On counting special Lagrangian homology 3-spheres

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1 Introduction

Recently, Strominger, Yau and Zaslow [17] proposed an interpretation of mirror symmetry for Calabi-Yau 3-folds \( M \) involving fibrations of \( M \) by special Lagrangian 3-tori, allowing singular fibres. This has aroused a great deal of interest in special Lagrangian 3-folds amongst mathematicians and physicists – see for instance the papers of Gross [4, 5], or Ruan [15]. Probably the most difficult mathematical problems in verifying the SYZ conjecture lie in understanding the singularities of special Lagrangian 3-folds and fibrations, and their compactness properties.

General singularities occurring in families of special Lagrangian 3-folds are at present rather poorly understood. In this paper we study two kinds of singularities of special Lagrangian 3-folds. We first describe these locally in \( \mathbb{C}^3 \), and then we explain what happens when a compact special Lagrangian 3-fold \( N \) in a Calabi-Yau 3-fold \( M \) develops a singularity with this local model, and how this depends on the topology of \( N \).

Our aim in doing this is to contribute to the understanding of special Lagrangian 3-fold singularities by providing some well worked out examples of local and global behaviour. Therefore we try to be quite explicit and detailed. The local models we use are mainly due to Harvey and Lawson [7], in 1982. So it is likely that parts of the picture we present have also been worked out by other researchers in the field; however, at the time of writing the author knows of no papers which overlap significantly with this one.

But our paper has a second goal, which is much more speculative and poorly supported by proofs. It is to define an invariant of Calabi-Yau 3-folds \( M \) by counting special Lagrangian homology 3-spheres in each homology
class, and consider how it behaves under deformations of $M$. This is inspired by the theory of Gromov-Witten invariants and quantum cohomology, where one defines an invariant of symplectic manifolds by counting pseudo-holomorphic curves.

Let $M$ be a Calabi-Yau 3-fold. For each $\delta \in H_3(M, \mathbb{Z})$, let $S(\delta)$ be the set of special Lagrangian rational homology 3-spheres in $M$ with homology class $\delta$. Suppose $S(\delta)$ is finite. Then we define

$$I(\delta) = \sum_{N \in S(\delta)} w(N),$$

(1)

where $w(N)$ is a integer-valued weight function depending on the topology of $N$. Actually, when $N$ is a $d$-fold cover of another special Lagrangian 3-fold $N'$, the correct weight is not $w(N)$ but $w(N)/d$, and thus $I(\delta)$ is a rational number. So we have a map $I : H_3(M, \mathbb{Z}) \to \mathbb{Q}$, which is our invariant.

In the moduli space of Calabi-Yau structures on $M$ there are certain special real hypersurfaces, determined using the homology of $M$. At such a hypersurface, some of the special Lagrangian 3-folds in $M$ will become singular. A special Lagrangian 3-fold may exist in $M$ only on one side of the hypersurface, and become singular at the hypersurface. More generally, as we approach the hypersurface from one side one or more special Lagrangian 3-folds may collapse down to a single singular 3-fold, and then on the other side this singular 3-fold is replaced by a different collection of one or more special Lagrangian 3-folds.

Now our invariant $I$ will only be really interesting if it is stable under deformations of the underlying Calabi-Yau 3-fold $M$. So we wish to ensure that as we pass through such a hypersurface in the Calabi-Yau moduli space, $I$ is either unchanged, or at least transforms according to some rigid set of rules.

Using our models of singularities of special Lagrangian 3-folds, we explicitly describe two kinds of transition of the set of special Lagrangian 3-folds as we pass through a real hypersurface in the Calabi-Yau moduli space. We use these transitions to calculate identities which the weight function $w(N)$ in (1) must satisfy for $I$ to be invariant, or transform nicely, as we pass through the hypersurface. It turns out that the simple weight function $w(N) = |H_1(N, \mathbb{Z})|$ satisfies these identities.

Motivated by this, we fix $w(N) = |H_1(N, \mathbb{Z})|$, and formulate a conjecture, Conjecture 8.3, on how $I$ transforms under deformations of the Calabi-Yau
3-fold $M$. This is based on examining in detail only two kinds of transition; there are probably lots of other kinds of transitions, and these may invalidate the conjecture.

I freely admit that Conjecture 8.3 is based on very slim evidence, and may turn out to be entirely wrong. The thing which persuaded me to publish it is the very neat way in which the transitions I studied seemed to determine $w$ more or less uniquely, and then finding that this weight function counts something meaningful in String Theory.

We begin in §2 with an introduction to special Lagrangian geometry. Our definition of a special Lagrangian 3-fold in a Calabi-Yau 3-fold is not quite conventional: we allow the special Lagrangian 3-fold to have a phase $e^{i\theta}$, which is important because later we will consider configurations of special Lagrangian 3-folds with different phases.

Sections 3-5 study singular special Lagrangian 3-folds locally modelled on a cone on $T^2$ in $\mathbb{C}^3$. Such singular 3-folds $N_0$ can appear as the limit of a family of nonsingular 3-folds $N_t, t > 0$, in which a circle in $N_t$ shrinks to a point. We show that there are not one but three different ways of ‘resolving’ $N_0$ with such a family $N_{j,t}$, for $t > 0$ and $j = 1, 2, 3$, and we investigate the topology of $N_{j,t}$ and determine on which side of the real hypersurface it exists.

Sections 6 and 7 give a similar analysis of another kind of singularity, locally modelled on the union of two special Lagrangian 3-planes in $\mathbb{C}^3$. Such singular 3-folds can appear as the limit of a family of nonsingular 3-folds in which a 2-sphere $S^2$ shrinks to a point. However, in this case the singular limit can often be regarded as the union of two nonsingular special Lagrangian 3-folds.

Therefore we ask if, given special Lagrangian 3-folds $N^+$ and $N^-$ intersecting at a point $p$, we can find a family of special Lagrangian 3-folds $N$ diffeomorphic to $N^+ \# N^-$, and converging to the singular union $N^+ \cup N^-$. The answer turns out to depend on the phases of $N^+, N^-$ and $N$ in an interesting way.

Finally, in §8 we use the ideas of §3-§7 to guess a good definition for our invariant $I$, and make a conjecture about its behaviour. We also speculate about the relevance of our ideas to String Theory, about possible generalizations of our invariant, and about its applications to the study of the global structure of the moduli space of complex structures on $M$.

The author hopes to publish a subsequent paper [9] giving constructions of families of special Lagrangian 3-folds in $\mathbb{C}^3$, and more general calibrated
submanifolds in $\mathbb{R}^n$. These will give a number of other local models for how families of special Lagrangian 3-folds can develop singularities.

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2 Calabi-Yau and special Lagrangian geometry

2.1 Calabi-Yau manifolds

Definition 2.1 Let $(M, J)$ be a compact complex manifold of dimension $m \geq 2$, and $g$ a Kähler metric on $M$ with Kähler form $\omega$. Then the holonomy group $\text{Hol}(g)$ is a subgroup of $\text{U}(m)$. We call $(M, J, g)$ a Calabi-Yau manifold if $\text{Hol}(g) \subseteq \text{SU}(m)$.

The canonical bundle $K_M = \Lambda^{m,0}$ of a Calabi-Yau manifold $(M, J, g)$ is isomorphic to the trivial holomorphic line bundle $M \times \mathbb{C}$. Thus $c_1(M) = 0$ in $H^2(M, \mathbb{Z})$, where $c_1(M)$ is the first Chern class. Also, $M$ has a holomorphic volume form $\Omega$, which is a non-vanishing holomorphic section of $K_M$, and satisfies $\nabla \Omega = 0$, where $\nabla$ is the Levi-Civita connection of $g$. This $\Omega$ is unique up to a complex factor, but we can normalize $\Omega$ by requiring that

$$\omega^m/m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}. \quad (2)$$

Then $\Omega$ is unique up to a factor $e^{i\theta}$ for $\theta \in \mathbb{R}$.

Every Calabi-Yau manifold is Ricci-flat. Conversely, a compact, simply-connected Ricci-flat Kähler manifold $(M, J, g)$ is a Calabi-Yau manifold; but if $M$ is not simply-connected then $K_M$ may not be trivial, so that $\Omega$ exists locally but not globally on $M$. From Yau’s solution of the Calabi conjecture [19] we deduce an existence theorem for Calabi-Yau metrics.

Theorem 2.2 (Yau) Let $(M, J)$ be a compact complex manifold with trivial canonical bundle $K_M$, admitting Kähler metrics. Then there is a unique Ricci-flat Kähler metric $g$ in each Kähler class on $M$, and then $(M, J, g)$ is a Calabi-Yau manifold.

Using algebraic geometry one can find many examples of projective compact complex manifolds with trivial canonical bundle, such as the quintic in $\mathbb{CP}^4$, and all of these admit families of Calabi-Yau metrics.
2.2 Calibrated geometry

Next we discuss the theory of calibrated geometry, as in Harvey and Lawson [7].

Definition 2.3 Let \((M, g)\) be a Riemannian manifold. An oriented tangent k-plane \(V\) on \(M\) is a vector subspace \(V\) of some tangent space \(T_pM\) to \(M\) with \(\dim V = k\), equipped with an orientation. If \(V\) is an oriented tangent k-plane on \(M\) then \(g|_V\) is a Euclidean metric on \(V\), so combining \(g|_V\) with the orientation on \(V\) gives a natural volume form \(\text{vol}_V\) on \(V\), which is a \(k\)-form on \(V\).

Now let \(\varphi\) be a closed \(k\)-form on \(M\). We say that \(\varphi\) is a calibration on \(M\) if for every oriented \(k\)-plane \(V\) on \(M\) we have \(\varphi|_V \leq \text{vol}_V\). Here \(\varphi|_V = \alpha \cdot \text{vol}_V\) for some \(\alpha \in \mathbb{R}\), and \(\varphi|_V \leq \text{vol}_V\) if \(\alpha \leq 1\). Let \(N\) be an oriented submanifold of \(M\) with dimension \(k\). Then each tangent space \(T_pN\) for \(p \in N\) is an oriented tangent \(k\)-plane. We say that \(N\) is a calibrated submanifold or \(\varphi\)-submanifold if \(\varphi|_{T_pN} = \text{vol}_{T_pN}\) for all \(p \in N\).

If \((M, g)\) is a Riemannian manifold, \(\varphi\) a calibration on \(M\) of degree \(k\), and \(N\) a compact oriented submanifold of \(M\) of dimension \(k\), then

\[
[\varphi] \cdot [N] = \int_{p \in N} \varphi|_{T_pN} \leq \int_{p \in N} \text{vol}_{T_pN} = \text{vol}(N),
\]

where \([\varphi] \cdot [N]\) is the product between the de Rham cohomology class \([\varphi] \in H^k(M, \mathbb{R})\) and the homology class \([N] \in H_k(M, \mathbb{R})\). Thus \(\text{vol}(N) \geq [\varphi] \cdot [N]\), and equality holds if and only if \(N\) is a calibrated submanifold. So a compact calibrated submanifold is a minimal submanifold, as it is volume-minimizing in its homology class. Noncompact calibrated submanifolds are also locally minimizing.

Since we shall be discussing families of submanifolds, convergence of submanifolds and singular submanifolds in this paper, we had better be clear about what we mean by submanifold. Recall that there are two main concepts of nonsingular submanifold: immersed and embedded submanifolds.

Definition 2.4 Let \(N, M\) be manifolds and \(f : N \to M\) a smooth map. We say that \(f\) is an immersion if \(df|_n : T_nN \to T_{f(n)}M\) is injective for each \(n \in N\), and that \(f\) is an embedding if it is an injective immersion. Also, \(f\) is proper if whenever \(C \subseteq M\) is compact then \(f^{-1}(C)\) is compact.

An immersed submanifold of \(M\) is the image \(f(N)\) of a proper immersion \(f : N \to M\), and an embedded submanifold \(f(N)\) of \(M\) is the image of a proper
embedding $f : N \to M$. Requiring $f$ to be proper means that submanifolds $f(N)$ are automatically closed in $M$.

For most of the paper it will not matter whether we consider submanifolds to be immersed or embedded, as we will be interested only in local questions. But in §8 we find it natural to count immersed special Lagrangian homology 3-spheres, rather than embedded ones. We will also need to discuss singular submanifolds. There is an elegant and powerful theory of singular submanifolds called geometric measure theory, which is described in Federer [1] and Morgan [13]. We define a singular submanifold to be an integral current, in the sense of geometric measure theory.

Harvey and Lawson [7, §II] use integral and rectifiable currents in their theory of calibrated geometry. Integral currents are important tools in the study of minimal submanifolds (and hence calibrated submanifolds), because there are compactness theorems for integral currents which enable one to easily construct volume-minimizing currents. There are also regularity results due to Almgren which show that the singularities of a volume-minimizing current have codimension at least two.

We will not actually use any geometric measure theory in this paper — we will simply suppose that we know a singular submanifold when we see one. But anyone who wants a formal definition of singular submanifold, or to know exactly what we mean when we say that a family of submanifolds $N_t$ converges to a singular submanifold $N_0$ as $t \to 0$, is advised to look at Morgan [13].

### 2.3 Special Lagrangian submanifolds in $\mathbb{C}^m$

We shall be interested in special Lagrangian submanifolds in $\mathbb{C}^m$ and in Calabi-Yau manifolds. For reasons explained later, we will define special Lagrangian submanifolds slightly differently in these two situations. We begin with special Lagrangian submanifolds in $\mathbb{C}^m$.

**Definition 2.5** Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric $g_0$, a real 2-form $\omega_0$ and a complex $m$-form $\Omega_0$ on $\mathbb{C}^m$ by

$$g_0 = |dz_1|^2 + \cdots + |dz_m|^2, \quad \omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m),$$

and

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_m.$$

Then $\text{Re}\, \Omega_0$ and $\text{Im}\, \Omega_0$ are real $m$-forms on $\mathbb{C}^m$. Let $L$ be an oriented real submanifold of $\mathbb{C}^m$ of real dimension $m$. We say that $L$ is a special Lagrangian
submanifold of $\mathbb{C}^m$ if $L$ is calibrated w.r.t. $\text{Re}\, \Omega_0$, in the sense of Definition 2.3.

Harvey and Lawson [7, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

**Proposition 2.6** Let $L$ be a real $m$-dimensional submanifold of $\mathbb{C}^m$. Then $L$ admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\omega_0|_L \equiv 0$ and $\text{Im}\, \Omega_0|_L \equiv 0$.

Note that an $m$-dimensional submanifold $L$ in $\mathbb{C}^m$ is called Lagrangian if $\omega_0|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\text{Im}\, \Omega_0|_L \equiv 0$, which is how they get their name.

**Definition 2.7** Let $L$ be a (singular) special Lagrangian submanifold in $\mathbb{C}^m$. We say that $L$ is a cone in $\mathbb{C}^m$ if $L = tL$ for all $t > 0$, where $tL = \{tx : x \in L\}$. Let $L$ be a special Lagrangian cone in $\mathbb{C}^m$. Then either $L$ is an $m$-plane $\mathbb{R}^m$ in $\mathbb{C}^m$, or $0$ is a singular point of $L$.

We are particularly interested in cones in which $0$ is the only singular point. Suppose $L$ is nonsingular except at $0$, and define $\Sigma$ to be $L \cap S^{2m-1}$, the intersection of $L$ with the unit sphere $S^{2m-1}$ in $\mathbb{C}^m$. Then $\Sigma$ is a compact, nonsingular manifold of dimension $m-1$, and we refer to $L$ as the cone on $\Sigma$. For example, we shall later consider special Lagrangian cones on $T^2$ in $\mathbb{C}^3$.

Now let $L$ be a nonsingular special Lagrangian submanifold in $\mathbb{C}^m$. We say that $L$ is Asymptotically Conical, or AC for short, if there exists a special Lagrangian cone $L_0$ in $\mathbb{C}^m$, nonsingular except at $0$, such that $L$ is asymptotic to $L_0$ to order $O(r^{-1})$ as $r \to \infty$, where $r$ is the radius function in $\mathbb{C}^m$. We call $L_0$ the asymptotic cone of $L$.

Here is why we are interested in conical and asymptotically conical special Lagrangian submanifolds. Let $L$ and $L_0$ be as above, with $L_0$ singular at $0$. It is easy to see that if $t > 0$ then $tL$ is also AC, and that $tL \to L_0$ as $t \to 0_+$. Thus the singular special Lagrangian submanifold $L_0$ is the limit of the family of nonsingular special Lagrangian submanifolds $\{tL : t > 0\}$. So AC special Lagrangian submanifolds provide local models for how singularities can develop in families of nonsingular special Lagrangian submanifolds.

By combining the ideas of McLean [12, §3] on the deformation theory of compact special Lagrangian submanifolds with some analysis on noncompact manifolds with asymptotically conical ends, one can prove the following result on deformation of AC special Lagrangian manifolds.
Theorem 2.8  Let $L$ be a nonsingular, AC special Lagrangian submanifold in $\mathbb{C}^m$, asymptotic to the special Lagrangian cone $L_0$. Then the moduli space of AC special Lagrangian submanifolds in $\mathbb{C}^m$ asymptotic to $L_0$ is near $L$ a smooth manifold of dimension $b_1^c(L) + b_{m-1}^c(L)$, where $b_j^c(L) = \dim H^j_c(L, \mathbb{R})$, and $H^*_c(L, \mathbb{R})$ is the compactly-supported de Rham cohomology of $L$.

Note that the compactly-supported de Rham cohomology group $H^k_c(L, \mathbb{R})$ is naturally isomorphic to the homology group $H_{m-k}(L, \mathbb{R})$, where homology is as usual calculated using chains with compact support. Thus $b_1^c(L) + b_{m-1}^c(L) = b_{m-1}(L) + b_1(L)$.

Suppose $L$ is an AC special Lagrangian $m$-fold in $\mathbb{C}^m$, asymptotic to the cone $L_0$. Then for any $\gamma \in SU(m)$ we see that $\gamma \cdot L$ is another AC special Lagrangian $m$-fold, asymptotic to $\gamma \cdot L_0$. Let $G$ be the Lie subgroup of $SU(m)$ with $\gamma \cdot L_0 = L_0$, and $G_0$ the connected component of $G$ containing 1.

Then $\{ \gamma \cdot L : \gamma \in G_0 \}$ is a connected family of AC special Lagrangian submanifolds asymptotic to $L_0$. We can regard it as a family of deformations of $L$. However, using the ideas of McLean [12, §3], it can be shown that all these deformations are actually trivial. Thus $\gamma \cdot L = L$ for all $\gamma \in G_0$, and we have proved:

**Proposition 2.9**  Let $L$ be a nonsingular AC special Lagrangian submanifold in $\mathbb{C}^m$, asymptotic to the cone $L_0$. Let $G$ be the Lie subgroup of $SU(m)$ preserving $L_0$, and $G_0$ the connected component of $G$ containing 1. Then $L$ is preserved by $G_0$.

### 2.4 Special Lagrangian submanifolds in Calabi-Yau manifolds

Next we define special Lagrangian submanifolds in Calabi-Yau manifolds.

