Scalar Field Theory in the AdS/CFT Correspondence Revisited

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Abstract

We consider the role of boundary conditions in the $AdS_{d+1}/CFT_d$ correspondence for the scalar field theory. Also a careful analysis of some limiting cases is presented. We study three possible types of boundary conditions, Dirichlet, Neumann and mixed. We compute the two-point functions of the conformal operators on the boundary for each type of boundary condition. We show how particular choices of the mass require different treatments. In the Dirichlet case we find that there is no double zero in the two-point function of the operator with conformal dimension $\frac{d}{2}$. The Neumann case leads to new multiplicative coefficients for the boundary two-point functions. In the massless case we show that the conformal dimension of the boundary conformal operator is precisely the unitarity bound for scalar operators. We find a one parameter family of boundary conditions in the mixed case. There are again new multiplicative coefficients for the boundary two-point functions. For a particular choice of the mixed boundary condition and with the mass squared in the range $-d^2/4 < m^2 < -d^2/4 + 1$ the boundary operator has conformal dimension comprised in the interval $\left[\frac{d^2}{4} - 1, \frac{d^2}{4}\right]$. For mass squared $m^2 > -d^2/4 + 1$ the same choice of mixed boundary condition leads to a boundary operator whose conformal dimension is the unitarity bound.

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1 Introduction

Since the proposal of Maldacena’s conjecture, which gives a correspondence between a field theory on anti-de Sitter space (AdS) and a conformal field theory (CFT) on its boundary [1], an intensive work has been devoted to get a deeper understanding of its implications. In particular, a precise form to the conjecture has been given in [2][3]. It reads

\[
Z_{\text{AdS}}[\phi_0] = \int_{\phi_0} D\phi \exp(-I[\phi]) \equiv Z_{\text{CFT}}[\phi_0] = \left< \exp \left( \int_{\partial\Omega} d^dx O \phi_0 \right) \right>,
\]

where \(\phi_0\) is the boundary value of the bulk field \(\phi\) which couples to the boundary CFT operator \(O\). This allows us to obtain the correlation functions of the boundary CFT theory in \(d\) dimensions by calculating the partition function on the AdS\(_{d+1}\) side. The AdS/CFT correspondence has been studied for the scalar field [3][4][5][6], the vector field [3][5][7][8], the spinor field [7][9][10], the Rarita-Schwinger field [11][12][13], the graviton field [14][15], the massive symmetric tensor field [16] and the antisymmetric \(p\)-form field [17][18]. In all cases Dirichlet boundary conditions were used. Several subtle points have been clarified in these papers and all results lend support to the conjecture.

In a broader sense Maldacena’s conjecture is a concrete realization of the holographic principle [19][20]. We expect that any field theory relationship in AdS space must be reflected in the border CFT. An example of this is the well known equivalence between Maxwell-Chern-Simons theory and the self-dual model in three dimensional Minkowski space [21]. This equivalence holds also in AdS\(_3\) and using the AdS/CFT correspondence we have shown that the corresponding boundary operators have the same conformal dimensions [8]. Another situation involves massive scalar fields in AdS spaces. If the scalar field has mass-squared in the range \(-d^2/4 < m^2 < -d^2/4 + 1\) then there are two possible quantum field theories in the bulk [22]. The AdS/CFT correspondence with Dirichlet boundary condition can easily account for one of the theories. The other one appears in a very subtle way by identifying a conjugate field through a Legendre transform as the source of the boundary conformal operator [23].

Since a field theory is determined not only by its Lagrangian but also by its boundary terms in the action we expect that the AdS/CFT correspondence must be sensitive to these boundary terms. This is easily seen to be true by computing the left-hand side of Eq.(1) for a classical field configuration. All that is left is a boundary term. If we start with different boundary terms in the action then we obtain different correlation functions on the right-hand side.
The origin of boundary terms in the action is due to the variational principle. In order to have a stationary action boundary terms, which will depend on the choice of the boundary conditions, must be introduced. The importance of these boundary terms for the AdS/CFT correspondence was recognized in the case of spinor fields where the action is of first order in derivatives and the classical action vanishes on-shell [24]. They also played an important role in the case of Chern-Simons theory [8]. Therefore it is crucial to understand the implications of different types of boundary conditions for the same theory since they in general imply different boundary terms.

In this work we will study the role of different types of boundary condition for the scalar field theory. We will consider Dirichlet, Neumann and mixed boundary conditions. Each type of boundary condition requires a different boundary term. We will show that the mixed boundary conditions are parametrized by a real number so that there is a one parameter family of boundary terms consistent with the variational principle.

