. Relativistically, time and coordinate variables play similar roles. The usual formulation of a quantum field theory uses the fact that the system contains both positive and negative energy states which move forward and backward in time respectively. Propagators are constructed as symmetric linear combinations of these states. The conventional choice of energy eigenstates as basis states gives a physical non-relativistic limit. In the non-relativistic limit, the positive and negative eigenstates are separated, and the non-relativistic theory contains energy states of one sign only. This restriction corresponds to the fact that in the non-relativistic world, time runs only in the forward direction, and physical states are positive energy states. Relativistically, the special nature of time emerges only in statistical situations. When a system consists of many degrees of freedom which form a heat bath, then the system will move towards equilibrium, and the forward time direction is singled out as the physical direction.

Now consider our situation. The CSP formalism is mathematically similar to the CTP formalism with the roles of space and time reversed. The rigorous procedure for constructing the CSP formalism is presented in detail in Section 3. Propagators are constructed as symmetric combinations of positive and negative momentum eigenstates, moving forward and backward in coordinate space. The imaginary time box of finite temperature field theory is replaced by a box in real space, and the parameter \( \beta = 1/T \) which characterizes the length of the box in imaginary time is replaced with a parameter \( iL \) where \( L \) is the length of the box in coordinate space. The Boltzmann factor \( e^{-\beta p_0} \) which gives the exponential decay that leads to equilibration, is replaced by a factor \( e^{-iL p} \) which gives rise to oscillatory behaviour. Thus, in contrast to the finite temperature case, the system maintains its ability to evolve in both spatial directions, even after statistical effects have been included.

This paper is organized as follows. We work with a simple scalar theory with a Lagrangian of the form,

\[
\mathcal{L} = \frac{1}{2}[(\partial_t \phi_1)^2 - (\partial_x \phi_1)^2 - m_1^2] + \frac{1}{2}[(\partial_t \phi_2)^2 - (\partial_x \phi_2)^2 - m_2^2] - \frac{\lambda}{3!}(\phi_1 + \phi_2)^3
\]

In section 2 we discuss standard field theory in 1 + 1 dimensions at finite temperature. In section 3 we derive the CSP formulation of field theory in a box in 1 + 1 dimensions, working in analogy with the standard finite temperature formulation outlined in section 2. We calculate the self energy using this CSP formalism. In section 4 we calculate the self energy using the box technique. In section 5 we compare the results of the two calculations, and discuss our conclusions.

**II. FIELD THEORY IN 1 + 1 DIMENSIONS**

In the next two sections we consider a non-interacting scalar field of mass \( m \).
A. Quantization

We start from the Klein–Gordon equation of motion,

\[
(\partial_t^2 - \partial_x^2 + m^2)\phi = 0
\]  \hspace{1cm} (2)

The solution of the Klein–Gordon equation has the form

\[
\phi \sim e^{i(\omega_k t \pm kx)}; \quad \omega_k = \sqrt{k^2 + m^2}
\]

and describes the propagation of a plane wave of energy \(\omega\). This equation of motion should be obtainable from a least action principle. If the action is defined in terms of a Lagrange density as,

\[
I = \int L dt
\]

then the equation of motion is determined from

\[
\delta I = 0 \rightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \phi} - \frac{\partial L}{\partial \phi} = 0
\]

which allows us to identify,

\[
L = \frac{1}{2} [(\partial_t \phi)^2 - (\partial_x \phi)^2 - m^2 \phi^2]
\]  \hspace{1cm} (4)

To quantize the system we go to the Hamiltonian formalism. We define canonical momenta

\[
\Pi = \frac{\partial L}{\partial (\partial_t \phi)} = \partial_t \phi
\]  \hspace{1cm} (3)

and take the Legendre transform,

\[
H = \Pi \partial_x \phi - L
\]

\[
= \frac{1}{2} [(\partial_t \phi)^2 + \Pi^2 + m^2 \phi^2]
\]  \hspace{1cm} (4)

