Picard-Fuchs Equations and Whitham Hierarchy in
\( N = 2 \) \( \text{Supersymmetric } SU(r + 1) \) Yang-Mills Theory

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Abstract

In general, Whitham dynamics involves infinitely many parameters called Whitham times, but in the context of \( N = 2 \) supersymmetric Yang-Mills theory it can be regarded as a finite system by restricting the number of Whitham times appropriately. For example, in the case of \( SU(r + 1) \) gauge theory without hypermultiplets, there are \( r \) Whitham times and they play an essential role in the theory. In this situation, the generating meromorphic 1-form of the Whitham hierarchy on Seiberg-Witten curve is represented by a finite linear combination of meromorphic 1-forms associated with these Whitham times, but it turns out that there are various differential relations among these differentials. Since these relations can be written only in terms of the Seiberg-Witten 1-form, their consistency conditions are found to give the Picard-Fuchs equations for the Seiberg-Witten periods.

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I. INTRODUCTION

Thanks to the study of electro-magnetic duality initiated by Seiberg and Witten, the prepotential of the low energy effective action of $N = 2$ supersymmetric Yang-Mills theory was turned out to be viewed as a function on a complex projective space having singularities when the masses of charged particles vanish. This complex projective space can be identified with the moduli space of a Riemann surface determined by several physical requirements, thus the effective theory can be considered to be controlled by the geometry of moduli space of a Riemann surface. According to this observation, since the effective coupling constants of the theory is interpreted as the period matrix of a Riemann surface, determining the period matrix from calculation of periods becomes equivalent to evaluate effective coupling constants. It is interesting that the instanton contributions to prepotential can be obtained from the evaluation of periods and the prepotentials obtained in this way are known to be consistent to the instanton calculus. In these studies, the method based on Picard-Fuchs equations played a crucial role.

However, on the one hand, the theory of prepotential often shows unexpected aspects behind the effective theory. For example, it is known that the Seiberg-Witten solutions can be understood in the framework of Whitham theory. Gorsky et al. noticed that the Whitham dynamics in $N = 2$ Yang-Mills theory could be written essentially by only finite number of Whitham times and found that the second-order derivatives of prepotential over the Whitham times could be represented by an elliptic function associated with Seiberg-Witten curve.

However, we can further learn more aspects of Whitham hierarchy in gauge theory from the basic idea of Gorsky et al. For instance, note that since the number of time variables of the hierarchy is restricted to be finite the generating meromorphic 1-form of the Whitham hierarchy is represented by a finite linear combination of meromorphic 1-forms associated with these Whitham times. Then we can expect that there must be closed differential relations among these meromorphic differentials associated with Whitham times. In fact, a detailed study supports this observation and the aim of the paper is to show the consequence of these relations, especially, a connection to Picard-Fuchs equations for the Seiberg-Witten periods.

The paper is organized as follows. In Sec. II, we briefly summarize the Whitham dynamics in SU($r + 1$) gauge theory. In addition, following to Gorsky et al., we consider the situation that
the number of Whitham times is finite. Since the meromorphic 1-forms on Seiberg-Witten curve consisting of the Whitham hierarchy must be always represented by simply a linear combination of Abelian differentials, we can expect the existence of differential relations among these meromorphic 1-forms. In Sec. III, it is shown that such relations can be in fact found and as a result Picard-Fuchs equations for the Seiberg-Witten periods are available from this viewpoint. It should be noted that the generating meromorphic differential of the Whitham hierarchy can be written in terms of the Seiberg-Witten 1-form. This indicates that it is sufficient to consider only the Seiberg-Witten periods in order to calculate the periods of the Whitham hierarchy. In Sec. IV, it is shown that the SU(3) Picard-Fuchs equations for the Seiberg-Witten periods can be obtained from the Picard-Fuchs equations with Whitham times for the periods of the Whitham hierarchy by considering the specialization condition to Seiberg-Witten model. Sec. V is a brief summary.

