Mean Magnetic Field Generation in Sheared Rotators

Eric G. Blackman\textsuperscript{1,2}

1. Theoretical Astrophysics, Caltech 130-33 Pasadena CA, 91125
2. Institute for Theoretical Physics, UCSB, Santa Barbara, CA, 93106

ABSTRACT

A generalized mean magnetic field induction equation for differential rotators is derived, including a compressibility, and the anisotropy induced on the turbulent quantities from the mean magnetic field itself and a mean velocity shear. Derivations of the mean field equations often do not emphasize that there must be anisotropy and inhomogeneity in the turbulence for mean field growth. The anisotropy from shear is the source of a term involving the product of the mean velocity gradient and the cross-helicity correlation of the isotropic parts of the fluctuating velocity and magnetic field, \langle v \cdot b \rangle^{(0)}. The full mean field equations are derived to linear order in mean fields, but it is also shown that the cross-helicity term survives to all orders in the velocity shear. This cross-helicity term can obviate the need for a pre-existing seed mean magnetic field for mean field growth: though a fluctuating seed field is necessary for a non-vanishing cross-helicity, the term can produce linear (in time) mean field growth of the toroidal field from zero mean field. After one vertical diffusion time, the cross-helicity term becomes sub-dominant and dynamo exponential amplification/sustenance of the mean field can subsequently ensue. The cross-helicity term should produce odd symmetry in the mean magnetic field, in contrast to the usually favored even modes of the dynamo amplification in sheared discs. This may be important for the observed mean field geometries of spiral galaxies. The strength of the mean seed field provided by the cross-helicity depends linearly on the magnitude of the cross-helicity.

Subject Headings: magnetic fields; galaxies: magnetic fields; turbulence; accretion discs.
1. Introduction

Mean magnetic field dynamo theory has been a leading formalism to characterize the evolution and origin of large-scale magnetic fields in stars and Galaxies (e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983). The mean field dynamo appeals to a combination of helical turbulence, differential rotation, and turbulent diffusion to exponentiate an initial seed mean magnetic field. The standard textbook kinematic theory is widely known to be incomplete, because it ignores the back reaction of the growing magnetic field on the turbulence driving the field growth.

The extent of dynamo growth quenching from the backreaction is not agreed upon (Cowling 1957; Piddington 1981; Zeldovich et al. 1983; Vainshtein & Cattaneo 1992; Cattaneo & Hughes 1996; Vainshtein 1998; Field et al. 1999). Field et al. (1999) and Blackman & Field (1999ab) have suggested that some previous criticisms of enhanced suppression might be challenged. Blackman & Field 1999b argue that simulations which employ periodic boundary conditions (e.g. Cattaneo & Hughes 1996) cannot test for suppression of dynamo theory because an upper limit on the required dynamo quantities can be shown to be strongly restricted when boundary terms are ignored. While some simulations done with diffusive boundary conditions do show evidence for a mean-field dynamo in fully non-linear MHD (Brandenburg & Donner 1997), there also exist non-linear analytic models (Vainshtein 1998) which show more extreme suppression. The restrictions with respect to actual astrophysical systems needs to be investigated and more analytic and numerical studies are pending.

Since the backreaction consequences are not fully resolved, and because some results of backreaction studies do not support catastrophic quenching of the dynamo theory, there remains ample motivation to continue studying the solutions of the mean-field dynamo equations in astrophysical systems. An industry continues to do so, motivated by some success in the solar (c.f. Parker 1979; Belvedere 1990), galactic (c.f. Ruzmaikin et al 1988; Beck et al 1996), and accretion disc (Brandenburg & Donner 1997) cases and the simplicity of the theory (c.f. Moffatt 1978; Parker 1979). The focus is often on the solutions, rather than the derivation of the equations being solved.

In this paper, I derive a generalized dynamo equation, including a restricted compressibility and the anisotropy induced from the backreaction of the mean magnetic field and from a mean velocity shear. The inclusion of the shear induced anisotropy leads to an important term in the mean field evolution equations, involving the product of mean velocity gradients and the cross-helicity. This term should be included in dynamo models of sheared systems such as galaxies, stars, and accretion discs, as it turns out to play a fundamental role in the mean field growth as long as the cross-helicity is non-vanishing. In particular, it can generate a mean magnetic field when the initial mean field is zero as first pointed out by

Previous studies have recognized the potential importance of the cross-helicity term. Yoshizawa and Yokoi (1993) and Yokoi (1996) first discussed the term in the astrophysical context. Brandenburg & Urpin (1998) have produced a nice analysis of the dynamo equation with shear. The results herein are closer to those of Brandenburg & Urpin (1998). However, there is an ambiguity in their derivation of the employed dynamo equations of the type emphasized by Blackman & Field (1999a). It is important to distinguish the isotropic component of the turbulence from the anisotropic component, both of which must be present. The formalism herein explicitly avoids this ambiguity and provides a more complete derivation of the generalized dynamo equations with the important shear term. In addition, it will be shown that the cross-helicity term survives to all orders in the shear not just to linear order.

In section 2 the basic mean field equations are derived including anisotropies induced both from the mean magnetic field and the velocity shear and compressibility. In section 3, I show that the term which will lead to linear mean field growth from zero mean field survives all orders in the shear anisotropy. In section 4 the mean magnetic field equation is solved in the early time limit when the shear term dominates. After one vertical diffusion time from the chosen \( t = 0 \), this term becomes sub-dominant. The resulting solutions are then given for both \( \alpha - \Omega \) and \( \alpha^2 \) dynamos. Section 5 discusses the implications of these solutions, with emphasis on galaxies, and section 6 is the conclusion.

