Constraints on the bulk Lorentz factor in the internal shock scenario for gamma–ray bursts

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\textbf{ABSTRACT}

We investigate, independently of specific emission models, the constraints on the value of the bulk Lorentz factor $\Gamma$ of a fireball. We assume that the burst emission comes from internal shocks in a region transparent to Thomson scattering and before deceleration due to the swept up external matter is effective. We consider the role of Compton drag in decelerating fast moving shells before they interact with slower ones, thus limiting the possible differences in bulk Lorentz factor of shells. Tighter constraints on the possible range of $\Gamma$ are derived by requiring that the internal shocks transform more than a few per cent of the bulk energy into radiation. Efficient bursts may require a hierarchical scenario, where a shell undergoes multiple interactions with other shells. We conclude that fireballs with average Lorentz factors larger than 1000 are unlikely to give rise to the observed bursts.

\textbf{Key words:} gamma rays: bursts — X-rays: general —

\section{INTRODUCTION}

In the last few years, increasing evidence in favor of the fireball model (Rees \& Mészáros 1992) for gamma-ray bursts has been gathered, thanks to the observations of BeppoSAX (Boella et al. 1997). The power law decay of the optical afterglow of several bursts has lasted for time–scales of a year, in striking agreement with the simplest fireball scenario (Wijers, Rees \& Mészáros 1997), in which a shock wave propagates in the interstellar medium (ISM), accelerating particles which then emit by the synchrotron process (Sari, Piran \& Narayan 1998). However a single shock wave cannot account for both the temporal behavior of the $\gamma$–ray emission (Fenimore et al. 1999a) and for the requirement of a high efficiency in converting its kinetic energy into radiation (Sari \& Piran 1997). In fact the time variability structure of bursts appears to be constant during the whole of the high energy emission (Fenimore et al. 1999b), while the deceleration of a shock slowed down by interactions with the ISM would produce a time dilation between the first and the last spike of the burst. Moreover the observed variability cannot be explained as the consequence of inhomogeneities in the ISM since a radiative efficiency of less than 1% would be expected (Sari \& Piran 1997; see however Dermer, Böttcher \& Chiang 1999).

In the internal shock scenario, put forward by Rees \& Mészáros (1994), the inner engine produces many relativistic expanding shells (or an unsteady wind) with a distribution of Lorentz factors centered on a mean value $\langle \Gamma \rangle$ and width $\Delta \Gamma \sim \langle \Gamma \rangle$. The burst radiation is produced through the dissipation occurring when a faster shell catches up a slower one. No matter the physical mechanism producing the observed photons, the inner (hidden) engine is responsible for the temporal structure of the observed burst through the time history of the shell emission (Kobayashi, Piran \& Sari 1997). Even if we may have information on the time–scale of the shell ejection, it is hard to estimate the bulk Lorentz factor $\langle \Gamma \rangle$ of the relativistic outflow. This is however a critical parameter to unveil the mechanism(s) that powers the outflow and the radiating process that produces $\gamma$–rays. In fact, the knowledge of $\langle \Gamma \rangle$ allows us to estimate the amount of baryon loading of the fireball and the intrinsic frequency of the emitted photons. The constraints related to the compactness problem (see e.g. Piran 1992) give a lower limit $\langle \Gamma \rangle \gtrsim 100$, while an upper limit $\langle \Gamma \rangle \lesssim 10^5$ is inferred from the need of having the fireball opaque till the acceleration stage is completed. This poorly constrains the properties of the outflow, leaving a broad interval for physically possible $\langle \Gamma \rangle$.

In this letter we analyze the kinematic evolution of an inhomogeneous fireball, in which the flow is approximated by discrete shells with different intrinsic properties (baryon load, energy and Lorentz factor) and the outflow energy is dissipated through binary shell–shell collisions. We derive general constraints on the relativistic properties of the out-
flow both in the simplest internal shock scenario, that requires a single interaction for each couple of shells, and in the case of hierarchically developed internal shocks, in which successive shell mergers end up in a single (or a few) more massive shells, whose interaction with the ISM produces the afterglow. The latter scenario is strongly favored by the need of dissipating a significant fraction of ordered outflow energy (see also Kobayashi et al. 1997).