II. WHITHAM HIERARCHY IN GAUGE THEORY

In this section, we briefly sketch the relation between Seiberg-Witten solution and Whitham dynamics in the context of $N = 2$ supersymmetric Yang-Mills theory.$^{39-41}$

A. Seiberg-Witten solution

To begin with, let us recall that the Seiberg-Witten curve in $SU(r+1)$ gauge theory without matter hypermultiplets$^{2-4}$ is given by the characteristic equation

$$\det[x - L(\omega)] = 0$$

of the Lax operator $L(\omega)$ for Toda chain with $r+1$ sites,$^{39}$ where $x$ is the eigenvalue of $L(\omega)$ and $\omega$ is the spectral parameter. (2.1) can be rewritten in the form of spectral curve

$$P(x) = \Lambda_{SU(r+1)}^{r+1} \left( \omega + \frac{1}{\omega} \right),$$

where $\Lambda_{SU(r+1)}$ is the dynamical mass parameter and

$$P(x) := x^{r+1} - \sum_{i=2}^{r+1} u_i x^{r+1-i}$$

$$= x^{r+1} - \sum_{i=2}^{r+1} u_i x^{r+1-i}$$
represents the simple singularity of type $A_r$ with moduli $u_i$. This spectral curve (2.2) can be further rewritten in the familiar hyperelliptic form\textsuperscript{2-4}

$$y^2 = P^2 - 4\Lambda^2,$$

(2.4)

where $\Lambda := \Lambda_{r+1}^{SU(r+1)}$ and we have introduced

$$y := \Lambda_{SU(r+1)}^{r+1} \left( \omega - \frac{1}{\omega} \right).$$

(2.5)

Note that the hyperelliptic curve (2.4) is a Riemann surface of genus $r$.

For a study of Riemann surface, it is often useful to consider the periods of Abelian differentials over the 1-cycles on the surface. In the case at hand, we can take $2r$ 1-cycles $(A_i, B_i)$ ($i = 1, \ldots, r$) on (2.4) as a canonical basis ($B_i$ are symplectic duals of $A_i$), which can be expressed by using the branching points of (2.4).

On the other hand, in order to interpret the components of period matrix constructed from periods of Abelian differentials as the effective coupling constants, the combination of Abelian differentials must be fixed uniquely up to total derivatives. In addition, in general, there are three kinds of Abelian differentials on a Riemann surface, but that of the third kind is not required here because we are considering a pure gauge theory. Therefore, the expected meromorphic differential 1-form is expressed by the Abelian differentials of the first and second kinds, and the one satisfying these requirements is called Seiberg-Witten differential $dS_{SW}$, given by

$$dS_{SW} := x \frac{d\omega}{\omega} = x \frac{\partial_x P}{y} dx,$$

(2.6)

where we have ignored the numerical normalization for simplicity, and then the Seiberg-Witten periods are given by the loop integrals over the canonical cycles

$$a_i := \oint_{A_i} dS_{SW}, \quad a_{D_i} := \oint_{B_i} dS_{SW}.$$  

(2.7)

Note that $dS_{SW}$ can be viewed as the canonical 1-form of the integrable system. In this way, we can see the relation between Seiberg-Witten solution and integrable system.

B. Whitham hierarchy
We have seen that the Seiberg-Witten solution has a connection to integrable system, but it can be also viewed as a part of Whitham theory of solitons on a Riemann surface.

To see this, let us recall that in general Whitham theory consists of the following three ingredients:

- Riemann surface of genus $g$.
- Punctures on the surface.
- Existence of local coordinates near the punctures.

Gorsky et al.\(^\text{41}\) noticed that the meromorphic differentials of the second kind $d\Omega_n$ of $(n + 1)$-th order punctures ($n > 0$) on a Riemann surface was defined up to a linear combination of $g$ holomorphic differentials $d\omega_i$ and considered how to fix this combination by taking two basic requirements. The first one was to require

$$\oint_{A_i} d\Omega_n = 0 \quad (2.8)$$

and the second one was to introduce new meromorphic differentials $d\hat{\Omega}_n$ which enjoy the property that their differentiations over the moduli coincide with holomorphic differentials.

According to their result,\(^\text{41}\) the differential

$$dS := \sum_{n=1}^{\infty} T_n d\hat{\Omega}_n = \sum_{i=1}^g \alpha_i d\omega_i + \sum_{n=1}^{\infty} T_n d\Omega_n \quad (2.9)$$

with infinitely many parameters $T_n$ called Whitham times is found to be the expected solution which is suitable for applications to gauge theory. For this new meromorphic differential $dS$, the periods

$$\alpha_i := \oint_{A_i} dS, \quad \alpha_{D_i} := \oint_{B_i} dS \quad (2.10)$$
can be defined in a natural way.