We will also show that different types of boundary condition give rise to different conformal field theories at the border. For the scalar field this was somehow expected. The two solutions found in [22] correspond to two different choices of energy-momentum tensor. Both of them are conserved and their difference gives a surface contribution to the isometry generators. Although these two solutions were found in the Hamiltonian context by requiring finiteness of the energy they will reappear here by considering different types of boundary condition which amounts to different boundary terms in the action. We can also look for the asymptotic behavior of the scalar field near the boundary according to the chosen type of boundary condition. For the Dirichlet boundary condition it is well known that the scalar field behaves as \( x_0^{d/2 - \sqrt{d^2/4 + m^2}} \) near the border at \( x_0 = 0 \). There is no upper restriction on the mass in this case. It corresponds to one of the solutions found in [22] and gives rise to a boundary conformal operator with conformal dimension \( d/2 + \sqrt{d^2/4 + m^2} \). For a particular choice of mixed boundary condition and when the mass squared is in the range \(-d^2/4 < m^2 < -d^2/4 + 1\) the scalar field behaves as \( x_0^{d/2 + \sqrt{d^2/4 + m^2}} \) near the border. It corresponds precisely to the second solution of [22] and gives rise to a boundary conformal operator with conformal dimension \( d/2 - \sqrt{d^2/4 + m^2} \). Note that the upper limit for the mass squared \(-d^2/4 + 1\) is consistent with the unitarity bound \((d - 2)/2\).

Another important point that we will show is the existence of boundary conditions which give rise to boundary conformal operators for which the unitarity bound \((d - 2)/2\) is reached. They correspond to a massless scalar field with Neumann boundary conditions.
condition or to a scalar field with \( m^2 > -d^2/4 + 1 \) with a particular choice of the mixed boundary condition (the same choice which gives the boundary operator with conformal dimension \( d/2 - \sqrt{d^2/4 + m^2} \)). In this way, using different boundary conditions, we obtain all scalar conformal field theories allowed by the unitarity bound.

We will also analyze carefully two cases where the mass of the scalar field takes special values. In some cases the usual expansion of the modified Bessel functions in powers of \( x_0 \) breaks down and we must use expansions involving logarithms. When \( m^2 = -d^2/4 \) it gives rise to the asymptotic behavior \( x_0^{d/2} \ln x_0 \) and the two-point function is obtained without troubles. This is to be contrasted with the usual limiting procedure where the mass goes to \( m^2 = -d^2/4 \) but the two-point function has a double zero in the limit [23]. The other case corresponds to \( \sqrt{d^2/4 + m^2} \) integer but non-zero. In this case we just reproduce the known results.

We should stress the fact that the use of different types of boundary condition (for given values of \( m^2 \) and \( d \)) allows us in general to get boundary two-point functions with different multiplicative coefficients. This will affect the three-point and higher-point functions. Maybe this is related to the fact that AdS and field theory calculations agree up to some dimension dependent normalization factors [25] but we will not discuss this further.

The paper is organized as follows. In section 2 we find the boundary terms corresponding to each type of boundary condition. In section 3 we consider the Dirichlet case while in Section 4 we treat Neumann boundary conditions. Finally in section 5 we consider mixed boundary conditions. In appendix A we list all boundary two-point functions that we computed and appendix B contains some useful formulae.

## 2 The Variational Principle

We take the usual Euclidean representation of the \( \text{AdS}_{d+1} \) in Poincaré coordinates described by the half space \( x_0 > 0, x_i \in \mathbb{R} \) with metric

\[
ds^2 = \frac{1}{x_0^2} \sum_{\mu=0}^{d} dx^\mu dx^\mu.\]

