Entropy from Conformal Field Theory at Killing Horizons

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Abstract

On a manifold with boundary, the constraint algebra of general relativity may acquire a central extension, which can be computed using covariant phase space techniques. When the boundary is a (local) Killing horizon, a natural set of boundary conditions leads to a Virasoro subalgebra with a calculable central charge. Conformal field theory methods may then be used to determine the density of states at the boundary. I consider a number of cases—black holes, Rindler space, de Sitter space, Taub-NUT and Taub-Bolt spaces, and dilaton gravity—and show that the resulting density of states yields the expected Bekenstein-Hawking entropy. The statistical mechanics of black hole entropy may thus be fixed by symmetry arguments, independent of details of quantum gravity.

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In attempting to understand the thermodynamic properties of black holes, physicists are pulled in two directions. On the one hand, we would like to find a microscopic “statistical mechanical” description of black hole thermodynamics in terms of some specific set of quantum states. But we also recognize that the existing derivations of black hole temperature and entropy use only broad features of quantum field theory and semiclassical gravity, suggesting that a microscopic explanation cannot be too sensitive to the details of quantum gravity. We seem to need a kind of “quantum gravity without quantum gravity”: a general principle that governs the density of states in quantum gravity and yet is independent of the details of the theory.

A natural candidate for such a general principle is a symmetry. The idea of using symmetry arguments to count states in quantum gravity was originally suggested by Strominger [1] in the context of the (2+1)-dimensional black hole. Brown and Henneaux [2] had noted in 1986 that (2+1)-dimensional gravity with a negative cosmological constant has an asymptotic symmetry consisting of a pair of Virasoro algebras, implying that any microscopic quantum theory should be a conformal field theory. But conformal field theories have a peculiar property: the Cardy formula [3] determines the asymptotic density of states entirely in terms of the Virasoro algebra, independent of other details of the theory. Strominger observed that if one uses the central charge of ref. [2] in the Cardy formula, the resulting density of states reproduces the standard Bekenstein-Hawking entropy for the (2+1)-dimensional black hole, thus providing the sort of universal mechanism we need.

Unfortunately, there are two basic limitations to Strominger’s approach. First, of course, it works only in 2+1 dimensions. This is less restrictive than it might appear, since many of the higher dimensional black holes in string theory have near-horizon geometries that reduce that of the (2+1)-dimensional black hole [4]. Still, it seems unnatural to depend on particular features of 2+1 dimensions for what ought to be a universal computation. Second, since Strominger’s argument is based on an algebra of transformations at infinity, it is insensitive to important details of the structure of the interior of spacetime, and yields only a sort of “maximum possible entropy.” For example, the entropy computation of ref. [1] applies equally well to a black hole of mass $m$ and a spherical star of the same mass. It may be that finer details of the conformal field theory at infinity can distinguish such cases [5], but presumably one would like to be able to count the states of a black hole more directly.

In ref. [6], I proposed a generalization of the Brown-Henneaux-Strominger construction based on the algebra of deformations at a black hole horizon. If the horizon is treated as a boundary, the algebra of constraints in general relativity acquires a central extension. Given a plausible set of boundary conditions, this extended algebra contains a natural Virasoro subalgebra, and the Cardy formula can again be used to obtain the correct entropy. This construction is valid for black holes in any dimension. Unfortunately, however, the derivation is tied to a particular “Schwarzschild-like” coordinate system, and some rather arbitrary restrictions on diffeomorphisms are required.

In this paper, I rederive the central extension of the constraint algebra of general
relativity using manifestly covariant phase space methods. If one takes the boundary to be a surface that looks locally like a Killing horizon, the resulting algebra again contains a natural Virasoro algebra with a calculable central charge. I consider a variety of spacetimes—rotating black holes, Rindler space, de Sitter space, and Taub-NUT and Taub-Bolt spaces—as well as the extension to two-dimensional dilaton gravity. In each case, the Cardy formula leads to a density of states that yields the expected Bekenstein-Hawking entropy.

1 The General Argument

Since much of this paper is rather technical, I will begin with a summary of the general argument and a discussion of some of the broader physical issues. My starting point is the investigation of the algebra of constraints in general relativity in the presence of a boundary. Naively, one would expect this algebra to be equivalent to the algebra of diffeomorphisms of the spacetime $\mathcal{M}$. But as Brown and Henneaux have stressed [2], boundary terms in the constraints can lead to a central extension of $\text{Diff}\mathcal{M}$.

Such a central extension is of interest in its own right, but it becomes especially important if a subalgebra isomorphic to $\text{Diff} S^1$ or $\text{Diff} \mathbb{R}$ acquires a central term. A centrally extended algebra of this type is known as a Virasoro algebra, and such algebras play a fundamental role in conformal field theory. The algebra we are considering is a classical Poisson algebra, but any quantum theory of gravity should presumably inherit this structure, perhaps with order $\hbar$ corrections. This means that we can use powerful techniques developed in conformal field theory to obtain useful information about quantum gravity. In particular, the Cardy formula determines the asymptotic behavior of the density of states in terms of quantities fixed almost uniquely by the Virasoro algebra. (I discuss the derivation and applicability of the Cardy formula in Appendix C.)

To investigate the constraint algebra, I use covariant phase space methods [7–9], which exploit the isomorphism between phase space and the space of solutions of the field equations to provide a canonical formalism that is manifestly covariant. In particular, a formalism developed by Wald and his collaborators [10–13] is especially well suited for my purposes. The results are summarized by eqns. (3.3) and (3.5), which give the central extension of the algebra of constraints for an arbitrary covariant theory.

The central term in the constraint algebra arises from boundary terms in the generators. Since I am interested in black holes, I specify boundary conditions that reflect the presence of a horizon. There are a number of ways of imposing such a requirement. I choose the simplest, though not the most general, which is to demand that the boundary look locally like a Killing horizon. This condition, along with a somewhat more mysterious restriction on the average surface gravity, is sufficient to determine a central extension of the constraint algebra. In particular, diffeomorphisms of the “$r - t$ plane,” which play the central role in Euclidean path integral computation of black hole entropy, acquire a central term with the structure one expects for a Virasoro algebra. The general form of this algebra is given by eqns. (4.21)–(4.22); the specialization to a one-parameter
subalgebra is given by eqn. (5.9). If one now employs the Cardy formula, the resulting density of states (5.13) is precisely what is needed to reproduce the Bekenstein-Hawking entropy.

It is worth pausing for a moment to discuss the sense in which one can treat a horizon as a boundary. A black hole horizon is not, after all, a true “edge” of spacetime; there is nothing to stop an external observer from passing through the horizon.

Consider, however, an arbitrary quantum mechanical question about a black hole. Such a question automatically calls for the computation of a conditional probability: for instance, “What is the probability of observing a photon of Hawking radiation of frequency $\nu$, given the presence of a black hole with a horizon of area $A$?”

Now, in semiclassical gravity, such a condition can be imposed by fixing the background metric to be that of a prescribed black hole. In a true quantum theory, however, this is no longer possible, since the metric is itself a quantum field that cannot be precisely specified. Instead, the most direct way to ask such a conditional question is to require the presence of a surface with suitable properties to ensure it is a horizon. Note that the amount of boundary data one is allowed to impose is “half the phase space,” exactly the amount compatible with the uncertainty principle. This means that whether or not we treat the horizon as a physical boundary, we must treat it as a surface upon which we impose boundary conditions. The existence of such boundary conditions and the consequent restrictions on variations of the fields are sufficient to justify the methods of this paper.

The analysis I have just described was developed for black holes in ordinary general relativity, but its extension to other configurations and other theories is straightforward. In section 6 I discuss some generalizations, and show that in each case one obtains a density of states that reproduces the expected entropy. It thus appears that there may be a universal statistical mechanical picture of entropy associated with horizons: regardless of the details of a quantum theory of gravity, symmetries inherited from the classical theory may be sufficient to determine the asymptotic behavior of the density of states.

2 Constraint Algebras and Covariant Phase Space

Let us begin with a brief review of covariant phase space methods, in the formalism developed by Wald et al. [10–13]. Consider a general diffeomorphism-invariant field theory in $n$ spacetime dimensions with a Lagrangian $L[\phi]$, where $L$ is viewed as an $n$-form and $\phi$ denotes an arbitrary collection of dynamical fields. The variation of $L$ takes the form

$$\delta L = E \cdot \delta \phi + d\Theta$$

(2.1)

where the field equations are given by $E = 0$ and the symplectic potential $\Theta[\phi, \delta \phi]$ is an $(n - 1)$-form determined by the “surface terms” in the variation of $L$. The symplectic current $(n - 1)$-form $\omega$ is defined by

$$\omega[\phi, \delta_1 \phi, \delta_2 \phi] = \delta_1 \Theta[\phi, \delta_2 \phi] - \delta_2 \Theta[\phi, \delta_1 \phi]$$

(2.2)
and its integral over a Cauchy surface $C$,

$$\Omega[\phi, \delta_1 \phi, \delta_2 \phi] = \int_C \omega[\phi, \delta_1 \phi, \delta_2 \phi]$$  \hfill (2.3)

gives a presymplectic form on the space of solutions of the field equations. This space, in turn, can be identified with the usual phase space, and $\Omega$ becomes the standard presymplectic form of Hamiltonian mechanics [9, 10].

