Hawking Radiation As Tunneling

Maulik K. Parikh

*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA*

*Spinoza Institute, University of Utrecht, 3584 CE Utrecht, The Netherlands*

Frank Wilczek

*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*

Abstract

We present a short and direct derivation of Hawking radiation as a tunneling process, based on particles in a dynamical geometry. The imaginary part of the action for the classically forbidden process is related to the Boltzmann factor for emission at the Hawking temperature. Because the derivation respects conservation laws, the exact spectrum is no longer precisely thermal.

I. INTRODUCTION

Broadly speaking, there are two standard approaches to Hawking radiation within quantum field theory; both have roots in the very early literature [1,2]. In the first, one considers a collapse geometry [1]. The response of external fields to this can be done explicitly, or implicitly by abstracting appropriate boundary conditions. In the second, one treats the black hole immersed in a thermal bath [2]. In this approach, one shows that (in general, metastable) equilibrium is possible. By detailed balance, this implies the possibility of emission from the hole.
Neither of these approaches corresponds very directly to a heuristic picture often proposed to visualize the source of the radiation, as tunneling. According to this picture, the radiation arises by a process analogous to electron-positron pair creation in a constant electric field. The idea is that, since the energy of a particle changes sign as it crosses the horizon, a pair created just inside or just outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side.

Here we shall show, expanding on ideas in [3,4], that this heuristic can be used in a short, direct semi-classical derivation of black hole radiance (a related approach is pursued in [5]). In what follows, energy conservation plays a fundamental role: one must make a transition between states with the same total energy, and the mass of the residual hole must go down as it radiates. Indeed, it is precisely the possibility of lowering the black hole mass that drives the dynamics. This supports the idea that, in quantum gravity, black holes are properly regarded as highly excited states.

We will consider a hole in empty Schwarzschild space, but with a dynamical geometry. Note that the geometry is not truly static even classically since there is no global Killing vector. By contrast, in both the standard calculations, the background geometry is considered fixed, and energy conservation is not enforced during the emission process. Because we are treating this aspect more realistically, we must – and do – find corrections to the standard results. These become qualitatively significant when the quantum of radiation carries away a substantial fraction of the mass of the hole.

II. Calculation

To describe across-horizon phenomena, it is necessary to choose coordinates that, unlike Schwarzschild coordinates, are not singular at the horizon. A particularly suitable choice is obtained by introducing a time coordinate,

\[ t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2}M}{\sqrt{r} + \sqrt{2}M}, \]

where \( t_s \) is Schwarzschild time. With this choice, the line element reads
\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + 2 \sqrt{\frac{2M}{r}} \ dt \ dr + dr^2 + r^2 d\Omega^2 . \]  

(2)

There is now no singularity at \( r = 2M \), and the true character of the spacetime, as being stationary but not static, is manifest. These coordinates were first introduced by Painlevé [6] (who used them to criticize general relativity, for allowing singularities to come and go!). Their utility for studies of black hole quantum mechanics was emphasized more recently in [7].

For our purposes, the crucial features of these coordinates are that they are stationary and nonsingular through the horizon. Thus it is possible to define an effective “vacuum” state of a quantum field by requiring that it annihilate modes which carry negative frequency with respect to \( t \). Such a state will look essentially empty (in any case, nonsingular) to a freely-falling observer as he or she passes through the horizon. This vacuum differs strictly from the standard Unruh vacuum, defined by requiring positive frequency with respect to the Kruskal coordinate \( U = -\sqrt{r - 2M} \exp \left( -\frac{t - r}{4M} \right) \) [8]. The difference, however, shows up only in transients, and does not affect the late-time radiation.

The radial null geodesics are given by

\[ \dot{r} \equiv \frac{dr}{dt} = \pm \sqrt{\frac{2M}{r}} , \]

(3)

with the upper (lower) sign in Eq. (3) corresponding to outgoing (ingoing) geodesics, under the implicit assumption that \( t \) increases towards the future. These equations are modified when the particle’s self-gravitation is taken into account [3]. When the black hole mass is held fixed and the total ADM mass allowed to vary, a shell of energy \( \omega \) moves in the geodesics of a spacetime with \( M \) replaced by \( M + \omega \). If instead we fix the total mass and allow the hole mass to fluctuate, then the shell of energy \( \omega \) travels on the geodesics given by the line element

\[ ds^2 = - \left( 1 - \frac{2(M - \omega)}{r} \right) dt^2 + 2 \sqrt{\frac{2(M - \omega)}{r}} \ dt \ dr + dr^2 + r^2 d\Omega^2 , \]