**Definition 2.10**  Let $(M, J, g)$ be a Calabi-Yau manifold of complex dimension $m$ with Kähler form $\omega$, and let $\Omega$ be a holomorphic volume form on $M$, normalized to satisfy (2). Then Re $\Omega$ and Im $\Omega$ are real, closed $m$-forms on $M$. Fix $\theta \in \mathbb{R}$. Then $\cos \theta \ \text{Re} \ \Omega + \sin \theta \ \text{Im} \ \Omega$ is a calibration on $M$. Let $N$ be a oriented real $m$-dimensional submanifold of $M$. We call $N$ a special Lagrangian submanifold with phase $e^{i\theta}$ if $N$ is calibrated w.r.t. $\cos \theta \ \text{Re} \ \Omega + \sin \theta \ \text{Im} \ \Omega$.

This is not the conventional definition: it is usual simply to define a special Lagrangian manifold to be calibrated w.r.t. Re $\Omega$, as in Definition 2.5. We
have changed the definition because we will later need to consider several special Lagrangian manifolds $N_1, N_2, \ldots$ in different homology classes in $M$ with different phases, so we can discuss what happens to special Lagrangian submanifolds as we vary $[\Omega] \in H^m(M, \mathbb{C})$.

From Proposition 2.6 we deduce:

**Proposition 2.11** Let $(M, J, g)$, $m$, $\omega$ and $\Omega$ be as above, and choose $\theta \in \mathbb{R}$. Suppose $N$ is a real $m$-dimensional submanifold of $M$. Then the following are equivalent:

(i) there is a unique orientation on $N$ making $N$ into a special Lagrangian submanifold with phase $e^{i\theta}$, and

(ii) $\omega_N \equiv 0$ and $(\sin \theta \Re \Omega - \cos \theta \Im \Omega)_N \equiv 0$.

Thus, if $N$ is a special Lagrangian submanifold in $M$ with phase $e^{i\theta}$, then

$$\int_N \cos \theta \Re \Omega + \sin \theta \Im \Omega = \text{vol}(N) \quad \text{and} \quad \int_N \sin \theta \Re \Omega - \cos \theta \Im \Omega = 0.$$

So we see that:

**Corollary 2.12** Let $(M, J, g)$ be a Calabi-Yau manifold with holomorphic volume form $\Omega$, and $N$ a compact special Lagrangian submanifold of $M$ with phase $e^{i\theta}$. Then

$$[\Re \Omega] \cdot [N] = \text{vol}(N) \cos \theta \quad \text{and} \quad [\Im \Omega] \cdot [N] = \text{vol}(N) \sin \theta,$$

where $[\Re \Omega], [\Im \Omega] \in H^m(M, \mathbb{R})$ and $[N] \in H_m(M, \mathbb{R})$. Thus the homology class $[N]$ determines both the phase $e^{i\theta}$ and the volume $\text{vol}(N)$ of $N$.

The deformation theory of special Lagrangian submanifolds was studied by McLean [12, §3], who proved the following result.

**Theorem 2.13** Let $(M, J, g)$ be a Calabi-Yau manifold, and $N$ a compact special Lagrangian submanifold of $M$. Then the moduli space $\mathcal{M}$ of special Lagrangian submanifolds in $M$ is near $N$ a smooth manifold of dimension $b^1(N)$, the first Betti number of $N$. 
The idea in the proof of this theorem is that an infinitesimal deformation of \( N \) as a submanifold in \( M \) corresponds to a section of the normal bundle \( \nu \) of \( N \) in \( M \). But because \( N \) is Lagrangian, contracting with \( \omega \) gives an isomorphism between the vector bundles \( \nu \) and \( T^*N \) over \( N \). So there is a 1-1 correspondence between infinitesimal deformations of \( N \) in \( M \) and 1-forms \( \alpha \) on \( N \).

McLean shows that \( \alpha \) corresponds to an infinitesimal deformation of \( N \) as a special Lagrangian submanifold if and only if \( d\alpha = d^*\alpha = 0 \). But as \( N \) is compact, by Hodge theory the vector space of 1-forms \( \alpha \) with \( d\alpha = d^*\alpha = 0 \) is isomorphic to \( H^1(N, \mathbb{R}) \), and so has dimension \( b^1(N) \).

For simplicity we will suppose that the moduli space \( \mathcal{M} \) of special Lagrangian submanifolds in \( M \) near \( N \) is connected and simply-connected, and that each member of \( \mathcal{M} \) is an embedded submanifold diffeomorphic to \( N \). It is well-known that \( \mathcal{M} \) not only has dimension \( b^1(N) \), but is actually naturally isomorphic to an open subset of an affine space \( A_{M,N} \) modelled on the vector space \( H^1(N, \mathbb{R}) \). An account of this can be found in Hitchin [8, §4].

Write \( N' \) for a general member of \( \mathcal{M} \). One way to define coordinates on \( A_{M,N} \) is to use the relative homology groups \( H_2(M; N', \mathbb{Z}) \). Each element \( \sigma \in H_2(M; N', \mathbb{Z}) \) is represented by a compact, oriented surface \( \Sigma \) in \( M \), with boundary in \( N' \). As \( \omega|_{N'} \equiv 0 \) it turns out that \( \int_{\Sigma} \omega \) depends only on \( \sigma \in H_2(M; N', \mathbb{Z}) \), and not on the choice of representative \( \Sigma \). In fact we may regard \( \omega \) as defining a class \([\omega]\) in the relative de Rham cohomology group \( H^2(M; N', \mathbb{R}) \), and \( \int_{\Sigma} \omega = [\omega] \cdot \sigma \).

As \( \mathcal{M} \) is connected and simply-connected and all \( N' \in \mathcal{M} \) are embedded, we can naturally identify the groups \( H_2(M; N', \mathbb{Z}) \) for all \( N' \in \mathcal{M} \). This gives an isomorphism \( H_2(M; N', \mathbb{Z}) \cong H_2(M; N, \mathbb{Z}) \) for each \( N' \in \mathcal{M} \). For each \( \sigma \in H_2(M; N, \mathbb{Z}) \), define a function \( f_{\sigma} : \mathcal{M} \to \mathbb{R} \) by \( f_{\sigma}(N') = \int_{\Sigma} \omega \), where \( \Sigma \) is a compact, oriented surface in \( M \) with boundary in \( N' \), whose homology class \([\Sigma]\) in \( H_2(M; N', \mathbb{Z}) \) is identified with \( \sigma \in H_2(M; N, \mathbb{Z}) \).

It can be shown that \( \mathcal{M} \) has the structure of an open set in an affine space \( A_{M,N} \) modelled on \( H^1(N, \mathbb{R}) \), and that each \( f_{\sigma} \) is the restriction to \( \mathcal{M} \) of an affine linear function on \( A_{M,N} \). If the image of \( H_1(N, \mathbb{R}) \) in \( H_1(M, \mathbb{R}) \) is zero, then we can use a collection of the functions \( f_{\sigma} \) as affine coordinates on \( \mathcal{M} \).

In fact there is a second way to regard \( \mathcal{M} \) as a subspace of an affine space. As \( N \) is a compact oriented \( m \)-manifold, we have \( b^1(N) = b^{m-1}(N) \) by Poincaré duality, and so \( \mathcal{M} \) is a manifold of dimension \( b^{m-1}(N) \). It turns out that we can embed \( \mathcal{M} \) as an open subset in a different affine
space $A^*_M,N$ modelled on $H^{m-1}(N, \mathbb{R})$. Furthermore, the product $A_M,N \times A^*_M,N$ is naturally a symplectic manifold (as $H^1(N, \mathbb{R})$ and $H^{m-1}(N, \mathbb{R})$ are dual vector spaces), and $\mathcal{M}$ can be regarded as a Lagrangian submanifold of $A_M,N \times A^*_M,N$.

Here is how to define affine linear coordinates on $A^*_M,N$. Suppose for simplicity that $N$ has phase 1. Then $\text{Im} \Omega|_N \equiv 0$. As above we can identify the relative homology groups $H_m(M; N', \mathbb{Z})$ with $H_m(M; N, \mathbb{Z})$ for all $N' \in \mathcal{M}$. Fix $\sigma \in H_m(M; N, \mathbb{Z})$. Then the corresponding element of $H_m(M; N', \mathbb{Z})$ is represented by an $m$-chain $\Sigma$ in $M$, with boundary in $N'$. As $\text{Im} \Omega|_{N'} \equiv 0$ we see that $\int_\Sigma \text{Im} \Omega$ depends only on $\sigma$, and not on the choice of representative $\Sigma$.

So define a function $F_\sigma : \mathcal{M} \to \mathbb{R}$ by $F_\sigma(N') = \int_\Sigma \text{Im} \Omega$, where $\Sigma$ is any $m$-chain representing $\sigma$ in $H_m(M; N', \mathbb{Z})$. These functions $F_\sigma$ for $\sigma \in H_m(M; N, \mathbb{Z})$ are affine linear on $A^*_M,N$, and if the image of $H_{m-1}(N, \mathbb{R})$ in $H_{m-1}(M, \mathbb{R})$ is zero, then we can use a collection of the functions $F_\sigma$ as coordinates on $\mathcal{M}$.

Our next result concerns the stability of compact special Lagrangian submanifolds $N$ under small deformations of the underlying Calabi-Yau $m$-fold $(M, J, g)$. McLean does not discuss this question, but it can be answered by the same techniques used to prove Theorem 2.13.

**Theorem 2.14** Suppose $(M, J, g)$ is a Calabi-Yau manifold, and $N$ a compact, nonsingular special Lagrangian submanifold in $M$. Let $(M, \tilde{J}, \tilde{g})$ be a nearby Calabi-Yau structure on $M$, with Kähler form $\tilde{\omega}$. Provided $(M, \tilde{J}, \tilde{g})$ is sufficiently close to $(M, J, g)$, there exists a special Lagrangian submanifold $\tilde{N}$ in $(M, \tilde{J}, \tilde{g})$ close to $N$ in $M$ if and only if $[\tilde{\omega}|_N] = 0$ in $H^2(N, \mathbb{R})$.

Now consider the case in which $N$ is a compact special Lagrangian 3-manifold with $b^1(N) = 0$. Then $H_1(N, \mathbb{Q}) = H_2(N, \mathbb{Q}) = 0$. It follows that $H_*(N, \mathbb{Q}) \cong H_*(S^3, \mathbb{Q})$, and so $N$ is called a homology 3-sphere. (By this we mean a rational homology 3-sphere, not an integral homology 3-sphere). Theorem 2.13 shows that the moduli space of deformations of $N$ is of dimension 0, and Theorem 2.14 shows that $N$ persists under small deformations $(M, \tilde{J}, \tilde{g})$ of the underlying Calabi-Yau 3-fold $(M, J, g)$.

**Corollary 2.15** Let $(M, J, g)$ be a Calabi-Yau 3-fold, and $N$ a compact special Lagrangian homology 3-sphere in $M$. Then $N$ is rigid, that is, it admits no deformations as a special Lagrangian submanifold, and it is also stable under small deformations $(M, \tilde{J}, \tilde{g})$ of the underlying Calabi-Yau 3-fold $(M, J, g)$. 

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This is important to us, for the following reason. Since special Lagrangian homology 3-spheres are rigid, it seems plausible that there should exist only finitely many special Lagrangian homology 3-spheres in any given homology class in \( H_3(M, \mathbb{Z}) \). If this is so, then the number of special Lagrangian homology 3-spheres \( N \) in this homology class (perhaps counted with some sign or weight depending on \( N \)) would be an invariant of the Calabi-Yau 3-fold. We will discuss this in §8.

### 3 A family of special Lagrangian 3-folds in \( \mathbb{C}^3 \)

We begin with a proposition taken from Harvey and Lawson [7, §III.3.A], defining an explicit family of special Lagrangian 3-folds in \( \mathbb{C}^3 \).

**Proposition 3.1** Let \( a_1, a_2, a_3 \) and \( b \) be real numbers, and define a subset \( L_{a_1,a_2,a_3,b} \) in \( \mathbb{C}^3 \) by

\[
L_{a_1,a_2,a_3,b} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - a_1 = |z_2|^2 - a_2 = |z_3|^2 - a_3, \quad \text{Im}(z_1z_2z_3) = b \}.
\]

Then \( L_{a_1,a_2,a_3,b} \) is a special Lagrangian 3-fold in \( \mathbb{C}^3 \), wherever it is nonsingular.

Observe that adding a constant to \( a_1, a_2 \) and \( a_3 \) does not change \( L_{a_1,a_2,a_3,b} \), so \( L_{a_1,a_2,a_3,b} \) only really depends on 3 real parameters. We find it convenient to fix the additive constant by requiring that \( \min(a_1, a_2, a_3) = 0 \).

The reason one can find these examples explicitly is that they have a large symmetry group. Let \( G \cong T^2 \) be the group of diagonal matrices in \( \text{SU}(3) \), so that each \( \gamma \in G \) acts on \( \mathbb{C}^3 \) by

\[ \gamma : (z_1, z_2, z_3) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_3}z_3) \]

for some \( \theta_1, \theta_2, \theta_3 \in \mathbb{R} \) with \( \theta_1 + \theta_2 + \theta_3 = 0 \). Then each \( L_{a_1,a_2,a_3,b} \) is invariant under \( G \). In fact, if \( L \) is any connected, \( G \)-invariant special Lagrangian 3-fold in \( \mathbb{C}^3 \), then \( L \) is a subset of some \( L_{a_1,a_2,a_3,b} \). Our next result gives the singularities of \( L_{a_1,a_2,a_3,b} \).

**Proposition 3.2** Let \( L_{a_1,a_2,a_3,b} \) be defined by (4), with \( \min(a_1, a_2, a_3) = 0 \). Then
(i) $L_{0,0,0,0}$ has one singular point at 0.

(ii) If $a_1 > 0$ then $L_{a_1,0,0,0}$ has singular set $\{(\sqrt{a_1} e^{i\theta}, 0, 0) : \theta \in [0, 2\pi)\}$.

(iii) If $a_2 > 0$ then $L_{0,a_2,0,0}$ has singular set $\{(0, \sqrt{a_2} e^{i\theta}, 0) : \theta \in [0, 2\pi)\}$.

(iv) If $a_3 > 0$ then $L_{0,0,a_3,0}$ has singular set $\{(0, 0, \sqrt{a_3} e^{i\theta}) : \theta \in [0, 2\pi)\}$.

(v) If $b \neq 0$, or if two of $a_1, a_2, a_3$ are nonzero, then $L_{a_1,a_2,a_3,b}$ is nonsingular.

Here is a sketch of the proof. Define $f_1, f_2, f_3 : \mathbb{C}^3 \to \mathbb{R}$ by

$$f_1(z_1, z_2, z_3) = |z_1|^2 - |z_3|^2, \quad f_2(z_1, z_2, z_3) = |z_2|^2 - |z_3|^2$$

and

$$f_3(z_1, z_2, z_3) = \text{Im}(z_1 z_2 z_3).$$

Then $L_{a_1,a_2,a_3,b}$ is defined by the equations $f_1 = a_1 - a_3$, $f_2 = a_2 - a_3$ and $f_3 = b$. One can show that $L_{a_1,a_2,a_3,b}$ is singular at $(z_1, z_2, z_3)$ if and only if the real 1-forms $df_1, df_2, df_3$ are linearly dependent at $(z_1, z_2, z_3)$, and this is a simple calculation.

We now describe the topology of $L_{a_1,a_2,a_3,b}$ in cases (i)--(v).

**Case (i).** Define subsets $L_0^\pm$ in $\mathbb{C}^3$ by

$$L_0^+ = \{(r e^{i\theta_1}, r e^{i\theta_2}, r e^{i\theta_3}) : r \geq 0 \text{ and } \theta_1, \theta_2, \theta_3 \in \mathbb{R} \text{ with } \theta_1 + \theta_2 + \theta_3 = 0\},$$

$$L_0^- = \{(r e^{i\theta_1}, r e^{i\theta_2}, r e^{i\theta_3}) : r \geq 0 \text{ and } \theta_1, \theta_2, \theta_3 \in \mathbb{R} \text{ with } \theta_1 + \theta_2 + \theta_3 = \pi\}.$$

Then $L_0^\pm$ are both special Lagrangian cones on $T^2$ in the sense of Definition 2.7, which intersect only at 0, their common singular point. But $L_{0,0,0,0} = L_0^+ \cup L_0^-$. Thus in this case $L_{a_1,a_2,a_3,b}$ splits into two pieces $L_0^\pm$. Harvey and Lawson remark [7, p. 97] that $L_0^\pm$ are not real analytic.

**Case (ii).** Let $a_1 > 0$, write $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$, and define maps $\phi_{1,a_1}^\pm : S^1 \times \mathbb{C} \to \mathbb{C}^3$ by

$$\phi_{1,a_1}^+ : (e^{i\theta}, z) \mapsto ((|z|^2 + a_1)^{1/2} e^{i\theta}, z, e^{-i\theta} \bar{z}),$$

$$\phi_{1,a_1}^- : (e^{i\theta}, z) \mapsto ((|z|^2 + a_1)^{1/2} e^{i\theta}, z, -e^{-i\theta} \bar{z}).$$

Now $\phi_{1,a_1}^\pm$ are smooth, injective maps $S^1 \times \mathbb{C} \to \mathbb{C}^3$, whose first derivatives have full rank at every point. Therefore the images of $\phi_{1,a_1}^\pm$ are nonsingular submanifolds of $\mathbb{C}^3$, which are embedded and closed.
So define $L_{1,a_1}^\pm = \phi_{1,a_1}^\pm (\mathcal{S}^1 \times \mathbb{C})$. An equivalent definition is

$$
L_{1,a_1}^+ = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - a_1 = |z_2|^2 = |z_3|^2, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\},
$$

$$
L_{1,a_1}^- = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - a_1 = |z_2|^2 = |z_3|^2, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \leq 0\}. \tag{5}
$$

Then $L_{1,a_1}^+$ and $L_{1,a_1}^-$ are both nonsingular, 3-dimensional submanifolds of $\mathbb{C}^3$ diffeomorphic to $\mathcal{S}^1 \times \mathbb{C}$. Comparing (4) and (5) we see that $L_{a_1,0,0,0} = L_{1,a_1}^+ \cup L_{1,a_1}^-$. Since $L_{a_1,0,0,0}$ is a special Lagrangian 3-fold we deduce that $L_{1,a_1}^\pm$ are also special Lagrangian 3-folds, which is easy to verify directly. It can be shown that $L_{1,a_1}^\pm$ are asymptotically conical in the sense of Definition 2.7, asymptotic to $L_0^\pm$.

Observe that $L_{1,a_1}^+ \cap L_{1,a_1}^- = \{(\sqrt{a_1 e^{i\theta}}, 0, 0) : \theta \in [0, 2\pi)\}$, which is the singular set of $L_{a_1,0,0,0}$ given in Proposition 3.2. Thus $L_{a_1,0,0,0}$ is the union of two nonsingular special Lagrangian 3-folds $L_{1,a_1}^+$ and $L_{1,a_1}^-$, and the singularities of $L_{a_1,0,0,0}$ occur at their intersection. Note that we could consider $L_{a_1,0,0,0}$ to be a nonsingular, immersed submanifold.