Obtaining Hamiltonian’s equations in the usual way we get,

\[
\partial_t \phi = \frac{\partial H}{\partial \Pi}; \quad \partial_t \Pi = -\frac{\partial H}{\partial \phi}
\]

We can write these results as Poisson Brackets:

\[
\partial_t \phi = \{\phi, H\}; \quad \partial_t \Pi = \{\Pi, H\}
\]

To quantize we supplement the initial conditions by the equal time commutation relations (ETCRs)

\[
[\phi(x, t), \Pi(x', t)] = i\delta(x - x')
\]  \hspace{1cm} (5)

and replace the Poisson brackets with commutation relations,

\[
i\partial_t \phi = [\phi, H]; \quad i\partial_t \Pi = [\Pi, H]; \quad H(t) = \int dx H(x, t)
\]
B. Symmetries

The energy momentum tensor,

\[
\Theta_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial^\nu \phi)} \partial_\mu \phi - g_{\mu\nu} \mathcal{L}
\]  

has for its zero-zero component:

\[
\Theta_{00} = \mathcal{H}
\]

From the conservation equation \(\partial_\mu \Theta_{\mu\nu} = 0\) we obtain,

\[
\frac{d}{dt} \int dx \mathcal{H} = \frac{d}{dt} H = 0
\]

which means that the eigenvalues of \(H\) are conserved quantities, and the eigenstates of \(H\) are stationary states. Note that (6) has solution

\[
\phi(x, t) = e^{iHt} \phi(x, 0)e^{-iHt}; \quad \Pi(x, t) = e^{iHt}\Pi(x, 0)e^{-iHt},
\]

which identifies \(H\) as the generator of translations in time.

C. Propagators

Next we consider momentum-space expansions of the field variables. We use the stationary eigenstates of the Hamiltonian as basis states and expand around solutions of the Klein–Gordon equation (2) by writing,

\[
\phi(x, t) = \int \frac{dk_0}{2\pi} \frac{2}{2\omega_k} \left[ a(k)e^{-i(\omega_k t - kx)} + a^\dagger(k)e^{i(\omega_k t - kx)} \right]
\]  

where \(\omega_k = \sqrt{k^2 + m^2}\). The creation and annihilation operators satisfy the commutation relation,

\[
[a(k), a^\dagger(p)] = 2\pi(2\omega_k)\delta(k - p)
\]

as a consequence of (3) and (5). The interpretation of this expression is as follows. The frequency \(\omega_k\) is always positive. The operator \(a^\dagger(k)\) creates a mode of energy \(\omega_k\) which propagates forward in time, and the operator \(a(k)\) destroys a mode of energy \(\omega_k\), which is equivalent to creating a mode with negative energy that propagates backwards in time (an anti-particle). Thus, by expanding in this set of creation and annihilation operators we use a basis set that separates positive and negative energy eigenstates. This basis choice allows us to define time causal propagators,

\[
iD_R(x, t; y, t') = \theta(t - t')[\phi(x, t), \phi(y, t')]
\]

\[
D_R(K) = \frac{1}{(k_0 + i\epsilon)^2 - k^2 - m^2} = \frac{1}{(k_0 + i\epsilon)^2 - \omega_k^2}
\]

where translation invariance has been used in taking the Fourier transform.
D. Finite Temperature Field Theory

Finite temperature field theory involves expectation values of operators weighted with Boltzmann factors,

\[ \langle O \rangle = \sum_n \langle \phi_n | e^{-\beta H} O | \phi_n \rangle \]

The Boltzmann factor is equivalent to an imaginary time evolution operator which gives,

\[ \langle O \rangle = \sum_n \langle \phi_n(x); t - i\beta | O | \phi_n(x); t \rangle \]

We impose periodic boundary conditions in imaginary time,

\[ \phi_n(t - i\beta) = \phi_n(t) \]