For any diffeomorphism generated by a smooth vector field $\xi^a$, one can define a conserved Noether current $(n - 1)$-form $J$ by

$$J[\xi] = \Theta[\phi, L_\xi \phi] - \xi \cdot L,$$  \hfill (2.4)

where $L_\xi$ denotes the Lie derivative in the direction $\xi$ and the dot $\cdot$ means contraction of a vector with the first index of a form. On shell, the Noether current is closed, and can be written in terms of an $(n - 2)$-form $Q$, the Noether charge, as

$$J = \delta Q.$$  \hfill (2.5)

Now consider a vector field $\xi^a$, and the corresponding generator of diffeomorphisms $H[\xi]$. In the covariant phase space formalism, Hamilton’s equations of motion become

$$\delta H[\xi] = \Omega[\phi, \delta \phi, L_\xi \phi].$$  \hfill (2.6)

It is easy to see that when $\phi$ satisfies the equations of motion,

$$\omega[\phi, \delta \phi, L_\xi \phi] = \delta J[\xi] - d(\xi \cdot \Theta[\phi, \delta \phi]),$$  \hfill (2.7)

so by eqns. (2.3) and (2.5),

$$H[\xi] = \int_{\partial C} (Q[\xi] - \xi \cdot B),$$  \hfill (2.8)

where the $(n - 1)$-form $B$ is defined by the requirement that

$$\delta \int_{\partial C} \xi \cdot B[\phi] = \int_{\partial C} \xi \cdot \Theta[\phi, \delta \phi].$$  \hfill (2.9)

Given a choice of boundary conditions at $\partial C$, finding $B$ is roughly equivalent to finding the appropriate boundary terms for the Hamiltonian constraint in the standard ADM formalism of general relativity. It should be emphasized that $B$ may not always exist; given a choice of boundary conditions, eqn. (2.9) may not have a solution for every vector field $\xi^a$.

For general relativity in $n$ spacetime dimensions, the Lagrangian $n$-form is

$$L_{a_1...a_n} = \frac{1}{16\pi G} \epsilon_{a_1...a_n} R,$$  \hfill (2.10)

which yields a symplectic potential $(n - 1)$-form [12]

$$\Theta_{a_1...a_{n-1}} \rightarrow g, \delta g = \frac{1}{16\pi G} \epsilon_{b a_1...a_{n-1}} \left( g^{bc} \nabla_c (g_{de} \delta g^{de}) - \nabla_c \delta g^{bc} \right).$$  \hfill (2.11)

The corresponding Noether charge, evaluated when the vacuum field equations hold, is

$$Q_{a_1...a_{n-2}} \rightarrow g, \xi = -\frac{1}{16\pi G} \epsilon_{b a_1...a_{n-2}} \nabla^b \xi^c.$$  \hfill (2.12)
3 Central Terms in the Algebra of Diffeomorphisms

In the absence of a boundary, the Poisson brackets of the generators \( H[\xi] \) form the standard “surface deformation algebra” [14], equivalent on shell to the algebra of diffeomorphisms. On a manifold with boundary, however, the addition of boundary terms can alter the Poisson brackets, leading to a central extension of the surface deformation algebra [2]. That is, the Poisson algebra may take the form

\[
\{ H[\xi_1], H[\xi_2] \} = H[\{\xi_1, \xi_2\}] + K[\xi_1, \xi_2],
\]

where the central term \( K[\xi_1, \xi_2] \) depends on the dynamical fields only through their (fixed) boundary values. This phenomenon is not peculiar to gravity [15]; it occurs because the generators are unique only up to the addition of constants, and the constant term in the boundary contribution to \( H[\{\xi_1, \xi_2\}] \) may not match the corresponding term in \( \{ H[\xi_1], H[\xi_2] \} \). The existence of such a central extension has been studied extensively in (2+1)-dimensional gravity, both in the metric formulation [2,16] and in the Chern-Simons formulation [17,18]. Here we wish to investigate it in a more general setting.

Consider the Poisson brackets of the generators of diffeomorphisms in the covariant phase space formalism of the preceding section. Let \( \xi_1^a \) and \( \xi_2^a \) be two vector fields, and suppose the fields \( \phi \) solve the equations of motion (so, in particular, the “bulk” constraints are all zero). Denote by \( \delta_\xi \) the variation corresponding to a diffeomorphism generated by \( \xi \). For the Noether current \( J[\xi_1] \),

\[
\delta_\xi J[\xi_1] = \mathcal{L}_\xi J[\xi_1] = \xi_2 \cdot dJ[\xi_1] + d(\xi_2 \cdot J[\xi_1]) = d \left[ \xi_2 \cdot (\Theta[\phi, \mathcal{L}_\xi \phi] - \xi_1 \cdot L) \right],
\]

where I have used the fact that \( dJ = 0 \) on shell. Hence from eqns. (2.6) and (2.7),

\[
\delta_\xi H[\xi_1] = \int_C \delta_\xi J[\xi_1] - d(\xi_1 \cdot \Theta[\phi, \mathcal{L}_\xi \phi])
= \int_{\partial C} (\xi_2 \cdot \Theta[\phi, \mathcal{L}_\xi \phi] - \xi_1 \cdot \Theta[\phi, \mathcal{L}_\xi \phi] - \xi_2 \cdot (\xi_1 \cdot L)).
\]

We can now take advantage of an observation due to Brown and Henneaux [2]. Since eqn. (3.3) was evaluated on shell, the “bulk” part of the generator \( H[\xi_1] \) on the left-hand side, which consists entirely of a sum of constraints, vanishes. Hence the left-hand side can be interpreted as the variation \( \delta_\xi J[\xi_1] \), where \( J \) is the boundary term in the constraint. [I show in Appendix B by direct computation that \( \delta_\xi J[\xi_1] \) agrees with the right-hand side of eqn. (3.3).] On the other hand, the Dirac bracket \( \{ J[\xi_1], J[\xi_2] \}^* \) means precisely the change in \( J[\xi_1] \) under a surface deformation generated by \( J[\xi_2] \); that is,

\[
\delta_\xi J[\xi_1] = \{ J[\xi_1], J[\xi_2] \}^*.
\]

Comparing eqn. (3.1), evaluated on shell, we see that

\[
K[\xi_1, \xi_2] = \delta_\xi J[\xi_1] - J[\{\xi_1, \xi_2\}],
\]

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where $\delta_{\xi_2}J[\xi_1]$ is given by eqn. (3.3). Given a suitable set of boundary conditions, this permits a simple determination of the central term $K[\xi_1, \xi_2]$.

In particular, for vacuum general relativity, the Lagrangian $\mathbf{L}$ vanishes on shell, and the right-hand side of eqn. (3.3) can be computed from eqn. (2.11). One obtains

$$\{J[\xi_1], J[\xi_2]\}^* = \frac{1}{16\pi G} \int_{\partial C} \epsilon_{bca_{1...a_{n-2}}} \left[ \xi^d_2 \nabla_d (\nabla^d \xi^c_1 - \nabla^c \xi^d_1) - \xi^d_1 \nabla_d (\nabla^d \xi^c_2 - \nabla^c \xi^d_2) \right].$$

(3.6)

## 4 Local Killing Horizons

To proceed further, we must specify boundary conditions at $\partial C$ more precisely. We are interested in the entropy associated with horizons, either black hole and cosmological, and should thus choose boundary conditions that reflect the presence of a horizon.

Ashtekar et al. have recently discussed a general set of boundary conditions for isolated horizons [19], and these, or their generalization to rotating horizons, may ultimately be the appropriate ones to use. For now, however, I will take a more conservative (and easier) approach, and look for boundary conditions that imply the presence of a local Killing horizon.

Consider an $n$-dimensional spacetime $M$ with boundary $\partial M$, such that a neighborhood of $\partial M$ admits a Killing vector $\chi^a$ that satisfies $\chi^2 = g_{ab} \chi^a \chi^b = 0$ at $\partial M$. $M$ need not be “all of spacetime”—we are not restricting our attention to eternal stationary black holes—but can be a small region containing a momentarily stationary black hole or cosmological horizon; the condition $\chi^2 = 0$ can then be viewed as determining the location of the relevant boundary.