The action for the massive scalar field theory is given by

\[
I_0 = -\frac{1}{2} \int d^{d+1}x \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right),
\]
and the corresponding equation of motion is
\[
\left( \nabla^2 - m^2 \right) \phi = 0. \tag{4}
\]
The solution which is regular at \( x_0 \to \infty \) reads \[5\]
\[
\phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{-i \vec{k} \cdot \vec{x}} x_0^\frac{d}{2} a(\vec{k}) K_{\nu}(k x_0), \tag{5}
\]
where \( \vec{x} = (x^1, \ldots, x^d) \), \( k = |\vec{k}| \), \( K_{\nu} \) is the modified Bessel function, and
\[
\nu = \sqrt{\frac{d^2}{4} + m^2}. \tag{6}
\]
From Eq.(5) we also get
\[
\partial_0 \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{-i \vec{k} \cdot \vec{x}} x_0^{\frac{d}{2} - 1} a(\vec{k}) \left[ \left( \frac{d}{2} + \nu \right) K_{\nu}(k x_0) - k x_0 K_{\nu+1}(k x_0) \right]. \tag{7}
\]
In order to have a stationary action we must supplement the action \( I_0 \) with a boundary term \( I_S \) which cancels its variation. The appropriate action is then
\[
I = I_0 + I_S. \tag{8}
\]
In order to capture the effect of the Minkowski boundary of the \( AdS_{d+1} \), situated at \( x_0 = 0 \), we first consider a boundary value problem on the boundary surface \( x_0 = \epsilon > 0 \) and then take the limit \( \epsilon \to 0 \) at the very end. Then the variational principle applied to the action \( I \) gives
\[
\delta I = -\int d^{d+1}x \, \epsilon^{-d+1} \partial_0 \phi_\epsilon \, \delta \phi_\epsilon + \delta I_S = 0, \tag{9}
\]
where \( \phi_\epsilon \) and \( \partial_0 \phi_\epsilon \) are the value of the field and its derivate at \( x_0 = \epsilon \) respectively. This equation will be used below to find out the appropriate boundary term \( I_S \) for each type of boundary condition.

For Dirichlet boundary condition the variation of the field at the border vanishes so that the first term in Eq.(9) also vanishes and the usual action \( I_0 \) is already stationary. Making use of the field equation the action \( I \) takes the form
\[
I_D = -\frac{1}{2} \int d^{d+1}x \, \partial_\mu (\sqrt{g} \, \phi \, \partial^\nu \phi) = -\frac{1}{2} \int d^d x \, \epsilon^{-d+1} \phi_\epsilon \partial_0 \phi_\epsilon. \tag{10}
\]
It is to be understood that $\partial_0 \phi_\epsilon$ in Eq.(10) is evaluated in terms of the Dirichlet data $\phi_\epsilon$.

To consider Neumann boundary conditions we first take a unitary vector which is inward normal to the boundary $n^\mu(x_0) = (x_0, 0)$. The Neumann boundary condition then fixes the value of $n^\mu(\epsilon) \partial_\mu \phi_\epsilon \equiv \partial_n \phi_\epsilon$. The boundary term to be added to the action reads

$$I_S = \int d^{d+1} x \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) = \int d^{d} x \epsilon^{-d+1} \phi_\epsilon \partial_0 \phi_\epsilon , \quad (11)$$

so that we find the following expression for the action at the boundary

$$I_N = -\frac{1}{2} d^{d+1} x \epsilon^{-d} \phi_\epsilon \partial_n \phi_\epsilon . \quad (12)$$

Here $\phi_\epsilon$ is to be expressed in terms of the Neumann value $\partial_n \phi_\epsilon$. Notice that the on-shell value of the action with Neumann boundary condition Eq.(12) differs by a sign from the corresponding action with Dirichlet boundary condition Eq.(10).

In the case of mixed boundary conditions we fix the value of

$$\phi(x) + \alpha n^\mu(x_0) \partial_\mu \phi(x) \equiv \psi^\alpha(x) \quad (13)$$

at the border. Here $\alpha$ is an arbitrary real but non-zero coefficient. In this case the surface term to be added to the action is

$$I^\alpha_S = -\frac{\alpha}{2} \int d^{d+1} x \partial_\mu (\sqrt{g} g^{\mu\nu} \phi \partial_\nu \phi) = -\frac{\alpha}{2} \int d^{d} x \epsilon^{-d+2} \partial_0 \phi_\epsilon \partial_0 \phi_\epsilon , \quad (14)$$

and we find the following expression for the action at the boundary

$$I^\alpha_M = -\frac{1}{2} d^{d+1} x \epsilon^{-d+1} \psi^\alpha_\epsilon \partial_0 \phi_\epsilon . \quad (15)$$

Clearly $\partial_0 \phi_\epsilon$ in the above expression must be written in terms of the boundary data $\psi^\alpha_\epsilon$. We then have a one parameter family of surface terms since the variational principle does not impose any condition on $\alpha$. In this way the value of the on-shell action Eq.(15) also depends on $\alpha$.

In the following sections we will consider each boundary condition separately.