This constraint leads to Feynman rules which are the same as the zero temperature rules except that the zero momentum components take discrete values \( k_0 = 2\pi n T \), and the part of the loop integral that corresponds to integrals over the zeroth momentum components are replaced by summations. In the evaluation of Feynman integrals, these sums are replaced by contour integrals that contain distribution functions which have poles at the same discrete values \( k_0 = 2\pi n T \). These contour integrals can be rotated so that the integration contour encloses the real axis, running from negative infinity to positive infinity above the real axis (the branch \( C_1 \)), and backwards from positive infinity to negative infinity below the real axis (the branch \( C_2 \)). This contour is called the closed time path (CTP) contour. The propagator can be written as a 2 × 2 matrix of the form,

\[ G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \] (10)

where \( G_{11} \) is the propagator for fields moving along \( C_1 \), \( G_{12} \) is the propagator for fields moving from \( C_1 \) to \( C_2 \), etc. The four components are given by,

\[ G_{11}(X - Y) = -i\langle T\phi(X)\phi(Y) \rangle \]
\[ G_{12}(X - Y) = -i\langle \phi(Y)\phi(X) \rangle \]
\[ G_{21}(X - Y) = -i\langle \phi(X)\phi(Y) \rangle \]
\[ G_{22}(X - Y) = -i\langle \tilde{T}\phi(X)\phi(Y) \rangle \] (11)

where \( T \) is the usual time ordering operator and \( \tilde{T} \) is the anti-chronological time ordering operator. These four components are related by the condition,

\[ G_{11} + G_{12} + G_{21} + G_{22} = 0 \] (12)

and therefore only three are independent. We define the “physical” combinations,

\[ G_R = G_{11} - G_{12} = -i\theta(t - t')\langle [\phi(x,t), \phi(y,t')] \rangle \]
\[ G_A = G_{11} + G_{21} = i\theta(t' - t)\langle [\phi(x,t), \phi(y,t')] \rangle \]
\[ G_F = G_{11} + G_{22} = -i\langle \{\phi(x,t), \phi(y,t') \} \rangle \]. (13)
In momentum space the cyclicity of the trace allows us to obtain the relationship between these propagators known as the KMS condition,

\[ G_F(P) = N(p_0)(G_R(P) - G_A(P)) \]  

(14)

\[ N(p_0) = 1 + 2n(p_0); \quad n(p_0) = \frac{1}{e^{\beta p_0} - 1}. \]

III. THE CLOSED SPACE PATH

A. Formulation of the theory

We start with a Klein–Gordon equation of the form,

\[ (\partial_t^2 - \partial_x^2 - m^2)\phi(x, t) = 0 \]  

(15)

which has solutions,

\[ \phi \sim e^{i(k_0t\pm\omega_{k_0}x)}; \quad \omega_{k_0} = \sqrt{k_0^2 + m^2} \]

that describe the propagation of a plane wave with momentum \( \omega_{k_0} \). We should be able to obtain the equation of motion from a least action principle of the form,

\[ \delta \tilde{I} = 0 \rightarrow \frac{\partial}{\partial t} \frac{\partial \tilde{L}}{\partial (\partial_t \phi)} - \frac{\partial \tilde{L}}{\partial \phi} = 0 \]

The Lagrangian that produces the correct equation of motion is,

\[ \tilde{L} = \frac{1}{2}[\partial_x \phi]^2 - (\partial_t \phi)^2 - m^2 \phi^2] \]  

(16)

To obtain the Hamiltonian we define canonical momenta

\[ \tilde{\Pi} = \frac{\partial L}{\partial (\partial_x \phi)} = \partial_x \phi \]

and take the Legendre transform,

\[ \tilde{H} = \tilde{\Pi} \partial_x \phi - \mathcal{L} = \frac{1}{2}[(\partial_\phi \phi)^2 + (\tilde{\Pi})^2 + m^2] \]  

(17)

Obtaining Hamiltonian’s equations in the usual way we get,

\[ \partial_x \phi = \frac{\partial \tilde{H}}{\partial \tilde{\Pi}}; \quad -\partial_x \tilde{\Pi} = \frac{\partial \tilde{H}}{\partial \phi} \]