In practice, it will be useful to work at a “stretched horizon” $\chi^2 = \epsilon$, taking $\epsilon$ to zero at the end of the computation. Near this stretched horizon, one can define a vector orthogonal to the orbits of $\chi^a$ by

$$\nabla_a \chi^2 = -2\kappa \rho_a,$$

(4.1)

where $\kappa$ is the surface gravity at the horizon. Note that

$$\chi^a \rho_a = -\frac{1}{\kappa} \chi^a \nabla_a \chi^b = 0.$$

(4.2)

At the horizon, $\chi^a$ and $\rho^a$ become null, and the normalization in eqn. (4.1) has been chosen so that $\rho^a \to \chi^a$. Away from the horizon, however, $\chi^a$ and $\rho^a$ define two orthogonal directions.

If we now vary the metric, we will typically find ourselves in a spacetime that admits no Killing vector even near $\partial M$. In order for the boundary condition $\chi^2 = 0$ to continue to make sense, we must at least require that $\chi^a \chi^b \delta g_{ab} = 0$, where $\chi^a$ is now viewed as a fixed vector field. I will impose slightly stronger conditions, to preserve the “asymptotic” structure at the horizon:

$$\chi^a \chi^b \chi^2 \delta g_{ab} \to 0, \quad \chi^a \chi^b \delta g_{ab} \to 0 \quad \text{as} \, \chi^2 \to 0,$$

(4.3)
where \( t^b \) is any unit spacelike vector tangent to \( \partial M \). Equivalently, we require that

\[
\delta \chi^2 = 0, \quad \chi^a t^b \delta g_{ab} = 0, \quad \text{and} \quad \delta \rho_a = -\frac{1}{2\kappa} \nabla_a (\delta \chi^2) = 0 \quad \text{at} \quad \chi^2 = 0. \quad (4.4)
\]

Note that these conditions guarantee that the boundary \( \chi^2 = 0 \) remains null, and that \( \chi^a \) continues to be the null normal to this boundary. Indeed, the normal \( \rho_a \) satisfies

\[
\delta \rho^2 = \rho_a \rho_b \delta g_{ab} = \chi^a \chi^b \delta g_{ab}, \quad \text{which vanishes at the boundary, and it is not hard to see that} \quad \delta (\rho^2 - \chi^2) = \rho_b \delta g^{ab} \text{has components only along} \chi^a \text{at} \partial M.
\]

In viewing these boundary conditions, it may be helpful to keep a specific example in mind. For a Kerr black hole in Boyer-Lindquist coordinates, \( \chi^2 \) is equal to the lapse function \( N^2 \), and \( \chi^2 \sim h(\theta)(r - r_+) \) near the horizon \( r = r_+ \), where

\[
h(\theta) = \frac{1}{m} \left( \frac{m^2 - a^2}{m^2 + (m^2 - a^2)^{1/2}} \right)^{1/2} (1 - a\Omega_H \sin \theta).
\]

The first condition in eqn. (4.3) requires that the boundary remain at \( r = r_+ \), and that \( h(\theta) \) remain fixed at the boundary [6]. The second condition then requires that the shift function \( N^\phi \) be fixed at the boundary, or equivalently that the angular velocity \( \Omega_H \) of the horizon be held fixed. In this sense, the conditions (4.3) are horizon analogs to the fall-off conditions one usually imposes at infinity.

For a diffeomorphism generated by a vector field \( \xi^a \), eqn. (4.3) implies that

\[
\frac{\chi^a \chi^b}{\chi^2} \nabla_a \xi_b = \chi^a \nabla_a \left( \frac{\chi^b \xi^b}{\chi^2} \right) - \kappa \frac{\rho_b \xi^b}{\chi^2} = 0. \quad \text{(4.5)}
\]

This suggests that we focus on vector fields of the form

\[
\xi^a = R \rho^a + T \chi^a. \quad \text{(4.6)}
\]

The corresponding diffeomorphisms are, in a reasonable sense, deformations in the “\( r-t \) plane,” which are known to play a crucial role in the Euclidean approach to black hole thermodynamics [20].

The appearance of a term in the “radial” direction \( \rho^a \) may at first seem surprising, since we are ultimately interested in diffeomorphisms that preserve the horizon. This term may be easily understood, though: the boundary condition \( \chi^2 = 0 \) is not quite diffeomorphism invariant, since \( \chi^a \) is held fixed, and an extra transformation is necessary to restore our gauge condition at the boundary. If we write the action as \( I = \int \hat{\theta}(\chi^2) L \), where \( \hat{\theta} \) is a step function, it is not hard to see that, in the notation of eqn. (2.1),

\[
\delta I = \int_M \hat{\theta}(\chi^2) E \cdot \delta \phi + \int_{\chi^2 = 0} \left( \Theta[\phi, \delta \phi] - \frac{1}{2\kappa} \frac{\delta \chi^2}{\rho^2} \rho \cdot L \right), \quad \text{(4.7)}
\]

where the last term comes from varying \( \chi^2 \) in the step function. The role of \( R \) is essentially to remove this term, allowing us to work with a fixed boundary even as \( \chi^2 \) varies.
For vector fields of the form (4.6), condition (4.5) becomes

\[ R = \frac{1}{\kappa} \frac{\chi}{\rho^2} \chi^a \nabla_a T. \]  

(4.8)

We must now check whether the diffeomorphisms characterized by eqns. (4.6) and (4.8) form a closed subalgebra. It is not hard to see that closure requires a new condition,

\[ \rho^a \nabla_a T = 0 \]  

(4.9)

at the horizon.

The need for this restriction can be traced back to the fact that eqn. (4.8) depends on the metric, thus making the parameters \( \xi^a \) functions on phase space that must themselves be transformed. Now, it is certainly possible to work out the algebra of surface deformations when the \( \xi^a \) are functions on phase space rather than fixed parameters. To do so would require adding terms in eqn. (3.4) and similar relations to reflect this additional dependence. For now, however, I will restrict myself to diffeomorphisms that satisfy condition (4.9). For the Kerr black hole in Boyer-Lindquist coordinates, \( \rho^a \nabla_a \sim (r - r_+) [F_1(r, \theta) \partial_r + F_2(r, \theta) \partial_\theta] \), where \( F_1 \) and \( F_2 \) are well-behaved functions, so this is essentially a requirement that spatial derivatives not blow up at the horizon.

The boundary conditions imposed so far are fairly straightforward. However, I show in Appendix B that they are not sufficient to guarantee the existence of Hamiltonians \( H[\xi] \) for diffeomorphisms satisfying eqns. (4.6), (4.8), and (4.9). To ensure integrability of eqn. (2.9), a further, somewhat less transparent condition is needed.

One possible new condition can be obtained by considering the quantity \( \tilde{\kappa} \) defined by

\[ \tilde{\kappa}^2 = - \frac{a^2}{\chi^2}, \]  

(4.10)

where \( a^a = \chi^b \nabla_b \chi^a \) is the acceleration of an orbit of \( \chi^a \). When \( \chi^a \) is a Killing vector, it is easy to see that \( \tilde{\kappa} \) approaches \( \kappa \), the surface gravity, as \( \chi^2 \to 0 \), and that away from the horizon, \( \tilde{\kappa} = \kappa \rho / |\chi| \). Under variations of the metric, however, this will no longer be the case, and we cannot even demand that \( \tilde{\kappa} \) be a constant. We can, however, fix the average value of \( \tilde{\kappa} \) over a cross section of the horizon, by requiring that

\[ \delta \int_{\partial C} \hat{\epsilon} \left( \tilde{\kappa} - \frac{\rho}{|\chi|} \kappa \right) = 0, \]  

(4.11)

where \( \hat{\epsilon} \) is the induced volume element on \( \partial C \).

The technical role of this condition is discussed more fully in Appendix B, where it is shown that it guarantees the existence of generators \( H[\xi] \). For now, let us merely note that for a diffeomorphism of the type we are considering, condition (4.11) requires that

\[ \int_{\partial C} \hat{\epsilon} D^3 T = 0, \]  

(4.12)
where \( D = \chi^a \nabla_a \). For a one-parameter group of diffeomorphisms such that \( DT_\alpha = \lambda_\alpha T_\alpha \), this in turn implies an orthogonality relation
\[
\int_{\partial C} \hat{\epsilon} T_\alpha T_\beta \sim \delta_{\alpha+\beta}, \tag{4.13}
\]
which will be important later in our derivation of the central charge.