Next, in order to make a contact with Seiberg-Witten solution, Gorsky et al.\(^\text{41}\) regarded the Riemann surface used here as the Seiberg-Witten hyperelliptic curve (2.4).

In such a situation, they found that the Whitham hierarchy could be actually written by only first $r$ time variables and gave an explicit expression of $dS$. In particular, in the case of SU($r + 1$) gauge theory, $n$ is restricted to $n < r + 1$. Namely, in this situation, the periods (2.10) reduce to
\[ \alpha_i = \sum_{n=1}^{r} T_n \int_{A_i} d\hat{\Omega}_n, \quad \alpha_{D_i} = \sum_{n=1}^{r} T_n \int_{B_i} d\hat{\Omega}_n \]  

(2.11)

and \( d\hat{\Omega}_n \) are given by

\[ d\hat{\Omega}_n = R_n \frac{d\omega}{\omega}, \quad R_n := P_n/(r+1). \]  

(2.12)

In this expression, \( P_n/(r+1) \) means the non-negative terms in the expansion of \( P_n/(r+1) \) for a large \( x \), and in general, \( P_n/(r+1) \) in SU\((r+1)\) gauge theory is easily found to give

\[ P_n/(r+1) = x^n - \frac{n}{r+1} u_2 x^{n-2} - \frac{n}{r+1} u_3 x^{n-3} - \frac{n}{r+1} \left[ u_4 + \frac{u_5^2}{2} \left( 1 - \frac{n}{r+1} \right) \right] x^{n-4} - \cdots. \]  

(2.13)

Note that the periods are now represented by a finite linear combination of \( d\hat{\Omega}_n \) because we are considering only for \( n \leq r+1 \) case. In addition, from (2.12), it is immediate to see that the Seiberg-Witten solution is recovered at the point

\[ (T_1, T_2, T_3, \cdots, T_r) = (1, 0, 0, \cdots, 0). \]  

(2.14)

In fact, we find \( d\hat{\Omega}_1 = dS_{SW} \). Of course, in this case, we have \( dS = dS_{SW} \).

III. PICARD-FUCHS STRUCTURE BEHIND WHITHAM HIERARCHY

A. Relations among meromorphic differentials

We have seen that \( dS \) is represented by a linear combination of \( d\hat{\Omega}_n \) and also seen that \( d\hat{\Omega}_1 = dS_{SW} \). Then, are \( d\hat{\Omega}_n \) for \( n \neq 1 \) related to \( dS_{SW} \)? If we can find any relation among them, the role of the Seiberg-Witten solution in the Whitham dynamics will be clarified.

To find an answer to this question, let us notice that any meromorphic differential on a Riemann surface must be always written in terms of the basis of Abelian differentials on the surface. Of course, this must be true also for \( d\hat{\Omega}_n \) for all \( n \). Therefore, if we consider a differentiation of \( d\hat{\Omega}_n \) over moduli, it will be ultimately represented by a linear combination of various \( d\hat{\Omega}_n \) and their derivatives. However, actually, in the case of Seiberg-Witten Riemann surface, we can show that the derivatives of \( d\hat{\Omega}_n \) for \( n > 1 \) are obtained from the Seiberg-Witten differential \( d\hat{\Omega}_1 \). Thus as the
result, we can conclude that $dS$ is generated from $d\hat{\Omega}_1$ and accordingly the periods of $dS$ can be directly determined through the Seiberg-Witten periods themselves.

To see this more concretely, let us consider the case of $d\hat{\Omega}_2$ as an example. Since the differentiations of $d\hat{\Omega}_2$ over moduli are

$$\frac{\partial d\hat{\Omega}_2}{\partial u_i} = \frac{dx}{y} \left[ -2\delta_{2,i} x^r + \delta_{2,i} \sum_{j=2}^{r+1} (r+1-j) u_j x^{r-j} + 2x^{r+2} \right], \quad (3.1)$$

where $\delta_{i,j}$ are the Kronecker’s delta symbols, and those for $d\hat{\Omega}_1$ are

$$\frac{\partial d\hat{\Omega}_1}{\partial u_i} = \frac{x^{r+1-i}}{y} dx, \quad (3.2)$$

it is easy to see that

$$\frac{\partial d\hat{\Omega}_2}{\partial u_2} = \frac{2}{r+1} \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_{i+1}} \frac{\partial d\hat{\Omega}_2}{\partial u_i} = 2 \frac{\partial d\hat{\Omega}_1}{\partial u_{i-1}} (i \neq 2). \quad (3.3)$$