We can write these results as Poisson Brackets:

\[ \partial_x \phi = \{ \phi, \tilde{H} \}; \quad \partial_x \tilde{\Pi} = \{ \tilde{\Pi}, \tilde{H} \} \]

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To quantize we supplement the initial conditions with the equal place commutation relations (EPCRs)

\[ [\phi(x, t), \Pi(x, t')] = i\delta(t - t') \]  

(18)

and replace the Poisson brackets with commutation relations,

\[ i\partial_x \phi = [\phi, \tilde{H}]; \quad i\partial_x \Pi = [\Pi, \tilde{H}]; \quad \tilde{H}(x) = \int dt \tilde{H}(x, t). \]  

(19)

The energy momentum tensor is obtained from (6) as before. The 1–1 component is \( \Theta_{11} = -\tilde{H} \) and the constraint \( \partial^\mu \theta_{\mu\nu} = 0 \) leads to

\[ \frac{d}{dx} \int dt \tilde{H} = 0 \]

which means that the eigenvalues of \( \tilde{H} \) are conserved and that the eigenstates are stationary states. Note that (19) has solutions,

\[ \phi(x) = e^{i\tilde{H}x} \phi(0) e^{-i\tilde{H}x}; \quad \Pi(x) = e^{i\tilde{H}x} \Pi(0) e^{-i\tilde{H}x} \]

which identifies \( \tilde{H} \) as the generator of translations in space.

To obtain an expression for the propagators, we use a basis set of the stationary eigenstates of the Hamiltonian \( \tilde{H} \) and expand around solutions of the equation of motion (15) by writing,

\[ \phi(x, t) = \int \frac{dk_0}{2\pi(2\tilde{\omega}k_0)} [a(k_0)e^{-i(\tilde{\omega}k_0x - k_0t)} + a^\dagger(k_0)e^{i(\tilde{\omega}k_0x - k_0t)}]. \]  

(20)

The creation and annihilation operators satisfy the commutation relation,

\[ [a(k_0), a^\dagger(p_0)] = 2\pi(2\tilde{\omega}k_0)\delta(k_0 - p_0) \]  

(21)

The interpretation of this expansion is as follows. The operator \( a^\dagger(k) \) creates a mode of momentum \( \tilde{\omega}k_0 \) which propagates forward in space, and the operator \( a(k) \) destroys a mode of momentum \( \tilde{\omega}k_0 \), which is equivalent to creating a mode with negative momentum that propagates backwards in space. By expanding in momentum eigenstates, instead of energy eigenstates, we have obtained a formalism in which it is simple to obtain propagators that are retarded in space.

The space-retarded propagator is defined as,

\[ iD_R(x; y, t') = D_{11} - D_{12} = \theta(x - y)[\phi(x, t), \phi(y, t')] \]

Substituting in the expressions for the momentum expansions of the fields (20) and using the EPCR’s (21) we obtain, in momentum space,

\[ D_R(K) = \frac{1}{k_0^2 - (k + i\epsilon)^2 + m^2} = \frac{1}{(k + i\epsilon)^2 - \tilde{\omega}^2k_0^2} \]  

(22)

which can be compared to the result for the time-retarded propagator (9).
We can write the propagator in matrix form on the CSP contour. The result is analogous to the CTP form (10),

\[ D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \]

with (compare (11)),

\[
D_{11}(X - Y) = -i\langle S\phi(X)\phi(Y) \rangle \\
D_{12}(X - Y) = -i\langle \phi(Y)\phi(X) \rangle \\
D_{21}(X - Y) = -i\langle \phi(X)\phi(Y) \rangle \\
D_{22}(X - Y) = -i\langle \tilde{S}\phi(X)\phi(Y) \rangle.
\]

The time ordering operator is replaced with a space ordering operator \( S \), and \( \tilde{S} \) is the reverse space ordering operator. The four components are related by the condition (compare (12)),

\[ D_{11} + D_{12} + D_{21} + D_{22} = 0 \]