**Cases (iii) and (iv).** We can treat these exactly like case (ii), but with a cyclic permutation of $z_1, z_2$ and $z_3$. In particular, if for $a_2 > 0$ and $a_3 > 0$ we define $L_{2,a_2}^\pm$ and $L_{3,a_3}^\pm$ by

$$
L_{2,a_2}^+ = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - a_2 = |z_3|^2, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\},
$$

$$
L_{2,a_2}^- = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - a_2 = |z_3|^2, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \leq 0\},
$$

$$
L_{3,a_3}^+ = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - a_3, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\},
$$

$$
L_{3,a_3}^- = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - a_3, \\
\text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \leq 0\}. \tag{6}
$$

Then $L_{2,a_2}^\pm$ and $L_{3,a_3}^\pm$ are all nonsingular special Lagrangian 3-folds diffeomorphic to $\mathcal{S}^1 \times \mathbb{C}$, with $L_{0,a_2,0,0} = L_{2,a_2}^+ \cup L_{2,a_2}^-$ and $L_{0,0,a_3,0} = L_{3,a_3}^+ \cup L_{3,a_3}^-$. Also, $L_{2,a_2}^\pm$ approach $L_0^+$ as $a_2 \to 0$ and $L_{3,a_3}^\pm$ approach $L_0^+$ as $a_3 \to 0$, and for fixed $a_2, a_3 > 0$ both $L_{2,a_2}^\pm$ and $L_{3,a_3}^\pm$ are asymptotic to $L_0^\pm$ at infinity in $\mathbb{C}^3$. 

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Thus the $L_{j,a_j}^+$ for $j = 1, 2, 3$ are three different families of AC special Lagrangian 3-folds asymptotic to the singular cone $L_0^+$, each family depending on 1 positive real parameter $a_j$. Similarly, the $L_{j,a_j}^-$ for $j = 1, 2, 3$ are three different families of AC special Lagrangian 3-folds asymptotic to $L_0^-$. This suggests that if $M$ is a Calabi-Yau 3-fold, and $N$ a singular special Lagrangian 3-fold in $M$ with one singular point modelled on $L_0^+$ or $L_0^-$, then there may be three different ways to ‘resolve’ $N$ as a nonsingular special Lagrangian 3-fold.

Case (v). If $b \neq 0$, or if two of $a_1, a_2, a_3$ are nonzero, then it can be shown that $L_{a_1,a_2,a_3,b}$ is nonsingular and diffeomorphic to $T^2 \times \mathbb{R}$. In particular, $L_{a_1,a_2,a_3,b}$ is connected, and it does not split naturally into two pieces, as in cases (i)–(iv). It has two ends, which are asymptotic to $L_0^\pm$ at infinity.

Here are some remarks on these examples.

- As $L_0^+$ is invariant under $G = T^2$, which is connected, we see by Proposition 2.9 that any AC special Lagrangian 3-fold $L$ asymptotic to $L_0^+$ is also $G$-invariant. But from above, any connected $G$-invariant special Lagrangian 3-fold is a subset of some $L_{a_1,a_2,a_3,b}$. So by the analysis of the $L_{a_1,a_2,a_3,b}$ above, we see that the $L_{j,a_j}^+$ for $j = 1, 2, 3$ and $t > 0$ are the only AC special Lagrangian 3-folds in $\mathbb{C}^3$ asymptotic to $L_0^+$.

- From above, $L_{1,t}^+$ is diffeomorphic to $S^1 \times \mathbb{C}$ when $t > 0$, which has $b_1 = 1$ and $b_2 = 0$. So by Theorem 2.8, the moduli space of AC special Lagrangian 3-folds asymptotic to $L_0^+$ is near $L_{1,t}^+$ a manifold of dimension 1. This agrees with the fact that $L_{1,t}^+$ depends on one real parameter $t$.

- Similarly, if $b \neq 0$ or two of $a_1, a_2, a_3$ are nonzero, then $L_{a_1,a_2,a_3,b}$ is diffeomorphic to $T^2 \times \mathbb{R}$, which has $b_1 = 2$ and $b_2 = 1$. So by Theorem 2.8, the moduli space of AC special Lagrangian 3-folds asymptotic to $L_0^+ \cup L_0^-$ is near $L_{a_1,a_2,a_3,b}$ a manifold of dimension 3. This agrees with the fact that $L_{a_1,a_2,a_3,b}$ depends on 3 real parameters, that is, the 4 parameters $a_1, a_2, a_3, b$ subject to the condition $\min(a_1, a_2, a_3) = 0$.

- Let $t > 0$. Then $L_{1,t}^\pm$ is diffeomorphic to $S^1 \times \mathbb{C}$, and so $H_1(L_{1,t}^\pm, \mathbb{Z}) \cong \mathbb{Z}$. Define subsets $D_{1,t}$ and $\gamma_{1,t}$ in $\mathbb{C}^3$ by

$$
\gamma_{1,t} = \{(t^{1/2} e^{i\theta}, 0, 0) : \theta \in [0, 2\pi)\},
$$

$$
D_{1,t} = \{(z_1, 0, 0) : z_1 \in \mathbb{C}, \quad |z_1|^2 \leq t\}.
$$

(7)
Then $\gamma_{1,t}$ is a smooth, oriented $S^1$ in $L^+_{1,t}$, and $D_{1,t}$ is a closed, oriented holomorphic disc in $\mathbb{C}^3$ with area $\pi t$ and boundary $\gamma_{1,t}$. The homology class of $\gamma_{1,t}$ generates $H_1(L^+_{1,t}, \mathbb{Z})$. There are similar holomorphic discs $D_{2,t}$ with boundary $\gamma_{2,t}$ in $L^+_{2,t}$, and $D_{3,t}$ with boundary $\gamma_{3,t}$ in $L^+_{3,t}$. We will explain why this is interesting in §4.

4 A model degeneration of special Lagrangian 3-folds

In §3 we defined a 1-parameter family of special Lagrangian 3-folds $L^+_{1,t}$ in $\mathbb{C}^3$, which converged to a singular $T^2$-cone $L^+_0$ as $t \to 0$. We shall now treat this as a local model for how a 1-parameter family $\{N_t : t \in (0, \epsilon)\}$ of compact, nonsingular special Lagrangian 3-folds in a Calabi-Yau 3-fold $M$ can converge to a singular special Lagrangian 3-fold $N_0$ in $M$.

Let $(M,J,g)$ be a Calabi-Yau 3-fold with Kähler form $\omega$, let $\Omega$ be a holomorphic 3-form on $M$, and $N$ a compact special Lagrangian 3-fold in $M$ with phase $e^{i\theta}$. By replacing $\Omega$ by $e^{-i\theta}\Omega$ we may suppose that $N$ has phase 1. Suppose that there exist a small open ball $U$ in $M$, holomorphic coordinates $(z_1,z_2,z_3)$ on $U$, and a small constant $\rho > 0$ satisfying the following conditions:

- In coordinates $U = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 < \rho^2\}$. That is, $U$ is the open ball of radius $\rho$ about 0 in $\mathbb{C}^3$.
- On $U$ the metric $g$, Kähler form $\omega$ and 3-form $\Omega$ on $M$ are very close to the Euclidean versions $g_0$, $\omega_0$, $\Omega_0$ on $\mathbb{C}^3$ defined by (3).
- In $U$, the 3-fold $N$ in $M$ is very close to the 3-fold $L^+_{1,t}$ in $\mathbb{C}^3$ defined by (5), for some $t$ with $0 < t \ll \rho^2$.

Let $\gamma_{1,t}$ and $D_{1,t}$ be as in (7). Since by assumption $N$ is close to $L^+_{1,t}$ in $U$, we can choose an oriented circle $\gamma$ in $N$ which is close to $\gamma_{1,t}$, and a closed, oriented disc $D$ in $U$ with boundary $\gamma$ which is close to $D_{1,t}$. Let $\alpha^Z$ be the homology class of $\gamma$ in $H_1(N, \mathbb{Z})$, and $\alpha^\mathbb{R}$ the homology class of $\gamma$ in $H_1(N, \mathbb{R})$. Let $\beta$ be the homology class of $D$ in the relative homology group $H_2(M; N, \mathbb{Z})$.

Suppose $\mathcal{M}$ is a connected and simply-connected open neighbourhood of $N$ in the moduli space of special Lagrangian 3-folds in $M$. Then $\mathcal{M}$ is a
manifold of dimension $b^1(N)$ by Theorem 2.13. Furthermore, we explained in §2.4 why $\mathcal{M}$ is isomorphic to an open subset of an affine space $A_{M,N}$ modelled on $H^1(N,\mathbb{R})$, and how to associate a function $f_\sigma : \mathcal{M} \to \mathbb{R}$ to each class $\sigma \in H_2(M;N,\mathbb{Z})$.

Let $f_\beta : \mathcal{M} \to \mathbb{R}$ be the function associated to $\beta = [D] \in H_2(M;N,\mathbb{Z})$. Then $f_\beta(N) = \int_D \omega$. Since $D$ is close to $D_{1,t}$, and $\omega$ close to $\omega_0$, it follows that $\int_D \omega \approx \int_{D_{1,t}} \omega_0$. But $\int_{D_{1,t}} \omega_0 = \pi t$, as it is the area of $D_{1,t}$, which is a disc of radius $t^{1/2}$. Therefore $f_\beta(N) \approx \pi t$. Since we are free to change $t$ a little bit, we can assume that $f_\beta(N) = \pi t$.

Let $N, N'$ be special Lagrangian 3-folds in $\mathcal{M}$, and $x, x'$ the corresponding points in the affine space $A_{M,N}$. Since $A_{M,N}$ is modelled on the vector space $H^1(N,\mathbb{R})$, we can regard $x - x'$ as an element of $H^1(N,\mathbb{R})$. Now $\alpha^x = [\gamma]$ in $H_1(N,\mathbb{Z})$ and $\beta = [D]$ in $H_2(M,N,\mathbb{Z})$, and $\gamma = \partial D$. Thus $\alpha^x = \partial \beta$, where $\partial : H_2(M;N,\mathbb{Z}) \to H_1(N,\mathbb{Z})$ is the boundary map. Using this it is easy to show that

$$f_\beta(N) - f_\beta(N') = f_\beta(x) - f_\beta(x') = \alpha^x \cdot (x - x') = \alpha^x \cdot (x - x'),$$

(8)

where $\cdot$ is the natural pairing between $H_1(N,\mathbb{Z})$ or $H_1(N,\mathbb{R})$ and $H^1(N,\mathbb{R})$.

Here is a conjecture on the behaviour of special Lagrangian deformations of $N$.

**Conjecture 4.1** Suppose $N'$ is close to $N$ in $\mathcal{M}$, and define $t'$ by $f_\beta(N') = \pi t'$. Then there are holomorphic coordinates $(z_1', z_2', z_3')$ in $U$ such that $N'$ is close to the 3-fold $L_{1,t'}^+$ in $\mathbb{C}^3$ defined by (5), in the coordinates $(z_1', z_2', z_3')$. The new coordinates $(z_1', z_2', z_3')$ are close to the old coordinates $(z_1, z_2, z_3)$, and differ from them by a small translation and a rotation in $SU(3)$ close to the identity.

In particular, as $L_{1,t'}^+$ is only well-defined when $t' \geq 0$ and nonsingular when $t' > 0$, we conjecture that $f_\beta(N') > 0$ for all nonsingular $N'$ near $N$ in $\mathcal{M}$, and on the boundary $f_\beta(N') = 0$, the 3-folds $N'$ have a singular point modelled locally on $L_0^+$.

I have stated this as a conjecture because I have not yet proved it. However, I am confident it is true, and I can see the outlines of a proof using analytic methods. I am beginning work on the proof of this conjecture (and similar results) in collaboration with my student, Stephen Marshall. The basic idea behind the conjecture is if $N$ is locally modelled on $L_{1,t}^+$ then every nearby $N'$ in $\mathcal{M}$ should be locally modelled on some $L_{1,t'}^+$ for some $t' > 0$. 

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But we can find $t'$ explicitly using relative homology, and this is the purpose of the function $f_\beta$.

We now work out some of the consequences of Conjecture 4.1. It turns out that these are rather different when $\alpha^R \neq 0$ and when $\alpha^R = 0$, so we consider the two cases separately.

### 4.1 The case $\alpha^R \neq 0$

We have $H_1(N, \mathbb{R}) \cong H^1(N, \mathbb{R})^*$ as $N$ is a compact manifold. Thus, as $\alpha^R \neq 0$ in $H_1(N, \mathbb{R})$, we see that $H_1(N, \mathbb{R})$ and $H^1(N, \mathbb{R})$ are both nonzero vector spaces. Furthermore, as the function $f_\beta : A_{M,N} \to \mathbb{R}$ satisfies $f_\beta(x) - f_\beta(x') = \alpha^R \cdot (x - x')$ by (8), regarding $x - x'$ as an element of $H^1(N, \mathbb{R})$, we see that $f_\beta$ is not constant on $A_{M,N}$.

Therefore $\{x' \in A_{M,N} : f_\beta(x') = 0\}$ is a real hyperplane in $A_{M,N}$, which divides $A_{M,N}$ into two halves, the ‘upper half-plane’ $f_\beta > 0$, and the ‘lower half-plane’ $f_\beta < 0$. According to Conjecture 4.1, near $N$ the moduli space $\mathcal{M}$ of special Lagrangian deformations of $N$ lies solely in the upper half-plane $f_\beta > 0$. At the dividing wall $f_\beta = 0$ they become singular, with a $T^2$-cone singularity modelled on $L_0^+$, where a circle in $N$ has been crushed to a point.

It is important to note that these singularities happen in real codimension one in the moduli space $\mathcal{M}$ of special Lagrangian 3-folds in $M$, and therefore that $\mathcal{M}$ should be thought of as a manifold with boundary, where the interior of $\mathcal{M}$ corresponds to nonsingular 3-folds, and the boundary to singular 3-folds. Moreover, the singularities develop in a very orderly and predictable way, on a real hyperplane in $A_{M,N}$.

This is very different to the case of special Lagrangian submanifolds of dimension 2, which are essentially holomorphic curves w.r.t. another complex structure. Here singularities develop in real codimension two (that is, complex codimension one) in the moduli space of special Lagrangian 2-folds. Thus, a generic path in the moduli space meets only nonsingular 2-folds. This means that for many purposes we can ignore the singularities, and treat the moduli space of special Lagrangian 2-folds as a manifold without boundary.

However, for special Lagrangian 3-folds, these singularities in real codimension one cannot be ignored or perturbed away. This suggests that special Lagrangian 3-folds should have weaker existence and compactness properties than special Lagrangian 2-folds.
4.2 The case $\alpha^R = 0$

By (8), if $\alpha^R = 0$ and $N' \in \mathcal{M}$ then $f_\beta(N) - f_\beta(N') = \alpha^R \cdot (x - x') = 0$. Thus $f_\beta(N') = f_\beta(N)$ for all $N' \in \mathcal{M}$, and $f_\beta$ is constant on $A_{M,N}$, with value $\pi t$. So $f_\beta > 0$ on all of $A_{M,N}$, and the inequality $f_\beta > 0$ does not constrain the moduli space $\mathcal{M}$ in $A_{M,N}$ in this case. Also Conjecture 4.1 (if true) implies that all $N'$ close to $N$ in $\mathcal{M}$ are locally modelled on $L^t_{1,t}$, for the same value of $t$.

Let $\pi : H_1(N,\mathbb{Z}) \rightarrow H_1(N,\mathbb{R})$ be the natural projection. Then $\pi(\alpha^2) = \alpha^2 = 0$, and $\alpha^2 \in \text{Ker} \, \pi$. But Ker $\pi$ is the subgroup of elements of $H_1(N,\mathbb{Z})$ with finite order (the torsion subgroup). Thus $\alpha^2$ has finite order, that is, $k\alpha = 0$ in $H_1(N,\mathbb{Z})$ for some positive integer $k$. Since $\alpha = [\gamma]$, this means that we can find an immersed surface $\Sigma$ in $N$ with boundary $k\gamma$.

Therefore $kD - \Sigma$ is a 2-chain in $M$ without boundary, and defines a homology class $\chi$ in $H_2(M,\mathbb{Z})$. Moreover

$$[\omega] \cdot \chi = k \int_D \omega - \int_\Sigma \omega = k f_\beta(N), \quad \text{(9)}$$

since $f_\beta(N) = \int_D \omega$, and $\omega|_\Sigma \equiv 0$, as $\Sigma \subset N$ and $\omega|_N = 0$. Here $[\omega] \in H^2(M,\mathbb{R})$, and $\cdot$ is the natural pairing between $H^2(M,\mathbb{R})$ and $H_2(M,\mathbb{Z})$.

Observe that $[\omega] \cdot \chi > 0$ as $k > 0$ and $f_\beta(N) > 0$, so $\chi \neq 0$ in $H_2(M,\mathbb{Z})$.

Now consider what happens if we deform the underlying Calabi-Yau manifold $(M,J,g)$, in a way that changes the cohomology class $[\omega] \in H^2(M,\mathbb{R})$. Choose a smooth 1-parameter family of Calabi-Yau metrics $g_s$ on $M$ for $s \in (-\epsilon,\epsilon)$, with associated complex structures $J_s$, Kähler forms $\omega_s$ and holomorphic volume forms $\Omega_s$, such that $g_0 = g$, $J_0 = J$ and $\Omega_0 = \Omega$.

We wish to find a smooth family of 3-folds $N_s$ in $M$ for $s \in (-\epsilon,\epsilon)$ such that $N_s$ is special Lagrangian with respect to $(M,J_s,g_s)$, and $N_0 = N$. By Theorem 2.14, $N_s$ exists for small $s$ if and only if $[\omega_s]|_N = 0$ in $H^2(N,\mathbb{R})$. Assume that this is so. Let $\mathcal{M}_s$ be the moduli space of special Lagrangian deformations of $N_s$. Then we can define the function $f_\beta$ on $\mathcal{M}_s$ as before. From (9) we deduce that

$$f_\beta(N'_s) = \frac{1}{k} [\omega_s] \cdot \chi \quad \text{for all } N'_s \in \mathcal{M}_s.$$

Therefore, by Conjecture 4.1 (if true), we expect that near $N$, each $N'_s$ in $\mathcal{M}_s$ should be locally modelled on $L^t_{1,t'}$, for $t' > 0$ satisfying $k\pi t' = [\omega_s] \cdot \chi$.  

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Thus we expect that the family \( \mathcal{M}_s \) of nonsingular special Lagrangian 3-folds exists near \( N \) only when \( [\omega_s] \cdot \chi > 0 \). When \( [\omega_s] \cdot \chi = 0 \), every element of \( \mathcal{M}_s \) has a singular point modelled on \( L_0^+ \). When \( [\omega_s] \cdot \chi < 0 \), there are no special Lagrangian 3-folds locally isotopic to \( N \), and so \( \mathcal{M}_s \) is empty.