2 KINEMATIC OF THE FLOW

We describe the relativistic outflow as constituted of $N_S$ shells, each with its own relativistic Lorentz factor $\Gamma_i$, mass $M_i$ and energy $E_i = \Gamma_i M_i c^2$. The global properties of the flow are described by the averages $\langle \Gamma \rangle$, $\langle M \rangle$ and $\langle E \rangle$ and their dispersions, by the total duration of the ejection of shells $T$ (as measured in the rest frame of the inner engine) and by the time interval between the ejection of two successive shells $\Delta t$. If the ejection time of a typical shell equals the time of quiescent phase between the production of two consecutive shells, $\Delta t = T/2N_S$ (it is thus assumed that the time necessary to eject a fast or a slow shell is the same, see however Panaiteanu, Spada and Mészáros 1999).

The average mass, energy and relativistic factor are related as

$$\left\langle M \right\rangle = \frac{E}{N_S \langle \Gamma \rangle c^2} \simeq 5.6 \times 10^{-7} E_{52} \langle \Gamma \rangle_{2}^{-1} N_{S,2}^{-1} M_{\odot}, \quad (1)$$

where $E = 10^{52} E_{52}$ erg is the total energy of the outflow $^*$. The dynamics of each shell is characterized by an initial phase in which it is accelerated to its final Lorentz factor as $\Gamma \sim R/R_0$, where $R_0$ is the distance of the shell from the center of expansion at the ejection time (see e.g. Piran 1998). This phase ends when the final Lorentz factor is reached at the ‘acceleration’ radius:

$$R_A = \Gamma_i R_0 \simeq 10^3 \Gamma_{i,2} R_{0,7} \text{ cm} \quad (2)$$

During the following evolution, the shell coasts with constant velocity until an interaction with another shell or with the ambient medium takes place. Assuming that the acceleration phase lasts for a time interval negligible with respect to the coasting phase, the interaction between two shells with Lorentz factors $\Gamma_1$ and $\Gamma_2 > \Gamma_1$ and initially spaced by a time interval $\Delta t$ occurs at a radius:

$$R_t = 2 \frac{\alpha_t^2}{\alpha_i^2 - 1} \Gamma_i^2 c \Delta t = 4.0 \times 10^{23} \Gamma_{i,2}^2 T_1 N_{(8,2)}^{-1} \text{ cm} \quad (3)$$

where the numerical value has been computed for $\alpha_i = \Gamma_2/\Gamma_1 = 2$ and assuming $\Delta t = T/2N_S$.

The interaction of the shells with the ambient medium is a continuous process and does not happen at a well determined radius. However a typical scale can be estimated as the distance at which the shell Lorentz factor is half its initial value, which is reached when the shell has swept up an external mass $m$ equal to its rest mass divided by the initial Lorentz factor (see Mészáros & Rees 1997). Hence the deceleration radius can be approximated as $M_i/T_1 = (4/3) \pi R_{0,7}^2 n_{mp} m$, i.e.

$$R_D \simeq 1.2 \times 10^{16} E_{52}^{-1/3} n_{50}^{-1/3} \Gamma_{i,2}^{-2/3} N_{(8,2)}^{-1/3} \text{ cm} \quad (4)$$

where $n$ is the density (assumed uniform) of the ambient medium and shells of equal energy have been considered. Therefore, the more the shell is relativistic, the smaller the deceleration radius is (Eq. 4), while, on the contrary, the largest the radius at which internal shocks occur (see Eq. 3). This allows us to put some constraints on the average relativistic factor of the flow, since observationally the internal shocks set up before the afterglow, i.e. before the development of the external shock.

To fully describe the kinematic of the flow, a fourth transition radius is important, i.e. the radius at which the shell becomes transparent (to Thomson scattering). If, in fact, the shell becomes optically thin before the acceleration phase is completed, the internal energy can escape from the shell and the acceleration process is damped. On the other hand, if two shells collide when still opaque, they give rise to a single shell that is reaccelerated to an intermediate Lorentz factor. The transparency radius corresponds to:

$$R_t = \left( \frac{M_i \sigma_T}{4 \pi m_{mp}} \right)^{1/2} \approx 5.9 \times 10^{12} E_{52}^{1/2} T_{1,2}^{-1/2} N_{(8,2)}^{-1/2} \text{ cm} \quad (5)$$

3 COMPARING THE RADII

Fig. 1 shows a comparison of the four critical radii for a typical burst with energy $E = 10^{52}$ erg and duration $T = 10$ s. The ISM has been assumed uniform with density $n = 1$ cm$^{-3}$. Two cases have been considered: the left panel refers to a burst made by a (relatively) small number of shells ($N_S = 100$), while for the right panel $N_S = 25000$. The smaller value has been derived assuming that each of the burst pulses fitted by Norris et al. (1996) is produced by the interaction of a shell pair. Norris et al. (1996) find a number of pulses $N_p \lesssim 50$, and hence we have $N_S = 2 N_p \approx 100$. In the latter case, instead, the central engine emits shells at regular time intervals, equally spaced by the smallest variability time-scale observed in GRB light-curves (200µs, Scharer & Walker 1999). In this situation each of the Norris et al. (1996) pulses is considered as a blend of sub-pulses, with an envelope corresponding to the varying efficiency of the central engine: indeed, Walker, Scharer & Fenimore (1999) have shown that millisecond variability on top of the larger time-scale modulation is a common feature of GRB light-curves.