3 Dirichlet Boundary Condition

We begin by recalling the main results for the Dirichlet case [4][5]. Let $\phi_\epsilon(\vec{k})$ be the Fourier transform of the Dirichlet boundary value of the field $\phi(x)$. From Eq.(5) we
get

\[ a(\vec{k}) = \frac{\epsilon^{-\frac{d}{2}} \phi_{\epsilon}(\vec{k})}{K_{\nu}(k\epsilon)} , \]  

(16)

and inserting this into Eq.(7) we find

\[ \partial_0 \phi_{\epsilon}(\vec{x}) = \int d^d y \, \phi_{\epsilon}(\vec{y}) \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \epsilon^{-1} \left[ \frac{d}{2} + \nu - k\epsilon \frac{K_{\nu+1}(k\epsilon)}{K_{\nu}(k\epsilon)} \right] . \]  

(17)

Then the action Eq.(10) reads

\[ I_{D} = -\frac{1}{2} \int d^d x \, d^d y \, \phi_{\epsilon}(\vec{x}) \phi_{\epsilon}(\vec{y}) \epsilon^{-d} \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \left[ \frac{d}{2} + \nu - k\epsilon \frac{K_{\nu+1}(k\epsilon)}{K_{\nu}(k\epsilon)} \right] . \]  

(18)

The next step is to keep the relevant terms in the series expansions of the Bessel functions and to integrate in $\vec{k}$. We consider first the case $\nu \neq 0$ that is $m^2 \neq -\frac{d^2}{4}$. For completeness we list the relevant modified Bessel functions in Appendix B. For $\nu$ not integer we make use of Eqs.(90,94), whereas for $\nu$ integer but non-zero we use Eqs.(91,95). In both cases we get the same result

\[ I_{D}^{\nu \neq 0} = -\frac{\nu}{\pi^2} \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(\nu)} \int d^d x \, d^d y \, \phi_{\epsilon}(\vec{x}) \phi_{\epsilon}(\vec{y}) e^{2(\nu - \frac{d}{4})} \frac{1}{|\vec{x} - \vec{y}|^{2(\frac{d}{2} + \nu)}} + \cdots , \]  

(19)

where the dots stand for either contact terms or higher order terms in $\epsilon$.

Since the metric is singular in the border the action is divergent and the limit $\epsilon \to 0$ has to be taken carefully [5]. In order to have a finite action we take the limit

\[ \lim_{\epsilon \to 0} \epsilon^{\nu - \frac{d}{2}} \phi_{\epsilon}(\vec{x}) = \phi_0(\vec{x}) . \]  

(20)

Then we make use of the AdS/CFT equivalence in the form

\[ \exp \left( -I_{AdS} \right) \equiv \left\langle \exp \left( \int d^d x \, \mathcal{O}(\vec{x}) \, \phi_0(\vec{x}) \right) \right\rangle , \]  

(21)

and we find the following two-point function

\[ \left\langle \mathcal{O}_D^{\nu \neq 0}(\vec{x}) \mathcal{O}_D^{\nu \neq 0}(\vec{y}) \right\rangle = \frac{2\nu}{\pi^2} \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(\nu)} \frac{1}{|\vec{x} - \vec{y}|^{2(d + \nu)}} . \]  

(22)
Then the conformal operator $\mathcal{O}^{\nu=0}_D$ on the boundary CFT has conformal dimension $\frac{d}{2} + \nu$. From Eq.(20) we find that near the border $\phi$ behaves as $x_0^{d/2-\nu} \phi_0(\vec{x})$ as expected. In this way we have extended the results of [4][5] to the case $\nu$ integer but non-zero.

For future reference we note that in the particular case $m=0$, that is $\nu = \frac{d}{2}$, Eq.(22) reads

$$\left\langle \mathcal{O}^{\nu=\frac{d}{2}}_D(\vec{x}) \mathcal{O}^{\nu=\frac{d}{2}}_D(\vec{y}) \right\rangle = \frac{d}{\pi^{\frac{d}{2}}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \left\{ \frac{1}{|\vec{x} - \vec{y}|^d} \right\},$$

(23)

so that the operator $\mathcal{O}^{\nu=\frac{d}{2}}_D$ has conformal dimension $d$.