Note that in the derivation of (3.1) and (3.2) we have used the general formulæ

$$\frac{\partial d\hat{\Omega}_n}{\partial u_i} = \frac{dx}{y} [\partial_{u_i} R_n \cdot \partial_x P - \partial_x R_n \cdot \partial_{u_i} P] + d \left( R_n \partial_{u_i} P \right). \quad (3.4)$$

In a similar way, we can obtain differential relations between $d\hat{\Omega}_n$ for $n > 1$ and $d\hat{\Omega}_1$, but we omit the derivations for them and show only the result for $n = 3$ and 4 cases here.

For $d\hat{\Omega}_3$:

$$\frac{\partial d\hat{\Omega}_3}{\partial u_2} = -\frac{3}{r+1} \left[ u_2 \frac{\partial d\hat{\Omega}_1}{\partial u_2} - \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_i} \right],$$

$$\frac{\partial d\hat{\Omega}_3}{\partial u_3} = -\frac{3}{r+1} \left[ u_2 \frac{\partial d\hat{\Omega}_1}{\partial u_3} - \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_i+1} \right],$$

$$\frac{\partial d\hat{\Omega}_3}{\partial u_i} = 3 \left[ \frac{\partial d\hat{\Omega}_1}{\partial u_{i-2}} - \frac{u_2}{r+1} \frac{\partial d\hat{\Omega}_1}{\partial u_i} \right] (i \neq 2, 3). \quad (3.5)$$

For $d\hat{\Omega}_4$:

$$\frac{\partial d\hat{\Omega}_4}{\partial u_2} = -\frac{4}{r+1} \left[ u_3 \frac{\partial d\hat{\Omega}_1}{\partial u_2} - \sum_{i=3}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_{i-1}} - \frac{r-3}{r+1} u_2 \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_{i+1}} \right],$$

$$\frac{\partial d\hat{\Omega}_4}{\partial u_3} = -\frac{4}{r+1} \left[ 2u_2 \frac{\partial d\hat{\Omega}_1}{\partial u_3} + u_3 \frac{\partial d\hat{\Omega}_1}{\partial u_3} - \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_i} \right],$$

$$\frac{\partial d\hat{\Omega}_4}{\partial u_4} = -\frac{4}{r+1} \left[ 2u_2 \frac{\partial d\hat{\Omega}_1}{\partial u_4} + u_3 \frac{\partial d\hat{\Omega}_1}{\partial u_4} - \sum_{i=2}^{r+1} (r+1-i) u_i \frac{\partial d\hat{\Omega}_1}{\partial u_{i+1}} \right],$$

$$\frac{\partial d\hat{\Omega}_4}{\partial u_i} = 4 \left[ \frac{\partial d\hat{\Omega}_1}{\partial u_{i-3}} - \frac{2u_2}{r+1} \frac{\partial d\hat{\Omega}_1}{\partial u_{i-1}} - \frac{u_3}{r+1} \frac{\partial d\hat{\Omega}_1}{\partial u_i} \right] (i \neq 2, 3, 4). \quad (3.6)$$
B. Picard-Fuchs equations from Whitham hierarchy

If the derivatives of $d\hat{\Omega}_n$ over moduli for $n > 1$ are eliminated from the relations (3.3), (3.5) and (3.6) by using differentiations, the equations satisfied by $d\hat{\Omega}_1$ will be obtained. Furthermore, since $d\hat{\Omega}_1 = dS_{SW}$, we can identify such equations as Picard-Fuchs equations for Seiberg-Witten periods.