2. Mean field evolution equation

Mean field dynamo theory characterizes the growth of magnetic fields on scales larger than the input turbulence. The induction equation describing the evolution of the magnetic field is given by

\[
\partial_t B = \nabla \times (V \times B) + \lambda \nabla^2 B, \tag{1}
\]

where \( \lambda \) is the micro-physical diffusivity and the pressure gradient and is taken parallel to the density gradient to ignore the Biermann-Battery type term (Biermann 1950). The Navier-Stokes equation describing the velocity evolution is given by

\[
\rho \partial_t V = -V \cdot \nabla V - \nabla P - \nabla (B^2/8\pi) + B \cdot \nabla B/(4\pi) + F(x,t) + \nabla \phi, \tag{2}
\]

where \( V \) is the velocity, \( B \) is the magnetic field, \( P \) is the pressure, and \( F \) is a forcing function, and \( \nabla \phi \) is a gravitation term. Splitting the magnetic field into mean and fluctuating components we write \( \bar{B} = b + \bar{B} \) where \( \langle b \rangle = 0 \), and the velocity \( V \) is similarly defined. (Note that since astrophysical rotators usually have turbulent input scales that are a non-trivial fraction of the mean scales, it is important to distinguish between fluctuating quantities and small scale quantities. Their magnitudes differ by a quantity that varies as the 3/2 power of the ratio of the turbulent input scale to the mean scale.)
We assume that the mean velocity is time independent, so we do not consider the mean velocity equation any further. For the mean induction equation we have (e.g. Moffatt 1978)

\[ \partial_t \bar{\mathbf{B}} = \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \nabla \times (\mathbf{\bar{V}} \times \mathbf{\bar{B}}), \]  

(3)

where the brackets and the over-bar indicate mean values and the lower case indicates fluctuating values.

To write the fluctuating components, I follow the approach of Field et al. (1999) and Blackman & Chou (1997). The fluctuating fields are written in terms of the zeroth order isotropic base state and higher order anisotropic components which are due to the presence of a mean field. Here we consider the influence of both the mean magnetic field as well as the mean velocity field. This approach can be motivated when the magnitude of the fluctuating anisotropic component of the velocity and magnetic field are < the mean velocity and mean magnetic field respectively.

The zeroth order quantities are not solved for but have the properties of isotropic homogeneous turbulence. The base state is also taken to have a reflection asymmetry. In astrophysical systems this property can result from the presence of an underlying rotation and a density gradient (Moffatt 1978; Krause & Rädler 1980). One can take the liberty of presuming that the forcing function acts only on the zeroth order state and feeds it with isotropic but reflection asymmetric turbulence. Since we do not solve the zeroth order equations, we do not need to specify the explicit form of the forcing function. The anisotropy induced from the mean velocity and mean magnetic field will be considered in the higher order components. Compressibility is allowed for in a restricted sense: the density will be taken to have a spatially dependent mean component \( \bar{\rho} \), and a fluctuating component which is free of influence of the mean fields.

Subtracting (3) from (1) we have for the fluctuating field

\[ \partial_t \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}) - \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\mathbf{\bar{V}} \times \mathbf{b}) \]  

(4)

For the fluctuating Navier-Stokes equation we have

\[ \bar{\rho} \partial_t \mathbf{v} = -\mathbf{v} \cdot \nabla \mathbf{\bar{V}} + \langle \mathbf{v} \cdot \nabla \mathbf{v} \rangle - \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p - \nabla (\mathbf{\bar{B}} \cdot \mathbf{b})/4\pi - \nabla b^2/8\pi + \nabla \langle b^2 \rangle/8\pi \]

\[ + F(x,t) + \mathbf{B} \cdot \nabla \mathbf{b}/4\pi + \mathbf{b} \cdot \nabla \mathbf{B}/4\pi + \mathbf{b} \cdot \nabla \mathbf{b}/4\pi - \langle \mathbf{b} \cdot \nabla \mathbf{b} \rangle/4\pi. \]  

(5)

We have ignored the microphysical viscosities for present purposes, as they are taken to be small. Write \( \mathbf{b} = \mathbf{b}^{(0)} + \mathbf{b}^{(A)} \) and similarly \( \mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(A)} \) where \( \mathbf{b}^{(A)} \) and \( \mathbf{v}^{(A)} \) are the anisotropic parts of the fluctuating components. As long as \( |\mathbf{b}^{(A)}|/(\sim 4\pi \bar{\rho} |\mathbf{v}^{(A)}|) < |\mathbf{B}| \) and \( |\mathbf{V}| \), then \( \mathbf{v}^{(A)} \) and \( \mathbf{b}^{(A)} \) can in principle, be solved for explicitly in terms of all orders in \( \mathbf{\bar{V}} \) and \( \mathbf{\bar{B}} \) (see Field et al. 1999 for the solution to all orders in \( \mathbf{\bar{B}} \) ignoring mean field gradients).
In this section I restrict the calculation to linear order in both $|\mathbf{B}|/|\mathbf{b}|$ and $(|\mathbf{V}|/R)/(|\mathbf{b}|/L)$, which is sufficient for the main points, and then in the next section I expand to all order in $(|\mathbf{V}|/R)/(|\mathbf{b}|/L)$, since this quantity is of order 1 for many applications.