Now we consider the case $\nu = 0$, that is $m^2 = -\frac{d^2}{4}$. Since the two-point function Eq.(22) has a double zero for $\nu = 0$ it was argued [23] that the correct result can be found by introducing a normalization on the boundary operator. Instead we will make use of the expansion for the Bessel function $K_0$. Using Eqs.(91,93) we get

$$k \epsilon K_1(k \epsilon) K_0(k \epsilon) = -\frac{1}{(k \epsilon)^{d/2}} \ln \epsilon \left[ 1 + \frac{(k \epsilon)^2}{2} \ln \epsilon + O(\epsilon^2) \right],$$

(24)

where the dots denote all other terms representing either contact terms in the two-point function or terms of higher order in $\epsilon$. Notice that it is essential to separate the contributions of $k$ and $\epsilon$ in the terms $\ln k \epsilon$ in order to identify the relevant contributions. Substituting in Eq.(18) we find

$$I^{\nu=0}_D = \frac{1}{2} \int d^d x \, d^d y \, \phi_\epsilon(\vec{x}) \phi_\epsilon(\vec{y}) \frac{e^{-d}}{ln^2 \epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|\vec{x} - \vec{y}|^d} \ln k + \cdots.$$ 

(25)

The integration in $k$ is carried out by making use of Eq.(95) yielding

$$I^{\nu=0}_D = -\frac{\Gamma\left(\frac{d}{2}\right)}{4\pi^{\frac{d}{2}}} \int d^d x \, d^d y \, \phi_\epsilon(\vec{x}) \phi_\epsilon(\vec{y}) \frac{e^{-d}}{ln^2 \epsilon} \frac{1}{|\vec{x} - \vec{y}|^d} + \cdots.$$ 

(26)

Now taking the limit

$$\lim_{\epsilon \to 0} (\epsilon^{\frac{d}{2}} \ln \epsilon)^{-1} \phi_\epsilon(\vec{x}) = \phi_0(\vec{x}),$$

(27)

and making use of the AdS/CFT equivalence Eq.(21) we find the following two-point function

$$\left\langle \mathcal{O}^{\nu=0}_D(\vec{x}) \mathcal{O}^{\nu=0}_D(\vec{y}) \right\rangle = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^d}.$$ 

(28)
Then the conformal operator $O^{\nu=0}_D$ on the boundary CFT has conformal dimension $\frac{d}{2}$ as expected. As anticipated in [5] the scalar field approaches the boundary as $x_0^{d/2}\ln x_0 \phi_0(\vec{x})$ due to the logarithm appearing in the expansion of the Bessel function.

4 Neumann Boundary Condition

Using the Neumann boundary condition we get from Eq.(7)

$$a(\vec{k}) = \frac{e^{-\frac{d}{2}} \partial_n \phi_\nu(\vec{k})}{(\frac{d}{2} + \nu)K_\nu(k\epsilon) - k\epsilon K_{\nu+1}(k\epsilon)},$$

and substituting this in Eq.(5) we find

$$\phi_\nu(\vec{x}) = \int d^d y \partial_n \phi_\nu(\vec{y}) \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{1}{\frac{d}{2} + \nu - k\epsilon \frac{K_{\nu+1}(k\epsilon)}{K_\nu(k\epsilon)}}.$$  

Then the action Eq.(12) reads

$$I_N = \frac{1}{2} \int d^d x \ d^d y \partial_n \phi_\nu(\vec{x}) \partial_n \phi_\nu(\vec{y}) \epsilon^{-d} \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{1}{\frac{d}{2} + \nu - k\epsilon \frac{K_{\nu+1}(k\epsilon)}{K_\nu(k\epsilon)}}.$$  

In order to keep the relevant terms in the series expansions of the Bessel functions we must consider the massive and massless cases separately.

In the massless case we have $\nu = \frac{d}{2}$. For $d$ odd we make use of Eq.(90), whereas for $d$ even we use Eq.(91). In both cases we get for $d > 2$

$$\frac{1}{\frac{d}{2} + \nu - k\epsilon \frac{K_{\nu+1}(k\epsilon)}{K_\nu(k\epsilon)}} = -(d - 2)(k\epsilon)^{-2} + \cdots,$$

up to contact terms and higher order terms in $\epsilon$. Substituting this in Eq.(31) we find

$$I_N^{\nu=\frac{d}{2}} = -\frac{d - 2}{2} \int d^d x \ d^d y \partial_n \phi_\nu(\vec{x}) \partial_n \phi_\nu(\vec{y}) \epsilon^{-d-2} \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} k^{-2} + \cdots,$$

and performing the integral in $\vec{k}$ we get

$$I_N^{\nu=\frac{d}{2}} = -\frac{\Gamma\left(\frac{d}{2}\right)}{4\pi^2} \int d^d x \ d^d y \partial_n \phi_\nu(\vec{x}) \partial_n \phi_\nu(\vec{y}) \frac{\epsilon^{-d-2}}{|\vec{x} - \vec{y}|^{d-2}} + \cdots,$$
where the dots stand for either contact terms or higher order terms in $\epsilon$.

