The energetics of crystallizing white dwarfs revisited again

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ABSTRACT

The evolution of white dwarfs is a cooling process that depends on the energy stored in the core and on the way in which it is transferred through the envelope. In this paper we show that despite some (erroneous) claims, the redistribution of chemical elements ensuing the crystallization of C/O white dwarfs provides between the 10% and the 20% of the total energy released during the crystallization process, depending on the internal chemical composition, which is not negligible at all, given the present state of the art of the white dwarf cooling theory.

Subject headings: stars: interiors — white dwarfs.
1. Introduction

The evolution of white dwarfs is essentially a cooling process. From this point of view, the star can be described as an energy reservoir (the inner core) surrounded by an opaque envelope (the outer partially degenerate layers) that controls the rate of cooling (Mestel, 1952). It is clear that an accurate description of the process demands a good knowledge of all the sources of energy as well as of the “transparency degree” of the atmosphere.

One of the possible sources of energy is related with solidification. Crystallization not only induces a release of latent heat (Van Horn, 1968; Lamb & Van Horn, 1975) but also a release of gravitational energy associated with a change of chemical composition during the phase transition (Stevenson, 1980; Mochkovitch, 1983; Isern et al., 1991; Xu & Van Horn, 1992). Although the importance of solidification of alloys was recognized long ago in geophysical circles, this is not yet the case in astrophysics and some misleading concepts and formulae are still appearing in the literature. For instance, it has been recently claimed (Hansen, 1999) that the amount of energy due to chemical differentiation at crystallization in white dwarf interiors is negligible, and much smaller than the latent heat contribution. This claim, that is incorrect, stems from an erroneous estimation of this differentiation contribution.

There are several factors that influence the computed delay introduced by the change of the chemical composition, namely: i) the phase diagram of the C/O binary mixture, ii) the chemical profile at the beginning of the white dwarf phase, and iii) the transparency of the envelope. Obviously, any change in any one of these inputs directly translates into a change of the computed delay, but the underlying physical picture is the same and the role played by this phenomenon has to be clarified if white dwarfs have to be used as effective tools for deriving important information about the history of the Galaxy.

In this paper we show how to correctly compute the release of gravitational energy
associated to solidification (within the current limits of our knowledge of the physics of Coulomb plasmas) and we discuss the importance of this process for different assumptions about white dwarfs.

2. The amount of energy released

The local energy budget of the white dwarf can be written as (Isern et al., 1997):

\[-\frac{dL_t}{dm} + \epsilon_\nu = C_v \frac{dT}{dt} + \frac{dV}{dt} \frac{\partial P}{\partial T} v, n_0 \frac{dM_s}{dt} \delta(m - M_s) + \frac{\partial E}{\partial X_0} \frac{dX_0}{dt} (1)\]

where \(E\) is the internal energy per unit mass, \(V = 1/\rho\), \(l_s\) is the latent heat of crystallization and \(M_s\) is the rate at which the solid core grows; the delta function indicates that the latent heat is released at the solidification front. We are assuming that the white dwarf is made of two chemical species with atomic numbers \(Z_0\) and \(Z_1\), mass numbers \(A_0\) and \(A_1\) and abundances by mass \(X_0\) and \(X_1\), respectively (\(X_0 + X_1 = 1\), and the suffix 0 refers to the heaviest element). The chemical differentiation contributes to the luminosity not only through compressional work, which is almost negligible during the crystallization phase, but also through energy due to the change in the chemical abundances or, from a first-principle point of view, to a variation of the chemical potentials, which leads to the last term of this equation (Chabrier, 1997). It is worthwhile to notice here that when the star has exhausted all the internal sources of energy, the compression of the outer layers prevents the star to disappear to undetectable luminosities (D’Antona & Mazzitelli, 1990; Isern et al., 1997, 1998) and that a rigorous treatment of these layers is crucial to understand the properties of very cool white dwarfs.

A useful way to include the energetics of crystallization in numerical codes can be derived, under the assumption that the cooling process is slow enough, which is almost always the case. Then, a rate of release of energy per gram of crystallized matter, \(\epsilon_g\), related
to the change in chemical composition, can be defined as follows (see equation (12) in Isern et al., 1997):

\[ \epsilon_g = -\Delta X_0 \left[ \left( \frac{\partial E}{\partial X_0} \right)_{M_s} - \left\langle \frac{\partial E}{\partial X_0} \right\rangle \right] \]  

(2)

where \( \Delta X_0 = X_0^{\text{sol}} - X_0^{\text{liq}} \). The first term represents the energy released in the layer that is crystallizing, as a consequence of the increase of concentration of the heaviest species, whereas the second term, in angular brackets, represents the energy that is absorbed in average in the convective layer that appears just above the crystallization front, as a consequence of the decrease of concentration of the heaviest species.