To see this, it is enough to consider the cross derivatives of $d\hat{\Omega}_2$. For example, for $d\hat{\Omega}_2$, from $[\partial_2 \partial_i - \partial_i \partial_2] d\hat{\Omega}_2 = 0$, where $\partial_i := \partial / \partial u_i$, we get

$$\left( r + 1 \right) \partial_2 \partial_{t-1} - \left( r + 1 - i \right) \partial_{t+1} - \sum_{j=2}^{r+1} (r + 1 - j) u_j \partial_i \partial_j d\hat{\Omega}_1 = 0 \quad (i \neq 2),$$

which are the Picard-Fuchs equations obtained by several authors.\(^{26,37}\) For other $d\hat{\Omega}_n$, we can construct similar equations and, in fact, we can obtain the “hierarchy” of Picard-Fuchs equations as follows:

$$\sum_{i=2}^{r+1} (r + 1 - i) u_i \left( \partial_3 \partial_i - \partial_i \partial_{+1} \right) d\hat{\Omega}_1 = 0,$$

$$\left( r + 1 - i \right) \partial_{t+1} - \left( r + 1 \right) \partial_3 \partial_{t-2} + \sum_{j=2}^{r+1} (r + 1 - j) u_j \partial_i \partial_j d\hat{\Omega}_1 = 0 \quad (i \neq 2, 3),$$

$$\left( r + 1 \right) \partial_2 \partial_{t-2} - \left( r + 2 - i \right) \partial_i - \sum_{j=2}^{r+1} (r + 1 - j) u_j \partial_i \partial_j d\hat{\Omega}_1 = 0 \quad (i \neq 2, 3),$$

$$2(r + 1) u_2 \partial_2^2 + (r - 3)(r - 2) u_2 \partial_3 + (r + 1) \sum_{i=3}^{r+1} (r + 1 - i) u_i \partial_{t-1} \partial_3$$

$$+ (r - 3) u_2 \sum_{i=2}^{r+1} (r + 1 - i) u_i \partial_{t+1} \partial_3 - (r + 1) \sum_{i=2}^{r+1} (r + 1 - i) u_i \partial_i \partial_2 d\hat{\Omega}_1 = 0,$$

$$2(r + 1) u_2 \partial_2 \partial_3 + (r - 3)^2 u_2 \partial_5 + (r + 1) \sum_{i=3}^{r+1} (r + 1 - i) u_i \partial_{t-1} \partial_4$$

$$- (r + 1) \sum_{i=2}^{r+1} (r + 1 - i) u_i \partial_{t+1} \partial_3 + (r - 3) u_2 \sum_{i=2}^{r+1} (r + 1 - i) u_i \partial_{t+1} \partial_4 d\hat{\Omega}_1 = 0,$$

$$\left( r + 1 \right)^2 \partial_2 \partial_{t-3} - 2(r + 1) u_2 \partial_{t-1} \partial_2 - (r + 1)(r + 3 - i) \partial_{t-1}$$

$$- (r - 3)(r + 1 - i) u_2 \partial_{t+1} - (r + 1) \sum_{j=3}^{r+1} (r + 1 - j) u_j \partial_i \partial_j d\hat{\Omega}_1 = 0 \quad (i \neq 2, 3, 4),$$

$$2 u_2 (\partial_2 \partial_4 - \partial_3^2) - \sum_{i=2}^{r+1} (r + 1 - i) u_i (\partial_4 \partial_i - \partial_3 \partial_{i+1}) d\hat{\Omega}_1 = 0,$$

$$2 u_2 (\partial_3 \partial_i - \partial_4 \partial_{i-1}) + (r + 1) \partial_4 \partial_{t-3} - (r + 1 - i) \partial_{i+1}$$

$$d\hat{\Omega}_1 = 0.$$
\[-\sum_{j=2}^{r+1} (r+1-j)u_j \partial_i \partial_{j+1} d\hat{\Omega}_1 = 0 \quad (i \neq 2, 3, 4),
\[
\left[ 2u_2(\partial_i \partial_2 - \partial_3 \partial_{i-1}) + (r+1)\partial_3 \partial_{i-3} - (r+2-i)\partial_i 
- \sum_{j=2}^{r+1} (r+1-j)u_j \partial_i \partial_j \right] d\hat{\Omega}_1 = 0 \quad (i \neq 2, 3, 4).
\]

Note that the equations in (3.8) are all second-order equations and in some cases we can simplify
them by using \((\partial_i \partial_j - \partial_p \partial_q) d\hat{\Omega}_1 = 0\), where \(i + j = p + q\).\(^{26,37}\)

C. Picard-Fuchs equations as a complete system

Of course, as a complete Picard-Fuchs system, it is not necessary to consider all equations in
(3.7) and (3.8). In general, since there are \(r\) moduli parameters in the SU\((r+1)\) gauge theory, it is
sufficient to extract at least \(r\) independent equations from them.