Taking the limit
\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{d}{2}-1} \partial_n \phi_{\epsilon}(\vec{x}) = \partial_n \phi_0(\vec{x}),
\]
and making use of the AdS/CFT equivalence of the form
\[
\exp(-I_{AdS}) \equiv \left\langle \exp \left( \int d^d x \ \mathcal{O}(\vec{x}) \ \partial_n \phi_0(\vec{x}) \right) \right\rangle,
\]
we find the following boundary two-point function
\[
\left\langle \mathcal{O}_{\vec{x}}^{\nu-\frac{d}{2}} \right\rangle = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^{2\frac{d}{2}+\nu}}.
\]

Then for $d > 2$, even or odd, the conformal dimension of the operator $\mathcal{O}_{\vec{x}}^{\nu-\frac{d}{2}}$ is precisely the unitarity bound $\frac{d}{2} + \nu$. From Eq.(35) we find that near the border the scalar field goes as $x_0^{d/2+\nu} \partial_n \phi_0(\vec{x})$. Comparing Eqs.(23,37) we see that the conformal dimensions of the boundary operators for the massless Dirichlet and Neumann cases are different and for the later case the unitarity bound is reached.

For the massive scalar field, that is $\nu \neq \frac{d}{2}$, we first consider the case $\nu \neq 0$ i.e. $m^2 \neq -\frac{d^2}{4}$. We have again to consider separately the cases with $\nu$ not integer and $\nu$ integer but non-zero. In both cases we find
\[
I_{\vec{x}}^{\nu-\frac{d}{2}} = -\frac{\nu}{\pi^{\frac{d}{2}}} \frac{1}{\left(\frac{d}{2} - \nu\right)^2} \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(\nu)} \int d^d x \ d^d y \ \partial_n \phi_{\nu}(\vec{x}) \ \partial_n \phi_{\nu}(\vec{y}) \ \frac{\epsilon^{2(\nu-\frac{d}{2})}}{|\vec{x} - \vec{y}|^{2\nu}} + \cdots.
\]

Taking the limit
\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{d}{2}} \ \partial_n \phi_{\epsilon}(\vec{x}) = \partial_n \phi_0(\vec{x}),
\]
and making use of the AdS/CFT equivalence Eq.(36) we find the following boundary two-point function
\[
\left\langle \mathcal{O}_{\vec{x}}^{\nu \neq 0, \frac{d}{2}} \right\rangle = \frac{2\nu}{\pi^{\frac{d}{2}}} \frac{1}{\left(\frac{d}{2} - \nu\right)^2} \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(\nu)} \frac{1}{|\vec{x} - \vec{y}|^{2\frac{d}{2}+\nu}}.
\]

Then the operator $\mathcal{O}_{\vec{x}}^{\nu \neq 0, \frac{d}{2}}$ has conformal dimension $\frac{d}{2} + \nu$ and the field $\phi$ goes to the border as $x_0^{d/2-\nu} \partial_n \phi_0(\vec{x})$. Comparing Eqs.(22,40) we notice that the multiplicative
coefficients of the boundary two-point functions corresponding to the massive $\nu \neq 0$ Dirichlet and Neumann cases are in general different.

Now we consider the case $\nu = 0$ that is $m^2 = -\frac{d^2}{4}$. Following the now usual steps we get

$$I_N^{\nu=0} = -\frac{\Gamma\left(\frac{d}{2}\right)}{d^{2} \pi^{\frac{d}{2}}} \int d^d x \ d^d y \ \partial_n \phi_\epsilon(x) \ \partial_n \phi_\epsilon(y) \ \frac{\epsilon^{-d}}{\ln \epsilon} \frac{1}{|\vec{x} - \vec{y}|^d} + \cdots. \quad (41)$$

Taking the limit

$$\lim_{\epsilon \to 0} (\epsilon^{\frac{d}{2}} \ln \epsilon)^{-1} \ \partial_n \phi_\epsilon(x) = \partial_n \phi_0(x), \quad (42)$$

and making use of the AdS/CFT equivalence Eq.(36) we find the following boundary two-point function

$$\langle \mathcal{O}_N^{\nu=0}(\vec{x}) \mathcal{O}_N^{\nu=0}(\vec{y}) \rangle = \frac{2 \Gamma\left(\frac{d}{2}\right)}{d^{2} \pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^d}. \quad (43)$$

Then the conformal operator $\mathcal{O}_N^{\nu=0}$ on the boundary CFT has conformal dimension $\frac{d}{2}$. Near the border the scalar field has a logarithmic behavior $x_0^{d/2} \ln x_0 \ \partial_n \phi_0(\vec{x})$. Again we find that the multiplicative coefficients of the boundary two-point functions corresponding to the $\nu = 0$ Dirichlet and Neumann cases are in general different.