In order to compute \((\partial E/\partial X_0)_{V,T}\), we should take into account that the internal energy per unit mass can be divided into the electronic and the ionic components. Although in Salaris et al. (1997) we used the complete expressions, here, for the sake of simplicity, we are only going to use the completely degenerate nonrelativistic expression for the electrons

\[ E_e = \frac{3}{2} K_1 \rho^{2/3} Y_e^{5/3} \]  

(3)

where \( K_1 = 1.004 \times 10^{13} \) (cgs units), and the ideal contribution plus the Madelung term of the Coulomb energy for the ions

\[ E_i \simeq E_{\text{id}} + E_{\text{Mad}} \simeq \frac{3}{2} K_1 \rho^{2/3} Y_e^{5/3} \]  

(4)

where \( \mathcal{R} \) is the gas constant and \( \Gamma = \Gamma_e Z_e^{5/3} \) is the Coulomb coupling constant, with \( \Gamma_e = 2.272 \times 10^5 (\rho Y_e)^{1/3}/T \) and \( Z_e^{5/3} = \mu [X_0 Z_0^{5/3}/A_0 + (1 - X_0) Z_1^{5/3}/A_1] \), being \( \mu \) the mean molecular weight per ion. Therefore, the derivative \((\partial E/\partial X_0)_{T,V}\) is the sum of three contributions:

\[ \left( \frac{\partial E_e}{\partial X_0} \right)_{T,V} = \frac{5}{2} K_1 \rho^{2/3} Y_e^{2/3} (Z_0/A_0 - Z_1/A_1) \]  

(5)

\[ \left( \frac{\partial E_{\text{id}}}{\partial X_0} \right)_{T,V} = \frac{3}{2} \mathcal{R} T \left( \frac{1}{A_0} - \frac{1}{A_1} \right) \]  

(6)
\[
\frac{\partial E_{\text{Mad}}}{\partial X_0}_{T,V} = -0.9\mathcal{RT}_e \left[ \frac{1}{3Y_e} \left( \frac{Z_0}{A_0} - \frac{Z_1}{A_1} \right) \left( \frac{Z_0^{5/3}}{A_0} - \frac{Z_1^{5/3}}{A_1} \right) \right] + \left( \frac{Z_0^{5/3}}{A_0} - \frac{Z_1^{5/3}}{A_1} \right) \]  
\]  

(7)

Since the contributions to the electron mole number, \( Y_e \), of carbon and oxygen are very similar, the derivative \( \frac{\partial E}{\partial X_0}_{T,V} \) is dominated by the ionic contribution (and in particular by the second term of the Madelung term). It is important to remind that the electronic term (5) is not negligible when neutron-rich species are considered (i.e. \(^{22}\text{Ne}, ^{56}\text{Fe}\)). In the case of C/O mixtures, we can retain only the second term in (7):

\[
\frac{\partial E}{\partial X_0}_{T,V} \simeq -0.9\mathcal{RT}_e \left( \frac{Z_0^{5/3}}{A_0} - \frac{Z_1^{5/3}}{A_1} \right) \]  
\]  

(8)

This expression can be directly substituted in equation (1), which is well suited for numerical codes. Note as well that this term when substituted in equation (1) is local and, therefore, generally speaking, can be considered as a source or a sink of energy, depending on the local change of chemical composition.

In order to guess the importance of the process of redistribution related to crystallization, we can compare with the latent heat \( l_s \sim k_B T \) per ion (Hansen, 1999). In this case, it is convenient to deal with the energy released per ion \( \Delta u \):

\[
\frac{\Delta u}{k_BT} \simeq 0.9\mu\mathcal{R}_e \left( \frac{Z_0^{5/3}}{A_0} - \frac{Z_1^{5/3}}{A_1} \right) \Delta X_0 \simeq \frac{\Delta u}{l_s} \]  
\]  

(9)

Note that equations (8) and (9) are exactly the same as equations (7) and (8) of Chabrier (1997), where the energy is given explicitly per unit mass, which implies the factor \( \mu \) in the ratio \( \Delta u/k_BT \). Typically, \( \Delta X_0 = 0.2 \), leading to \( \Delta u/l_s = 6 \), which is different from the 0.3 value quoted by Hansen (1999). The reason of this large difference is that in equation (1) of Hansen (1999), a factor \( \mu \) is missing. Therefore, the energy released by the separation of
carbon and oxygen is underestimated in Hansen (1999) by more than an order of magnitude, leading to erroneous conclusions.