In both cases the maximum value of $\langle \Gamma \rangle$ is bound by the requirement that internal shocks must happen before external ones, while the minimum value is constrained by the transparency condition. This second constraint is less severe since, as already mentioned, collisions between opaque shells would simply cause a reacceleration of the merged shell. Note however that this could play a role in preventing the presence of very slow shells outside the transparency radius $R_t$. It should be also stressed that numerically the limit given by the transparency requirement is similar to the minimum value $\Gamma \gtrsim 100$, already obtained from the compactness argument (e.g. Piran et al. 1996). The acceleration radius does not impose any significant constraint, showing that from the kinematic and radiative points of view, shells could be potentially accelerated even to very high Lorentz factors $\Gamma \sim 10^5$.

Here and in the following we parameterize a quantity $Q$ as $Q = 10^n Q_n$ and adopt CGS units.

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From the condition $R_I < R_D$ we get an upper limit:
\[
\Gamma_{\text{lim}} = 800 \frac{T_1^{3/8} \rho_2^{1/8} n_0^{-1/8}}{P_5^{1/4} n_{-3}^{1/4}}.
\] (6)
This limit is quite robust with respect to $E$ and $n$ while it is more dependent of $T$ and $N_5$. A variation of three orders of magnitude of the total energy changes it by a factor $\sim$ 3 only.

The density of the ISM has been assumed uniform and $\sim$ one proton cm$^{-3}$. This is probably a lower limit. However, even in the case of the hypernova scenario (Paczynski 1998), in which GRB would occur in a much denser environment ($n \sim 10^4$ cm$^{-3}$), $\Gamma_{\text{lim}}$ decreases only by a factor of $\sim$ 3. Finally, the total burst duration of 10 seconds corresponds to the mean T90 parameter of the long GRBs, and is appropriate for the majority of bursts. Its small uncertainty influences $\Gamma_{\text{lim}}$ rather weakly.

4 EFFICIENCY OF INTERNAL SHOCKS

The efficiency of internal shocks in converting the bulk outflow energy into internal energy can be easily estimated since each collision satisfies energy and momentum conservation (see also Kobayashi et al. 1997). Consider two shells of rest masses $m_1$ and $m_2$ and Lorenz factors $\Gamma_1$ and $\Gamma_2$ ($\Gamma_2 > \Gamma_1$), respectively. Calling $\epsilon$ the internal (random) energy of the merged shell after the interaction, we have:

\[
\begin{align*}
\Gamma_1 m_1 + \Gamma_2 m_2 &= \Gamma_f \left(m_1 + m_2 + \epsilon/c^2\right) \\
\Gamma_1 \beta_1 m_1 + \Gamma_2 \beta_2 m_2 &= \Gamma_f \beta_f \left(m_1 + m_2 + \epsilon/c^2\right),
\end{align*}
\] (7)

where the subscript $f$ refers to quantities after the interaction and $\Gamma = (1 - \beta^2)^{-1/2}$. If we assume that all of the internal energy is converted into radiation, we obtain - independently of the emission mechanism - an upper limit for the efficiency $\eta = \epsilon/(\Gamma_1 m_1 + \Gamma_2 m_2)$. From the conservation equations an implicit solution for the final bulk Lorentz factor of the merged shells and the maximum radiative efficiency can then be derived:

\[
\begin{align*}
\beta_f &= \frac{\beta_1 + \alpha_\Gamma \alpha_m \beta_2}{1 + \alpha_\Gamma \alpha_m} \\
\eta &= 1 - \frac{\Gamma_f (1 + \alpha_m)}{\Gamma_1 (1 + \alpha_\Gamma \alpha_m)}.
\end{align*}
\] (8)

where $\alpha_m = m_2/m_1$. The above relations give an upper limit to the fraction of energy that can be radiated in photons. In fact, strictly speaking, $\eta$ is the fraction of bulk kinetic energy converted into internal energy. In the standard synchrotron shock model, this random energy is equally shared among protons, electrons and magnetic field, and only one third of this energy (the fraction going to electrons) can be radiated (see e.g. Panaitescu et al. 1999). The major features of Eq. 7 are that for a large difference in the Lorentz factors ($\alpha_\Gamma \gg 1$) the efficiency can approach unity and that for a fixed value of $\alpha_\Gamma$ the maximum efficiency is reached when $m_1 = m_2$.