So we have the picture that as we deform the Kähler class \( [\omega] \) of the underlying Calabi-Yau 3-fold \((M, J, g)\) through the real hyperplane \( [\omega] \cdot \chi = 0 \) in the Kähler cone, every special Lagrangian 3-fold in \( \mathcal{M} \) simultaneously develops a \( T^2 \)-cone singularity, at least near \( N \). On the other side \( [\omega] \cdot \chi < 0 \) of the real hyperplane, the special Lagrangian 3-folds no longer exist. We can think of this process as being like a bubble bursting: the 3-fold suddenly becomes singular, and then disappears. Thus special Lagrangian 3-folds should not be thought of as particularly stable under large deformations of the Calabi-Yau 3-fold.

### 4.3 Interpretation in terms of ‘relative Gromov invariants’

The *Gromov invariants* of a symplectic manifold \((M, \omega)\) count the pseudo-holomorphic curves in \( M \) w.r.t. an almost complex structure \( J \) tamed by \( \omega \), satisfying certain homological conditions. They were introduced by Gromov [3], who proved the basic existence and compactness theorems for pseudo-holomorphic curves. The *Gromov-Witten invariants* and *quantum cohomology* are closely-related ideas. An introduction to this subject may be found in McDuff and Salamon [11].

If \( M \) is a symplectic manifold and \( N \) a Lagrangian submanifold, then we can consider pseudo-holomorphic curves in \( M \) with boundary in \( N \). Gromov shows [3, §0.4] that such curves have the same string existence and compactness properties as pseudo-holomorphic curves without boundary. Thus it is natural to try and define *relative Gromov invariants* of the pair \((M, N)\) by counting pseudo-holomorphic curves in \( M \) with boundary in \( N \), satisfying homological conditions. This idea is used by Oh [14, §10] to define *relative quantum cohomology*.

Now we assumed above that \( N \) is close to \( L_{1,t}^+ \) in \( U \), and we chose a disc \( D \) in \( M \) with boundary in \( N \) close to the disc \( D_{1,t} \) in \( \mathbb{C}^3 \) with boundary in \( L_{1,t}^+ \). But \( D_{1,t} \) is in fact holomorphic. As holomorphic curves with boundary in Lagrangian submanifolds are stable under small perturbations, we should be able to choose \( D \) to be a holomorphic disc, and this choice should be unique.
Thus we expect there to exist a holomorphic curve $D$ in $M$, with boundary in $N$, such that $[D] = \beta \in H_2(M; N, \mathbb{Z})$. So the ‘relative Gromov invariant’ counting holomorphic curves with relative homology class $\beta$ should be nonzero, provided there are no other holomorphic curves with relative homology class $\beta$. If there were other such curves we would have to consider the possibility that the total number, counted with signs, is zero; and then under deformations the holomorphic curves might all cancel out in pairs, to leave none.

Let us assume for the moment that such a ‘relative Gromov invariant’ is well-defined and nonzero, and is invariant under deformations of $N$ through nonsingular immersed Lagrangian submanifolds. If this is so, then we expect that for any nonsingular special Lagrangian deformation $N'$ of $N$ (and not just for $N'$ close to $N$) there should exist a holomorphic curve $\Sigma$ with boundary in $N'$ and $[\Sigma] = \beta$, under the identification $H_2(M; N', \mathbb{Z}) \cong H_2(M; N, \mathbb{Z})$.

But by calibrated geometry the area of $\Sigma$ is $\pi f_\beta(N')$, and clearly the area of a holomorphic curve must be positive. So we make the following conjecture, which strengthens part of Conjecture 4.1.

**Conjecture 4.2** Suppose that $D$ is the only holomorphic curve in the relative homology class $\beta$. Let $\alpha^R \neq 0$, and let $\mathcal{M}'$ be the connected component containing $N$ of the moduli space of nonsingular special Lagrangian 3-folds in $M$. Then $f_\beta > 0$ holds globally on $\mathcal{M}'$, and not just near $N$. Similarly, when $\alpha^R = 0$ the entire moduli space $\mathcal{M}'$ becomes singular when $f_\beta = 0$, and not just the part near $N$.

The idea here is that holomorphic curves $\Sigma$ with boundary on a special Lagrangian 3-fold $N$ can act as a global obstruction to deforming $N$, because the area of $\Sigma$ must remain positive. As the area of $\Sigma$ tends to zero the 3-fold $N$ becomes singular, because the boundary of $\Sigma$ (presumably a circle) shrinks to a point. When $\alpha^R \neq 0$ these singularities develop on a real hyperplane $f_\beta = 0$ in $A_{M,N}$, so the moduli space is confined to the ‘upper half-plane’ $f_\beta > 0$.

**Remark.** Holomorphic curves with boundary on special Lagrangian 3-folds are familiar to String Theorists as ‘world sheet instantons’. That is, they are imagined to be the world sheet (in Euclidean time) of an open string moving in $M$, with its end-points on $N$.

For example, Strominger, Yau and Zaslow [17, §2] explain that the leading order expression [17, eqn (2.10)] they derive for the mirror metric receives corrections due to ‘open string disc instantons’ – holomorphic discs whose
boundaries are nontrivial 1-cycles in $N$. These corrections are of order $e^{-\kappa A}$, where $A$ is the area of the disc and $\kappa > 0$ is large, and so the corrections are only significant when the discs have small area.

Thus String Theorists already know that holomorphic discs with boundary on special Lagrangian 3-folds are important, and that things go wrong when the area of the discs approaches zero. This fits very neatly with the picture we have outlined above.

5 Three special Lagrangian 3-folds for the price of one

Let $(M, J, g)$ be a Calabi-Yau 3-fold, and $N_1$ a compact special Lagrangian 3-fold in $M$. Suppose, as in §4, that $N_1$ is locally modelled on $L^{+}_{1,t_1}$ in $\mathbb{C}^3$ in some open set $U$ in $M$, for $t_1 > 0$. Now in §2 we wrote down three distinct families $L^{+}_{j,t}$ for $j = 1, 2, 3$, of special Lagrangian 3-folds in $\mathbb{C}^3$ asymptotic to the same $T^2$-cone $L^+_0$. In this section we shall consider whether there also exist special Lagrangian 3-folds $N_2, N_3$ in $M$, which are close to $N_1$ outside $U$, but in $U$ are modelled on $L^{+}_{2,t_2}$ and $L^{+}_{3,t_3}$ respectively, for some $t_2, t_3 > 0$.

We begin by describing how the 3-manifolds $N_1, N_2, N_3$ are related topologically. Recall that $U$ is isomorphic to an open ball in $\mathbb{C}^3$ of radius $\rho$, and let $\bar{U}$ be its closure. Define $P = N_1 \setminus U$, $Q_1 = N_1 \cap \bar{U}$ and $T = P \cap Q_1$. Then $T$ is a 2-torus $T^2$, isomorphic to the intersection of $L^{+}_{1,t_1}$ with the sphere $S^5$ of radius $\rho$ in $\mathbb{C}^3$. It is convenient to identify $T$ with the 2-torus

$$\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 = \rho^2, \quad z_1 z_2 z_3 = \rho^3 \} , \quad (10)$$

which is the intersection of $L^+_0$ with the sphere $S^5$ of radius $\rho$. This is close to the intersection of $L^{+}_{1,t_1}$ with $S^5$, as $t_1 \ll \rho^3$ by assumption.

Then $P$ and $Q_1$ are both 3-manifolds with common boundary $T$. Also, $Q_1$ has the topology of a ‘solid torus’ $S^1 \times D^2$, where $D^2$ is the closed disc in $\mathbb{R}^2$. As 3-manifolds we may write $N_2 = P \cup_T Q_2$ and $N_3 = P \cup_T Q_3$, where $Q_2$ and $Q_3$ are solid tori with boundary $T$. That is, we make $N_2, N_3$ by cutting out the solid torus $Q_1$ from $N_1$, and then gluing in two other solid tori $Q_2, Q_3$ with the same boundary. Now $Q_j$ is constructed from $L^{+}_{j,t_j}$ for $j = 1, 2, 3$, and cyclic permutation of $z_1, z_2, z_3$ permutes the families $L^{+}_{j,t_j}$ in the obvious way. We use this to relate the solid tori $Q_1, Q_2$ and $Q_3$.

Consider the map $\tau : T \to T$ given by $\tau : (z_1, z_2, z_3) \mapsto (z_3, z_1, z_2)$, where $T$ is given by (10). Then $\tau$ is a diffeomorphism of $T$, with $\tau^3 = 1$. We can
think of $Q_2, Q_3$ as differing from $Q_1$ by automorphisms $\tau, \tau^2$ of the boundary torus $T$. So we make $N_2$ or $N_3$ by cutting out the solid torus $Q_1$, and then gluing it back in again using the diffeomorphism $\tau$ or $\tau^2$ to identify the cut surfaces. This is called a Dehn twist.

By assumption, $Q_j$ is close to $L_{j,tj}^+$ in $\mathbb{C}^3$ for $j = 1, 2, 3$. So, as in §4, let $\gamma_j$ be an oriented $S^1$ in $Q_j$ close to $\gamma_{j,tj}$, and $D_j$ an oriented disc in $U$ close to $D_{j,tj}$, with boundary $\gamma_j$. Let $\alpha_j^z$ be the homology class of $\gamma_j$ in $H_1(N_j, \mathbb{Z})$, and $\alpha_j^z$ the homology class of $\gamma_j$ in $H_1(N_j, \mathbb{R})$. Let $\beta_j$ be the relative homology class of $D_j$ in $H_2(M; N_j, \mathbb{Z})$. Then $Q_j$ contracts onto the circle $\gamma_j$, and so $H_1(Q_j, \mathbb{Z}) \cong \mathbb{Z}$, and $\alpha_j^z = [\gamma_j]$ is a generator of $H_1(Q_j, \mathbb{Z})$.

In the next three propositions we consider the topology of $N_1, N_2$ and $N_3$.

**Proposition 5.1** The inclusion $\iota : T \to P$ induces a map $\iota_\ast : H_1(T, \mathbb{Z}) \to H_1(P, \mathbb{Z})$, which has $\text{Ker}(\iota_\ast) \cong \mathbb{Z}$. Let $\zeta$ be a generator for $\text{Ker}(\iota_\ast)$. Define integers $k_1, k_2, k_3$ by $\iota_j(\zeta) = k_j[\gamma_j]$, where $\iota_j : H_1(T, \mathbb{Z}) \to H_1(Q_j, \mathbb{Z})$ is induced by the inclusion $T \to Q_j$. Then $k_1 + k_2 + k_3 = 0$, and no more than one of $k_1, k_2, k_3$ is zero.

**Proof.** As $H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$ and $\text{Ker}(\iota_\ast)$ is a subgroup of $H_1(T, \mathbb{Z})$, we see that $\text{Ker}(\iota_\ast)$ is isomorphic to 0, $\mathbb{Z}$ or $\mathbb{Z}^2$. But using Poincaré duality ideas for manifolds with boundary, we can show that the map $H_1(T, \mathbb{R}) \to H_1(P, \mathbb{R})$ must have image and kernel $\mathbb{R}$, and this forces $\text{Ker}(\iota_\ast) \cong \mathbb{Z}$.

From above, $Q_2$ differs from $Q_1$ by the automorphism $\tau$ of its boundary $T$. This means that $\iota_1 : H_1(T, \mathbb{Z}) \to H_1(Q_1, \mathbb{Z})$ and $\iota_2 : H_1(T, \mathbb{Z}) \to H_1(Q_2, \mathbb{Z})$ are related by $\iota_2 = \iota_1 \circ \tau_\ast$, where $\tau_\ast : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$ is induced by $\tau$, and we identify $H_1(Q_1, \mathbb{Z})$ and $H_1(Q_2, \mathbb{Z})$ with $\mathbb{Z}$ in the obvious way. Similarly, $\iota_1 : H_1(T, \mathbb{Z}) \to H_1(Q_1, \mathbb{Z})$ and $\iota_3 : H_1(T, \mathbb{Z}) \to H_1(Q_3, \mathbb{Z})$ are related by $\iota_3 = \iota_1 \circ \tau^2_\ast$.

So $\iota_1(\zeta) = k_1[\gamma_1], \iota_2(\tau_\ast\zeta) = k_2[\gamma_1]$ and $\iota_3(\tau^2_\ast\zeta) = k_3[\gamma_1]$ by definition of $k_1, k_2$ and $k_3$. Thus

$$\iota_1(\zeta + \tau_\ast\zeta + \tau^2_\ast\zeta) = (k_1 + k_2 + k_3)[\gamma_1].$$

As $\tau^3 = 1$ we can think of $\tau_\ast$ as a 120° rotation on $H_1(T, \mathbb{Z})$, and thus $1 + \tau_\ast + \tau^2_\ast = 0$, as a map from $H_1(T, \mathbb{Z})$ to itself. Therefore $\zeta + \tau_\ast\zeta + \tau^2_\ast\zeta = 0$, and so $k_1 + k_2 + k_3 = 0$ by (11).

As $\zeta \neq 0$, we see that $\zeta$ and $\tau_\ast(\zeta)$ are linearly independent in $H_1(T, \mathbb{Z})$. But $\iota_1 : H_1(T, \mathbb{Z}) \to H_1(Q_1, \mathbb{Z})$ has kernel $\mathbb{Z}$, so $\zeta$ and $\tau_\ast(\zeta)$ cannot both be
in Ker $\iota_1$. Therefore $k_1$ and $k_2$ cannot both be zero. But if two of $k_1, k_2, k_3$ are zero then so is the third, as $k_1 + k_2 + k_3 = 0$. Thus no more than one of $k_1, k_2, k_3$ can be zero.

**Proposition 5.2** There exists $\chi \in H_2(M, \mathbb{Z})$ with the following properties for $j = 1, 2, 3$.

(i) If $k_j = 0$ then $\alpha_j^3 \neq 0$, and $N_j$ only exists as a special Lagrangian 3-fold if $[\omega] \cdot \chi = 0$.

(ii) If $k_j \neq 0$ then $\alpha_j^3 = 0$, and $\alpha_j^2$ has order $|k_j|$ in $H_1(N_j, \mathbb{Z})$. If $N_j$ exists as a special Lagrangian 3-fold locally modelled on $L^+_{j,t_j}$, then $t_j$ satisfies $[\omega] \cdot \chi = k_j \pi t_j$. In particular, as we need $t_j > 0$, this means that $N_j$ only exists if either $k_j > 0$ and $[\omega] \cdot \chi > 0$, or $k_j < 0$ and $[\omega] \cdot \chi < 0$.

**Proof.** The class $\zeta$ in $H_1(T, \mathbb{Z})$ is homologous to 0 in $P$, by definition. So there exists a 2-chain $\Sigma$ in $P$ with boundary in $T$, and $[\partial \Sigma] = \zeta$. But $\zeta$ is homologous to $k_j[\gamma_j]$ in $Q_j$, as $\iota_j(\zeta) = k_j[\gamma_j]$. Thus we can find a 2-chain $\Sigma_j$ in $N_j$ with $\partial \Sigma_j = k_j \gamma_j$, such that $\Sigma_j$ agrees with $\Sigma$ in $P$ for $j = 1, 2, 3$. Thus $k_jD_j - \Sigma_j$ is a 2-chain in $M$ without boundary, as $\gamma_j = \partial D_j$, and defines a homology class $\chi$ in $H_2(M, \mathbb{Z})$. This $\chi$ is independent of $j$, as the chains $k_jD_j - \Sigma_j$ coincide outside the open ball $U$ for $j = 1, 2, 3$.

Suppose $N_j$ is indeed a special Lagrangian 3-fold modelled on $L^+_{j,t_j}$. Then we have

$$[\omega] \cdot \chi = \int_{k_jD_j} \omega - \int_{\Sigma_j} \omega = k_j \int_{D_j} \omega \approx k_j \pi t_j,$$

since $\int_{D_j} \omega \approx \int_{D_j,t_j} \omega_0 = \pi t_j$ as $D_j, \omega$ are close to $D_j,t, \omega_0$ in $U$, and $\omega|_{\Sigma_j} \equiv 0$ because $\Sigma_j \subset N_j$ and $\omega|_{N_j} \equiv 0$. As we are free to change $t_j$ a little bit, we can take $[\omega] \cdot \chi = k_j \pi t_j$.

Now suppose that $n\gamma_j$ is homologous to zero in $N_j$ for some $n \in \mathbb{Z}$. Then there exists a 2-chain $C$ in $N_j$ with $\partial C = n\gamma_j$. By choosing $C$ generically we can arrange that $C \cap T$ is a 1-chain in $T$. Let $\eta = [C \cap T]$ in $H_1(T, \mathbb{Z})$. As $C \cap T = \partial(C \cap P)$ it follows that $\iota_*(\eta) = 0$, where $\iota_* : H_1(T, \mathbb{Z}) \rightarrow H_1(P, \mathbb{Z})$ is the natural projection.

But Ker $\iota_*$ is generated by $\zeta$, so $\eta = l \zeta$ for some $l \in \mathbb{Z}$. Also $Q_j \cap C$ is a homology between $C \cap T$ and $n\gamma_j$, and thus $\iota_j(\eta) = n[\gamma_j]$. Substituting
Proposition 5.3 In the situation above $b_1(P) \geq 1$ and $b_1(P) - 1 \leq b_1(N_j) \leq b_1(P)$, so $b_1(N_j) = 0$ for any $j$ only if $b_1(P) = 1$. Suppose $b_1(P) = 1$. Then $H_1(P; T, \mathbb{Z})$ is finite, and for $j = 1, 2, 3$ we have

(i) If $k_j = 0$ then $b_1(N_j) = 1$, and $H_1(N_j, \mathbb{Z})$ is infinite.

(ii) If $k_j \neq 0$ then $b_1(N_j) = 0$, so that $N_j$ is a (rational) homology sphere, and $H_1(N_j, \mathbb{Z})$ is finite with $|H_1(N_j, \mathbb{Z})| = |k_j| \cdot |H_1(P; T, \mathbb{Z})|$.

Proof. We argued in the proof of Proposition 5.1 that the map $H_1(T, \mathbb{R}) \to H_1(P, \mathbb{R})$ has image $\mathbb{R}$. Hence $H_1(P, \mathbb{R})$ contains a copy of $\mathbb{R}$, and $b_1(P) \geq 1$. The exact sequence $H_1(T, \mathbb{R}) \to H_1(P, \mathbb{R}) \to H_1(P; T, \mathbb{R}) \to 0$ then shows that $\dim H_1(P; T, \mathbb{R}) = b_1(P) - 1$. But $H_1(P; T, \mathbb{R}) \cong H_1(N_j; Q_j, \mathbb{R})$ by excision.