Now, given any one-parameter group of diffeomorphisms satisfying conditions (4.6), (4.8), and (4.9), with or without (4.12), it is easy to check that
\[
\{\xi_1, \xi_2\}^a = (T_1 DT_2 - T_2 DT_1)\chi^a + \frac{1}{\kappa} \lambda^2 D(T_1 DT_2 - T_2 DT_1)\rho^a. \tag{4.14}
\]
This is isomorphic to the standard algebra of diffeomorphisms of the circle or the real line. The question before us is whether the algebra of constraints merely reproduces this \( \text{Diff}S^1 \) or \( \text{Diff} \mathbb{R} \) algebra, or whether it acquires a central extension.

To compute the possible central term in the this algebra, we return to eqns. (3.5) and (3.6). Let us first consider the integration measure in (3.6). Let \( H \) denote the \((n-2)\)-dimensional intersection of the Cauchy surface \( C \) with the Killing horizon \( \chi^2 = 0 \). The vector \( \chi^a \) is, of course, one of the null normals to \( H \); denote the other future-directed null normal by \( N^a \), with a normalization \( N_a \chi^a = -1 \). Then
\[
\epsilon_{ba_1...a_{n-2}} = \hat{\epsilon}_{a_1...a_{n-2}} (\chi_b N_c - \chi_c N_b) + \ldots, \tag{4.15}
\]
where \( \hat{\epsilon} \) is the induced measure on \( \mathcal{H} \) and the omitted terms do not contribute to the integral. In general, we do not know much about \( N^a \). However, consider the vector
\[
k^a = -\frac{1}{\chi^2} \left( \chi^a - \frac{|\chi|}{\rho} \rho^a \right). \tag{4.16}
\]
This vector is defined even in the limit \( \chi^2 \to 0 \); it is null everywhere, and is normalized so that \( k_a \chi^a = -1 \). It follows that \( N^a = k^a - \alpha \chi^a - t^a \), where \( t^a \) is tangent to \( \mathcal{H} \) and has a norm \( t^2 = 2\alpha - \alpha^2 \chi^2 \). It is then easy to see that
\[
\chi^b (\chi_b N_c - \chi_c N_b) = \frac{|\chi|}{\rho} \rho_c - \chi^2 t_c
\]
\[
\rho^b (\chi_b N_c - \chi_c N_b) = \left( \frac{\rho}{|\chi|} + t \cdot \rho \right) \chi_c. \tag{4.17}
\]
Thus for a vector of the form (4.6),
\[
\xi^b \epsilon_{ba_1...a_{n-2}} = \hat{\epsilon}_{a_1...a_{n-2}} \left[ \frac{|\chi|}{\rho} T \rho_c + \left( \frac{\rho}{|\chi|} + t \cdot \rho \right) R \chi_c \right] + O(\chi^2). \tag{4.18}
\]

The computation of the remainder of the integrand in eqn. (3.6) is straightforward. It turns out that the term proportional to \( R \) in eqn. (4.18) gives a contribution of order
\( \chi^2 \), so the vector \( t \) drops out of the result. Using the identities in Appendix A, one finds that

\[
\{ J[\xi_1], J[\xi_2] \}^* = -\frac{1}{16 \pi G} \int_{\mathcal{H}} \hat{e}_{a_1 \ldots a_{n-2}} \left[ \frac{1}{\kappa} (T_1 D^3 T_2 - T_2 D^3 T_1) - 2\kappa (T_1 D T_2 - T_2 D T_1) \right],
\]  

(4.19)

where terms of order \( \chi^2 \) have been omitted.

This expression has the characteristic three-derivative structure of the central term of a Virasoro algebra. According to eqn. (3.5), though, we must also compute the surface term \( J[\{ \xi_1, \xi_2 \}] \) of the Hamiltonian to obtain the complete expression for the central term in the constraint algebra. From eqn. (2.8), this Hamiltonian consists of two terms. The first is straightforward to compute: using the same methods that led to eqn. (4.18), one finds that

\[
Q_{a_1 \ldots a_{n-2}} = \frac{1}{16 \pi G} \hat{e}_{a_1 \ldots a_{n-2}} \left( 2\kappa T - \frac{1}{\kappa} D^2 T \right) + O(\chi^2).
\]  

(4.20)

The second term is more complicated, and is discussed in detail in Appendix B, where it is shown that it makes no further contribution. Hence combining eqns. (4.19) and (4.20), we obtain a central term

\[
K[\xi_1, \xi_2] = \frac{1}{16 \pi G} \int_{\mathcal{H}} \hat{e}_{a_1 \ldots a_{n-2}} \frac{1}{\kappa} \left( DT_1 D^2 T_2 - DT_2 D^2 T_1 \right),
\]  

(4.21)

and a centrally extended constraint algebra

\[
\{ J[\xi_1], J[\xi_2] \}^* = J[\{ \xi_1, \xi_2 \}] + K[\xi_1, \xi_2].
\]  

(4.22)

5 Counting States

Equations (4.14) and (4.21)–(4.22) are almost the standard Virasoro algebra for diffeomorphisms of the circle or the real line. This algebra consists of vectors \( \xi(z) \) and generators \( L[\xi] \) with Poisson brackets

\[
i \{ L[\xi_1], L[\xi_2] \} = L[\{ \xi_1, \xi_2 \}] + \frac{c}{24} \int \frac{dz}{2\pi i} (\xi_1'' \xi_2' - \xi_1' \xi_2'')
\]  

(5.1)

for a constant \( c \), the central charge. The only essential difference between (4.21)–(4.22) and (5.1) is the form of the integral on the right-hand side of eqn. (4.21). If we let \( v \) denote a parameter along the orbits of the Killing vector \( \chi^a \), normalized so that \( \chi^a \nabla_a v = 1 \), and consider \( T_1 \) and \( T_2 \) to be functions of \( v \) and of “angular” coordinates \( \theta^i \) on \( \mathcal{H} \), we must require that

\[
\int_{\mathcal{H}} \hat{e} T_1(v, \theta^i) T_2(v, \theta^i) = \text{const.} \int dv T_1(v, \theta^i) T_2(v, \theta^i)
\]  

(5.2)

to recover the algebra (5.1).
Note that the left-hand side of this expression involves integration only over the cross section \( H \), and not along the orbits of \( \chi^a \). This mismatch of integrations was first noticed by Cadoni and Mignemi in the context of boundary algebras in two-dimensional gravity [21]. They proposed defining new generators, which in the notation of this paper are essentially integrals \( \int dv J \), which then form a standard Virasoro algebra. In the present context, though, the meaning of such an additional \( v \) integration is not clear.

In the absence of such an additional integration, we must choose an “angular” dependence of the functions \( T_i \)—that is, a dependence on coordinates of \( H \)—to enforce eqn. (5.2). This is precisely what the orthogonality condition (4.13) does for us. If, for example, we consider functions of \( v \) with period \( 2\pi/\kappa \), as suggested by the Euclidean theory, and write our modes as

\[
T_n(v, \theta^i) = \frac{1}{\kappa} e^{i\pi v} f_n(\theta^i),
\]

then (4.13) requires that

\[
\int_H \hat{\epsilon} f_m f_n \sim \delta_{m+n},
\]

which in turn reproduces eqn. (5.2).

In particular, for a rotating stationary black hole, the Killing vector \( \chi^a \) that becomes null at the horizon is

\[
\chi^a = t^a + \sum_{(a)} \Omega (\alpha) \psi^a (\alpha),
\]

where \( t^a \) is the Killing vector corresponding to time translation invariance, \( \psi^a (\alpha) \) are the Killing vectors for rotational symmetry\(^{\ast} \) with corresponding angles \( \phi (\alpha) \), and the \( \Omega (\alpha) \) are angular velocities of the horizon. A one-parameter group of diffeomorphisms satisfying (5.4) that closes under the brackets (4.14) is then given by

\[
T_n = \frac{1}{\kappa} \exp \left\{ i n \left( \kappa v + \sum_{(a)} \ell_a (\phi (\alpha) - \Omega (\alpha) v) \right) \right\},
\]

where the \( \ell_a \) are arbitrary integers, at least one of which must be nonzero, and the normalization has been chosen so that

\[
\{T_m, T_n\} = -i(m - n) T_{m+n}
\]

in the brackets (4.14).

Diffeomorphisms of this form were first considered in ref. [6], where the angular dependence was introduced as an ad hoc requirement. At first sight, the specialization to this particular subgroup of diffeomorphisms seems artificial, but it is shown in Appendix B that such a restriction—or more properly, the orthogonality relation (5.4)—is forced upon us by the requirement that the generator \( H [\xi] \) be well defined. This is perhaps not too surprising: the conventional Virasoro algebra is essentially the only central extension

\(^{\ast}\)In four spacetime dimensions, there is only one \( \psi^a \), but in higher dimensions, rotations in orthogonal planes can commute, and distinct axial symmetries are allowed [22].
of $\text{Diff}\, S^1$, so one should expect the requirement of consistency to lead to such an algebra. We saw in section 4 that a suitable orthogonality condition can arise naturally from a boundary condition like (4.11) that restricts the horizon integral of the surface gravity. But the origin of such a boundary condition remains unclear. I will return to this issue in the conclusion.