Working in the local standard of rest (LSR) frame (Field et al. 1999) in which $\mathbf{V} = 0$, but not $\nabla \tilde{V}$ to first order in the mean fields we have

$$
\partial_t \mathbf{b}^{(1)} = \mathbf{b}^{(0)} \cdot \nabla \mathbf{v}^{(1)} + \mathbf{b}^{(1)} \cdot \nabla \mathbf{v}^{(0)} - \mathbf{b}^{(0)} \nabla \cdot \mathbf{v}^{(1)} - \mathbf{b}^{(1)} \nabla \cdot \mathbf{v}^{(0)} - \nabla \times (\mathbf{v} \times \mathbf{b})^{(1)} - \mathbf{v}^{(0)} \cdot \nabla \mathbf{b}^{(1)} - \mathbf{v}^{(1)} \cdot \nabla \mathbf{b}^{(0)} + \mathbf{v}^{(0)} \nabla \cdot \mathbf{b}^{(1)} + \mathbf{v}^{(1)} \nabla \cdot \mathbf{b}^{(0)}
$$

(6)

and

$$
\tilde{\rho} \partial_t \mathbf{v}^{(1)} = -\mathbf{v}^{(1)} \cdot \nabla \tilde{V} + \langle \mathbf{v} \cdot \nabla \mathbf{v}^{(1)} \rangle - \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(1)} - \mathbf{v}^{(1)} \cdot \nabla \mathbf{v}^{(0)} - \nabla \times \langle \mathbf{v} \times \mathbf{b}^{(1)} \rangle - \tilde{B}_l \nabla b_{l}^{(0)}/4\pi - b_{m}^{(0)}/4\pi \nabla b_{m}^{(0)}/4\pi - b_{m}^{(1)} \nabla b_{m}^{(0)}/4\pi - \nabla \langle \mathbf{v}^2 \rangle^{(1)}/4\pi + \tilde{B} \cdot \nabla \mathbf{b}^{(0)}/4\pi + \mathbf{b}^{(0)} \cdot \nabla \tilde{B} \cdot 4\pi + \langle \mathbf{b} \cdot \nabla \mathbf{b}^{(1)} \rangle^{(1)}/4\pi - \langle \mathbf{b} \cdot \nabla \mathbf{b}^{(1)} \rangle^{(0)}/4\pi.
$$

(7)

Now

$$
\langle \mathbf{v} \times \mathbf{b}^{(1)} \rangle = \langle \mathbf{v}^{(0)} \rangle \times \int \partial_t \mathbf{b}^{(1)} dt - \langle \mathbf{b}^{(0)} \rangle \times \int \partial_t \mathbf{v}^{(1)} dt.
$$

(8)

Using (6) and (7) in (8) we have for the first term on the right of (8)

$$
\langle \mathbf{v}^{(0)} \rangle \times \int \partial_t \mathbf{b}^{(1)} dt = \frac{\tau_c}{3} \langle \mathbf{v}^{(0)} \cdot \mathbf{b}^{(0)} \rangle \nabla \times \tilde{\mathbf{V}} - \langle \mathbf{v}^{(0)} \cdot \mathbf{v}^{(0)} \rangle \nabla \times \tilde{\mathbf{B}} - \langle \mathbf{v}^{(0)} \cdot \nabla \times \mathbf{v}^{(0)} \rangle \tilde{\mathbf{B}}
$$

(9)

Following the same procedure as above, I obtain for the second term on the right of (8)

$$
-\langle \mathbf{b}^{(0)} \rangle \times \int \partial_t \mathbf{v}^{(1)} dt = \frac{\tau_c}{3} \langle \mathbf{b}^{(0)} \cdot \mathbf{v}^{(0)} \rangle \nabla \times \tilde{\mathbf{V}} + 2 \langle \mathbf{b}^{(0)} \cdot \nabla \times \mathbf{b}^{(0)} \rangle \tilde{\mathbf{B}}/(4\pi \tilde{\rho}) - 2 \langle \mathbf{b}^{(0)} \cdot \mathbf{b}^{(0)} \rangle \nabla \times \tilde{\mathbf{B}}/(4\pi \tilde{\rho}) + \tilde{\rho}^{-1} \langle \mathbf{b} \cdot \nabla \rho^{(1)} \rangle.
$$

(10)

Expand the last term in (11) to first order: the energy equation can be written

$$
\partial_t P + \mathbf{V} \cdot \nabla P = \gamma P \nabla \cdot \mathbf{v} + \bar{C},
$$

(12)
where $\gamma$ is the adiabatic index and $\bar{C}$ is a cooling or heating term which we assume to have only a mean contribution. Writing the fluctuating part of this equation to first order in the mean fields then gives

$$\partial_t p^{(1)} = -\nabla \cdot \nabla \bar{P} + \langle \nabla \cdot \nabla p \rangle^{(1)} - (\nabla \cdot \nabla p)^{(1)} + \gamma \bar{P} \nabla \cdot \nabla \bar{v} + \gamma (p \nabla \cdot \nabla)^{(1)} - \gamma (p \nabla \cdot \nabla)^{(1)}.$$  

(13)

Plugging this back into the last term of (11) and using $\nabla \cdot \bar{v} = 0$ gives

$$\bar{\rho}^{-1} \langle \mathbf{b} \times \nabla p^{(1)} \rangle_i = (\langle \epsilon_{ijk} b_j^{(0)} \partial_k \partial_i p^{(1)} \rangle + \gamma_\epsilon \langle \nabla \cdot \nabla \rangle^{(1)} \bar{v}_m - \epsilon_{ijk} \langle b_j^{(0)} \partial_k v_m^{(0)} \rangle \partial_m \bar{P})$$

$$+ \gamma \epsilon_{ijk} \langle b_j^{(0)} \partial_k v_m^{(0)} \rangle \partial_k \bar{P} + \gamma \epsilon_{ijk} \langle b_j^{(0)} \partial_k v_m^{(0)} \rangle \partial_k \bar{P}.$$  