Thus in the exact sequence $H_1(Q_j, \mathbb{R}) \to H_1(N_j, \mathbb{R}) \to H_1(N_j; Q_j, \mathbb{R}) \to 0$ we have $\dim H_1(Q_j, \mathbb{R}) = 1$ and $\dim H_1(N_j; Q_j, \mathbb{R}) = b_1(P) - 1$. The image of $H_1(Q_j, \mathbb{R}) \to H_1(N_j, \mathbb{R})$ is generated by $\alpha_j$. Therefore $b_1(N_j) = b_1(P)$ if $\alpha_j \neq 0$, and $b_1(N_j) = b_1(P) - 1$ if $\alpha_j = 0$. Thus $b_1(P) - 1 \leq b_1(N_j) \leq b_1(P)$, as we want.

Now suppose $b_1(P) = 1$. Then $\dim H_1(P; T, \mathbb{R}) = 0$ from above, and thus $H_1(P; T, \mathbb{Z})$ is finite, as we have to prove. If $k_j = 0$ then $\alpha_j \neq 0$ by Proposition 5.2, and so $b_1(N_j) = b_1(P) = 1$, which implies that $H_1(N_j, \mathbb{Z})$ is infinite. This proves part (i). For part (ii), if $k_j \neq 0$ then $\alpha_j = 0$ by Proposition 5.2, so $b_1(N_j) = b_1(P) - 1 = 0$. This implies that $H_1(N_j, \mathbb{Z})$ is finite, and as $N_j$ is also connected and oriented, that $N_j$ is a rational homology 3-sphere.
It remains to show that \(|H_1(N_j; \mathbb{Z})| = |k_j| \cdot |H_1(P; T, \mathbb{Z})|\). Consider the exact sequence

\[
H_1(Q_j, \mathbb{Z}) \to H_1(N_j, \mathbb{Z}) \to H_1(N_j; Q_j, \mathbb{Z}) \to 0.
\] (12)

Here \(H_1(Q_j, \mathbb{Z})\) is isomorphic to \(\mathbb{Z}\) and is generated by \([\gamma_j]\). But the image of \([\gamma_j]\) in \(H_1(N_j, \mathbb{Z})\) is \(\alpha^\mathbb{R}_j\), which has finite order \(|k_j|\) in \(H_1(N_j, \mathbb{Z})\) by Proposition 5.2. Thus the image of \(H_1(Q_j, \mathbb{Z})\) in \(H_1(N_j, \mathbb{Z})\) is \(\mathbb{Z}_{|k_j|}\), the finite cyclic group of order \(|k_j|\). It follows from this, the exact sequence (12) and the excision isomorphism 

\[
H_1(N_j; Q_j, \mathbb{Z}) \cong H_1(P; T, \mathbb{Z})
\]

that \(|H_1(N_j, \mathbb{Z})| = |k_j| \cdot |H_1(P; T, \mathbb{Z})|\), and the proof is complete.

We can now draw some conclusions about the behaviour of families of special Lagrangian 3-folds. As in §4, we consider the cases \(\alpha^\mathbb{R} \neq 0\) and \(\alpha^\mathbb{R} = 0\) separately.

### 5.1 The case \(\alpha^\mathbb{R} \neq 0\)

Suppose, as in §4.1, that \(N\) is a special Lagrangian 3-fold in \(M\) modelled locally on \(L^+_1, t\) with \(\alpha^\mathbb{R} \neq 0\), which is part of a family \(\mathcal{M}\) of special Lagrangian 3-folds. Then we expect \(\mathcal{M}\) to be contained in the upper half-plane \(f \beta > 0\) in \(A_{M,N}\), and that at the boundary \(f \beta = 0\) the special Lagrangian 3-folds in \(\mathcal{M}\) develop singularities locally modelled on the \(T^2\)-cone \(L^+_0\).

Write \(N_1 = N\). Then in the notation of this section we have \(\alpha^\mathbb{R}_1 \neq 0\), so \(k_1 = 0\) and \([\omega] \cdot \chi = 0\) by part (i) of Proposition 5.2. But only one of \(k_1, k_2, k_3\) can be zero by Proposition 5.1, and so \(k_2 \neq 0\) and \(k_3 \neq 0\). So by part (ii) of Proposition 5.2 we see that \(N_2, N_3\) do not exist as special Lagrangian 3-folds, as this would require \([\omega] \cdot \chi > 0\) or \([\omega] \cdot \chi < 0\), but we know that \([\omega] \cdot \chi = 0\).

Let \(N_0\) be a singular special Lagrangian 3-fold at the boundary \(f \beta = 0\) of \(\mathcal{M}\), and suppose \(N_0\) has only one singular point, which is modelled locally on \(L^+_0\). Then there is a family \(\mathcal{M}\) of nonsingular special Lagrangian 3-folds \(N_1\) converging to \(N_0\), locally modelled on \(L^+_1, t_1\). However, we have shown that there do not exist other families of special Lagrangian 3-folds \(N_2, N_3\) in \(M\) converging to \(N_0\), locally modelled on \(L^+_2, t_2\) and \(L^+_3, t_3\). This is not obvious, but depends on the assumption that \(N\) is a compact nonsingular 3-manifold.

In fact we can take this argument further. Let us assume that any family of nonsingular special Lagrangian 3-folds in \(M\) converging to \(N_0\) must be locally modelled on AC special Lagrangian 3-folds in \(\mathbb{C}^3\) asymptotic to \(L^+_0\).
But we showed in §3 that any AC special Lagrangian 3-fold asymptotic to $L_0$ is one of the $L_{j,t}$. We have ruled out families modelled on $L_{2,t_2}$ and $L_{3,t_3}$. So, provided our assumption is correct, we deduce that $M$ is the only family of nonsingular special Lagrangian 3-folds in $M$ converging to $N_0$.

This is something which should be taken into account in discussions about compactifying moduli spaces of special Lagrangian 3-folds, or fibrations of Calabi-Yau 3-folds by special Lagrangian 3-folds, as suggested by Strominger, Yau and Zaslow [17]. We have shown (provided our assumptions are correct) that families of special Lagrangian 3-folds can develop singularities in codimension one, which cannot be repaired any other way.

That is, moduli spaces $M$ of special Lagrangian 3-folds should be expected to be manifolds with boundary, and it is too much to hope that in general singularities will develop only in codimension two in $M$. Nevertheless, the author is at the moment quite optimistic that special Lagrangian 3-folds will turn out to have quite good ‘compactness’ properties, and that the ideas of Strominger, Yau and Zaslow will turn out to be correct near the ‘large complex structure limit’.

5.2 The case $a^R = 0$

Suppose that $N$ is a special Lagrangian 3-fold in $M$ modelled locally on $L_{1,t}^+$ with $a^R = 0$. As in §4.2 we have $[\omega] \cdot \chi = k_{j=N}(N) = k\pi t > 0$. Moreover, if $N'$ is close to $N$ in the moduli space $M$ of special Lagrangian deformations of $N$, then we expect $N'$ also to be locally modelled on $L_{1,t}^+$, with the same value of $t$.

Write $N_1 = N$ and $t_1 = t$, and use the notation of this section. Consider, as above, whether there exist special Lagrangian 3-folds $N_2, N_3$ which are close to $N_1$ outside $U$, but inside $U$ are locally modelled on $L_{2,t_2}^+$ and $L_{3,t_3}^+$ for some $t_2, t_3 > 0$. As $[\omega] \cdot \chi > 0$, Proposition 5.2 shows that $N_j$ can only exist if $k_j > 0$. But $k_1 + k_2 + k_3 = 0$ by Proposition 5.1, and $k_1 = k$ by definition, so $k_1 > 0$. Thus at least one of $k_2$ and $k_3$ are less than zero, and so at most one of $N_2, N_3$ exists as a special Lagrangian 3-fold.

Now deform the underlying Calabi-Yau 3-fold $(M, J, g)$, changing the cohomology class $[\omega]$ of $\omega$, so that $[\omega] \cdot \chi$ passes through zero. Then, when $[\omega] \cdot \chi < 0$, Proposition 5.2 shows that $N_j$ exists only if $k_j < 0$. In fact we conjecture that under sufficiently small deformations of $(M, J, g)$, the conditions $k_j > 0$ when $[\omega] \cdot \chi > 0$ and $k_j < 0$ when $[\omega] \cdot \chi < 0$ are also sufficient conditions for $N_j$ to exist as a special Lagrangian 3-fold.
In particular, we have the following three cases.

(a) Suppose \( k_1, k_2 > 0 \) and \( k_3 < 0 \). When \( [\omega] \cdot \chi > 0 \) we expect that \( N_1, N_2 \) exist as special Lagrangian 3-folds but \( N_3 \) does not, but when \( [\omega] \cdot \chi < 0 \) then \( N_3 \) exists as a special Lagrangian 3-fold but \( N_1, N_2 \) do not.

(b) Suppose \( k_1 > 0, k_2 = 0 \) and \( k_3 < 0 \). When \( [\omega] \cdot \chi > 0 \) we expect that \( N_1 \) exists as a special Lagrangian 3-fold but \( N_2, N_3 \) do not, but when \( [\omega] \cdot \chi < 0 \) then \( N_3 \) exists as a special Lagrangian 3-fold but \( N_1, N_2 \) do not.

(c) Suppose \( k_1 > 0 \) and \( k_2, k_3 < 0 \). When \( [\omega] \cdot \chi > 0 \) we expect that \( N_1 \) exists as a special Lagrangian 3-fold but \( N_2, N_3 \) do not, but when \( [\omega] \cdot \chi < 0 \) then \( N_2, N_3 \) exist as special Lagrangian 3-folds but \( N_1 \) does not.

Let us now restrict our attention to the case that \( b_1(N_1) = 0 \), that is, \( N_1 \) is a (rational) homology 3-sphere. Then by Proposition 5.3, \( N_j \) is a homology 3-sphere if and only if \( k_j \neq 0 \). Also, by Corollary 2.15, if \( N_j \) is a homology 3-sphere then it is rigid as a special Lagrangian 3-fold, and has no deformations.

Cases (a)–(c) above show us that as we deform \((M, J, g)\), changing the cohomology class \([\omega]\) of \(\omega\), then (if our assumptions are correct) it can happen that two special Lagrangian homology 3-spheres disappear, and one reappears; or that one disappears, and another reappears; or that one disappears, and two reappear. In particular, this shows that the number of special Lagrangian homology 3-spheres in \(M\) is not invariant under deformations of \((M, J, g)\) changing the Kähler class \([\omega]\), even if counted with signs.

6 A second family of special Lagrangian 3-folds in \(\mathbb{C}^3\)

Most of the material in this section is not original; it is taken from Harvey and Lawson [7, §III.3.B], Lawlor [10] and Harvey [6, p. 139–140]. We begin with a proposition adapted from Harvey and Lawson [7, §III.3.B].
Proposition 6.1. Let \( c \in \mathbb{R} \), and define a submanifold \( K_c \) in \( \mathbb{C}^3 \) by

\[
K_c = \left\{ (zx_1, zx_2, zx_3) \in \mathbb{C}^3 : x_1, x_2, x_3 \in \mathbb{R} \text{ with } x_1^2 + x_2^2 + x_3^2 = 1 \right. \text{ and } z \in \mathbb{C} \text{ with } \text{Im}(z^3) = c \right\}. \tag{13}
\]

Then \( K_c \) is a special Lagrangian 3-fold in \( \mathbb{C}^3 \).

These submanifolds are clearly invariant under the obvious action of \( \text{SO}(3) \) on \( \mathbb{C}^3 \), and this is how Harvey and Lawson constructed them.

Let \( c > 0 \), and consider the equation \( \text{Im}(z^3) = c \). Writing \( z = re^{i\theta} \) gives \( r^3 \sin 3\theta = c \). So \( \sin 3\theta > 0 \), which holds when \( \theta \) lies in \((0, \pi/3)\) or \((2\pi/3, \pi)\) or \((4\pi/3, 5\pi/3)\), and then \( r \) is given explicitly by \( r = c^{1/3}(\sin 3\theta)^{-1/3} \). Thus \( \text{Im}(z^3) = c \) defines a curve in \( \mathbb{C} \) with 3 connected components. So \( K_c \) also divides into 3 connected components \( K_{12,c}, K_{13,c} \) and \( K_{23,c} \), each diffeomorphic to \( S^2 \times \mathbb{R} \), given explicitly by

\[
K_{12,c} = \left\{ (zx_1, zx_2, zx_3) \in \mathbb{C}^3 : (x_1, x_2, x_3) \in S^2 \text{ and } z = re^{i\theta} \right. \text{ with } \theta \in (0, \pi/3) \text{ and } r = c^{1/3}(\sin 3\theta)^{-1/3} \right\}, \tag{14}
\]

\[
K_{13,c} = \left\{ (zx_1, zx_2, zx_3) \in \mathbb{C}^3 : (x_1, x_2, x_3) \in S^2 \text{ and } z = re^{i\theta} \right. \text{ with } \theta \in (2\pi/3, \pi) \text{ and } r = c^{1/3}(\sin 3\theta)^{-1/3} \right\}, \tag{15}
\]

\[
K_{23,c} = \left\{ (zx_1, zx_2, zx_3) \in \mathbb{C}^3 : (x_1, x_2, x_3) \in S^2 \text{ and } z = re^{i\theta} \right. \text{ with } \theta \in (4\pi/3, 5\pi/3) \text{ and } r = c^{1/3}(\sin 3\theta)^{-1/3} \right\}. \tag{16}
\]

Observe that \( K_c = K_{-c} \), as if we simultaneously reverse the signs of \( z, x_1, x_2 \) and \( x_3 \) then \( (zx_1, zx_2, zx_3) \) is unchanged, but \( \text{Im}(z^3) \) changes to \( -\text{Im}(z^3) \). Thus \( K_c \) is also diffeomorphic to 3 copies of \( S^2 \times \mathbb{R} \) when \( c < 0 \).

However, \( K_0 \) is the (singular) union of the three special Lagrangian 3-planes

\[
\Pi_1 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R} \}, \quad \Pi_2 = \{(e^{\pi i/3} x_1, e^{\pi i/3} x_2, e^{\pi i/3} x_3) : x_j \in \mathbb{R} \} \quad \text{and} \quad \Pi_3 = \{(e^{2\pi i/3} x_1, e^{2\pi i/3} x_2, e^{2\pi i/3} x_3) : x_j \in \mathbb{R} \}. \tag{17}
\]

It is not difficult to show that each \( K_{jk,c} \) defined in (14)–(16) is asymptotically conical, and asymptotic at infinity to the union of two of \( \Pi_1, \Pi_2, \Pi_3 \). We have
chosen our notation such that $K_{jk,c}$ is asymptotic to $\Pi_j \cup \Pi_k$, which is a special Lagrangian cone on 2 copies of $S^2$.

Thus we have found a 1-parameter family \( \{ L_{12,c} : c > 0 \} \) of special Lagrangian 3-folds diffeomorphic to $S^2 \times \mathbb{R}$, which are asymptotic at infinity to the union of the two special Lagrangian 3-planes $\Pi_1$ and $\Pi_2$. It is natural to ask whether we can find a similar family of special Lagrangian 3-folds asymptotic to the union of \( \Pi_1 \) and \( \Pi_2' \) intersecting at 0. To answer this we first classify such pairs of 3-planes up to the action of SU(3).

**Proposition 6.2** Let $\Pi, \Pi'$ be special Lagrangian 3-planes in $\mathbb{C}^3$ which intersect only at 0. Then there exist $\theta_1, \theta_2, \theta_3 \in (0, \pi)$ and $A \in \text{SU}(3)$ such that

\[
A \cdot \Pi = \{ (x_1, x_2, x_3) : x_j \in \mathbb{R} \}, \quad A \cdot \Pi' = \{ (e^{i\theta_1}x_1, e^{i\theta_2}x_2, e^{i\theta_3}x_3) : x_j \in \mathbb{R} \}.
\]

Furthermore $\theta_1 + \theta_2 + \theta_3 = \pi$ if the intersection $\Pi \cap \Pi'$ is positive, and $\theta_1 + \theta_2 + \theta_3 = 2\pi$ if the intersection $\Pi \cap \Pi'$ is negative.

We leave the proof as an exercise. Here we mean that the intersection $\Pi \cap \Pi'$ is positive or negative in the sense of homology; note that as the dimension is odd, $\Pi \cap \Pi'$ has the opposite sign to $\Pi' \cap \Pi$, so we can arrange for $\Pi \cap \Pi'$ to be positive by swapping round $\Pi$ and $\Pi'$.

Next we define a family of special Lagrangian 3-folds $K_{\theta,A}$ which generalizes the family $K_{12,c}$ given in (14). This family was first found by Lawlor [10], and made more explicit by Harvey [6, p. 139–140]. Our treatment is based on that of Harvey.

Let $a_1, a_2, a_3 > 0$, and define polynomials $p(x), P(x)$ by

\[
p(x) = (1 + a_1x^2)(1 + a_2x^2)(1 + a_3x^2) - 1 \quad \text{and} \quad P(x) = \frac{p(x)}{x^2}.
\]

Define real numbers $\theta_1, \theta_2, \theta_3$ and $A$ by

\[
\theta_k = a_k \int_{-\infty}^{\infty} \frac{dx}{(1 + a_kx^2)\sqrt{P(x)}} \quad \text{and} \quad A = \frac{4\pi}{3} (a_1a_2a_3)^{-1/2}.
\]

Clearly $\theta_k > 0$ and $A > 0$. But writing $\theta_1 + \theta_2 + \theta_3$ as one integral and rearranging gives

\[
\theta_1 + \theta_2 + \theta_3 = \int_{-\infty}^{\infty} \frac{p'(x)dx}{(p(x) + 1)\sqrt{p(x)}} = 2 \int_{0}^{\infty} \frac{dw}{w^2 + 1} = \pi,
\]

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making the substitution $w = \sqrt{p(x)}$. So $\theta_k \in (0, \pi)$ and $\theta_1 + \theta_2 + \theta_3 = \pi$. It can be shown that this yields a 1-1 correspondence between triples $(a_1, a_2, a_3)$ with $a_k > 0$, and quadruples $(\theta_1, \theta_2, \theta_3, A)$ with $\theta_k \in (0, \pi)$, $\theta_1 + \theta_2 + \theta_3 = \pi$, and $A > 0$.

For $k = 1, 2, 3$ and $y \in \mathbb{R}$, define $z_k(y)$ by $z_k(y) = e^{i\psi_k(y)} \sqrt{a_k^{-1} + y^2}$, where

$$\psi_k(y) = a_k \int_{-\infty}^{y} \frac{dx}{(1 + a_k x^2)^{\frac{1}{2}}} P(x).$$

Now write $\theta = (\theta_1, \theta_2, \theta_3)$, and define a submanifold $K_{\theta,A}$ in $\mathbb{C}^3$ by

$$K_{\theta,A} = \{(z_1(y)x_1, z_2(y)x_2, z_3(y)x_3) : y \in \mathbb{R}, x_k \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}.$$  \hspace{1cm} (18)

The following proposition may be deduced from Harvey [6, Th. 7.78].