Assuming the orthogonality relation (5.4), it is easy to see that the algebra (4.22) is now a conventional Virasoro algebra. The microscopic degrees of freedom, whatever their detailed characteristics, must transform under a representation of this algebra. But as Strominger observed [1], this means that these degrees of freedom have a conformal field theoretic description, and powerful methods from conformal field theory are available to analyze their properties.

In particular, we can now use the Cardy formula to count states. If we consider modes of the form (5.6), the central term (4.21) is easily evaluated:

$$K[T_m, T_n] = -\frac{i A}{8\pi G} m^3 \delta_{m+n,0},$$

(5.8)

where $A$ is the area of the cross section $\mathcal{H}$. The algebra (4.22) thus becomes

$$i\{J[T_m], J[T_n]\}^* = (m - n) J[T_{m+n}] + \frac{A}{8\pi G} m^3 \delta_{m+n,0},$$

(5.9)

which is the standard form for a Virasoro algebra with central charge

$$\frac{c}{12} = \frac{A}{8\pi G}.$$  

(5.10)

The Cardy formula also requires that we know the value of the boundary term $J[T_0]$ of the Hamiltonian. This can be computed from eqn. (4.20):

$$J[T_0] = \frac{A}{8\pi G},$$

(5.11)

where I have used the results of Appendix B to justify neglecting the second term in eqn. (2.8).

The Cardy formula then tells us that for any conformal field theory that provides a representation of the Virasoro algebra (5.9)—modulo certain assumptions discussed in Appendix C—the number of states with a given eigenvalue $\Delta$ of $J[T_0]$ grows asymptotically for large $\Delta$ as

$$\rho(\Delta) \sim \exp \left\{ 2\pi \sqrt{\frac{c}{6} \left( \Delta - \frac{c}{24} \right)} \right\},$$

(5.12)

Inserting eqns. (5.10) and (5.11), we find that

$$\log \rho \sim \frac{A}{4G},$$

(5.13)

giving the expected behavior of the entropy of a black hole.
6 Some Examples

The derivation in the preceding section focused on the entropy of stationary black holes in ordinary general relativity. But it is easily generalized to a number of other interesting configurations. In this section I briefly discuss some of these.

a Rindler Space

A uniformly accelerated observer perceives a Killing horizon that is locally identical to that of a black hole. Since the derivation above required only local information about the horizon, it applies equally well to Rindler space. Subject to appropriate boundary conditions, quantum gravitational states in Rindler space must transform under a representation of the Virasoro algebra (5.9), and the density of states should again be governed by (5.13), which should now be interpreted as giving an entropy per unit horizon area.

Whether this is a reasonable result is a matter of debate in the literature. It seems inevitable, however, that any local description of black hole entropy in terms of horizon observables will apply to Rindler space as well. The advantage of the present approach is that entropy is defined relative to a boundary. For Rindler space, the degrees of freedom counted by eqn. (5.13) are relevant only if one imposes suitable boundary conditions at the horizon. Since these boundary conditions imply that information really is “lost” when it passes through the horizon, it is perhaps not unreasonable to attribute an entropy to the horizon.

This example illustrates a somewhat counterintuitive feature of quantum theory: the existence of a boundary can sometimes increase the number of degrees of freedom. For topological quantum field theories, this phenomenon has been studied in detail [24]. In Chern-Simons theories, for example, a “bulk” theory with only finitely many degrees of freedom can induce a Wess-Zumino-Witten model with infinitely many degrees of freedom on a boundary [25]. For these theories, the origin of the new degrees of freedom is understood: because boundary conditions limit admissible gauge transformations, quantities that would be considered “pure gauge” in the bulk become independent physical degrees of freedom on the boundary [26,27]. It is also possible to trace what happens when a boundary is eliminated, for example by “gluing” fields on two sides of a surface and summing over boundary values [28]. In that event, the full gauge invariance is restored, and the added symmetries lead to a reduction in the number of physical degrees of freedom.

While a full analysis of this sort is not yet available for quantum gravity, these examples suggest a way to make sense of the idea that Rindler space has a higher entropy than flat Minkowski space. Rindler space is equivalent to a wedge of Minkowski space, but with additional boundary conditions that are not present in the full Minkowski space. The resulting boundary degrees of freedom presumably disappear when one glues back the rest of Minkowski space and sums over boundary values, thereby eliminating the effect of the boundary conditions.
b de Sitter Space

The methods introduced here may also be applied to cosmological horizons in de Sitter space [29]. The de Sitter metric in stationary coordinates can be written as

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2d\Omega^2,$$

where $\Lambda = 3/\ell^2$ in four space time dimensions. The horizon at $r = \ell$ is a Killing horizon for the Killing vector

$$\chi^a = \left(\frac{\partial}{\partial t}\right)^a,$$

and the analysis of the preceding sections goes through with virtually no changes, yielding an entropy

$$S = \frac{A_{\text{hor}}}{4G} = \frac{3\pi}{G\Lambda}.$$

Note that here, as in Rindler space, the horizon is associated with a particular set of observers. For de Sitter space, however, the existence of an associated entropy seems to be less debated; in particular, standard Euclidean path integral methods [30] yield an entropy that agrees with that of eqn. (6.3).

c Taub-NUT and Taub-Bolt Spaces

Hawking and Hunter have recently investigated the entropies of Taub-NUT and Taub-Bolt spaces in the Euclidean path integral approach [31,32]. When analytically continued to Riemannian signature, these spaces have metrics of the form

$$ds^2 = V\left(dt + 4n\cos^2\frac{\theta}{2}d\phi\right)^2 + V^{-1}dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta\,d\phi^2),$$

where $V$ is a function of $r$ and $n$ is a constant, the NUT charge. The metric has a string singularity along the positive $z$ axis (i.e., at $\theta = 0$), a “Misner string,” whose existence is signaled by the fact that a small loop around the axis does not shrink to zero proper length. The Killing vector

$$\chi^a = \left(\frac{\partial}{\partial t}\right)^a - \frac{1}{4n}\left(\frac{\partial}{\partial \phi}\right)^a$$

has a norm that vanishes at $\theta = 0$, and its Killing horizon consequently has a one-dimensional component along the positive $z$ axis.

One can now define a stretched horizon around the Misner string at $\chi^2 = \epsilon$ and proceed exactly as above. The surface gravity $\kappa$ in eqn. (4.1) may be fixed by requiring that $\rho^2 + \chi^2 \to 0$ at the horizon; the result is that

$$\kappa = \frac{1}{4n},$$

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yielding the correct $8\pi n$ periodicity for the Euclidean theory. The remainder of the derivation is essentially unchanged. (Details will be published elsewhere [33].) Using the Cardy formula to count states on the string, one obtains a formal expression

$$S_{\text{string}} = \frac{A_{\text{string}}}{4G},$$

(6.7)

for the entropy, where the induced volume element at $\theta = 0, t = \text{const.}$ is

$$\hat{\epsilon} = 4ndr d\phi,$$

(6.8)

and correspondingly

$$A_{\text{string}} = 8\pi n \int_{r_0}^{\infty} dr.$$  (6.9)

An additional contribution to $S$ comes from the “bolt,” which is a horizon for the Killing vector $\tilde{\chi} = \partial / \partial t$. ($\chi$ and $\tilde{\chi}$ have in common that the lapse function vanishes at their horizons.)

As in ref. [31], expression (6.9) is divergent. Again, however, as in [31], one can compare the entropy of Taub-Bolt space to that of a reference Taub-NUT space. Combining contributions from the Misner string and the “bolt,” one finds

$$S_{\text{Taub-Bolt}} - S_{\text{Taub-NUT}} = \frac{1}{4G} \left[ A_{\text{bolt}}^{\text{Taub-Bolt}} + A_{\text{string}}^{\text{Taub-Bolt}} - A_{\text{string}}^{\text{Taub-NUT}} \right] = \frac{\pi n^2}{G},$$

(6.10)

in agreement with the results of Hawking and Hunter.

It should be possible to extend these results to the asymptotically anti-de Sitter case discussed in ref. [32]. There have also been several recent attempts to regulate the Taub-NUT and Taub-Bolt entropy by adding counterterms at infinity [34,35]; it would be interesting to understand these in the light of the conformal field theory methods described here. Work on these issues is in progress.

**d Dilaton Gravity**

So far, we have only looked at standard general relativity. But the methods of this paper can be easily extended to other covariant theories of gravity. As an example, let us consider a general two-dimensional dilaton gravity theory, as described by Gegenberg, Kunstatter, and Louis-Martinez [36].