We define the “physical” combinations (compare (13)),

\[
D_R = D_{11} - D_{12} = -i\theta(x - x')\langle [\phi(x, t), \phi(y, t')] \rangle \\
D_A = D_{11} - D_{21} = i\theta(x' - x)\langle [\phi(x, t), \phi(y, t')] \rangle \\
D_F = D_{11} + D_{22} = -i\langle \{\phi(x, t), \phi(y, t')\} \rangle.
\]

The KMS-like condition is obtained by analogy with (14),

\[
D_F(P) = N(p)(D_R(P) - D_A(P)) \\
N(p) = \cot\left(\frac{1}{2}Lp\right) = i(1 + 2n(p)); \quad n(p) = \frac{1}{e^{iLp} - 1}
\]

B. Calculation of the imaginary part of the retarded self energy

We consider the contribution to the self energy of the field \( \phi_1 \) arising from a loop with one \( \phi_1 \) propagator and one \( \phi_2 \) propagator on internal lines. Other contributions to the \( \phi_1 \) self energy can be obtained in a straightforward way by setting \( m_1 = m_2 \). The spatially retarded self energy is given by,

\[
\Sigma_R(P) = \Sigma(P)_{11} + \Sigma(P)_{12} \\
= i\lambda^2 \int dK(D_{11}(K)D_{11}(K + P) - D_{21}(K)D_{12}(K + P)) \\
= i\frac{\lambda^2}{2} \int dK(D_F(K)D_R(P + K) + D_A(K)D_F(P + K))
\]
\[ D_F(K) = N(k)2i\text{Im}D_R(K) = -2i\pi N(k)\epsilon(k)\delta(k^2 - \tilde{\omega}_k^2) \]

Using the delta functions to do the \( k \) integrals we obtain,
\[
\Sigma_R(P) = \frac{\lambda^2}{4\pi} \mathcal{E}_{m_1,m_2} \int \frac{dk_0}{\tilde{\omega}_k} N(\tilde{\omega}_k) \left( \frac{1}{(p + \tilde{\omega}_k + i\epsilon)^2 - (\tilde{\omega}'_{k_0+q_0})^2} + \frac{1}{(p - \tilde{\omega}_k + i\epsilon)^2 - (\tilde{\omega}'_{k_0+q_0})^2} \right)
\]
where \( \mathcal{E}_{m_1,m_2} f(m_1, m_2) = f(m_1, m_2) + f(m_2, m_1) \). To get the imaginary part, we take the imaginary part of the two fractions using
\[
\text{Im} \left( \frac{1}{(\alpha + i\epsilon)^2 - \beta^2} \right) = -i\pi \epsilon(\alpha^2 - \beta^2)
\]
The two terms can be combined using the symmetrization factor \( \mathcal{O}_p f(p) = f(p) - f(-p) \). The result is,
\[
i\text{Im}\Sigma_R(P) = -i\frac{\lambda^2}{8} \mathcal{E}_{m_1,m_2} \mathcal{O}_p \int dk_0 N(\tilde{\omega}_k) \frac{1}{\tilde{\omega}_k} \epsilon(p + \tilde{\omega}_k)\delta[(\tilde{\omega}_k + p)^2 - (\tilde{\omega}'_{k_0+p_0})^2] \quad (29)
\]
We define,
\[
x = P^2 - \delta m^2 + 2p_0k_0 \quad (30)
\]
\[
\alpha = 2p\tilde{\omega}_k
\]
which allows us to write,
\[
\delta[(\tilde{\omega}_k + p)^2 - (\tilde{\omega}'_{k_0+p_0})^2] = \delta(x - \alpha). \quad (31)
\]
We rewrite the delta function using,
\[
\delta(x - \alpha) = 2\alpha [\theta(x)\theta(\alpha) - \theta(-x)\theta(-\alpha)]\delta(x^2 - \alpha^2)
\]
We obtain,
\[
x^2 - \alpha^2 = 4P^2(k_0 - k_{01})(k_0 - k_{02})
\]
\[
k_{01,2} = \frac{1}{2P^2} (-p_0(P^2 - \delta m^2) \pm p\sqrt{R})
\]
\[
R = (P^2 + (\Sigma m)^2)(P^2 + (\delta m)^2)
\]
\[
\Sigma m = m_1 + m_2; \quad \delta m = m_1 - m_2; \quad P^2 = p_0^2 - p^2.
\]
from which we derive,
\[
\delta(x - \alpha) = \frac{1}{\sqrt{R}}\tilde{\omega}_k \theta(xp)(\delta(k_0 - k_{01}) + \delta(k_0 - k_{02}))
\]
Substituting into (29) we obtain,
\[
i\text{Im}\Sigma_R(p_0, p) = i\frac{\lambda^2}{8\sqrt{R}} \mathcal{E}_{m_1,m_2} \mathcal{O}_p \int dk_0 N(\tilde{\omega}_k) \epsilon(p + \tilde{\omega}_k)\theta(xp)(\delta(k_0 - k_{01}) + \delta(k_0 - k_{02})) \quad (33)
\]
We define,