**Proposition 6.3** The set $K_{\theta,A}$ defined above is an embedded special Lagrangian submanifold in $\mathbb{C}^3$ diffeomorphic to $S^2 \times \mathbb{R}$. It is asymptotically conical, and asymptotic at infinity to the union $\Pi_1 \cup \Pi_\theta$ of two special Lagrangian 3-planes $\Pi_1, \Pi_\theta$ given by

$$\Pi_1 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R}\} \quad \text{and} \quad \Pi_\theta = \{(e^{i\theta_1}x_1, e^{i\theta_2}x_2, e^{i\theta_3}x_3) : x_j \in \mathbb{R}\}. \hspace{1cm} (19)$$

It can be shown that when $\theta_1 = \theta_2 = \theta_3 = \pi/3$ and $A = 4\pi c/3$, the submanifolds $K_{12,c}$ and $K_{\theta,A}$ defined by (14) and (18) agree. Thus the $K_{\theta,A}$ do generalize the $K_{12,c}$. Combining Propositions 6.2 and 6.3 we prove:

**Corollary 6.4** Let $\Pi, \Pi'$ be any two special Lagrangian 3-planes in $\mathbb{C}^3$ intersecting only at 0. Then there exists a 1-parameter family $\{K_{\Pi\Pi',A} : A > 0\}$ of AC special Lagrangian 3-folds, which are all diffeomorphic to $S^2 \times \mathbb{R}$, and asymptotic to $\Pi \cup \Pi'$ at infinity.

Finally we interpret the constant $A$ in $K_{\theta,A}$. Using the above notation, define

$$D_{\theta,A} = \{(x_1 e^{i\theta_1/2}, x_2 e^{i\theta_2/2}, x_3 e^{i\theta_3/2}) : x_k \in \mathbb{R}, a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \leq 1\}. \hspace{1cm} (20)$$
Then $D_{\theta,A}$ is a solid ellipsoid in $\mathbb{C}^3$, with boundary in $K_{\theta,A}$. The axes of $D_{\theta,A}$ have lengths $a_k^{-1/2}$ for $k = 1, 2, 3$, and so the volume of $D_{\theta,A}$ is $A$. Furthermore, $D_{\theta,A}$ is calibrated with respect to $\text{Im}(\Omega_0)$. That is, we can regard $D_{\theta,A}$ as a special Lagrangian 3-fold of phase $e^{i\pi/2}$, whereas $K_{\theta,A}$ has phase 1; so that $D_{\theta,A}$ and $K_{\theta,A}$ are both special Lagrangian, but of perpendicular phase.

We met a very similar situation in §3. There we defined a special Lagrangian 3-fold $L^+_{j,t}$ depending on a real parameter $t > 0$, and a holomorphic 2-disc $D^+_{j,t}$ with boundary on $L^+_{j,t}$, and area $\pi t$. Here we define a special Lagrangian 3-fold $K_{\theta,A}$ depending on a real parameter $A > 0$, and a special Lagrangian 3-disc $D_{\theta,A}$ of perpendicular phase, with boundary on $K_{\theta,A}$ and area $A$.

Suppose $K$ is an AC special Lagrangian 3-fold asymptotic to the cone $\Pi_1 \cup \Pi_2$, where $\Pi_1, \Pi_2$ are given in (17). As $\Pi_1 \cup \Pi_2$ is invariant under the obvious action of SO(3) on $\mathbb{C}^3$, it follows from Proposition 2.9 that $K$ is also SO(3)-invariant. But it is not difficult to show that any connected, SO(3)-invariant special Lagrangian 3-fold is an open subset of one of the 3-folds $K_c$ of (13). Thus $K$ must be $K_{12,c}$ for some $c > 0$.

We have shown that any AC special Lagrangian 3-fold asymptotic to $\Pi_1 \cup \Pi_2$ is one of the 3-folds $K_{12,c}$ defined in (14). But the 3-folds $K_{\theta,A}$ are generalizations of the $K_{12,c}$, asymptotic to $\Pi_1 \cup \Pi_\theta$. The author conjectures that any AC special Lagrangian 3-fold asymptotic to $\Pi_1 \cup \Pi_\theta$ is one of the 3-folds $K_{\theta,A}$ defined in (18), for some $A > 0$.

7 Another degeneration of special Lagrangian 3-folds

We now consider families of compact special Lagrangian 3-folds $N$ in a Calabi-Yau 3-fold $(M, J, g)$ which are locally modelled on the 3-folds $K_{\theta,A}$ defined in §6. The picture we discover is similar to that in §4.

Let $(M, J, g)$ be a Calabi-Yau 3-fold with Kähler form $\omega$, let $\Omega$ be a holomorphic 3-form on $M$, and $N$ a compact, embedded special Lagrangian 3-fold in $M$ with phase $e^{i\vartheta}$. By replacing $\Omega$ by $e^{-i\vartheta}\Omega$ we may suppose that $N$ has phase 1. Suppose that there exist a small open ball $U$ in $M$, holomorphic coordinates $(z_1, z_2, z_3)$ on $U$, and a small constant $\rho > 0$ satisfying the following conditions:
• In coordinates \( U = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 < \rho^2\} \). That is, \( U \) is the open ball of radius \( \rho \) about 0 in \( \mathbb{C}^3 \).

• On \( U \) the metric \( g \), Kähler form \( \omega \) and 3-form \( \Omega \) on \( M \) are very close to the Euclidean versions \( g_0, \omega_0, \Omega_0 \) on \( \mathbb{C}^3 \) defined by (3).

• In \( U \), the 3-fold \( N \) in \( M \) is very close to the 3-fold \( K_{\theta, A} \) in \( \mathbb{C}^3 \) defined by (18), for some \( \theta_1, \theta_2, \theta_3 \in (0, \pi) \) with \( \theta_1 + \theta_2 + \theta_3 = \pi \) and \( A \) with \( 0 < A \ll \rho^3 \).

Let \( D_{\theta, A} \) be as in (20). Since by assumption \( N \) is close to \( K_{\theta, A} \) in \( U \), we can choose a closed, oriented 3-disc \( D \) in \( U \) with boundary in \( N \) which is close to \( D_{\theta, A} \). In fact, because special Lagrangian 3-discs with boundary on \( N \) are stable under small deformations, we can choose \( D \) to be special Lagrangian with phase \( e^{i\pi/2} \), that is, \( D \) is calibrated with respect to \( \text{Im} \Omega \).

Let \( S \) be the boundary of \( D \). Then \( S \) is an oriented 2-sphere in \( N \). Let \( \eta \) be the homology class \([S]\) of \( S \) in \( H_2(N, \mathbb{Z}) \), and \( \zeta \) the relative homology class \([D]\) in \( H_3(M; N, \mathbb{Z}) \). Because \( D \) is close to \( D_{\theta, A} \) and \( \Omega \) close to \( \Omega_0 \) on \( U \), we have

\[
\int_D \text{Im} \Omega \approx \int_{D_{\theta, A}} \text{Im} \Omega_0 = A,
\]

as \( D_{\theta, A} \) is calibrated w.r.t. \( \text{Im} \Omega_0 \) and has area \( A \). Since we are free to change \( A \) by a little bit, let us define \( A \) to be \( \int_D \text{Im} \Omega \).

Suppose \( \Sigma \) is a 3-chain in \( M \) with boundary in \( N \), and \([\Sigma] = \zeta\). As \( \text{Im} \Omega \equiv 0 \) on \( N \), it easily follows that \( \int_\Sigma \text{Im} \Omega \) depends only on the relative homology class \([\Sigma]\) of \( \Sigma \) in \( H_3(M; N, \mathbb{Z}) \), and so \( \int_\Sigma \text{Im} \Omega = A \), as \([\Sigma] = [D]\) and \( \int_D \text{Im} \Omega = A \). Thus the constant \( A > 0 \) is determined by integrating \( \text{Im} \Omega \) on the relative homology class \( \zeta \) in \( H_3(M; N, \mathbb{Z}) \). This is similar to the situation in §4, where \( \pi t \) was given by integrating \( \omega \) on \( \beta \) in \( H_2(M; N, \mathbb{Z}) \).

Now let \( \mathcal{M} \) be a connected and simply-connected open neighbourhood of \( N \) in the moduli space of special Lagrangian 3-folds in \( M \). We explained in §2.4 why \( \mathcal{M} \) is isomorphic to an open subset of an affine space \( A_{M,N}^* \) modelled on \( H^2(N, \mathbb{R}) \), and how to associate a function \( F_\sigma : \mathcal{M} \rightarrow \mathbb{R} \) to each class \( \sigma \in H_3(M; N, \mathbb{Z}) \). When \( \sigma = \zeta \), we have \( F_{\zeta}(N) = A \), since \([D] = \zeta\) and \( \int_D \text{Im} \Omega = A \).

Let \( N, N' \) be special Lagrangian 3-folds in \( \mathcal{M} \), and \( x, x' \) the corresponding points in the affine space \( A_{M,N}^* \). Since \( A_{M,N}^* \) is modelled on the vector space
we can regard \(x - x'\) as an element of \(H^2(N, \mathbb{R})\). By the same proof as (8) in §4, we see that

\[
F_\zeta(N) - F_\zeta(N') = F_\zeta(x) - F_\zeta(x') = \eta \cdot (x - x')
\]

(21)

where \(\cdot\) is the natural pairing between \(H^2(N, \mathbb{Z})\) and \(H^2(N, \mathbb{R})\).

Here is a conjecture on the behaviour of special Lagrangian deformations of \(N\), analogous to Conjecture 4.1. We hope to be able to prove it by the methods outlined in §4.

**Conjecture 7.1** Suppose \(N'\) is close to \(N\) in \(\mathcal{M}\), and define \(A' = F_\zeta(N')\). Then there are holomorphic coordinates \((z'_1, z'_2, z'_3)\) in \(U\) and \(\theta'_1, \theta'_2, \theta'_3 \in (0, \pi)\) with \(\theta'_1 + \theta'_2 + \theta'_3 = \pi\), such that \(N'\) is close to the 3-fold \(K_{\theta', A'}\) in \(\mathbb{C}^3\) defined by (18) in the coordinates \((z'_1, z'_2, z'_3)\), where \(\theta' = (\theta'_1, \theta'_2, \theta'_3)\). The new coordinates \((z'_1, z'_2, z'_3)\) are close to the old coordinates \((z_1, z_2, z_3)\), and differ from them by a small translation and a rotation in \(SU(3)\) close to the identity.

In particular, as \(K_{\theta', A'}\) is only well-defined when \(A' \geq 0\) and nonsingular when \(A' > 0\), we conjecture that \(F_\zeta(N') > 0\) for all nonsingular \(N'\) near \(N\) in \(\mathcal{M}\), and on the boundary \(F_\zeta(N') = 0\), the 3-folds \(N'\) have a singular point modelled locally on the intersection of two special Lagrangian 3-planes \(\Pi_1, \Pi_{\theta'}\) in \(\mathbb{C}^3\).

In §4 we found that things were different when \(\alpha^s \neq 0\) and when \(\alpha^s = 0\). Here things are different when \(\eta \neq 0\) and when \(\eta = 0\), so we consider the two cases separately.

### 7.1 The case \(\eta \neq 0\)

As \(N\) is compact and oriented and \(S\) is an embedded hypersurface in \(N\), we see that \([S] \neq 0\) in \(H_2(N, \mathbb{Z})\) implies that \([S] \neq 0\) in \(H_2(N, \mathbb{R})\). Thus \(H_2(N, \mathbb{R})\) and \(H^2(N, \mathbb{R})\) are both nonzero vector spaces, and (21) shows that the function \(F_\zeta : A^*_{M,N} \to \mathbb{R}\) is not constant on \(A^*_{M,N}\). Therefore \(\{x' \in A^*_{M,N} : F_\zeta(x') = 0\}\) is a real hyperplane in \(A^*_{M,N}\), which divides \(A^*_{M,N}\) into two halves, the ‘upper half-plane’ \(F_\zeta > 0\), and the ‘lower half-plane’ \(F_\zeta < 0\).

According to Conjecture 7.1, near \(N\) the moduli space \(\mathcal{M}\) of special Lagrangian deformations of \(N\) lies solely in the upper half-plane \(F_\zeta > 0\). At the dividing wall \(F_\zeta = 0\) they become singular, with a singularity modelled on the union of two 3-planes \(\Pi_1 \cap \Pi_{\theta'}\), where an \(S^2\) in \(N\) has been crushed.
to a point. Thus, these singularities happen in real codimension one in \( \mathcal{M} \), as in §4.2.

Let \( N' \) be one of these singular 3-folds at \( F_\zeta' = 0 \), with a singularity modelled on \( \Pi_1 \cap \Pi_{\theta'} \) at \( p \in M \). But \( \Pi_1 \cap \Pi_{\theta'} \) is nonsingular, as an immersed submanifold. Thus we can regard \( N' \) as a nonsingular, immersed special Lagrangian 3-fold in \( M \), but with a different topology to \( N \), and a point of self-intersection at \( p \). That is, \( N' \) is the image of an immersion \( f : N \to M \), where \( \hat{N} \) is a compact, connected, nonsingular 3-manifold with \( b_2(\hat{N}) = b_2(N) - 1 \).

### 7.2 The case \( \eta = 0 \)

We first discuss the topology of \( N \). If \( [S] = \eta = 0 \) in \( H_2(N, \mathbb{Z}) \) then \( N \setminus S \) splits into two connected components. Regard these components as 3-manifolds with boundary, with orientation induced from \( N \), and let \( C^+ \) be the component with \( \partial C^+ = -S \) and \( C^- \) the component with \( \partial C^- = S \), viewing \( C^\pm \) as 3-chains. Then \( C^+ \) corresponds to the part of \( K_{\theta,A} \) asymptotic to the plane \( \Pi_1 \) in \( U \), and \( C^- \) to the part of \( K_{\theta,A} \) asymptotic to the plane \( \Pi_{\theta} \) in \( U \), where \( \Pi_1, \Pi_{\theta} \) are given in (19). As \( \partial D = S \) we see that \( C^+ + D \) and \( C^- - D \) are 3-chains without boundary. Define classes \( \chi^\pm \) in \( H_3(M, \mathbb{Z}) \) by \( \chi^+ = [C^+ + D] \) and \( \chi^- = [C^- - D] \).

**Lemma 7.2** These classes \( \chi^\pm \) satisfy \( \chi^+ + \chi^- = [N] \in H_3(M, \mathbb{Z}) \), \( [\text{Im } \Omega] \cdot \chi^+ = A = F_\zeta(N) \) and \( [\text{Im } \Omega] \cdot \chi^- = -A \). Furthermore the images of \( \chi^\pm \) in \( H_3(M, \mathbb{R}) \) are linearly independent.

**Proof.** The 3-chain equation \( (C^+ + D) + (C^- - D) = N \) implies that \( \chi^+ + \chi^- = [N] \). As \( \text{Im } \Omega |_N \equiv 0 \) and \( \int_D \text{Im } \Omega = A \) from above, we get \( [\text{Im } \Omega] \cdot \chi^\pm = \pm A \). Now if \( \chi^\pm \) were linearly dependent in \( H_3(M, \mathbb{R}) \) then the equations \( [\text{Im } \Omega] \cdot \chi^\pm = \pm A \) and \( A > 0 \) would force \( \chi^+ + \chi^- = 0 \), and so \( [N] = 0 \). But this contradicts the fact that \( [\text{Re } \Omega] \cdot [N] = \text{vol}(N) > 0 \), and thus the \( \chi^\pm \) are linearly independent in \( H_3(M, \mathbb{R}) \).

By (21), if \( \eta = 0 \) and \( N' \in \mathcal{M} \) then \( F_\zeta(N) - F_\zeta(N') = \eta \cdot (x - x') = 0 \). Thus \( F_\zeta(N') = F_\zeta(N) = A \) for all \( N' \in \mathcal{M} \), and \( F_\zeta \) is constant on \( A^*_{M,N} \), with value \( A \). So Conjecture 7.1 (if true) implies that all \( N' \) close to \( N \) in \( \mathcal{M} \) are locally modelled on \( K_{\theta',A} \), where \( \theta' \) can vary but \( A > 0 \) is fixed.

Now consider what happens if we deform the underlying Calabi-Yau manifold \( (M, J, g) \), in a way that changes the cohomology class \( [\Omega] \in H^3(M, \mathbb{C}) \).
Choose a smooth 1-parameter family of Calabi-Yau metrics $g_s$ on $M$ for $s \in (-\epsilon, \epsilon)$, with associated complex structures $J_s$, Kähler forms $\omega_s$ and holomorphic volume forms $\Omega_s$, such that $g_0 = g$, $J_0 = J$ and $\Omega_0 = \Omega$. Fix the phase of $\Omega_s$ by requiring that $[\text{Im } \Omega_s] \cdot [N] = 0$ for all $s \in (-\epsilon, \epsilon)$.

We wish to find a smooth family of 3-folds $N_s$ in $M$ for $s \in (-\epsilon, \epsilon)$ such that $N_s$ is special Lagrangian with respect to $(M, J_s, g_s)$, and $N_0 = N$. By Theorem 2.14, $N_s$ exists for small $s$ if and only if $[\omega_s|_N] = 0$ in $H^2(N, \mathbb{R})$. Assume that this is so. Let $\mathcal{M}_s$ be the moduli space of special Lagrangian deformations of $N_s$. Then we can define the function $F_\zeta$ on $\mathcal{M}_s$ as before. As $F_\zeta(N) = [\text{Im } \Omega] \cdot \chi^+$ by Lemma 7.2, we deduce that $F_\zeta(N') = [\text{Im } \Omega_s] \cdot \chi^+$ for all $N' \in \mathcal{M}_s$. Thus Conjecture 7.1 (if true) implies that near $N$, each $N'$ in $\mathcal{M}_s$ should be locally modelled on $K_{\theta', A'}$, where $A' = [\text{Im } \Omega_s] \cdot \chi^+$.

Thus we expect that the family $\mathcal{M}_s$ of nonsingular special Lagrangian 3-folds exists near $N$ only when $[\text{Im } \Omega_s] \cdot \chi^+ > 0$. When $[\text{Im } \Omega_s] \cdot \chi^+ = 0$, every element of $\mathcal{M}_s$ near $N$ has a singular point modelled on the intersection of two special Lagrangian 3-planes. When $[\text{Im } \Omega_s] \cdot \chi^+ < 0$, there are no special Lagrangian 3-folds locally isotopic to $N$, and $\mathcal{M}_s$ is empty. As in §4.2, we have found a way for special Lagrangian 3-folds to become singular, and then disappear, under deformation.

Choose $t \in (-\epsilon, \epsilon)$ with $[\text{Im } \Omega_t] \cdot \chi^+ = 0$, and let $N_t$ be in $\mathcal{M}_t$ near $N$. Then $N_t$ should have one singular point at $p \in M$, with singularity modelled on the union $\Pi_1 \cup \Pi_{\theta'}$ of two special Lagrangian 3-planes in $\mathbb{C}^3$. It is easy to see that $N_t$ is actually the union of two nonsingular compact special Lagrangian 3-folds $N^+_t$ and $N^-_t$, which intersect at $p$. As a 3-manifold $N$ is the connected sum $N = N^+_t \# N^-_t$, which implies that $b^1(N) = b^1(N^+_t) + b^1(N^-_t)$.