Regarding the phase diagram of a C/O mixture, our first estimates were based on the phase diagram of Stevenson (1980) that predicted complete immiscibility of carbon and oxygen in the solid phase. Using equation (2) and this phase diagram, the energy released by the chemical differentiation was \( \Delta E_g = 6.6 \times 10^{46} \) erg in the case of a 0.606 \( M_\odot \) white dwarf with a flat initial chemical profile \( (X_C = X_O = 0.5) \). This value has to be compared with the total energy released during the crystallization phase — which is equal to the difference \( \Delta B \) of the binding energies — when chemical differentiation is neglected, \( \Delta B = 7.2 \times 10^{46} \) erg. Since then, the advances in the physics of dense plasmas led to a better understanding of the phase transition and the corresponding diagram turned out to be of the spindle form. Consequently, the degree of chemical separation was strongly reduced (Segretain et al., 1994). For the white dwarf with the aforementioned characteristics \( \Delta E_g = 2.3 \times 10^{46} \) erg is obtained, while the latent heat is \( \Delta E_l = 1.8 \times 10^{46} \) erg, which means that this effect is still significant. Since the latent heat has been computed assuming \( l_s \sim k_BT \) per ion, the quoted value is, nevertheless, a gross estimate.

The initial chemical profile plays also a very important role (Hernanz et al., 1994; Segretain et al., 1994; Salaris et al., 1997). For instance, if a high effective rate of the \(^{12}\text{C}(\alpha, \gamma)^{16}\text{O} \) reaction is adopted, the abundance of oxygen in the central layers is as high as 0.74 by mass in the case of a 0.606 \( M_\odot \) and the degree of mixing in the liquid layers is strongly reduced by the very steep gradients of chemical composition that appear. In this case, the total gravitational energy release upon crystallization obtained using equation (2) is \( \Delta E_g = 1.1 \times 10^{46} \) erg, which is still not negligible since it is of the same order as the latent heat. It is important to stress here that the complete version of equation (2) has to be used since the term in angular brackets, which depends on the degree of mixing of
the liquid region, reduces the net gravitational energy released by the process. Moreover, the total energy released during the crystallization phase for the same white dwarf without introducing any simplification in the equation of state is $\Delta B = 9.6 \times 10^{46}$ erg when phase separation is included and $\Delta B = 8.6 \times 10^{46}$ erg when phase separation is neglected. The difference is caused by the contribution of the chemical redistribution process and confirms the values obtained above. Notice also that the chemical profiles adopted here are those that predict the maximum abundances of oxygen in the center and, consequently, minimize the effects of phase separation.

3. The time delay in the cooling process

According to equation (38) in Isern et al. (1997) or (11) in Chabrier (1997) the time delay in the cooling sequences can be expressed as

$$\Delta t = \int_{0}^{M_{\text{WD}}} \frac{\epsilon_g(T_c)}{L(T_c)} dm$$

(10)

where $T_c(m)$ is the core temperature when the crystallization front is located at mass $m$ and $L$ is its corresponding luminosity. From this equation it is obvious that any change in the transparency of the envelope of the white dwarf and, thus, in the relationship between the luminosity and the core temperature, directly translates into the delay introduced by phase separation in the cooling times. In order to see the influence of the envelope in the computed time delays we have adopted the following white dwarf model atmospheres. Our first model atmosphere has been obtained from the DA model sequence of Wood (1995), which has a mass fraction of the helium layer of $q_{\text{He}} = 10^{-4}$ and a hydrogen layer of $q_{\text{H}} = 10^{-2}$; the second model atmosphere is the non-DA model sequence of Wood & Winget (1989) which has a helium layer of $q_{\text{He}} = 10^{-4}$. However it should be noted that between these two model sequences there was a substantial change in the opacities, and therefore the comparison is meaningless (i.e., the non-DA model is more opaque than the
DA one). The remaining two model atmospheres are those of Hansen (1999) for both DA and non-DA white dwarfs. These atmospheres have been computed with state of the art physical inputs for both the equation of state and the opacities for the range of densities and temperatures relevant for white dwarf envelopes (although it should be mentioned that the contributions to the opacity of H$^3+$ and H$^2+$ ions were neglected in this calculation) and have the same hydrogen and helium layer mass fractions as those of Wood (1995) and Wood & Winget (1989), respectively. For the sake of conciseness we will refer to the cooling sequences obtained with these two sets of envelopes as “MW” and “BH”, respectively. In all the cases, the mass of the white dwarf is 0.606 $M_\odot$ and the initial chemical profile of the C/O mixture is that of Salaris et al. (1997).

EDITOR: PLACE FIGURE 1 HERE.