For $\Gamma_1, \Gamma_2 \gg 1$, as for internal shocks, the expression for $\eta$ reduces to:

\[
\eta \simeq 1 - \frac{1 + \alpha_m}{\sqrt{1 + \alpha_m} \left(\alpha_m + \alpha_\Gamma + 1/\alpha_\Gamma\right)}.
\] (9)

An even simpler relation is found if the two shells have the same total energy (i.e. $\alpha_\Gamma = 1/\alpha_m$):

\[
\eta \simeq 1 - \frac{1 + \alpha_m}{\sqrt{2 + 2/\alpha_m}}.
\] (10)

In this latter case the efficiency is always lower than 30%, independently of $\alpha_\Gamma$. Fig. 2 shows the efficiency for low–intermediate values of $\alpha_\Gamma$ in two limits: the most efficient situation ($m_1 = m_2$, dashed line) and the equal energy case ($\Gamma_1 m_1 = \Gamma_2 m_2$, solid line). The limit $\Gamma_1, \Gamma_2 \gg 1$ has been assumed. Note that, in this case, the efficiency does not depend on the value of $\Gamma_1$. 

**Figure 1.** Value of the four relevant radii as a function of the bulk average Lorentz factor ($\Gamma$) in a burst with energy $E = 10^{52}$ erg which lasts 10 seconds that produces 100 shells (left panel) and 25000 shells (right panel). The ISM has been assumed uniform with density $n = 1$ cm$^{-3}$. The shaded areas correspond to the interval of ($\Gamma$) in which the four radii are correctly nested. If ($\Gamma$) < 35 (100 in the right panel) the shells interact before becoming transparent and cannot emit the observed thin spectrum. If instead ($\Gamma$) > 750 (2700 in the right panel) the shells are decelerated by the external medium before the mutual interaction occurs, producing an external and not an internal shock.
Let us now consider the global efficiency of the burst. In the 'standard' internal shock scenario (Rees & Mészáros 1994) the shells have a distribution of Lorentz factors with mean value (Γ) and width ΔΓ ∼ ⟨Γ⟩. This in turn corresponds to a distribution of αΓ with ⟨αΓ⟩ ∼ 2 within a factor of order unity (which accounts for the particular shape of the distribution). This implies (see Fig. 2) η ≃ 5%, independently of the mass ratio αm. A numerical simulation for a log-normal distribution of Γ with ⟨Γ⟩ ≃ ⟨Γ⟩ gives η = 6.8%.

These (upper limits on) efficiencies are problematic since, in the absence of extreme beaming, the total energy release of the most powerful GRBs would exceed by some orders of magnitude the maximum energy release achievable by current models, that involve a stellar mass black hole as the energy source. If, e.g., GRB 990123 had an efficiency of 5%, the required energy would be Eiso ≃ 5 × 10^{55} erg. An extremely narrow beaming angle θ ≃ 0.5° would be then needed to reduce the energy release to the value Eiso = 10^{55} erg. A higher efficiency can be only obtained by allowing for a broader distribution of relativistic factors Γ and imposing that all shells have roughly the same mass, somehow requiring a finely tuned variable engine. Moreover, a very broad distribution of Lorentz factors implies that the Compton drag effect becomes crucial, as described in the following section.

4.1 Compton drag

Let us assume that the emission produced in the interaction between two shells is isotropic in the shell comoving frame K′. Consider also a third shell, faster than the previous two, with Lorentz factor Γ′ in that frame. This moves in the radiation bath of total energy E′ ph produced in the interaction, and thus Compton scatters a fraction τΓ′ of the photons, increasing their energy by a factor Γ′2 (here τΓ′ is the Thomson optical depth of the fast shell). The total energy lost by the fast shell in the frame K′ is hence τΓ′ (Γ′2 − 1)E′ ph.

If τΓ′ (Γ′2 − 1) approaches unity, there are two major consequences: first a significant part of the energy of photons is due to the inverse Compton mechanism rather than to the internal shock dissipation process; second, the fast shell looses a significant fraction of its bulk kinetic energy to up-scatter the primary photons and this causes a braking of the fast shell (in the frame K′). Thus, even if the original distribution of Γ factors were very broad, the early interactions would give rise to an efficient inverse Compton drag on the faster shells, reducing the mean value of αΓ in the successive collisions and hence the global efficiency of internal shocks. Moreover, in this case the net energy produced through inverse Compton scattering would be equal or even larger than that directly due to shocks, and the primary emission mechanism would be different. We plan to investigate this issue in a forthcoming paper.