### 5 Mixed Boundary Condition

Using the mixed boundary condition Eq.(13) and again Eqs.(5,7) we get

$$a(\vec{k}) = \frac{\epsilon^{-\frac{d}{2}} \psi_\epsilon(\vec{k})}{[\beta(\alpha, \nu) + 2\alpha \nu]K_\nu(k\epsilon) - \alpha k\epsilon K_{\nu+1}(k\epsilon)}, \quad (44)$$

where $\beta(\alpha, \nu)$ is defined as

$$\beta(\alpha, \nu) = 1 + \alpha \left(\frac{d}{2} - \nu\right). \quad (45)$$

Substituting Eq.(44) into Eq.(7) we find

$$\partial_0 \phi_\epsilon(x) = \int d^d y \ \psi_\epsilon(y) \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot (\vec{x} - \vec{y})} \epsilon^{-1} \frac{\epsilon^{\frac{d}{2}} + \nu - k\epsilon}{\beta(\alpha, \nu) + 2\alpha \nu - \alpha k\epsilon} \ \frac{K_{\nu+1}(k\epsilon)}{K_{\nu}(k\epsilon)}. \quad (46)$$
Using this we can write the action Eq.(15) as

\[ I_M = -\frac{1}{2} \int d^d x \, d^d y \, \psi_\epsilon(\vec{x}) \, \psi_\epsilon(\vec{y}) \, e^{-d} \int \frac{d^d k}{(2\pi)^d} \, e^{-i \vec{k} \cdot (\vec{x} - \vec{y})} \, \frac{\frac{d}{2} + \nu - k \epsilon \frac{K_{\nu+1}(k \epsilon)}{K_\nu(k \epsilon)}}{\beta(\alpha, \nu) + 2\alpha \nu - \alpha k \epsilon \frac{K_{\nu+1}(k \epsilon)}{K_\nu(k \epsilon)}}. \]  

(47)

As we shall see it is important to consider the cases \( \beta = 0 \) and \( \beta \neq 0 \) separately in order to find out the relevant terms in the series expansions of the Bessel functions.

Let us start with the case \( \beta = 0 \). For \( \beta = 0 \) we have \( \alpha = -1/(d/2 - \nu) \) and \( m \neq 0 \). We first consider the massive case with \( \nu \neq 0 \), \( d/2 \). Again we have to study separately the cases with \( \nu \) not integer and \( \nu \) integer but non-zero. Let us first consider the case \( \nu \) not integer. Making use of Eq.(90) with \( \beta = 0 \) we get

\[ \frac{d}{2} + \nu - k \epsilon \frac{K_{\nu+1}(k \epsilon)}{K_\nu(k \epsilon)} = \frac{d}{2} - \nu + \cdots, \]  

(48)

and

\[ \frac{1}{\beta(\alpha, \nu) + 2\alpha \nu - \alpha k \epsilon \frac{K_{\nu+1}(k \epsilon)}{K_\nu(k \epsilon)}} = -\frac{\frac{d}{2} - \nu}{2(1-\nu)(k \epsilon)^2 - 2^{1-2\nu} \Gamma(1-\nu)(k \epsilon)^{2\nu} + \cdots}. \]  

(49)

Notice that for \( 0 < \nu < 1 \) the dominating term in the denominator of the r.h.s of Eq.(49) is \((k \epsilon)^{2\nu}\). Substituting in Eq.(47) we get

\[ I_M^{\beta=0,0<\nu<1} = -2^{2\nu-2} \left( \frac{d}{2} - \nu \right)^2 \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \times \int d^d x \, d^d y \, \psi_\epsilon(\vec{x}) \, \psi_\epsilon(\vec{y}) \, e^{-2^{1-\nu} d} \int \frac{d^d k}{(2\pi)^d} \, e^{-i \vec{k} \cdot (\vec{x} - \vec{y})} \, k^{-2\nu} + \cdots. \]  

(50)