To see this, let us notice the equations in (3.7). Since the number of the equations is \(r-1\), one
more equation is necessary. However, we can not obtain the expected equation from (3.8) because
the equations presented there do not have the instanton corrections. If the instanton correction
terms are not included in any one of Picard-Fuchs equations, the prepotential obtained from them
will not show the instanton corrections precisely. Therefore, we require that the remaining one must
include instanton terms.

Actually, such equation was recognized by Ito and Yang\(^{43}\) as the scaling relation. There, the
Picard-Fuchs system was realized by two kinds of equations, one of which is Gauss-Manin system
and the other is the scaling relation. Since the Gauss-Manin system does not involve instanton
corrections, the situation looks like our’s. Therefore, also for our case, the scaling relation may be
used as the remaining Picard-Fuchs equation.

For this, let us consider the Eulerian operator
\[
\mathcal{E} := \sum_{i=2}^{r+1} i u_i \partial_i + (r+1)\Lambda \partial_\Lambda,
\]
which acts as
\[
\mathcal{E} d\hat{\Omega}_n = nd\hat{\Omega}_n
\]
for all \( n > 0 \). (3.10) indicates that the degree of \( d\hat{\Omega}_n \) is \( n \). Realizing (3.10) as an equation only in terms of moduli derivatives can be easily accomplished by considering the squaring equation

\[(\mathcal{E} - n)^2 d\hat{\Omega}_n = 0.\]

In this way, we can associate \( r \) independent Picard-Fuchs equations for \( d\hat{\Omega}_1 \).

IV. PICARD-FUCHS EQUATIONS WITH WHITHAM TIMES

A. The SU(3) Picard-Fuchs equations

Next, let us consider Picard-Fuchs equations for the periods \((\alpha_i, \alpha_{D_i})\) of Whitham hierarchy. In the case of \( r = 1 \), the resulting Picard-Fuchs equation takes the same form with the usual one up to rescaling of \( T_1 \). For this reason, we do not discuss this case, and instead, let us consider \( r = 2 \) case in order to find a non-trivial example of Picard-Fuchs equations with Whitham times.

In this case, the Picard-Fuchs equations with the Whitham times are found to be in the form

\[
\mathcal{L}_j(\alpha_i, \alpha_{D_i}) = 0, \quad (4.1)
\]

where

\[
\mathcal{L}_1 = \frac{54u^2vT_1T_2 - (108\Lambda^2 - 4u^3 - 27v^2)(uT_2^2 - 3T_1^2)}{u(-3T_1^2 + 4uT_2^2)} \partial_u^2
\]

\[
-\frac{3[(-108\Lambda^2 + 4u^3 + 27v^2)T_1T_2 + 8uv(uT_2^2 - 3T_1^2)]}{2(3T_1^2 - 4uT_2^2)} \partial_u \partial_v
\]

\[
+ \frac{T_2[8(108\Lambda^2 - 4u^3 - 27v^2)(-3T_1^2 + uT_2^2)T_2 - 9uvT_1(15T_1^2 + 28uT_2^2)]}{2u(3T_1^2 - 4uT_2^2)^2} \partial_u
\]

\[
+ \frac{2(108\Lambda^2 - 4u^3 - 27v^2)(3T_1^2 - uT_2^2)T_1T_2 + 3uv(9T_1^4 + 27uT_2^2T_2^2 - 4u^2T_1^2)}{u(3T_1^2 - 4uT_2^2)^2} \partial_v + 1,
\]

\[
\mathcal{L}_2 = \frac{-54u^2vT_1T_2 + (108\Lambda^2 - 4u^3 - 27v^2)(uT_2^2 - 3T_1^2)}{3(3T_1^2 - 4uT_2^2)^2} \partial_v^2
\]

\[
-\frac{3[8uv(uT_2^2 - 3T_1^2)] + (-108\Lambda^2 + 4u^3 + 27v^2)T_1T_2)}{2(3T_1^2 - 4uT_2^2)} \partial_u \partial_v - \frac{45vT_1T_2}{2(3T_1^2 - 4uT_2^2)} \partial_u
\]

\[
+ \frac{3v(3T_1^2 + uT_2^2)}{3T_1^2 - 4uT_2^2} \partial_v + 1. \quad (4.2)
\]