(14)

The 1st 3rd and 4th terms on the right of (14) vanish from anti-symmetrization after isotropization. The 5th term vanishes directly from isotropy of zeroth order correlations. Thus the only remaining term is the 2nd term. Combining this term with (10) and (11) gives

$$\langle \mathbf{v} \times \mathbf{b} \rangle^{(1)} = \frac{\tau_\epsilon}{3 \bar{\rho}} \langle \mathbf{b} \cdot \nabla \times \mathbf{v} \rangle^{(0)} \nabla \bar{P} + (\tau_\epsilon \mathbf{B} / 3) \langle 2 (\mathbf{b} \cdot \nabla \times \mathbf{b}) / (4 \pi \bar{\rho}) \rangle$$

$$- \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle^{(0)} - \tau_\epsilon \nabla \times \langle \mathbf{B} \rangle (\mathbf{b} \cdot \mathbf{b}) / (6 \pi \bar{\rho}) + \frac{T_a}{3} \langle \mathbf{v} \cdot \mathbf{v} \rangle^{(0)} + \frac{2 \pi}{3} \langle \mathbf{v} \cdot \mathbf{v} \rangle^{(0)} + \frac{2 \pi}{3} \langle \mathbf{v} \cdot \mathbf{v} \rangle^{(0)}$$

$$\equiv \alpha_1^{(0)} \nabla \bar{P} + \alpha_2^{(0)} \mathbf{B} - \beta_1^{(0)} \nabla \times \bar{B} + \beta_2^{(0)} \nabla \times \bar{V}.$$  

(15)

Taking the curl of (15), plugging into (3), and again assuming $\nabla \bar{\rho} \times \nabla \bar{P} = 0$ the equation for the evolution of the mean magnetic field becomes

$$\partial_t \mathbf{B} = \tau_c (\mathbf{b} \cdot \nabla \times \mathbf{b}) / (6 \pi \bar{\rho}) - \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle^{(0)} / 3 \nabla \times \bar{B}$$

$$+ \tau_c (\mathbf{b} \cdot \mathbf{b}) / (6 \pi \bar{\rho}) - \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle^{(0)} / 3 \nabla^2 \bar{B} - \tau_c (\mathbf{b} \cdot \mathbf{b}) / (6 \pi \bar{\rho}) \mathbf{b} \cdot \nabla \times \bar{B}$$

$$+ \tau_c (\mathbf{b} \cdot \mathbf{b}) \nabla \bar{\rho} \times (\nabla \times \bar{B}) + \frac{2 \pi}{3} \langle \mathbf{b} \cdot \mathbf{v} \rangle^{(0)} \nabla \times \bar{V} + \nabla \times (\bar{V} \times \bar{B})$$

$$\equiv \alpha_2^{(0)} \nabla \times \bar{B} + \nabla \alpha_2^{(0)} \times \bar{B} + \beta_1^{(0)} \nabla^2 \bar{B} - \nabla \beta_1^{(0)} \times (\nabla \times \bar{B}) - \beta_2^{(0)} \nabla^2 \bar{V} + \bar{B} \cdot \nabla \bar{V}.$$  

(16)

The last term of (16) is the standard $\Omega$ shearing term and the $\alpha_2^{(0)}$ and $\beta_1^{(0)}$ terms are the standard $\alpha$ and $\beta$ dynamo coefficients (Moffatt 1978) except for the inclusion of a spatially dependent density. We will see that the penultimate term on the right can generate a seed field. The coupling coefficient $\beta_2^{(0)}$ is the cross-helicity. The importance of a cross-helicity term was first addressed by Yoshizawa and Yokoi (1993), Yokoi (1996) but not as a source of seed field for the standard dynamo. Blackman & Chou (1997) and Brandenburg & Urpin (1998) first pointed out the seed field role. Before solving the generalized mean-field equation which includes the role of the cross-helicity, I will show that the cross-helicity term survives to all orders in the shear.

### 3. Survival of the linear growth term to all orders in mean shear
The expansion in section 2, was taken to linear order in $|\bar{B}|/|b^{(0)}|$ and $(|\bar{V}|/R)/(|b^{(0)}|/L)$, where $R$ is the scale of variation of $\bar{V}$ and $L$ is the turbulent outer scale. While $|\bar{B}|/|b^{(0)}| < 1$, $<< 1$ for early dynamo evolution, and $< 1$ in the Galaxy at present, and arguably $< 1$ in accretion discs, the quantity $(|\bar{V}|/R)/(|b^{(0)}|/L) \simeq 1$ in accretion discs for when a shearing instability drives the turbulence (Balbus & Hawley 1998). This is simply the statement that the instability growth time is of order the rotation time. For this purpose it is necessary to expand to all orders in at least $(|\bar{V}|/R)/(|b^{(0)}|/L)$.