We take $N^+_t$ to be the component of $N_t$ close to $C^+$, and $N^-_t$ to be the component close to $C^-$. Clearly we have $[N^+_t] = \chi^+$ and $[N^-_t] = \chi^-$. At $p$, where $N_t$ is modelled on $\Pi_1 \cup \Pi_{\theta'}$, we find that $\Pi_1$ is the tangent plane to $N^+_t$, and $\Pi_{\theta'}$ the tangent plane to $N^-_t$. As $\theta' = (\theta'_1, \theta'_2, \theta'_3)$ with $\theta'_k \in (0, \pi)$ and $\theta'_1 + \theta'_2 + \theta'_3 = \pi$, by Proposition 6.2 the intersection $\Pi_1 \cap \Pi_{\theta'}$ is positive. Thus $N^+_t \cap N^-_t$ is a positive intersection at $p$. If $N$ is embedded in $M$ then $N^\pm_t$ intersect only at $p$, and thus $\chi^+ \cap \chi^- = 1$, where $\cap : H_3(M, \mathbb{Z}) \times H_3(M, \mathbb{Z}) \to \mathbb{Z}$ is the intersection pairing.

By Theorem 2.14, these special Lagrangian 3-folds $N^\pm_t$ in $(M, J_t, g_t)$ are stable under small deformations $(M, J_s, g_s)$ of the underlying Calabi-Yau 3-fold provided that $[\omega_s|_{N^\pm}] = 0$ in $H^2(N^\pm, \mathbb{R})$. But as $N = N^+_t \# N^-_t$ we see that $[\omega_s|_{N^+_t}] = [\omega_s|_{N^-_t}] = 0$ if and only if $[\omega_s|_N] = 0$, which we have already assumed is true.
So extend $N^\pm_t$ to smooth families $N^\pm_s$ of special Lagrangian 3-folds in $(M, J_s, g_s)$, and let $\mathcal{M}^\pm_s$ be the moduli space of special Lagrangian deformations of $N^\pm_s$ in $(M, J_s, g_s)$. In particular, when $s = 0$ we have $(M, J_0, g_0) = (M, J, g)$, and so $\mathcal{M}^\pm_0$ are families of special Lagrangian 3-folds in the Calabi-Yau 3-fold $(M, J, g)$ that we started with.

Let $N^\pm_s$ be elements of these families $\mathcal{M}^\pm_s$. Then $N^+_{-\epsilon}$ and $N^-_{-\epsilon}$ are compact nonsingular special Lagrangian 3-folds in $(M, J, g)$. Thus, in $(M, J, g)$ we expect not just one family $\mathcal{M}$ of special Lagrangian 3-folds close to $N$, but also two other families $\mathcal{M}^+_0$, $\mathcal{M}^-_0$ of special Lagrangian 3-folds close to $N^+$ and $N^-$. The key to understanding what is going on here is to think about the phases of $N$, $N^+$ and $N^-$. We fixed the phase of $N$ to be 1 above. Let $N^\pm_s$ have volumes $V^\pm$ and phases $e^{i\theta^\pm}$. Then

$$V^+ e^{i\theta^+} = [\Omega] \cdot [N^+] = [\Omega] \cdot \chi^+ \quad \text{and} \quad V^- e^{i\theta^-} = [\Omega] \cdot [N^-] = [\Omega] \cdot \chi^-.$$  
(22)

So the phases of $N^\pm$ are determined by the cohomology class $[\Omega]$. Moreover, as $\chi^\pm$ are linearly independent by Lemma 7.2, by varying $[\Omega]$ in $H^3(M, \mathbb{C})$ we can vary the phases of $N^\pm$ independently.

Define $B^\pm$ to be the areas of $C^\pm$. Then $B^\pm > 0$. Since $C^\pm$ are calibrated w.r.t. $\text{Re} \Omega$ and $D$ is calibrated w.r.t. $\text{Im} \Omega$, we have

$$\int_{C^+} \Omega = B^+, \quad \int_{C^-} \Omega = B^- \quad \text{and} \quad \int_{D} \Omega = iA.$$ 

As $\chi^+ = [C^+ + D]$ and $\chi^- = [C^- - D]$, this shows that $[\Omega] \cdot \chi^\pm = B^\pm \pm iA$. So (22) gives

$$V^+ e^{i\theta^+} = B^+ + iA \quad \text{and} \quad V^- e^{i\theta^-} = B^- - iA.$$  
(23)

As $V^\pm$, $B^\pm$ and $A$ are all positive, we see that $\theta^+ \in (0, \pi/2)$ and $\theta^- \in (-\pi/2, 0)$, modulo $2\pi$. And since $A$ is assumed small, we see that $\theta^+$ is small and positive, and $\theta^-$ is small and negative.

Now let us deform the Calabi-Yau 3-fold in the family $(M, J_s, g_s)$, and write $A_s = [\text{Im} \Omega_s] \cdot \chi^+$. From above we expect that special Lagrangian deformations of $N$ should exist in $(M, J_s, g_s)$ when $A_s > 0$, that they become singular when $A_s = 0$, and that they do not exist when $A_s < 0$.

But we expect special Lagrangian deformations $N^\pm_s$ of $N^\pm$ to exist in $(M, J_s, g_s)$ for all small $s \in (-\epsilon, \epsilon)$. Let $e^{i\theta^\pm_s}$ be the phases of $N^\pm_s$. Replacing
A by $A_s$ in (23), we see that when $A_s = 0$ the phases of $N^+_s$ and $N_s$ all coincide, at 1. And when $A_s < 0$ we have $\theta^+_s \in (-\pi/2, 0)$ and $\theta^-_s \in (0, \pi/2)$, so that $\theta^+_s$ is now small and negative, and $\theta^-_s$ is small and positive.

To express this more clearly, we make the following definition.

**Definition 7.3** Let $(M, J, g)$ be a Calabi-Yau 3-fold, and suppose $\chi^+, \chi^- \in H_3(M, \mathbb{Z})$ are linearly independent in $H_3(M, \mathbb{R})$. Define a subset $W(\chi^+, \chi^-)$ in $H^3(M, \mathbb{C})$ by

$$W(\chi^+, \chi^-) = \{ \Phi \in H^3(M, \mathbb{C}) : (\Phi \cdot \chi^+)(\bar{\Phi} \cdot \chi^-) \in (0, \infty) \},$$

where $\bar{\Phi}$ is the complex conjugate of $\Phi$. That is, $\Phi \cdot \chi^+$ and $\bar{\Phi} \cdot \chi^-$ are complex numbers, and their product must be a positive real number for $\Phi$ to lie in $W(\chi^+, \chi^-)$. Then $W(\chi^+, \chi^-)$ is a real hypersurface in $H^3(M, \mathbb{C})$, but not a hyperplane.

Fix some small $\epsilon \in (0, \pi)$. Let $\Phi \in H^3(M, \mathbb{C})$, and write $(\Phi \cdot \chi^+)(\bar{\Phi} \cdot \chi^-) = R \cdot e^{i\theta}$, where $R \geq 0$ and $\theta \in (-\pi, \pi)$. Then $\Phi$ lies in $W(\chi^+, \chi^-)$ if $R > 0$ and $0 < \theta < \epsilon$, and $\Phi$ lies on the negative side of $W(\chi^+, \chi^-)$ if $R > 0$ and $-\epsilon < \theta < 0$. Note that $W(\chi^+, \chi^-) = W(\chi^-, \chi^+)$, but the positive side of $W(\chi^+, \chi^-)$ is the negative side of $W(\chi^-, \chi^+)$, and vice versa.

Thus we have the following picture. We are given compact, nonsingular special Lagrangian 3-folds $N^+$ and $N^-$ in $M$ with phases $e^{i\theta^+}$, $e^{i\theta^-}$, which intersect at one point $p \in M$, such that $N^+ \cap N^-$. As we deform the underlying Calabi-Yau 3-fold $(M, J, g)$ so that cohomology class $[\Omega]$ varies in $H^3(M, \mathbb{C})$, the phases $e^{i\theta^+}$, $e^{i\theta^-}$ of $N^+$ and $N^-$ change.

Then we believe that when $[\Omega]$ is on the positive side of $W(\chi^+, \chi^-)$, there should exist a special Lagrangian 3-fold $N$ diffeomorphic to $N^+ \# N^-$, with $[N] = [N^+] + [N^-]$ in $H_3(M, \mathbb{Z})$, and locally modelled on some $K_{\theta, A}$ near the point of intersection $p$ of $N^+ \cap N^-$. As we deform $[\Omega]$ through $W(\chi^+, \chi^-)$ this $N$ converges to the singular union $N^+ \cup N^-$, and on the negative side of $W(\chi^+, \chi^-)$ we believe that $N$ does not exist.

Thus, as we deform $[\Omega]$ through $W(\chi^+, \chi^-)$, a new special Lagrangian 3-fold $N$ is created or destroyed. This is only a local picture, in a small region of $H^3(M, \mathbb{C})$ near a point of $W(\chi^+, \chi^-)$. To extend it to a global picture one would have to include other hypersurfaces and special Lagrangian 3-folds in other homology classes. Also, we would have to consider the fact that the moduli space of complex 3-folds with holomorphic volume forms is only
locally identified with $H^3(M, \mathbb{C})$, and globally it has singularities on complex hypersurfaces, with monodromy around them.

8 Counting special Lagrangian homology 3-spheres

It is now well known that one can define important invariants of symplectic manifolds by counting (with signs) the pseudo-holomorphic curves satisfying certain homological conditions. This idea was first introduced by Gromov [3], who proved the basic existence and compactness properties for pseudo-holomorphic curves. The invariants were generalized by Witten [18] using ideas from physics, and by Ruan and Tian [16] in their theory of ‘quantum cohomology’.

A good introduction to the subject is given by McDuff and Salamon [11]. They define two kinds of invariant of a compact symplectic manifold [11, Ch. 7], Gromov invariants and Gromov-Witten invariants, obtained by counting pseudo-holomorphic curves in slightly different ways. The important thing about these invariants is that they are very stable under deformations, both of the choice of metric used to define them, and also of the symplectic structure.

Now it seems a natural (but perhaps optimistic) question to ask whether we can define similar invariants of Calabi-Yau 3-folds $(M, J, g)$ by counting special Lagrangian 3-folds $N$ in $M$ satisfying suitable homological conditions. Probably the simplest such condition is to count 3-folds $N$ in some fixed homology class in $H_3(N, \mathbb{Z})$, and we will focus on this. We shall also restrict our attention to (rational) homology 3-spheres, to get zero-dimensional moduli spaces.

Thus we aim to define an invariant as follows. Let $(M, J, g)$ be a Calabi-Yau 3-fold, let $\delta \in H_3(M, \mathbb{Z})$, and let $S(\delta)$ be the set of special Lagrangian homology 3-spheres $N$ in $M$ with $[N] = \delta$. Suppose $S(\delta)$ is finite, and define

$$I(\delta) = \sum_{N \in S(\delta)} w(N),$$

where $w$ is a weight function taking values in a commutative ring $R$, and $w(N)$ depends only on the topology of $N$. In this way we define a map $I : H_3(M, \mathbb{Z}) \to R$, which we consider to be an analogue of the Gromov
invariant. For this invariant to be interesting, we would like it to be stable under deformations of the underlying Calabi-Yau 3-fold \((M, J, g)\), or at least to change in a predictable way as we make these deformations.

Thus we need to know what can happen to special Lagrangian homology 3-spheres as we deform \((M, J, g)\), and especially how they can become singular, appear or disappear. Each such transition may change the set of special Lagrangian homology 3-spheres, and thus the invariant \(I(\delta)\). For \(I(\delta)\) to be invariant or to transform nicely under these transitions, the weight function \(w\) must satisfy some topological identities.

We have already described models for two such transitions in §5 and §7. We will calculate the conditions on \(w\) for \(I(\delta)\) to be invariant under the change described in §5, and to transform in a certain simple way under the change described in §7. It turns out that the weight function \(w(N) = |H_1(N, \mathbb{Z})|\) satisfies both of these conditions. We also propose a correction to (24) for multiple covers of special Lagrangian homology 3-spheres.

Motivated by this, we shall define an invariant \(I : H_3(M, \mathbb{Z}) \rightarrow \mathbb{Q}\) and make a conjecture about its behaviour. We interpret our conjecture in terms of String Theory, and finally we consider what the invariants would tell us if they work.

### 8.1 Invariance of \(I(\delta)\) under the transitions of §5

We use the notation of §5. In §5.2 we explained how, as we deform the Calabi-Yau 3-fold \((M, J, g)\), three special Lagrangian 3-folds \(N_1, N_2, N_3\) can converge to the same singular special Lagrangian 3-fold \(N_0\) with a \(T^2\)-cone singularity. This happens on a hyperplane \([\omega] \cdot \chi = 0\) in the Kähler cone, and \(N_j\) exists as a nonsingular special Lagrangian 3-fold in \((M, J, g)\) if either \(k_j > 0\) and \([\omega] \cdot \chi > 0\), or \(k_j < 0\) and \([\omega] \cdot \chi < 0\). It is easy to see that the condition for the invariant \(I(\delta)\) given by (24) to be unchanged by this transition is

\[
\sum_{j \in \{1,2,3\}: k_j > 0} w(N_j) = \sum_{j \in \{1,2,3\}: k_j < 0} w(N_j). \tag{25}
\]

Now if \(N\) is a homology 3-sphere then \(H_1(N, \mathbb{Z})\) is a finite group, so \(|H_1(N, \mathbb{Z})|\) is a positive integer. Take the commutative ring \(R\) to be \(\mathbb{Z}\), and define \(w(N) = |H_1(N, \mathbb{Z})|\). Remarkably, it turns out that this weight function, perhaps the simplest nontrivial invariant of \(N\) there is, satisfies (25).
**Proposition 8.1** Define an integer-valued invariant $w$ of compact, non-singular 3-manifolds $N$ by $w(N) = \left| H_1(N, \mathbb{Z}) \right|$ if $H_1(N, \mathbb{Z})$ is finite, and $w(N) = 0$ if $H_1(N, \mathbb{Z})$ is infinite. Then (25) holds for all sets of 3-manifolds $N_1, N_2, N_3$ constructed as in §5.

**Proof.** If none of $N_1, N_2$ or $N_3$ is a homology 3-sphere then $w(N_j) = 0$ for $j = 1, 2, 3$, and (25) holds trivially. So suppose at least one $N_j$ is a homology 3-sphere. Then Proposition 5.3 shows that $b_1(P) = 1$ and $H_1(P; T, \mathbb{Z})$ is finite, that $N_j$ is a homology 3-sphere if and only if $k_j \neq 0$, and that if $k_j \neq 0$ then $\left| H_1(N_j, \mathbb{Z}) \right| = |k_j| \cdot \left| H_1(P; T, \mathbb{Z}) \right|$. Therefore (25) becomes

$$
\sum_{j \in \{1, 2, 3\}: k_j > 0} |k_j| \cdot \left| H_1(P; T, \mathbb{Z}) \right| = \sum_{j \in \{1, 2, 3\}: k_j < 0} |k_j| \cdot \left| H_1(P; T, \mathbb{Z}) \right|.
$$

But this is clearly true, as $k_1 + k_2 + k_3 = 0$ by Proposition 5.1. □

### 8.2 Transformation of $I(\delta)$ under the transitions of §7

We use the notation of §7. Let $(M, J, g)$ be a Calabi-Yau 3-fold, and $N^+$, $N^-$ compact special Lagrangian homology 3-spheres in $M$, such that $N^\pm$ has phase $e^{i\theta^\pm}$ and $[N^\pm] = \chi^\pm \in H_3(M, \mathbb{Z})$. Let $W(\chi^+, \chi^-)$ be as in Definition 7.3. Suppose first that $N^+, N^-$ intersect transversely at one point $p$, and that $N^+ \cap N^-$ is a positive intersection, so that $\chi^+ \cap \chi^- = 1$.

Then we explained in §7 that as we deform $(M, J, g)$ so that $[\Omega]$ passes through $W(\chi^+, \chi^-)$ from the negative side to the positive side, we create a new special Lagrangian 3-fold $N$ diffeomorphic to $N^+ \# N^-$. As $N^\pm$ are homology 3-spheres, it follows that $N$ is a homology 3-sphere. Therefore it makes a contribution to the invariant $I(\chi^+ + \chi^-)$, where $I$ is given in (24).

Now suppose $N^+, N^-$ intersect transversely in $k+l$ points, where $N^+ \cap N^-$ is positive at $k$ points $p_1, \ldots, p_k$, and negative at $l$ points $q_1, \ldots, q_l$. Then $\chi^+ \cap \chi^- = k - l$. On the positive side of $W(\chi^+, \chi^-)$ we have $k$ distinct immersed special Lagrangian homology 3-spheres diffeomorphic to $N^+ \# N^-$, that is, the connected sum of $N^+$ and $N^-$ at the points $p_1, \ldots, p_k$.

For the negative intersections $q_1, \ldots, q_l$ we exchange $\chi^+$ and $\chi^-$, and then the rôle of the positive and negative sides of $W(\chi^+, \chi^-)$ are reversed.
Thus, on the negative side of \( W(\chi^+, \chi^-) \) we have \( l \) distinct immersed special Lagrangian homology 3-spheres diffeomorphic to \( N^+ \# N^- \), that is, the connected sum of \( N^+ \) and \( N^- \) at \( q_1, \ldots, q_l \). Hence, as we pass through the \( W(\chi^+, \chi^-) \) going from the negative side to the positive side, we simultaneously create \( k \) immersed special Lagrangian copies of \( N^+ \# N^- \), and destroy \( l \) immersed special Lagrangian copies of \( N^+ \# N^- \).

Suppose for simplicity that \( N^\pm \) are the only special Lagrangian homology 3-spheres in their homology classes \( \chi^\pm \). Then \( I(\chi^+) = w(N^+) \) and \( I(\chi^-) = w(N^-) \). Write \( I(\chi^+ + \chi^-)^+ \) for the value of \( I(\chi^+ + \chi^-) \) at some point on the positive side of \( W(\chi^+, \chi^-) \), and \( I(\chi^+ + \chi^-)^- \) for its value at a nearby point on the negative side. Then we have

\[
I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (k - l) \cdot w(N^+ \# N^-). \tag{26}
\]

If the weight function \( w \) satisfies the identity

\[
w(N^+ \# N^-) = w(N^+) \cdot w(N^-) \quad \text{for all homology 3-spheres } N^\pm, \tag{27}
\]

where multiplication is in the commutative ring \( R \), then (26) can be written

\[
I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (\chi^+ \cap \chi^-) \cdot I(\chi^+) \cdot I(\chi^-), \tag{28}
\]

as \( \chi^+ \cap \chi^- = k - l \) and \( w(N^+ \# N^-) = w(N^+) \cdot w(N^-) = I(\chi^+) \cdot I(\chi^-) \).

Because of the bilinearity of the r.h.s. of (28) in \( I(\chi^+) \) and \( I(\chi^-) \), it is easy to see that (28) should also hold when there are finitely many special Lagrangian homology 3-spheres in the homology classes \( \chi^\pm \), and not just one in each.