Though the derivation of Picard-Fuchs equations for other higher \( r \) is straightforward, the result is turned out to be too lengthy and complicated, so we do not consider these cases in this paper.
B. Specializations of SU(3) Picard-Fuchs equations

It may be instructive to see specializations of (4.2). With the help of (2.14), it is straightforward to make sure that the equations in (4.2) yield the usual SU(3) Picard-Fuchs equations\textsuperscript{18} $L_j(a_i, a_{D_i}) = 0$ for the Seiberg-Witten periods, which can be identified with Appell’s $F_4$ system\textsuperscript{44–47}

\begin{align*}
L_1 \rightarrow L_1 &:= (4u^3 + 27v^2 - 108\Lambda^2)\partial_u^2 + 12u^2v\partial_u\partial_v + 3uv\partial_v + u, \\
L_2 \rightarrow L_2 &:= (4u^3 + 27v^2 - 108\Lambda^2)\partial_v^2 + 36uv\partial_u\partial_v + 9v\partial_v + 3. \quad (4.3)
\end{align*}

Note that the consistency condition of (4.3) leads to $[u\partial_v^2 - 3\partial_u^2](a_i, a_{D_i}) = 0$, which coincides with (3.7) for $r = 2$.

On the other hand, from (4.2) with $(T_1, T_2) = (0, 1)$, we can also consider Picard-Fuchs equations

\begin{align*}
\tilde{L}_j(f_A, d\tilde{\Omega}_2, f_B, d\tilde{\Omega}_2) = 0 \quad \text{for the periods of } d\tilde{\Omega}_2,
\end{align*}

where

\begin{align*}
L_1 \rightarrow \tilde{L}_1 &:= u(4u^3 + 27v^2 - 108\Lambda^2)\partial_u^2 + 12u^2v\partial_u\partial_v - (4u^3 + 27v^2 - 108\Lambda^2)\partial_u - 3u^2v\partial_v + 4u^2, \\
L_2 \rightarrow \tilde{L}_2 &:= (4u^3 + 27v^2 - 108\Lambda^2)\partial_v^2 + 36uv\partial_u\partial_v - 9v\partial_v + 12. \quad (4.4)
\end{align*}

From (4.4), we can obtain a relation like that from (4.3), but the same equation is also available from (3.3), provided $\partial d\tilde{\Omega}_1/\partial u_i$ are eliminated from (3.3).

Finally, note that we have

\begin{align*}
\tilde{L}_j(a_i, a_{D_i}) = T_1\tilde{L}_j\left(\int_{A_i} d\tilde{\Omega}_1, \int_{B_i} d\tilde{\Omega}_1\right), \quad L_j(a_i, a_{D_i}) = T_2L_j\left(\int_{A_i} d\tilde{\Omega}_2, \int_{B_i} d\tilde{\Omega}_2\right) \quad (4.5)
\end{align*}

from (2.9), (4.3) and (4.4).

V. SUMMARY

In this paper, we have discussed the SU($r + 1$) gauge theory in the standpoint of Whitham dynamics and realized $r - 1$ Picard-Fuchs equations for Seiberg-Witten periods as consistency equations among meromorphic differentials associated with Whitham times. In addition, we have used the scaling relation as the remaining independent equation in order to include the instanton corrections. Though the generalization to other cases except SU($r + 1$) group is straightforward, the case of exceptional gauge groups would be interesting because there are two types of Seiberg-Witten curves in these gauge theories.\textsuperscript{7,11–16} In particular, it may be interesting to know how the
differences of physics expected from these two curves\textsuperscript{12,16,24,25} are reflected in the Whitham theory and the Picard-Fuchs structure behind it.

Of course, our construction of Picard-Fuchs equations may provide helpful informations not only for these cases but also when we consider the relation among flat coordinates,\textsuperscript{48,49} Witten-Dijkgraaf-Verlinde-Verlinde equations\textsuperscript{50–55} and Whitham hierarchy.\textsuperscript{39–41} We are now planning a discussion respect to this point.
REFERENCES