In this section I show that the second last term found in (16) survives to all orders in the mean shear field when the $\bar{V}$ is dominated by 1 component ($\bar{V}_\phi$ in the present case). Splitting up the small scale fields into isotropic and anisotropic components we recall $b = b^{(0)} + b^{(A)}$ and $v = v^{(0)} + v^{(A)}$. We then have for the turbulent EMF

$$\langle v \times b \rangle = \langle v^{(0)} \times b^{(A)} \rangle + \langle v^{(A)} \times b^{(0)} \rangle + \langle v^{(A)} \times b^{(A)} \rangle.$$  \hspace{1cm} (17)

To simply show that the linear growth term survives to non-linear order in the shear, let us take the mean magnetic field to be identically zero and ignore the pressure term. (This will have to be generalized in the future, but the issue of survival of the term in question is not affected.) Then, the evolution equations for the $n$th order anisotropic components become (ignoring second order correlations in $\tau_c$) simply

$$\partial_t v_i^{(n)} = -v_j^{(n-1)} \partial_j \bar{V}_i,$$  \hspace{1cm} (18)

and

$$\partial_t b_i^{(n)} = b_j^{(n-1)} \partial_j \bar{V}_i.$$  \hspace{1cm} (19)

Summing over $n$ we have

$$\partial_t v_i^{(A)} + v_j^{(A)} \partial_j \bar{V}_i = -v_j^{(0)} \partial_j \bar{V}_i,$$  \hspace{1cm} (20)

and

$$\partial_t b_i^{(A)} - b_j^{(A)} \partial_j \bar{V}_i = b_j^{(0)} \partial_j \bar{V}_i.$$  \hspace{1cm} (21)

Multiplying by an integrating factor, and defining $M_{iq}(t) \equiv [e\nabla \bar{V}]_{iq}$ (where $\nabla \bar{V}$ is independent of time) these equations can be written

$$\partial_t (v_i^{(A)} M_{iq}(t)) = -v_j^{(0)} \partial_j \bar{V}_i M(t)_{iq},$$  \hspace{1cm} (22)

and

$$\partial_t (b_i^{(A)} M_{iq}(-t)) = b_j^{(0)} \partial_j \bar{V}_i M_{iq}(-t).$$  \hspace{1cm} (23)

The solutions to these equations are given by

$$v_s^{(A)} = v_i^{(A)} M_{iq}(t) M_{qs}^{-1}(t) = -\int_0^t v_j^{(0)} \partial_j \bar{V}_i M_{iq}(t') dt' M_{qs}^{-1}(t) + v_s(0),$$  \hspace{1cm} (24)
we then have
\[ b_s^{(A)} = b_s^{(A)} M_{qs}^{-1} M_s^{-1} = \int_0^t b_j^{(0)} \partial_j \tilde{V}_i M_{qs}^{(-t')} dt' M_{qs}^{(-t)} + b_s(0) \] (25)

The 3 terms in the turbulent EMF (17) then become
\[ \langle \mathbf{v}^{(0)} \times \mathbf{b}^{(A)} \rangle_p = \langle \epsilon_{pmx} v_m^{(0)} \int b_j^{(0)} M_{qs}^{(-t')} \partial_j \tilde{V}_i M_{qs}^{(-t)} \rangle = \langle \epsilon_{pmx} v_m^{(0)} b_j^{(0)} \partial_j \tilde{V}_i \rangle = \frac{\tau_c}{3} \langle \mathbf{v}^{(0)} \cdot \mathbf{b}^{(0)} \rangle (\nabla \times \tilde{V})_p \] (26)

and
\[ -\langle \mathbf{b}^{(0)} \times \mathbf{v}^{(A)} \rangle_p = \langle \epsilon_{pmx} b_m^{(0)} \int v_j^{(0)} M_{qs}^{(t')} \partial_j \tilde{V}_i M_{qs}^{(t)} \rangle = \langle \epsilon_{pmx} v_m^{(0)} b_j^{(0)} \partial_j \tilde{V}_i \rangle = \frac{\tau_c}{3} \langle \mathbf{v}^{(0)} \cdot \mathbf{b}^{(0)} \rangle (\nabla \times \tilde{V})_p \] (27)

and finally
\[ \langle \mathbf{v}^{(A)} \times \mathbf{b}^{(A)} \rangle_p = \langle \epsilon_{pmx} \int v_j^{(0)} \partial_j \tilde{V}_i M_{qs}^{(t')} \partial_j \tilde{V}_i M_{qs}^{(t)} \rangle + \langle \epsilon_{pmx} v_i^{(A)} \partial_i \tilde{V}_i M_{is}^{(t)} \rangle \] (28)

Notice that the sum of (26) and (27) is simply \( \beta_2^{(0)} \) which enters in (15) and thus (16). Thus (28) appears to produce an extra term in the EMF. However, it vanishes when \( \tilde{V} \) is dominated by 1 component (in our case \( \tilde{V}_j \approx \tilde{V}_T \), the toroidal mean velocity): the index \( j \) in \( M_{ij} \) always corresponds to the index of \( \tilde{V}_j \) when the exponential in \( M_{ij} \) is expanded. This means that \( m = s \) in (28) and due to the \( \epsilon_{mpx} \), both terms in (28) vanish. Thus, we see that in the case for which the mean velocity is dominated by one component, the shearing term contributes a term which will lead to linear growth from zero mean field, as per the next section, even when the anisotropy due to this shear is included to all orders.

3. Growth of the Mean Field

For simplicity, I will assume that the mean field is axisymmetric. All of the components of the magnetic field can then be expressed in terms of the toroidal field \( \mathbf{B}_T \) and the poloidal field \( \nabla \times \mathbf{A}_T \), where \( \mathbf{A} \) is the vector potential. The equation for the poloidal mean field can then be “uncurled” (since an arbitrary gradient term vanishes for axis-symmetry). From (16) we then have
\[ \partial_t \mathbf{A}_T = \alpha_2^{(0)} \mathbf{B}_T - \beta_1^{(0)} \nabla \times \mathbf{A}_T + \beta_2^{(0)} (\nabla \times \tilde{V})_T + (\tilde{V} \times \mathbf{B})_T. \] (29)

For the toroidal component of \( \mathbf{B} \) we have
\[ \partial_t \mathbf{B}_T = -\alpha_2^{(0)} \nabla^2 \mathbf{A}_T - (\nabla \alpha_2^{(0)} \cdot \mathbf{A}_T + \beta_1^{(0)} \nabla^2 \mathbf{B}_T + (\nabla \times \mathbf{A}_T) \cdot \nabla \tilde{V} + (\nabla \beta_1^{(0)} \cdot \nabla) \mathbf{B}_T - \beta_2^{(0)} \nabla^2 \tilde{V}_T. \] (30)

Let us now proceed to solve equations (29) and (30).