(4)

so we should use Eq. (3) with \( M \rightarrow M - \omega \).
One might be concerned that since the typical wavelength of the radiation is of the order of the size of the black hole, a point particle description is inappropriate. However, when the outgoing wave is traced back towards the horizon, its wavelength, as measured by local fiducial observers, is ever-increasingly blue-shifted. Near the horizon, the radial wavenumber approaches infinity and the point particle, or WKB, approximation is justified.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from $r_{\text{in}}$ to $r_{\text{out}}$ can be expressed as

$$\text{Im } S = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r \, dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp' \, dr.$$  \hspace{1cm} (5)

Remarkably, this can be evaluated without entering into the details of the solution, as follows (compare [4]). We multiply and divide the integrand by the two sides of Hamilton’s equation \( \dot{r} = + \frac{dH}{dp_r} \bigg|_r \), change variable from momentum to energy, and switch the order of integration to obtain

$$\text{Im } S = \text{Im} \int_0^{+\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M - \omega')}{r}}} (-d\omega') ,$$  \hspace{1cm} (6)

where the minus sign appears because $H = M - \omega'$. But now the integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time (that is, into the lower half \( \omega' \) plane). In this way we obtain

$$\text{Im } S = +4\pi \omega \left( M - \frac{\omega}{2} \right) ,$$  \hspace{1cm} (7)

provided $r_{\text{in}} > r_{\text{out}}$. To understand this ordering – which supplies the correct sign – we observe that when the integrals in Eq. (5) are not interchanged, and with the contour evaluated via the prescription $\omega \to \omega - i\epsilon$, we have

$$\text{Im } S = +\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{M-\omega}^{M} \frac{dM'}{1 - \sqrt{\frac{2M'}{r}}} \, dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} -\pi r \, dr .$$  \hspace{1cm} (8)

Hence $r_{\text{in}} = 2M$ and $r_{\text{out}} = 2(M - \omega)$. (Incidentally, comparing the above equation with Eq. (5), we also find that $\text{Im } p_r = -\pi r$.) Thus, over the course of the classically forbidden trajectory, the outgoing particle travels radially inward with the apparent horizon to materialize at the final location of the horizon, viz. $r = 2(M - \omega)$.\[4\]
Alternatively, and along the same lines, Hawking radiation can also be regarded as pair creation *outside* the horizon, with the negative energy particle tunneling into the black hole. Since such a particle propagates backwards in time, we have to reverse time in the equations of motion. From the line element, Eq. (2), we see that time-reversal corresponds to \( \sqrt{2M/r} \to -\sqrt{2M/r} \). Also, since the anti-particle sees a geometry of fixed black hole mass, the upshot of self-gravitation is to replace \( M \) by \( M + \omega \), rather than \( M - \omega \). Thus an ingoing negative energy particle has

\[
\text{Im } S = \text{Im } \int_0^{-\omega} \int_{r_{\text{out}}}^{r_{\text{in}}} \frac{dr}{1 + \sqrt{2(M + \omega^2}/r}} d\omega' = +4\pi\omega \left( M - \frac{\omega}{2} \right),
\]

where to obtain the last equation we have used Feynman’s “hole theory” deformation of the contour: \( \omega' \to \omega' + i\epsilon \).

In either treatment, the exponential part of the semi-classical emission rate, in agreement with [4], is

\[
\Gamma \sim e^{-2\text{Im } S} = e^{-8\pi\omega(M - \omega^2)} = e^{+\Delta S_{\text{B-H}}},
\]

where we have expressed the result more naturally in terms of the change in the hole’s Bekenstein-Hawking entropy, \( S_{\text{B-H}} \). When the quadratic term is neglected, Eq. (10) reduces to a Boltzmann factor for a particle with energy \( \omega \) at the inverse Hawking temperature \( 8\pi M \). The \( \omega^2 \) correction arises from the physics of energy conservation, which (roughly speaking) effectively raises the effective temperature of the hole as it radiates. A nice consistency check is to consider the limit in which the emitted particle carries away the entire mass and charge of the black hole (corresponding to the transmutation of the black hole into an outgoing shell). There can be only one such outgoing state. On the other hand, there are \( \exp \left( S_{\text{B-H}} \right) \) states in total. Statistical mechanics then asserts that the probability of finding a shell containing all the mass of the black hole is proportional to \( \exp \left( -S_{\text{B-H}} \right) \), as above.