It is interesting to compare the core temperature-luminosity relationship for both sets of sequences. This is done in figure 1 where it can be seen that the DA model atmospheres MW and BH are in very good agreement down to temperatures of the order of $\log(T_c) \simeq 6.5$, whereas at lower temperatures the model atmospheres BH predict significantly lower luminosities (that is, they are less transparent). This is undoubtedly due to the significant improvement in cool white dwarf atmosphere calculations done in Hansen (1999). In contrast, the non-DA model atmosphere BH is by far more transparent at any temperature than the corresponding MW model. This is clearly due to the fact that the Wood & Winget (1989) $L-T_c$ relation was based on the old Los Alamos opacities which include a finite contribution from metals whereas Hansen’s non-DA are pure helium. In any event, it is straightforward from our equation (10) to realize why the computed time delays are very different in absolute value for the model cooling sequences reported here, as was clearly stated in Isern et al. (1997).
With these two sets of model atmospheres we have computed the cooling sequences for the following two cases: 1) crystallization and no phase separation and 2) crystallization and phase separation. The results are shown in figure 2 and table 1. The left panel of figure 2 shows the cooling sequences for the non-DA model envelopes BH (solid lines) and MW (dotted lines). The sequences with phase separation (BH2 and MW2, respectively) correspond, obviously, to the cooling curves with larger cooling times for the same luminosity. The right panel of figure 2 shows the same set of calculations for hydrogen-dominated white dwarf envelopes. We emphasize that these figures highlight the relative time delays due to chemical differentiation at crystallization which are obtained with either Wood (1995), Wood & Winget (1989) or Hansen (1999) effective temperature-interior temperature relationships.

Finally in table 1 we show the cooling times (in Gyr) for the eight cooling sequences described above at the approximate position of the observed turn-off of the white dwarf luminosity function, \( \log(L/L_\odot) \approx -4.5 \). From a superficial analysis of the data shown in table 1 it can be concluded that although the absolute value of the delay introduced by chemical segregation changes appreciably due to the transparency of the envelope, the relative contribution remains roughly constant and of the order of 10% for \( \log(L/L_\odot) \approx -4.5 \), in agreement with our previous results, which is not at all negligible even in an astrophysical context. This behavior is easily explained by the fact that the lifetime of white dwarfs in the cut-off of the white dwarf luminosity function is dominated by the crystallization phase and any change in the luminosity has the same influence on all the contributing sources of energy. It is worth mentioning that these results, similar to the ones
obtained in Segretain et al. (1994) and Hernanz et al. (1994), have been confirmed recently by Montgomery et al. (1999) who computed white dwarf cooling sequences, including the afore-described complete treatment of crystallization, with a complete stellar evolutionary code. Finally, it should be mentioned as well that if a flat initial profile is assumed for the C/O mixture, then the delay for a 0.606 $M_\odot$ white dwarf amounts to 1.8 Gyr when the DA envelope of Hansen (1999) is adopted (i.e., a $\sim$20% effect), and 0.8 Gyr when the non–DA envelope is assumed.

4. Conclusions

We have derived the energy release due to the chemical redistribution associated to the process of crystallization of C/O white dwarfs. We reassert that this contribution cannot be by any means neglected, in contrast with the recent calculations of Hansen (1999). We have demonstrated that the reason of this discrepancy stems from an incorrect calculation of the aforementioned energy, due to a missing (molecular weight) factor. This led Hansen (1999) to claim that the bulk of the correction in the cooling times introduced by the redistribution of C/O upon crystallization was just the consequence of using an inaccurate envelope model. Although the importance of a correct envelope model to determine absolute ages is certainly crucial, meaningful comparisons of the different energy sources can only be done on the basis of relative contributions for the same kind of models (that is, the same envelopes). It is evident that any increase of the transparency of the atmosphere will globally reduce the time scales of white dwarf cooling, including the delay introduced by crystallization. When meaningful comparisons are done, we have shown that the contribution of chemical segregation to the duration of the crystallization phase is at least equivalent to the contribution of the latent heat itself and strongly depends on the initial chemical profile of the white dwarf. Therefore, the decision of neglecting or not this
phenomenon only depends on the demanded degree of accuracy in the models. For instance, should we want to determine the age of the Galaxy or the ages of globular clusters with a precision of tenths of Gyr this effect should be included.

Finally, it should be mentioned as well that the chemical profiles of Salaris et al. (1997) used here show a notoriously high abundance of oxygen in the central layers, which is caused by an enhanced $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$ effective reaction rate, which is still the subject of a strong debate. This, of course, minimizes the effect of chemical differentiation upon crystallization.

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REFERENCES


Fig. 1.— The relationships between the core temperature and the luminosity for the various model atmospheres adopted in this work, see text for details.

Fig. 2.— Cooling curves (time is in Gyr) for the white dwarf models described in the text and in Table 1, the thinner horizontal line corresponds to \( \log(L/L_\odot) = -4.5 \), which is the approximate position of the observed cut-off of the white dwarf luminosity function.
Table 1. Cooling times (in Gyr) at $\log(L/L_\odot) = -4.5$ for the eight models studied in this paper. See text for details.

<table>
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<th>non-DA</th>
<th>Model</th>
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