After suitable field redefinitions, the Lagrangian two-form for this model takes the form

$$L_{ab} = \frac{1}{2G} \epsilon_{ab} \left( \phi R + \frac{1}{\ell^2} V(\phi) \right),$$

(6.11)

where $V$ is an arbitrary function of the dilaton field $\phi$. (The kinetic term for $\phi$ has been absorbed into $\phi R$ by a Weyl rescaling of the metric.) Black hole solutions are characterized by a Killing vector

$$\chi^a = \frac{\ell}{\sqrt{-g}} \epsilon^{ab} \nabla_b \phi$$

(6.12)
whose norm vanishes at the horizon $\phi = \phi_0$, i.e.,

\[ \chi^2(\phi_0) = -\ell^2 g^{ab} \nabla_a \phi \nabla_b \phi \big|_{\phi = \phi_0} = 0. \]  

(6.13)

The vector $\rho^a$ is fixed near the horizon by the orthogonality condition $\rho_a \chi^a = 0$ and the requirement that $\rho^2 / \chi^2 \to -1$; it is

\[ \rho_a = \ell \nabla_a \phi + O(\chi^2). \]  

(6.14)

This can be checked explicitly from the solutions in ref. [36]. Note that

\[ \chi^a \nabla_a \phi = 0 \quad \text{everywhere} \]

\[ \rho^a \nabla_a \phi \to 0 \quad \text{at the horizon}, \]  

(6.15)

where the second line follows from eqn. (6.13).

The symplectic potential $\Theta_a$ is easily determined from the definition (2.1). One obtains

\[ \Theta_a = 8\pi \phi \Theta^{\text{grav}}_a + \frac{1}{2G} \epsilon_{ab} \left[ \nabla^a \phi g_{bc} \delta g^{bc} - \nabla_c \phi \delta g^{bc} \right], \]  

(6.16)

where $\Theta^{\text{grav}}$ is the symplectic potential (2.11) for Einstein gravity in two dimensions. Similarly, the Noether charge takes the form

\[ Q[\xi] = 8\pi \phi Q^{\text{grav}}[\xi] + \frac{1}{G} \xi^c \epsilon_{bc} \nabla^b \phi, \]  

(6.17)

where $Q^{\text{grav}}$ is given by eqn. (2.12).

We can now use eqns. (3.3)–(3.5) to evaluate the central term in the constraint algebra at the horizon. From eqn. (6.15), we see that the second term in (6.16) gives no contribution to $K[\xi_1, \xi_2]$. Similarly, the second term in eqn. (6.17) vanishes at the horizon. We thus find that

\[ K[\xi_1, \xi_2] = 8\pi \phi_0 K^{\text{grav}}[\xi_1, \xi_2] \]

\[ J[\xi_0] = 8\pi \phi_0 J^{\text{grav}}[\xi_0], \]  

(6.18)

where $K^{\text{grav}}$ is given by eqn. (4.21).

We must now confront the “orthogonality problem” discussed at the beginning of section 5. The boundary $\mathcal{H}$ is now a point, so there are no “angular” integrals with which to impose condition (5.2). This is precisely the problem faced by Cadoni and Mignemi [21] at the boundary of two-dimensional asymptotically anti-de Sitter space, and it presumably reflects some of the difficulties in applying the AdS/CFT correspondence in two dimensions [37]. We can proceed as in ref. [21], by either defining new integrated generators $\int dv J[\xi]$ or by interpreting the Lagrangian (6.11) as one coming from dimensional reduction, with hidden “angular” dependence.

With either choice, it is straightforward to repeat the analysis of section 5. Thanks to the relation (6.18), this last step is trivial; we can simply substitute (6.18) into our previous results, to find

\[ \log \rho \sim \frac{8\pi \phi_0}{4G}, \]  

(6.19)

which is precisely the entropy obtained by Gegenberg et al. [36].
7 Conclusions and Open Questions

This paper began with a puzzle: how can the microscopic states of responsible for black hole thermodynamics “know” about the results of semiclassical computations temperature and entropy? I have suggested a possible answer: the symmetries of classical general relativity may be powerful enough to determine the asymptotic behavior of the density of states in any quantum theory of gravity, independent of the microscopic details. This is perhaps an unusual role for a group of symmetries, but it is not unheard of; indeed, in two-dimensional conformal field theory it is commonplace to use the Virasoro algebra and the Cardy formula to count states.

Clearly, the most serious technical shortcoming of this analysis is the poor understanding of the orthogonality conditions (4.13) and (5.4), which are necessary for the existence of a canonical Hamiltonian and a Virasoro algebra. We saw that a boundary condition like that of eqn. (4.11), which fixes an integral over the horizon, can lead to such orthogonality relations, but the argument is rather indirect, and seems to break down for two-dimensional theories. It seems likely that conditions (4.13) and (5.4) have a deeper significance that is not yet understood. In string theory, similar relations arise because black holes are often really compactified black strings; the integration that leads to the orthogonality in (5.4) is an integration over a compact dimension. In standard general relativity, it would be interesting to investigate the algebra of constraints in a null surface formulation [38,39], in which integrals along the horizon like those appearing in eqn. (5.2) might arise more naturally. Unfortunately, such an extension is not easy, since the constraint algebra on a null surface involves second class constraints.

Several obvious generalizations of this work should be possible. First, the boundary condition I have chosen—the existence of a local Killing horizon—is by no means the most general; it would be interesting to understand the application of these techniques to, for example, Ashtekar’s “isolated horizons” [19]. It should also be straightforward to extend these methods to a much wider class of gravitational theories, perhaps obtaining the generalized entropy formula of ref. [12].

Finally, a crucial step would be to extend the methods developed here to dynamical black holes. By choosing as my boundary the Killing horizon for a fixed Killing vector, I have implicitly ruled out dynamical processes such as black hole evaporation that require an evolving horizon. Strominger’s approach [1], by way of contrast, leads to a single Virasoro algebra that incorporates states corresponding to black holes with many masses and spins, but it does so by imposing boundary conditions at infinity rather than at the horizon. Ideally, one would like to combine these two approaches, finding boundary conditions that refer to a particular horizon—thus isolating the degrees of freedom of a specific black hole—but that are also loose enough to allow that black hole to evolve in time.

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Appendix A  Some Useful Identities

In this appendix, I collect some useful identities involving $\chi^a$ and $\rho^a$. First,

$$\nabla_a \rho_b = -\frac{1}{2\kappa} \nabla_a \nabla_b \chi^2 = \nabla_b \rho_a.$$  \hspace{1cm} (A.1)

$$\rho^a \nabla_a \chi^b - \chi^a \nabla_a \rho^b = -\rho^a \nabla_b \chi_a - \chi^a \nabla_a \rho^b = \chi_a (\nabla^b \rho^a - \nabla^a \rho^b) = 0.$$  \hspace{1cm} (A.2)

$$\chi^a \nabla_a \chi^b = -\chi^a \nabla^b \chi_a = \kappa \rho^b.$$  \hspace{1cm} (A.3)

$$\frac{\chi^a \chi^b}{\chi^2} \nabla_a \rho_b = \frac{-\chi^b \rho^a}{\chi^2} \nabla_a \chi - \frac{\kappa \rho^2}{\chi^2}.$$  \hspace{1cm} (A.4)

Next, let

$$\chi[a \nabla_b \chi_c] = \omega_{abc}.$$  \hspace{1cm} (A.5)

Then

$$\omega_{abc} \omega^{abc} = \frac{1}{3} \chi^2 (\nabla_a \chi_b)(\nabla^a \chi^b) - \frac{2}{3} \kappa \rho^2,$$  \hspace{1cm} (A.6)

so

$$\nabla_a \rho^a = \frac{1}{\kappa} \nabla_a (\chi^b \nabla_b \chi^a)$$

$$= -\frac{1}{\kappa} (\nabla_a \chi_b)(\nabla^a \chi^b) + \frac{1}{\kappa} R_{ab} \chi^a \chi^b = -2\kappa \frac{\rho^2}{\chi^2} + O(\chi^2)$$  \hspace{1cm} (A.7)

where the last equality uses the fact that $\omega^2/\chi^2$ goes to zero at the horizon [23]. From eqns. (A.4) and (A.7), we see that

$$\frac{\rho^a \rho^b}{\rho^2} \nabla_a \rho_b = \left(g^{ab} - \frac{\chi^a \chi^b}{\chi^2} - \sigma^{ab}\right) \nabla_a \rho_b = -\kappa \frac{\rho^2}{\chi^2} + O(\chi^2),$$  \hspace{1cm} (A.8)

where

$$\sigma^{ab} = g^{ab} - \frac{\chi^a \chi^b}{\chi^2} - \frac{\rho^a \rho^b}{\rho^2},$$  \hspace{1cm} (A.9)

and I have assumed that “spatial” derivatives of $\rho^a$ and $\chi^a$, that is, derivatives projected by $\sigma$, are $O(\chi^2)$ near the horizon. Further, since $(\nabla_a \chi_b)(\nabla^a \chi^b) = -2\kappa^2$ and $\omega_{abc} \omega^{abc} = 0$ on the horizon [23], it follows from (A.6) that

$$\frac{\rho^2}{\chi^2} = -1 + O(\chi^2).$$  \hspace{1cm} (A.10)
Appendix B  The Hamiltonian for General Relativity

We know from eqn. (2.8) that the Hamiltonian for general relativity can be written as a sum of two terms. The first of these terms, \( \int Q \), was evaluated in section 4. In this appendix, I discuss the second term, which must be determined by solving eqn. (2.9) for the \((n-1)\)-form \(B\).