\[ a_{1,2} = \frac{1}{p^2}(p_0\delta m^2 \pm p\sqrt{R}) \]  

which allows us to write,

\[ k_{01,2} = \frac{1}{2}(-p_0 + a_{1,2}) \]

\[ \frac{x(k_{01,2})}{p} = \frac{x_{1,2}}{p} = -p + a_{1,2} \]

\[ \omega_{k0,1,2} = \frac{1}{2p^2}|px_{1,2}| \]

\[ \epsilon(p + \omega_{1,2}) = \epsilon(p + a_{1,2}) \]

Substituting these results into (33) we obtain,

\[ \text{Im} \Sigma_R(p_0, p) = -i \frac{\lambda^2}{8\sqrt{R}} \mathcal{E}_{m_1,m_2} O_p \]

\[ [N\left(\frac{1}{p}(-p + a_1)\right)\theta(-p + a_1)\epsilon(p + a_1) + N\left(\frac{1}{p}(-p + a_2)\right)\theta(-p + a_2)\epsilon(p + a_2)] \]  

To simplify this result we expand out the symmetrization factor \( O_p \) using,

\[ p \rightarrow -p \quad \Rightarrow \quad a_{1,2} \rightarrow -a_{2,1} \]

The result is,

\[ i\text{Im} \Sigma_R(p_0, p) = -i \frac{\lambda^2}{8\sqrt{R}} \mathcal{E}_{m_1,m_2}[N(p - a_1)\epsilon(p^2 - a_1^2) + N(p - a_2)\epsilon(p^2 - a_2^2)] \]  

**IV. THE BOX CALCULATION**

In this section we calculate the same contribution to the self energy using the standard Feynman rules of zero temperature field theory, and explicitly constrain the system to a box of length \( L \). The self energy is given by,

\[ \Sigma(P) = i\lambda^2 \int dK \frac{1}{(K^2 - m_1^2 + i\epsilon)} \frac{1}{((K + P)^2 - m_2^2 + i\epsilon)} \]  

We restrict the variable \( k \) to values \( k = 2\pi n/L \) where \( L \) is the length of the box, and \( n \) is an integer. The integral becomes,

\[ \Sigma(p_0, p_m) = i\lambda^2 \int dk_0 \frac{1}{L} \sum_n \left( \frac{1}{(k_0^2 - k_n^2 - m_1^2 + i\epsilon)} \left( \frac{1}{(k_0 + p_0)^2 - (k_n + p_m)^2 - m_2^2 + i\epsilon} \right) \right) \]  

We can replace the sum over the discrete index by a contour integration:  