5 HIERARCHICAL INTERNAL SHOCKS

The low efficiency predicted by Eq. 9 refers to a single interaction between two shells. However, each time this interaction takes place, a new shell is formed which can in turn catch up (or be caught by) another one. If the i-th interaction has efficiency ηi, after N of them the fraction of the initial bulk energy converted into photons will be:

\[ η_N = \frac{N}{i=1} η_i (1 - η_{i-1}) ... (1 - η_1) \frac{η_{i-1}}{η_1} η \sum_{i=1}^{N} (1 - η)^{i-1} \] (11)

If the efficiency is constant (ηi = η) and N is arbitrarily large, all the energy of the outflow can be converted into radiation, as η∞ = 1 for any value of η. For the typical efficiency η = 6.8% derived above, about 9 interactions are needed to obtain an overall 50% efficiency. Assuming that,
as sketched in Fig. 3, the collision ‘tree’ develops as a binary bifurcation, the number $N$ of shells that has to merge into a single one after $\sim 9$ interactions is $N = 2^9 = 512$.

This result has two main consequences. Firstly, it is unlikely that a burst with high efficiency is produced by a small ($< 100$) number of shells: a scenario in which the inner engine emits a large number of shells is thus favored. Secondly, in deriving $\Gamma_{\text{lim}}$ (Eq. 6) we assumed that a single interaction was enough to power a GRB. However, within the hierarchical scenario, a shell must interact with $N$ other shells and hence Eq. 6 must be modified as:

$$\Gamma_{\text{lim}} = 800 \frac{T_{1}^{3/8}}{E_{52}^{1/8}} n_{0}^{-1/8} N_{S,4}^{1/4} S_{2}^{-1/4}$$

which gives $\Gamma_{\text{lim}} \simeq 650$ for - say - $N_{S} = 25000$ and $N = 512$. Even if this is an illustrative value only (the actual number depending on the details of the initial distribution of the Lorentz factors), we conclude that values of $\langle \Gamma \rangle$ larger than $\sim 1000$ are unlikely.

6 DISCUSSION

We have analyzed the kinematic efficiency of internal shocks in a fireball made of many individual shells. Since the radius at which the internal shocks set up grows with the average Lorentz factor of the flow, while the development of an external shock is favored by a higher relativistic motion, we can put limits on the average Lorentz factor $\langle \Gamma \rangle$ of the flow if - as inferred from observations - internal shocks develop before the external one. Moreover (see also Kobayashi et al. 1997) we conclude that a simple internal shock scenario in which each shell is caught up by a single faster shell, suffers from a very low radiative efficiency, unless extremely different Lorentz factors are involved. These would however cause a dramatic Compton drag effect, which in turn constrains the possibility of a very broad $\Gamma$ distribution. A possible solution is to have a flow with moderately relativistic Lorentz factors $100 \lesssim \langle \Gamma \rangle \lesssim 600$ in which collisions develop until a single (or a few) massive shell is formed. For this to happen the shells have to be ejected at small time intervals, of the order of milliseconds. In this case each burst is made of more than a thousand spikes that merge to produce the broad peaks often observed in GRB lightcurves. These peaks would then reflect a modulation of the hidden central engine that powers the burst, rather than a single collision.

It is interesting to ask whether an outflow with significantly higher Lorentz factors produces a burst or not. In a flow with very high $\langle \Gamma \rangle$, the internal shocks set-up at a larger radius, and hence a hierarchical internal shock cannot develop. Since the efficiency is roughly proportional to the number of shocks a shell undergoes, a higher bulk gamma is linked to a lower efficiency. Observationally, this implies that the ratio between burst and afterglow fluences is a function of $\langle \Gamma \rangle$, being larger for smaller $\langle \Gamma \rangle$. A further consequence is that very short (millisecond) bursts should be characterized by smaller efficiencies and brighter afterglows. To date, this can be only a prediction since the BeppoSAX trigger works only for long burst.

As a final note, following the explosion of the bright burst GRB 990123, two values of $\langle \Gamma \rangle$ have been derived. Sari & Piran (1999) obtain $\langle \Gamma \rangle \sim 200$ from the optical flash intensity while Liang et al. (1999) estimate $\langle \Gamma \rangle \sim 350$ from the lightcurve properties. Despite the differences, both values are in agreement with the general limits derived in this letter.

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