Thus, (28) gives a formula for how we expect \( I(\chi^+ + \chi^-) \) to change as we pass through the hypersurface \( W(\chi^+, \chi^-) \). The important thing about this formula is that the values of \( I(\chi^+) \) on the positive side of \( W(\chi^+, \chi^-) \) determine the values of \( I(\chi^-) \) on the negative side, and vice versa. So although \( I \) is not invariant under deformations of \( (M, J, g) \) which alter the cohomology class \( [\Omega] \), it appears to transform in a completely determined way, and that is more or less as useful. However, we have only considered the simplest kind of transition taking place on \( W(\chi^+, \chi^-) \), and we will shortly describe others whose effects on \( I \) are more difficult to write down.

In §8.1 we proposed the weight function \( w(N) = |H_1(N, \mathbb{Z})| \), for \( N \) a homology 3-sphere. Now \( H_1(N^+ \# N^-, \mathbb{Z}) \cong H_1(N^+, \mathbb{Z}) \times H_1(N^-, \mathbb{Z}) \) as finite groups, and so

\[
|H_1(N^+ \# N^-, \mathbb{Z})| = |H_1(N^+, \mathbb{Z})| \cdot |H_1(N^-, \mathbb{Z})|.
\]

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when \( N^\pm \) are homology 3-spheres. Thus this weight function \( w \) satisfies (27), as we wish.

### 8.3 Including multiple covers

Let \( N' \) be a special Lagrangian homology 3-sphere in a Calabi-Yau 3-fold \((M, J, g)\) with \([N'] = \chi' \in H_3(M, \mathbb{Z})\), and suppose for simplicity that \( N' \) is embedded. Let \( N \) be a finite cover of \( N' \) of degree \( d > 1 \), with covering map \( \pi : N \to N' \). Then we can regard \( N \) as an immersed special Lagrangian 3-fold in \( M \), with immersion \( \iota \circ \pi : N \to M \), where \( \iota : N' \hookrightarrow M \) is the inclusion map. The homology class of \( N \) in \( M \) is \([N] = d \cdot [N'] = d\chi' \) in \( H_3(M, \mathbb{Z})\).

Now \( N \) may or may not be a homology 3-sphere. Suppose it is. Then \( N \) is an immersed special Lagrangian homology 3-sphere in the homology class \( d\chi' \), and so it should make a contribution to the invariant \( I(d\chi') \) we are trying to construct. But what should this contribution be? Following (24) we could just put it to be \( w(N) \), and ignore the fact that \( N \) is a cover of \( N' \). Or we could look for a more general weight involving the topology of \( N \) and \( N' \) and the covering map \( \pi \).

In fact the author believes that \( N \) should contribute \( w(N)/d \) to \( I(d\chi') \). Note that if \( w \) takes values in \( \mathbb{Z} \), then \( w(N)/d \) takes values in \( \mathbb{Q} \), so that our invariant \( I(d\chi') \) will be actually be a rational number — which makes its interpretation as the ‘number of special Lagrangian homology 3-spheres’ problematic, but never mind.

Here is why we want to assign the weight \( w(N)/d \) to \( N \). Suppose that \( N' \) and \( N'' \) are special Lagrangian homology 3-spheres in \((M, J, g)\) which intersect transversely at one point \( p \), and that \( N' \cap N'' \) is a positive intersection. Let \( \chi' \) and \( \chi'' \) be the homology classes of \( N', N'' \) in \( H_3(M, \mathbb{Z}) \). Let \( \pi : N \to N' \) be a finite cover of degree \( d > 1 \) as above, and suppose \( N \) is also a special Lagrangian homology 3-sphere.

Consider what happens to \( I(d\chi' + \chi'') \) as we deform \((M, J, g)\) so that \([\Omega]\) passes through \( W(\chi', \chi'') \) going from the negative to the positive side. Now \( N \) intersects \( N'' \) at \( d \) distinct points in \( N \) (although only one point in \( M \)), and the intersection \([N] \cap [N''] = d \) in homology. Therefore, from §8.2, we would expect to create \( d \) distinct new special Lagrangian homology 3-spheres as we pass through \( W(\chi', \chi'') \), all diffeomorphic to \( N \# N'' \), which are the connected sums of \( N \) and \( N'' \) at the \( d \) points of intersection of \( N \) and \( N'' \).

However, a little thought shows that we actually create only one special Lagrangian \( N \# N'' \) as we pass through \( W(\chi', \chi'') \). That is, the \( d \) copies of
We can think of the \( d \) copies as being related by automorphisms of \( N \) in the covering group of the cover \( \pi: N \to N' \).

Now from (28), we would like the change in \( I(d\chi' + \chi'') \) across \( W(\chi', \chi'') \) to be

\[
I(d\chi' + \chi'')^+ - I(d\chi' + \chi'')^- = d \cdot I(d\chi') \cdot I(\chi''),
\]

since \( \chi' \cap \chi'' = 1 \). Suppose for simplicity that \( N, N' \) and \( N'' \) are the only special Lagrangian homology 3-spheres in their homology classes. Then we have

\[
I(d\chi' + \chi'')^+ - I(d\chi' + \chi'')^- = w(N\#N'') \quad \text{and} \quad I(\chi'') = w(N'').
\]

As \( w(N\#N'') = w(N) \cdot w(N'') \) by (27), it follows that (29) holds if \( I(d\chi') = w(N)/d \). This is why \( N \) should contribute \( w(N)/d \) to \( I(d\chi') \).

The author has not fully sorted out how finite covers behave under the \( T^2 \)-cone transitions considered in §5, but hopes to understand this soon.

### 8.4 A preliminary conjecture

I am now ready to formulate a first guess as to how to define an invariant counting special Lagrangian homology 3-spheres, and what its properties should be under deformations of the underlying Calabi-Yau 3-fold. It is possible — even likely — that my guess is wrong, and that special Lagrangian 3-folds just do not behave well enough for any such invariant to have nice properties under deformation. Even if my guess is basically correct, it will probably need to be modified later, in the light of a clearer understanding of special Lagrangian 3-folds. But I hope that these ideas will in the meantime help mathematicians and string theorists to work out what the right answers are.

We begin by defining our invariant \( I \).

**Definition 8.2** Let \((M, J, g)\) be a Calabi-Yau 3-fold, with Kähler form \( \omega \) and holomorphic volume form \( \Omega \). We say that \([\omega]\) is *generic* in \( H^2(M, \mathbb{R}) \) if whenever \( \chi \in H_2(M, \mathbb{Z}) \) is nonzero in \( H_2(M, \mathbb{R}) \), then \([\omega] \cdot \chi \neq 0\). We say that \([\Omega]\) is *generic* in \( H^3(M, \mathbb{C}) \) if \([\Omega]\) does not lie on any of the hypersurfaces \( W(\chi^+, \chi^-) \) defined in Definition 7.3. These conditions hold in dense subsets of \( H^2(M, \mathbb{R}) \) and \( H^3(M, \mathbb{C}) \).
Suppose $[\omega]$ and $[\Omega]$ are generic. For each $\delta \in H_3(M, \mathbb{Z})$, define $S(\delta)$ to be

$$S(\delta) = \{ \text{isomorphism classes of pairs } (N, f):$$
$$N \text{ is an oriented rational homology 3-sphere,}$$
$$f : N \to M \text{ is an immersion, } f(N) \text{ is a special}$$
$$\text{Lagrangian 3-fold, and } f_*([N]) = \delta \in H_3(M, \mathbb{Z}) \}.$$  

Here we consider two pairs $(N_1, f_1)$ and $(N_2, f_2)$ to be isomorphic if there exists an orientation-preserving diffeomorphism $\phi : N_1 \to N_2$ such that $f_1 = f_2 \circ \phi$. For each $(N, f)$ in $S(\delta)$, define the symmetry group $G(N, f)$ by

$$G(N, f) = \{ \text{diffeomorphisms } \phi : N \to N \text{ with } f \circ \phi = \phi \}.$$  

Then $G(N, f)$ is a finite group. Suppose $S(\delta)$ is finite, and define a map $I : H_3(M, \mathbb{Z}) \to \mathbb{Q}$ by

$$I(\delta) = \sum_{(N, f) \in S(\delta)} \frac{|H_1(N, \mathbb{Z})|}{|G(N, f)|}. \quad (30)$$

Here are some remarks on this definition.

- The group $G(N, f)$ is $\{1\}$ unless $N$ is a finite cover of some other special Lagrangian homology 3-sphere $N'$. In this case $G(N, f)$ acts freely on $N$, and $N' = N/G(N, f)$. Thus $G(N, f)$ is the covering group of the cover $\pi : N \to N'$, and the degree of the cover is $d = |G(N, f)|$. Hence $|H_1(N, \mathbb{Z})|/|G(N, f)| = w(N)/d$, which is the contribution for a finite cover we proposed in §8.3.

- The special Lagrangian 3-fold $N' = N/G(N, f)$ has homology class $\delta' = \delta/|G(N, f)|$ in $H_3(M, \mathbb{Z})$. Thus, if $\delta$ is a primitive element of $H_3(M, \mathbb{Z})$ then $|G(N, f)| = 1$ for all $(N, f) \in S(\delta)$, since otherwise $\delta/|G(N, f)|$ would not lie in $H_3(M, \mathbb{Z})$. Therefore, if $\delta$ is a primitive element of $H_3(M, \mathbb{Z})$ then $I(\delta)$ is a nonnegative integer.

- The assumption that $[\omega]$ and $[\Omega]$ be generic is rather strong — we only expect problems in defining $I(\delta)$ on a small number of the hypersurfaces $[\omega] \cdot \chi = 0$ and $[\Omega] \in W(\chi^+, \chi^-)$.  

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• It seems likely that one could extend the definition to nongeneric \( [\omega] \) by including a sum over certain kinds of singular special Lagrangian 3-fold in (30). For example, the singular 3-fold \( N_0 \) with a \( T^2 \)-cone singularity considered in §8.1 should be counted, with weight the l.h.s. of (25). Note that this weight depends not only on the topology of \( N_0 \), but also on the local geometry of the singular point.

• Is \( S(\delta) \) necessarily finite for each \( \delta \in H_3(M, \mathbb{Z}) \)? The author does not know.

• Should we also include an appropriate sign \( \pm 1 \) in (30), so that special Lagrangian homology 3-spheres are counted with signs? The author believes not.

We can now state our conjecture on the behaviour of the invariant \( I \).

**Conjecture 8.3** Let \( (M, J, g) \) be a Calabi-Yau 3-fold with \( K\ddot{a}hler \) form \( \omega \) and holomorphic volume form \( \Omega \), and suppose \( [\omega] \) and \( [\Omega] \) are generic. Let \( I : H_3(M, \mathbb{Z}) \rightarrow \mathbb{Q} \) be as in Definition 8.2. Then

(a) \( I \) is invariant under deformations of \( (M, J, g) \) that change \( [\omega] \) but leave \( [\Omega] \) fixed, or that multiply \( [\Omega] \) by a nonzero complex number. Therefore \( I \) depends only on the complex structure \( J \) on \( M \), and not on the metric \( g \).

(b) When we deform \( (M, J, g) \) so that \( [\Omega] \) passes through one of the hypersurfaces \( W(\chi^+, \chi^-) \) in \( H^3(M, \mathbb{C}) \) given in Definition 7.3, the invariant \( I \) transforms according to a complicated set of rules that we are not yet able to write down. Some of these rules may involve other ‘higher order’ invariants defined by counting special Lagrangian homology 3-spheres. One of these rules should be that if \( \chi^+ \) and \( \chi^- \) are primitive elements of \( H_3(M, \mathbb{Z}) \) and \( I(\chi^+ + \chi^-) \) are the values of \( I(\chi^+ + \chi^-) \) at two nearby points on the positive and negative sides of \( W(\chi^+, \chi^-) \), then

\[
I(\chi^+ + \chi^-)^+ - I(\chi^+ + \chi^-)^- = (\chi^+ \cap \chi^-) \cdot I(\chi^+) \cdot I(\chi^-). \tag{31}
\]

Part (a) of this conjecture is a firm statement, and is clearly either true or false. But part (b) is rather vague. Here is the reason why. When we
pass through the hypersurface $W(\chi^+, \chi^-)$ we expect to create or destroy new special Lagrangian 3-folds with homology class $\chi^+ + \chi^-$, which are connected sums of 3-folds with homology classes $\chi^+$ and $\chi^-$. But this is only the simplest kind of transition which happens on $W(\chi^+, \chi^-)$.

For instance, if $N_1^+ \in S(\chi^+)$ and $N^- \in S(\chi^-)$, then on $W(\chi^+, \chi^-)$ we may create a new special Lagrangian homology 3-sphere with homology class $2\chi^+ + \chi^-$, diffeomorphic to the triple connected sum $N_1^+ \# N^- \# N_2^+$. So there should be some change to $I(2\chi^+ + \chi^-)$ on $W(\chi^+, \chi^-)$. Similarly, if $a, b$ are positive integers then we can try and take a multiple connected sum of $a$ elements of $S(\chi^+)$ and $b$ elements of $S(\chi^-)$ to get a new special Lagrangian homology 3-sphere with homology class $a\chi^+ + b\chi^-$. The obvious thing to do is to look for a formula analogous to (31) for the change to $I(a\chi^+ + b\chi^-)$, in terms of $a, b, \chi^+ \cap \chi^-, I(\chi^+)$ and $I(\chi^-)$. The leading term in this formula should be

$$I(a\chi^+ + b\chi^-)^+ - I(a\chi^+ + b\chi^-)^- = C(a, b) \cdot (\chi^+ \cap \chi^-)^{a+b-1} \cdot I(\chi^+)^a \cdot I(\chi^-)^b + \cdots,$$

where $C(a, b)$ is a rational number depending on $a, b$, which is the number of graphs of a certain kind, divided by $a!b!$. However, when we think about this in detail, we run into a problem: can we include a special Lagrangian 3-fold in our connected sum more than once, and if so, how? The answer to this appears to be rather complex, and the author does not yet understand it.

One last remark: to understand how $I(\delta)$ transforms under deformations of $(M, J, g)$, we only really need to consider the singularities of special Lagrangian homology 3-spheres that occur in real codimension one in the moduli space of Calabi-Yau 3-folds, because we can always avoid singularities of higher codimension by choosing a generic path in the moduli space.

It may be that there are only a few kinds of singularity which occur in real codimension one, and if so, this might make the task of understanding the behaviour of $I(\delta)$ easier than it at first appears. The author believes that the singularities considered in §4-§5 and §7 occur in real codimension one.

### 8.5 Relationships with String Theory

I am indebted to Bobby Acharya for several discussions on String Theory and special Lagrangian 3-folds, and most of the following ideas are based on sug-
gestions made by him. However, all of the mistakes and misunderstandings were put in by me.

In String Theory, special Lagrangian 3-folds correspond roughly to physical objects called 3-branes. But a 3-brane is not just a 3-dimensional submanifold $N$; it also carries with it a complex line bundle over $N$ with a flat $U(1)$ connection. We will call a 3-brane isolated if it admits no deformations, which happens when $N$ is a rational homology 3-sphere.

If $N$ is a compact 3-manifold, then flat $U(1)$ connections on $N$ are equivalent to group homomorphisms $H_1(N, \mathbb{Z}) \to U(1)$. But when $H_1(N, \mathbb{Z})$ is finite, it is easy to show using the theory of finite abelian groups that the number of group homomorphisms $H_1(N, \mathbb{Z}) \to U(1)$ is exactly $|H_1(N, \mathbb{Z})|$. Hence, if $N$ is a special Lagrangian homology 3-sphere, then there are exactly $|H_1(N, \mathbb{Z})|$ flat $U(1)$ connections over $N$, and so $N$ gives rise to $|H_1(N, \mathbb{Z})|$ isolated 3-branes.

Therefore, the invariant $I(\delta)$ defined in (30) counts the number of isolated 3-branes in the homology class $\delta$. So it is a natural thing to count from the String Theory point of view, although the author formulated Conjecture 8.3 without knowing this. Part (a) of the conjecture thus says that the number of isolated 3-branes is independent of the Kähler class $[\omega]$. However, when we include multiple covers of $N$ the invariant $I(\delta)$ may not be an integer, only a rational number, and so we can’t really describe it as a ‘number of branes’.

This raises the question of how to interpret multiple covers as 3-branes. In String Theory $d$ coincident 3-branes give a complex vector bundle with fibre $\mathbb{C}^d$ over $N$, equipped with a flat $U(d)$ connection. This is considered as one 3-brane wrapped $d$ times around $N$, and so we call them multiply-wrapped 3-branes. For example, Gopakumar and Vafa [2] consider a special type of multiply-wrapped 3-brane they call 3-brane bound states. These correspond to special Lagrangian homology 3-spheres with an irreducible flat $U(d)$ connection, for $d > 1$.

Motivated by this, one could try and define more general invariants counting numbers of isolated, multiply-wrapped 3-branes; and these may be the ‘higher-order’ invariants mentioned in part (b) of Conjecture 8.3, and contributing to the transformation formula for $I$. The author hopes to address this in a subsequent paper.

Another interesting question is whether it would be natural in String Theory to combine all our invariants $I(\delta)$ into a generating function called a superpotential, which would be a holomorphic (and thus continuous) function
on the moduli space of complex structures on \( M \). The requirement that the superpotential be continuous under transitions such as those considered in §7 should cast much light on the transformation of \( I \) under deformations of the complex structure.

### 8.6 What use are the invariants anyway?

Suppose this whole programme can be carried through, and we can define an invariant \( I : H_3(M, \mathbb{Z}) \to \mathbb{Q} \) counting special Lagrangian homology 3-spheres, which depends only on the complex structure, and transforms in a definite way under deformations of the complex structure. What would this invariant tell us?

The author expects that the invariant should encode a lot of information about the global structure of the moduli space of complex structures on \( M \), in particular its singularities, and the monodromies around them. If we deform \((M, J, g)\) so that the volume of a special Lagrangian 3-fold goes to zero, then the complex structure must become singular. Thus, if \( I(\delta) \neq 0 \) near the complex hyperplane \([\Omega] \cdot \delta = 0\), then we expect \((M, J)\) to become singular when \([\Omega] \cdot \delta = 0\). For instance, this happens when \((M, J)\) develops an ordinary double point.

At such singularities, the identification between the moduli space of complex structures and \( P(H^3(M, \mathbb{C})) \) breaks down. If we go around a loop in the moduli space encircling the singular hypersurface, then \( H_3(M, \mathbb{Z}) \) and \( H^3(M, \mathbb{C}) \) are transformed by an automorphism called a monodromy. If we know the invariant \( I \) and its transformation properties, we may able to deduce what the monodromy is.

For as we deform \([\Omega]\) round a small loop about \([\Omega] \cdot \delta = 0\), the transformation rules tell us how to change \( I \) as we pass through each hypersurface \( W(\chi^+, \chi^-) \). But when \([\Omega]\) returns to its starting point, we don’t get the original invariant \( I \), but instead the composition of \( I \) with the monodromy around the loop. By comparing the initial and final versions of \( I \) we get a condition on the monodromy, which may determine it entirely.

### References