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\[
\sum_n f(p = \frac{2\pi n}{L}) = \frac{1}{2\pi i} \int dp \cot(\frac{1}{2}LP) f(p)
\]

where the contour is a clockwise loop that encloses the poles of the cotangent function which lie along the real axis. This gives,

\[
\Sigma(p_0, p_m) = -i \frac{\lambda^2}{2} \int \frac{dk_0}{2\pi} \int \frac{dk}{2\pi i} \cot(\frac{1}{2}Lk) \left( \frac{1}{k_0^2 - k^2 - m_1^2 + i\epsilon} \right) \left( \frac{1}{(k_0 + p_0)^2 - (k + p_m)^2 - m_2^2 + i\epsilon} \right)
\]

Using (27) we rewrite the cotangent function,

\[
\cot(\frac{1}{2}LP) = i(1 + 2n(p)) = N(p)
\]

We deform the contour to two semi-circles in the upper and lower half planes. We do the \(k\) integral first, picking up contributions from four poles. These four terms can be combined using the factors \(E_p\) and \(E_{m_1, m_2}\) where, \(E_p f(p) = f(p) + f(-p)\). We use the notation \(\omega_{k_0} = \sqrt{k_0^2 - m_1^2}, \omega'_{k_0} = \sqrt{k_0^2 - m_2^2}\) etc. Note that these frequencies can be imaginary. When doing the contour integral, it doesn’t matter if the frequencies are real or imaginary, since both halves of the contour form counter clockwise loops, and thus pole contributions from both sides of the real axis have the same sign. The result is,

\[
\Sigma(p_0, p_m) = \frac{\lambda^2}{8\pi} E_{m_1, m_2} E_p \int dk_0 N(\omega_{k_0}) \frac{1}{\omega_{k_0}} \frac{1}{(\omega_{k_0} + p_m)^2 - \omega'^2_{k_0 + p_0} - i\epsilon}
\]

To find the imaginary part of the spatially retarded piece we follow the analogous procedure as that used for finite temperature field theory in the imaginary time formalism: we analytically continue \(p_m\) to the whole complex \(p\) plane and take the discontinuity,

\[
d\text{Im} \Sigma_R(p_0, p) = \frac{1}{2}(\Sigma(p + i\epsilon) - \Sigma(p - i\epsilon))
\]

\[= -i \frac{\lambda^2}{8} E_{m_1, m_2} \mathcal{O}_p \int dk_0 N(\omega_{k_0}) \frac{1}{\omega_{k_0}} \epsilon(p + \omega_{k_0}) \delta[(\omega_{k_0} + p)^2 - \omega'^2_{k_0 + p_0}]
\]

Comparison of (41) with (29) shows immediately that

\[
(\text{Im} \Sigma_R(m^2))^\text{Box} = (\text{Im} \Sigma_R(-m^2))^\text{CSP}
\]

Note that this result is what we would have expected from (15) in which the mass term has the opposite sign relative to (2).

V. CONCLUSIONS

The result for the imaginary part of the spatially retarded self energy contains a cotangent factor. This factor is a direct consequence of the fact that the system has been constrained to a box, and gives rise to the 1 + 1 dimensional analogue of the Rabi oscillations that have been calculated non-relativistically from the Jaynes-Cummings model. In our case the oscillations are given by a cotangent function, instead of a sine or cosine function, and
thus are discontinuous. It is expected that this behaviour is an artifact of working in $1 + 1$

dimensions.

We have shown that it is possible to obtain spatially retarded green functions for a
system confined to a box by using a formulation of field theory that is similar to the closed
time path formulation of finite temperature field theory. All of the powerful machinery of
the CTP method can be taken over when calculating in a one dimensional box by using the
CSP formalism derived in this paper. We have calculated the self energy in a simple scalar
theory as an example, and verified that the result is related in a straightforward way to the
result that would be obtained in the traditional ‘box’ calculation. The possible advantages
of our method are the same as those of the CTP method of finite temperature field theory.
In particular, it is typically easier to generalize to higher $n$-point functions using such an
approach. When calculating in the traditional manner, it is necessary to do an analytic
continuation from discrete momentum values to the full complex plane, and this analytic
continuation becomes extremely complicated for higher $n$-point functions.
REFERENCES