As in section 4, we shall treat \(\chi^a\) and \(\rho_a\) as fixed vectors, and require that variations satisfy the boundary conditions (4.3). This means that \(\delta \chi^2 = 0\) and \(\chi^a \delta \chi_a = 0\), up to terms of order \(\chi^2\) that will drop out at the horizon. In analogy with the boundary condition (4.9), let us also set the \(\rho^a\) derivatives of our variations to zero at the boundary:

\[
\rho^a \nabla_a (g_{bc} \delta g^{bc}) = 0, \quad \rho^a \nabla_a \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right) = \rho^a \nabla_a \left( \frac{\delta \rho^2}{\rho^2} \right) = 0 \quad \text{at} \quad \chi^2 = 0. \quad (B.1)
\]

As discussed in section 4, for the Kerr metric in Boyer-Lindquist coordinates this is the requirement that radial derivatives not blow up at the horizon.

Now let \(\xi^a\) be a vector of the form (4.6). From eqn. (2.11), it is not hard to show that

\[
16\pi G \xi^b \Theta_{a_1 \ldots a_{n-2}} = -\hat{\epsilon}_{a_1 \ldots a_{n-2}} (TA + RB) + O(\chi^2), \quad (B.2)
\]

with

\[
A = \frac{\vert \chi \vert}{\rho} \left( \rho^b \nabla_c (g_{ab} \delta g^{bc}) - \rho_b \nabla_c \delta g^{bc} \right) = \frac{\vert \chi \vert}{\rho} \left( \nabla_b \rho_c \delta g^{bc} - \nabla_c \delta \rho^a \right)
\]

\[
B = \frac{\rho}{\vert \chi \vert} \left( \chi^b \nabla_c (g_{ab} \delta g^{bc}) - \chi_b \nabla_c \delta g^{bc} \right) = \frac{\rho}{\vert \chi \vert} \left( \chi^b \nabla_c (g_{ab} \delta g^{ab}) + g^{bc} \nabla_b \delta \chi_c \right), \quad (B.3)
\]

where I have used an argument parallel that following eqn. (4.15) to eliminate some terms involving variations tangent to the horizon. Using identities from Appendix A, one finds that

\[
A = \frac{\vert \chi \vert}{\rho} \left\{ \kappa \frac{\delta \rho^2}{\chi^2} + \chi^a \nabla_a \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right) \right\} + O(\chi^2)
\]

\[
B = \frac{\rho}{\vert \chi \vert} \left\{ -2\delta (\nabla_a \chi^a) - 2\kappa \frac{\rho^b \delta \chi_b}{\chi^2} \right\} + O(\chi^2). \quad (B.4)
\]

For a diffeomorphism satisfying the boundary condition (4.8), eqn. (B.2) thus gives

\[
\xi^b \Theta_{a_1 \ldots a_{n-2}} = -\frac{1}{16\pi G} \hat{\xi}_{a_1 \ldots a_{n-2}} \frac{\vert \chi \vert}{\rho} \left\{ \kappa \frac{\delta \rho^2}{\chi^2} + D \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right) \right\} - \frac{1}{\kappa} DT \left\{ -2\delta (\nabla_a \chi^a) - 2\kappa \frac{\rho^b \delta \chi_b}{\chi^2} \right\} + O(\chi^2), \quad (B.5)
\]
or equivalently
\[
x^b \Theta_{ba_1 \ldots a_{n-2}} = -\frac{1}{16\pi G} \hat{\epsilon}_{a_1 \ldots a_{n-2}} \left\{ -T \left[ 2\kappa \frac{\delta \rho}{\rho} + \frac{\chi}{\rho} D \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right) \right] \right.
\]
\[+ \frac{2}{\kappa} \frac{\chi}{\rho} DT \delta(\nabla_a \chi^a) + 2D \left( \frac{\chi}{\rho} T \frac{\rho^b \delta \chi_b}{\chi^2} \right) \right\} + O(\chi^2).
\]

Now, if \( \hat{\epsilon} \) were fixed—that is, if we froze the induced metric on the boundary \( \mathcal{H} \)—then the variations in eqn. (B.6) could be pulled through the prefactor \( \hat{\epsilon} \), and one could write the entire expression as a variation \( \delta(\xi - \mathbf{B}) \). But this is the wrong boundary condition for a black hole horizon \([20, 40]\); one should rather hold fixed the momentum conjugate to the horizon metric.† In general, the variation \( \delta \hat{\epsilon} \) will be independent of the variations \( \delta \chi_a \) and \( \delta \rho_a \) appearing in eqn. (B.6). This means that \( \xi^b \Theta_b \) will be a total variation only if (B.6) can be written in the form \( \delta(\xi - \mathbf{B}) \times \text{terms that vanish on shell} \).

For the first two terms, this is not hard. It is straightforward to check that
\[
\delta \left( \frac{\chi^a \rho^b}{\chi^2} (\nabla_a \chi_b + \nabla_b \chi_a) \right) = D \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right)
\]
\[ - \frac{1}{2} \frac{\chi}{\rho} \delta \left( \frac{(\rho^a - \chi^a)(\rho_a - \chi_a)}{\chi^2} \right) = \frac{\delta \rho}{\chi}, \quad (B.7)
\]
and the left-hand side of both of these expressions vanishes at \( \mathcal{H} \) when the boundary is a Killing horizon. Similarly, \( \nabla_a \chi^a = 0 \) when \( \chi^a \) is a Killing vector. We can thus write
\[
x^b \Theta_{ba_1 \ldots a_{n-2}} = \delta \left( x^b \mathbf{B}_{ba_1 \ldots a_{n-2}} \right) - \frac{1}{8\pi G} \hat{\epsilon}_{a_1 \ldots a_{n-2}} D \left( \frac{\chi}{\rho} T \frac{\rho^b \delta \chi_b}{\chi^2} \right) + O(\chi^2) \quad (B.8)
\]
with
\[
x^b \mathbf{B}_{ba_1 \ldots a_{n-2}} = -\frac{1}{16\pi G} \hat{\epsilon}_{a_1 \ldots a_{n-2}} \left\{ \frac{\chi}{\rho} T \left[ \kappa \frac{(\rho^a - \chi^a)(\rho_a - \chi_a)}{\chi^2} \right] \right.
\]
\[ - \frac{\chi^a \rho^b}{\chi^2} (\nabla_a \chi_b + \nabla_b \chi_a) \right\} + \frac{2}{\kappa} \frac{\chi}{\rho} DT \nabla_a \chi^a \quad (B.9)
\]

It remains for us to deal with the last term in eqn. (B.8). In general, this expression cannot be written as a total variation unless we either strengthen the boundary conditions or further restrict the allowed variations of the metric. The basic problem is that the quantity \( \rho^b \delta \chi_b/\chi^2 \) is not itself the variation of a local function. Indeed, the commutator
\[
\delta_2 \left( \frac{\rho^b \delta \chi_b}{\chi^2} \right) - \delta_1 \left( \frac{\rho^b \delta_2 \chi_b}{\chi^2} \right) = \frac{\delta_2 \rho^2 \rho^b \delta \chi_b}{\chi^2} - \frac{\delta_1 \rho^2 \rho^b \delta \chi_b}{\chi^2} + O(\chi^2) \quad (B.10)
\]

†In the Euclidean theory, this conjugate variable is the deficit angle at the horizon. Note that variations of the horizon metric have dropped out of eqn. (B.6) because so far they have appeared only in terms of order \( \chi^2 \), not because they have been set to zero.
would have to vanish if $\rho^b \delta \chi_b / \chi^2$ were a total variation. But it is evident that this quantity is not in general zero, since $\delta \rho^2$ and $\rho^a \delta \chi_a$ can be specified independently.