3a. Linear Growth Regime
At early times, the dominant term on the right of (30) is the term $Q(r, z) \equiv \beta_2^{(0)} \nabla^2 \vec{V}$, taken to be independent of time. Solving (30) with only this term gives

$$\vec{B}_T = -Q(r, z)t,$$

if we assume the initial $\vec{B}$ is negligible. Plugging this solution into (29), and solving we obtain, to lowest order in $t$

$$\vec{A}_T \simeq 2H^2 \alpha_2^{(0)} Q(r, z)t/\beta_1^{(0)}.$$  (32)

where $H$ is the characteristic gradient length of the mean magnetic field. How long does this solution hold approximately? A typical value of the other terms in (29) and (30) is given by the diffusion terms. Thus plugging (31) or (32) into the respective diffusion terms, with the strongest variation of the magnetic field taken to be the vertical direction, one finds that the linear growth term becomes subdominant at $t \sim H^2/\beta_1^{(0)}$. At this time, the fields have grown to the values

$$\vec{B}_{T, crit} \simeq -(H^2/\beta_1^{(0)})\beta_2^{(0)} \nabla^2 \vec{V}_T,$$

and

$$\vec{A}_{T, crit} \simeq H^4/(\beta_1^{(0)})^2 \alpha_2^{(0)} \beta_2^{(0)} \nabla^2 \vec{V}_T.$$  (34)

Because the fluctuating magnetic field builds up to near equipartition with the turbulent kinetic motions with a growth rate $\sim$ eddy turnover time, $|b^{(0)}|/(4\pi \bar{\rho})^{1/2} \sim |v^{(0)}|$ at times of interest for long term mean field evolution. We then have

$$V_{A, crit} \sim |V_T|(H^2/R^2)\chi^{(0)},$$  (35)

where $V_{A, crit}$ is the Alfvén speed associated with the mean magnetic field, $\chi^{(0)} = [\beta_2^{(0)}(4\pi \bar{\rho})^{-1/2}]|\beta_1^{(0)}|$, and $V_{A, crit}$ is the Alfvén speed associated with the mean magnetic field at the time when the linear term becomes subdominant. Note the importance of the ratio of cross-helicity magnitude to turbulent diffusivity magnitude.

**3b. Dynamo Growth Regime**

Now I proceed to solve (29) and (30) in the $t > H^2/\beta_1^{(0)}$ limit, when the when the third term on the right of (29) and the last term on the right of (30) can be ignored. In this regime, writing (29) and (30) in terms of their scalar components in cylindrical coordinates gives

$$\partial_t \bar{A}_\phi = \alpha_2^{(0)} \bar{B}_\phi + \beta_1^{(0)} \nabla^2 \bar{A}_\phi - \beta_1^{(0)} \bar{A}_\phi/r^2,$$  (36)

where $\phi$ indicates the toroidal component, and

$$\partial_t \bar{B}_\phi = -\alpha_2^{(0)} \nabla^2 \bar{A}_\phi + \alpha_2^{(0)} \bar{A}_\phi/r^2 - \nabla \bar{A}_\phi + \beta_1^{(0)} \nabla^2 \bar{B}_\phi - \beta_1^{(0)} \bar{B}_\phi/r^2 - \partial_z \bar{A}_\phi \partial_r \bar{V}_\phi + \bar{V}_\phi \bar{B}_\phi.$$  (37)
I assume the case where the vertical gradients of $\mathbf{B}$ dominate and the radial gradients of $\mathbf{V}$ dominate and look for solutions of the form $A = \exp[nt + ikz\cdot\mathbf{x}]$, and $B = \exp[nt + ik\cdot\mathbf{x}]$.

(Given the spatial density gradients in a system, one can be more accurate, since the $A_{\phi,\text{crit}}$ and $B_{\phi,\text{crit}}$ have spatial dependences which depend only on the density gradient. For present purposes I take the simpler approach.) From (36) and (37), we obtain

$$(n + \beta_1^{(0)} k_z^2) \bar{A}_\phi - \alpha_2^{(0)} \bar{B}_\phi = 0,$$

and

$$(\alpha_2^{(0)} k_z^2 - ik_z \partial_z \alpha_2^{(0)} - ik_z \partial_z \mathbf{V}_\phi) \bar{A}_\phi - (n + \beta_1^{(0)} k_z^2 - ik_z \partial_z \beta_1^{(0)}) \bar{B}_\phi = 0.$$  \hspace{2cm} (38) \hspace{2cm} (39)

The dispersion relation resulting from (38) and (39) is

$$n^2 + n(2\beta_1^{(0)} k_z^2 - ik_z \partial_z \beta_1^{(0)}) + (k_z \beta_1^{(0)})^2 - ik_z \partial_z \beta_1^{(0)} - (k_z \alpha_2^{(0)})^2 + i\alpha_2^{(0)} k_z \partial_z \alpha_2^{(0)} + i\alpha_2^{(0)} k_z \partial_z \mathbf{V}_\phi.$$  \hspace{2cm} (40)