Following standard arguments, Eq. (10) with the quadratic term neglected implies the Planck spectral flux appropriate to an inverse temperature of \( 8\pi M \):

\[
\rho(\omega) = \frac{d\omega \left| T(\omega) \right|^2}{2\pi e^{+8\pi M\omega} - 1},
\]

(11)
where \(|T(\omega)|^2\) is the frequency-dependent (greybody) transmission coefficient for the outgoing particle to reach future infinity without back-scattering. It arises from a more complete treatment of the modes, whose semi-classical behavior near the turning point we have been discussing.

The preceding techniques can also be applied to emission from a charged black hole. However, when the outgoing radiation carries away the black hole’s charge, the calculations are complicated by the fact that the trajectories are now also subject to electromagnetic forces. Here we restrict ourselves to uncharged radiation coming from a Reissner-Nordström black hole. The derivation then proceeds much as above.

The charged counterpart to the Painlevé line element is

\[
\begin{align*}
 ds^2 &= - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + 2 \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dt \, dr + dr^2 + r^2 d\Omega^2, \\
\end{align*}
\]

(12)

which is obtained from the standard Reissner-Nordström line element by the coordinate transformation,

\[
\begin{align*}
 t &= t_r + 2 \sqrt{2Mr - Q^2} + M \ln \left( \frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}} \right) \\
 &\quad + \frac{Q^2 - M^2}{\sqrt{M^2 - Q^2}} \arctanh \left( \frac{\sqrt{M^2 - Q^2} \sqrt{2Mr - Q^2}}{Mr} \right), \\
\end{align*}
\]

(13)

where \(t_r\) is the Reissner time coordinate. The line element now manifestly displays the stationary, nonstatic, and nonsingular nature of the spacetime.

The equation of motion for an outgoing massless particle is

\[
\dot{r} \equiv \frac{dr}{dt} = +1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}},
\]

(14)

with \(M \to M - \omega\) when self-gravitation is included [9]. The imaginary part of the action for a positive energy outgoing particle is

\[
\text{Im } S = \text{Im} \int_0^{+\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M - \omega')}{r} - \frac{Q^2}{r^2}}} (-d\omega'), \\
\]

(15)

which is again evaluated by deforming the contour in accordance with Feynman’s \(w' \to w' - i\epsilon\) prescription. The residue at the pole can be read off by substituting \(u \equiv \sqrt{2(M - \omega') r - Q^2}\). This yields an emission rate of
\[ \Gamma \sim e^{-2 \text{Im} S} = e^{-4\pi \left(2\omega \left(M - \frac{\omega}{2}\right) - (M - \omega) \sqrt{(M - \omega)^2 - Q^2} + M \sqrt{M^2 - Q^2}\right)} = e^{+\Delta S_{\text{B-H}}} . \]  

(16)

To first order in \( \omega \), Eq. (16) is consistent with Hawking’s result of thermal emission at the Hawking temperature, \( T_H \), for a charged black hole:

\[ T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} . \]

(17)

But again, energy conservation implies that the exact result has corrections of higher order in \( \omega \); these can all be collected to express the emission rate as the exponent of the change in entropy.

Since the emission rate has to be real, the presence of the first square root in Eq. (16) ensures that radiation past extremality is problematic. In fact it does not arise in the present framework, since there is no acceptable (real) value of \( r_{\text{out}} \). Thus, the third law of black hole thermodynamics is manifest.

### III. CONCLUSION

We have derived Hawking radiation as a tunneling process. Because we enforce energy conservation, we are necessarily led to a modified emission spectrum that is not strictly thermal. The resulting corrected formula has physically reasonable limiting cases. By virtue of its nonthermality, it suggests the possibility of information-carrying correlations in the radiation.

Note that only local physics has entered into our derivations. There was neither an appeal to Euclideanization nor any need to invoke an explicit collapse phase. The time asymmetry leading to outgoing radiation arose instead from use of the “normal” (Feynman) local contour deformation prescription in terms of the nonstatic time coordinate \( t \), which is nonsingular through the horizon.

**Acknowledgement**

F.W. is supported in part by DOE grant DE-FG02-90ER-40542.
REFERENCES