We now have three choices. First, we can try to strengthen our boundary conditions, for instance by fixing $\chi_a$ at $\mathcal{H}$, to allow the Hamiltonian to be defined. Fixing $\chi_a$ is too strong a restriction, though—it is incompatible with the existence of diffeomorphisms of the form (4.8)—and it seems difficult to find an alternative weak enough to allow any interesting central extensions of Diff$M$ to remain. Second, we can consider “integrated generators” $\int dv H[\xi]$, as introduced by Cadoni and Mignemi [21] and discussed briefly in section 5. The $v$ integral would then eliminate the last term in eqn. (B.8). But as noted in section 5, the meaning of such generators is unclear in the present context.

Our third alternative is to restrict our field variations to bring the last term in eqn. (B.8) under control. To analyze this possibility, let us consider mode expansions of $T$ and $\rho^b \delta \chi_b / \chi^2$,

$$T = \sum_n T_n e^{i \kappa n v}, \quad \frac{\rho^b \delta \chi_b}{\chi^2} = \sum_n b_n e^{i \kappa n v},$$

where the period $2\pi/\kappa$ has been chosen for convenience. The term in question then consists of a sum of pieces of the form

$$\int_{\mathcal{H}} \hat{H} \epsilon^{b_1 \ldots b_{n-2}} \rho^b \delta \chi_b \chi^2 = \int_{\mathcal{H}} \hat{H} \epsilon D \left( \frac{\chi_b \delta \rho^b}{\chi^2} \right),$$

as can easily be seen from the identities in Appendix A. We must now consider what variations $\chi_b \delta \rho^b / \chi^2$ are allowed. We must certainly permit variations $\delta \xi \rho^a = (\delta \xi g^{ab}) \rho_b$ corresponding to diffeomorphisms generated by vector fields satisfying the conditions (4.6), (4.8), and (4.9). But for consistency, we must then also allow variations of the form $\delta(\delta \xi \rho^a)$ whenever $\delta \rho^a$ is itself allowed. For such variations, eqn. (B.13) becomes

$$\int_{\mathcal{H}} \hat{H} \epsilon D \left( DT_1 \frac{\rho^b \delta \chi_b}{\chi^2} \right) = 0,$$

Together with the mode expansion (B.11), eqns. (B.13) and (B.14) give (B.12), as required.
Finally, let us verify that (B.9) gives the appropriate contribution to the Dirac bracket (4.19). To check this, we must look at the variation of $B$ under a second diffeomorphism that satisfies the conditions (4.6), (4.8), and (4.9). Under such a variation,

$$g_{ab} \delta \xi^a g^{ab} = -2 \nabla_a \xi^a = 2DT_2$$

$$\delta \xi^a \rho^a = (\delta \xi^a g^{ab}) \rho_b = -\frac{1}{\kappa} D^2 T_2 \chi^a + 2DT_2 \rho^a + O(\chi^2),$$  \hfill (B.15)

and hence

$$\delta \xi^a (\nabla_a \chi^a) = -\frac{1}{2} D(g_{ab} \delta \xi^a g^{ab}) = -D^2 T_2$$

$$\delta \xi^a \left( \frac{\chi^a \rho^b}{\chi^2} (\nabla_a \chi_b + \nabla_b \chi_a) \right) = -D \left( \frac{\chi^a \delta \xi^b}{\chi^2} \right) = \frac{1}{\kappa} D^3 T_2 + O(\chi^2)$$

$$\delta \xi^a \left( \frac{\rho^a - \chi^a}{\chi^2} (\rho_a - \chi_a) \right) = \delta \xi^a \rho^a \frac{\rho^2}{\chi^2} = 2DT_2 + O(\chi^2).$$  \hfill (B.16)

Note that the vector $\chi^a$ and the one-form $\rho_a$ have been held fixed in these variations, since they are being treated as fixed, field-independent parameters, while $\delta \xi^a$ means the variation induced by the Poisson brackets on the phase space. Substituting eqn. (B.16) into eqn. (B.9), we see that

$$\delta \xi^a \int_{H^a} B_{ba_1...a_{n-2}}$$

$$= -\frac{1}{16\pi G} \int_{H^a} \hat{\epsilon}_{a_1...a_{n-2}} \left\{ -\frac{2}{\kappa} D \left( \frac{\rho^2}{\rho} T_1 D^2 T_2 \right) + \frac{1}{\kappa} D^3 T_2 - 2\kappa T_1 DT_2 \right\} + O(\chi^2).$$

By the orthogonality conditions discussed above, the first term gives no contribution, and we find exact agreement with the terms proportional to $T_1$ in eqn. (4.19).

**Appendix C  Does the Cardy Formula Apply?**

Begin with a conformal field theory on the plane. Such a theory is characterized by a pair of Virasoro algebras, one for left-moving modes and one for right-moving modes, and states will fall into representations of these algebras. Conversely, any theory whose states provide a representation of a Virasoro algebra has a conformal field theoretic description.

Since the plane is conformal to the cylinder, we can transform our theory to one on a cylinder; the central terms a conformal anomaly, but its effect on such a transformation is simply to shift the stress-energy tensor [41]. To count states, we can now use a standard trick: we first compute the partition function, and then obtain the density of states from a Legendre transformation. We therefore continue our theory to imaginary
time and compactify the cylinder to a torus of modulus $\tau$. The partition function is then

$$Z(\tau, \bar{\tau}) = \text{Tr} e^{2\pi i \tau L_0} e^{2\pi i \bar{\tau} \bar{L}_0} = \sum \rho(\Delta, \bar{\Delta}) e^{2\pi i \Delta} e^{2\pi i \bar{\Delta}}, \quad \text{(C.1)}$$

and if we can determine $Z$, we can extract the density of states $\rho(\Delta, \bar{\Delta})$ by means of a contour integral. It should be stressed that the transformation from the plane to the cylinder and the continuation to imaginary time are merely tricks to obtain the density of states; we are not assuming any fundamental role for compact spaces or Euclidean signature.

The derivation of the Cardy formula starts with the observation [3] that the quantity

$$Z_0(\tau, \bar{\tau}) = \text{Tr} e^{2\pi i (L_0 - \frac{c}{24})} e^{2\pi i (\bar{L}_0 - \frac{c}{24})} \quad \text{(C.2)}$$

is invariant under modular transformations, the large diffeomorphisms of the torus. In particular, $Z_0$ is invariant under the $S$ transformation $\tau \rightarrow -1/\tau$. Using this invariance, one can write $Z(\tau, \bar{\tau})$ in terms of $Z(-1/\tau, -1/\bar{\tau})$ and a rapidly varying phase, and use the method of steepest descents to extract $\rho(\Delta, \bar{\Delta})$. Details are given in ref. [4]; the general result is that

$$\rho(\Delta) \sim \exp \left\{ \frac{2\pi}{6} \left( \frac{c_{\text{eff}}}{c} - \frac{c_{\text{eff}}}{24} \right) \right\} \rho(\Delta_0), \quad \text{(C.3)}$$

where the “effective central charge” is

$$c_{\text{eff}} = c - 24\Delta_0. \quad \text{(C.4)}$$

and $\Delta_0$ is the lowest eigenvalue of $L_0$ in the trace (C.1).

To determine the applicability of this formula to the problem discussed in this paper, we must check several points. First, the conformal field theories for which the Cardy formula was developed are two-dimensional and have two Virasoro algebras, while we have no obviously important two-manifold and have only one Virasoro algebra. The derivation of the Cardy formula (C.3), however, requires few of the details of conformal field theory; all that is really needed is the existence of a Virasoro algebra and the diffeomorphism invariance expressed by eqn. (C.2). In essence, one may forget about the original physical motivation, and view the Cardy formula as a statement about representations of Diff$S^1$. In particular, left- and right-moving states in a conformal field theory effectively decouple, and the central extension described in this paper simply corresponds to a conformal field theory in which one sector is absent.

Second, the derivation described in this appendix implicitly assumed that $L_0$ had a discrete spectrum. While much of section 5 was also based on a discrete set of modes—see, for example, eqn. (5.3)—it is not clear that this is an appropriate assumption, and one might well want to recast the argument to describe a continuous set of modes. This presents no difficulty, however: the derivation of the Cardy formula requires only small modifications when $L_0$ has a continuous spectrum. In that case, one should understand $\rho(\Delta) d\Delta$ as a density of states in an interval $d\Delta$ of eigenvalues of $L_0$, but the interpretation of $S$ in eqn. (5.13) as entropy remains unchanged.
Finally, we must worry about the value of $\Delta_0$, and the difference between the central charge $c$ and the effective central charge $c_{\text{eff}}$ that appears in the Cardy formula. Here, the methods of this paper have nothing to say: one can determine $\Delta_0$ and $c_{\text{eff}}$ only when one has a concrete conformal field theory to represent the horizon degrees of freedom.

References