The solution to this quadratic is

$$2n = -k_z \beta_1^{(0)} + ik_z \partial_z \beta_1^{(0)} \pm [4k_z^2 \alpha_2^{(0)} - k_z^2 (\partial_z \beta_1^{(0)})^2 - i4k_z \alpha_2^{(0)} (\partial_z \alpha_2^{(0)} + \partial_z \mathbf{V}_\phi)]^{1/2}.$$  \hspace{2cm} (41)

Scaling the terms inside the square brackets on the right of (41) shows that the second term is down from the first by some constant $\ll 1$ times $(L/H)^2$, where $L$ is the dominant turbulent scale. We assume that this second term can be ignored for simplicity. The remaining real part of (41) is of interest. This can be found from the following: write the right side as $(c + di)^{1/2}$, where $c$ and $d$ are real and seek the real quantity $a$ such that $a + bi = (c + di)^{1/2}$. In general, this implies that

$$2a^2 = [c \pm (c^2 + d^2)^{1/2}].$$  \hspace{2cm} (42)

Applying (42) to (41) gives

$$\text{Re}[n] = -k_z^2 \beta_1^{(0)} \pm k_z \alpha_2^{(0)} / 2^{1/2} \pm \partial_z \mathbf{V}_\phi / \alpha_2^{(0)} / 2^{1/2}.$$  \hspace{2cm} (43)

There are two interesting limits of (43). In the limit that the differential rotation dominates the inner sum, we have

$$\text{Re}[n] \simeq -k_z^2 \beta_1^{(0)} \pm (k_z \alpha_2^{(0)})^{1/2} \partial_z \mathbf{V}_\phi / 2^{1/2}.$$  \hspace{2cm} (44)

This has growing modes for $k_z \lesssim (|\alpha_2^{(0)} \partial_z \mathbf{V}_\phi| / 2\beta_1^{(0)})^{1/3}$. The maximum growth wavenumber is $k_{z,\text{max}} = (|\alpha_2^{(0)} \partial_z \mathbf{V}_\phi| / 2\beta_1^{(0)})^{1/3}$ so that $v_{\text{max}} \sim 0.3(|\alpha_2^{(0)} \partial_z \mathbf{V}_\phi| / \beta_1^{(0)})^{1/3}$. This is an “$\alpha - \Omega$” type dynamo (c.f. Moffatt 1978; Ruzmaikin et al. 1988).

In the limit that $\partial_z \alpha_2^{(0)}$ dominates or is comparable to $\partial_z \mathbf{V}_\phi$, we have

$$\text{Re}[n] \simeq -k_z^2 \beta_1^{(0)} \pm k_z \alpha_2^{(0)} / 2^{1/2} \pm (k_z \alpha_2^{(0)})^{1/2} / 2^{1/2}.$$  \hspace{2cm} (45)
where the latter similarity follows when \(|(\partial_z a_2^{(0)})/(k_z a_2^{(0)})| \sim 1\). This has growing modes for \(k_z \leq 1.30\alpha_2^{(0)}/\beta_1^{(0)}\). The maximum growth wave number is at \(\sim 0.65\alpha_2^{(0)}/\beta_1^{(0)}\) so that \(n_{\text{max}} \sim 0.4(\alpha_2^{(0)})^2/\beta_1^{(0)}\). This is an “\(a^2\)” type dynamo (c.f. Moffatt 1978).

4. Discussion

The results of sections 3a and 3b show the two phases of mean field growth for a shearing (disc) rotator. The first regime does not require a seed mean field, only a fluctuating field whose mean can vanish. The toroidal field produced in the first phase (the linear growth regime) should be asymmetric with respect to the disc plane (like an A0 mode, e.g. Beck et al. 1996) since the cross-helicity \(\beta_2^{(0)} = \langle b^{(0)} \cdot v^{(0)} \rangle\) is a pseudo-scalar. The poloidal field growth feeds on this linear toroidal field. If the cross-helicity has the same sign throughout a hemisphere, the linear growth phase should not produce radial field reversals in that hemisphere since the mean field vector is determined only by the sign of the cross-helicity and the direction of the mean velocity and its gradient.

The initial growth phase is important since standard dynamo models generally consider the equations only in the second phase. In the second phase it is generally easier to amplify S0 type modes, whose toroidal field is symmetric with respect to the galactic plane (Ruzmaikin et al 1988; Beck et al 1996), though more complicated models are able to excite mixed modes e.g. A0-S0 (Moss & Tuominen 1990; Moss et al. 1993). The dominant observed field after amplification saturates also depends on which seed field modes were present to begin with (Ruzmaikin et al. 1988; Poezd et al. 1993). Suppose an asymmetric seed field is present with magnitude well in excess of the seed symmetric mode. Then even if the asymmetric mode grows more slowly than the symmetric mode, the difference of initial magnitudes can more than compensate and the asymmetric mode could dominate. Also, once equipartition is reached, different modes may oscillate with different frequencies.

In the Galaxy for example, the magnitude of the asymmetric seed resulting from the linear growth phase is, assuming no vertical shear and a radius of 8Kpc, from (35) \(B \sim 2.7 \times 10^{-9}(\chi^{(0)}/0.01)(H/0.5\text{Kpc})^2(R/8\text{Kpc})^{-2}(V_\phi/2 \times 10^7\text{cm/s})\text{G}\). This is a substantial seed field, which would arise in one vertical diffusion time. I have scaled with \(\chi^{(0)} \sim 0.01\). However, note in the Galaxy for example, the dynamo can exponentiate > 20 or so times in the Galactic lifetime. Thus even if \(\chi^{(0)} \sim 10^{-5}\) the seed from the linear growth phase could be important for seeding the subsequent dynamo growth up to its present value of few \(\times 10^{-6}\) G. If \(\chi^{(0)} \geq 0.01\), or if there were vertical shear, then (35) would imply an even larger seed, and a much shorter time for the cross-helicity + dynamo to produce an equipartition mean field. The exponential amplification rate of a symmetric mode from a seed field of \(10^{-9}\text{G}\) would have to be larger than that of the asymmetric mode to be competitive with that seeded from the asymmetric linear growth phase.
Han et al. (1998) suggest that the Faraday rotation data for M31 supports the presence of an even mode (S0). For our Milky way however, the data favor a dominant A0 mode (Han et al. 1997). While more galaxy data are needed, there appears to be the possibility that different modes dominate in different galaxies. The results herein suggest that the cross-helicity term can be important in determining the field geometry in the framework of in situ field generation models.

A number of simple but important complications affect the observational implications of the above results. First, inhomogeneity and large scale local structures can lead to coherent local structures. There may also be seed fields present from protogalactic, cosmological, or supernovae origins which are amplified concurrently. In addition, the mean field in a galaxy is likely never to be zero since the dominant input scale of the turbulence in galaxies is always a non-trivial fraction of the galactic radius. For example, at the solar location of our Galaxy, the scale ratio is at most (100pc/10kpc). Thus, just the random RMS field (Blackman 1998), or RMS contributions to dynamo coefficients that result (Vishniac & Brandenburg 1997) may be important in influencing the observation of a mean field. (It is important to note that mean field theory and Faraday rotation measurements are degenerate with respect to mean field topology: they cannot distinguish between a large scale field formed from averaging over small scale loops, or from that formed by averaging of a topologically connected field line with fluctuations. Spectral energy distribution approaches also share this degeneracy.)

The two phase growth described herein, as that of Brandenburg & Urpin (1998), could be testable in some non-linear MHD disc simulations. Note that periodic boundary condition simulations cannot test for mean field growth because the mean field is conserved by construction (c.f. Balbus & Hawley 1998). It is known that diffusion through the boundary is required for a working dynamo (Parker 1979; Ruzmaikin et al 1988), and it has also been realized that diffusive boundary conditions are required for a non-vanishing $\alpha^{(0)}$ effect even when dynamo action is not present (Blackman & Field 1999b). There is some evidence that a dynamo is operating in steady accretion disc simulations with the appropriate diffusive boundary conditions (Brandenburg & Donner 1997).

Brandenburg & Urpin (1998) point out that some previous simulations have estimated $\chi$. (Note that this is not $\chi^{(0)}$ which comes in directly into the present formalism, but the full $\chi$, to all orders in the mean fields. The two should be distinguished.) It is found that $\chi \sim 0.03$ in stratified convection simulations (Brandenburg et. al. 1996), $\chi \sim 3 \times 10^{-4}$ in magneto-shearing driven instability simulations (Brandenburg et. al. 1995), and $\chi \sim 5 \times 10^{-3}$ in supernova induced turbulent flows (Korpi et al. 1998; though these have not reached the saturated steady-state) potentially relevant for galaxies. Chandran & Rodriguez (1997) studied the evolution of the cross-helicity in the framework of the direct interaction
approximation, and do not rule out the possibility of significant cross-helicity on the turbulent input scale. Future simulations should give a better handle on $\chi$ and $\chi^{(0)}$, and a comparison of the two in the various settings. As pointed out earlier, even values of $10^{-5}$ might be significant. Note that it is also important to distinguish between an RMS value that may fluctuate in time and a systematic value that remains steady in time.

The use of a linear backreaction model is certainly incomplete (as are most studies of dynamo equation solutions). Both the shear and $\bar{B}$ should be included to all orders. Field et al. (1999) solved for all orders in $\bar{B}$ without field gradients. Ultimately, more analytic and numerical work are needed to understand how well the dynamo survives in the non-linear backreaction. Interestingly, dynamo theory in its standard form is the “complement” of analytic accretion on disc theory: the latter ignores the dynamics of the magnetic field, while the former does not fully include the magnetic backreaction on the velocity dynamics. Both require some closure approximation to turbulence to make analytic progress. Studies which focus on the solutions of the accretion disc or mean field dynamo equations often gloss over the approximations which led to the form of the equations being solved.

5. Conclusions

The equations for the evolution of the mean magnetic field in a sheared rotator have been derived, taking into account the anisotropy induced in the turbulence from the large scale magnetic field and the differential shear. A restricted compressibility has also been included. A shear term violating the homogeneity of the dynamo equations was shown to survive to all orders in the anisotropy. For finite cross-helicity, the mean magnetic field can grow in two phases. Linear growth ensues for one vertical diffusion time. The mean toroidal and poloidal fields grow without a seed mean field to a value whose Alfvén speed is of order $\sim (H/R)^2 \chi^{(0)} V_\phi$ in an A0 mode (toroidal field anti-symmetric with respect to the disc plane). After the vertical diffusion time, the second phase incurs, and the more “standard” dynamo evolution takes over. If the linear growth phase seed field dominated the field geometry, there would be a reversal in the toroidal field across the disc plane and no reversals in radius if the cross-helicity has the same sign in a given hemisphere. Determining the magnitude of $\chi^{(0)}$ and the full $\chi$ to all orders for various applications will continue to be aided by numerical simulations. For non-vanishing cross-helicity, equation (16) should replace the standard textbook dynamo equations for sheared rotators.

Acknowledgements: Thanks to the Black Hole Astrophysics Program at the ITP where part of this work was carried out and supported by NSF Grant PHY94-07194, and to the referee, editor, and G. Field for comments.
REFERENCES


Brandenburg, A., Jennings, R.L., Nordlund, A., Rieutord, M., Stein, R.F., Tuominen, I.