Integration over \( \vec{k} \) thus yields

\[ I_M^{\beta=0,0<\nu<1} = -\frac{1}{4\pi^2} \left( \frac{d}{2} - \nu \right)^2 \frac{\Gamma(\frac{d}{2} - \nu)}{\Gamma(1-\nu)} \int d^d x \, d^d y \, \psi_\epsilon(\vec{x}) \, \psi_\epsilon(\vec{y}) \frac{e^{-2(\nu+\frac{d}{2})}}{|\vec{x} - \vec{y}|^{2\nu}} + \cdots. \]  

(51)

For \( \nu > 1 \) the dominating term in the denominator of the r.h.s of Eq.(49) is \((k \epsilon)^{2\nu}\) and Eq.(47) reads

\[ I_M^{\beta=0,\nu>1} = -(\nu - 1) \left( \frac{d}{2} - \nu \right)^2. \]
\[
\times \int d^d x \, d^d y \, \overline{\psi}(\vec{x}) \, \psi(\vec{y}) \, e^{-\frac{d-2}{2} \int \frac{d^d k}{(2\pi)^d} \, e^{-i \vec{k} \cdot (\vec{x} - \vec{y})} \, k^{-2} + \cdots}. \tag{52}
\]

Integration over \( \vec{k} \) is carried out for \( d > 2 \) thus giving

\[
I^{\beta=0, \nu>1}_M = - (\nu - 1) \left( \frac{d}{2} - \nu \right)^2 \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}} \int d^d x \, d^d y \, \overline{\psi}(\vec{x}) \, \psi(\vec{y}) \left( \frac{e^{-\frac{d-2}{2} \left| \vec{x} - \vec{y} \right|^2}}{\left( \frac{d}{2} - \nu \right)^2} \right) + \cdots. \tag{53}
\]

For the case \( \nu \) integer and non-zero we make use of Eq.(91). The logarithmic terms vanish in the limit \( \epsilon \to 0 \) and we find that the same result Eq.(53) holds for \( \nu \) integer and \( \nu \) not integer.

Now in the action Eq.(51) we take the limit

\[
\lim_{\epsilon \to 0} \epsilon^{-\nu - \frac{d-2}{2}} \overline{\psi}(\vec{x}) = \psi_0(\vec{x}), \tag{54}
\]

whereas in the action Eq.(53) the limit to be taken is

\[
\lim_{\epsilon \to 0} \epsilon^{-\frac{d-2}{2} - 1} \overline{\psi}(\vec{x}) = \psi_0(\vec{x}). \tag{55}
\]

Using the AdS/CFT equivalence

\[
\exp(-I_{AdS}) \equiv \left\langle \exp\left( \int d^d x \, \mathcal{O}(\vec{x}) \, \psi_0(\vec{x}) \right) \right\rangle, \tag{56}
\]

we get the following boundary two-point functions

\[
\left\langle O^{\beta=0, 0<\nu<1}_M (\vec{x}) O^{\beta=0, 0<\nu<1}_M (\vec{y}) \right\rangle = \frac{1}{2\pi^{\frac{d}{2}}} \left( \frac{d}{2} - \nu \right)^2 \frac{\Gamma\left(\frac{d}{2} - \nu\right)}{\Gamma(1 - \nu)} \frac{1}{\left| \vec{x} - \vec{y} \right|^2(\frac{d}{2} - \nu)} , \tag{57}
\]

\[
\left\langle O^{\beta=0, \nu>1}_M (\vec{x}) O^{\beta=0, \nu>1}_M (\vec{y}) \right\rangle = (\nu - 1) \left( \frac{d}{2} - \nu \right)^2 \frac{\Gamma\left(\frac{d-2}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{1}{\left| \vec{x} - \vec{y} \right|^2(\frac{d}{2} - \nu)} . \tag{58}
\]

Then the operators \( O^{\beta=0, 0<\nu<1}_M \) and \( O^{\beta=0, \nu>1}_M \) have conformal dimensions \( \frac{d}{2} - \nu \) and \( \frac{d-2}{2} \) respectively. For \( 0 < \nu < 1 \) the field \( \phi \) approaches the boundary as \( x_0^{\frac{d-2}{2}+\nu} \psi_0(\vec{x}) \). The derivation of the conformal dimension \( \frac{d}{2} - \nu \) for its associated boundary operator \( O^{\beta=0, 0<\nu<1}_M \) is a rather striking novel feature. So far this conformal dimension had been found only by using the Legendre transformation approach developed in [23]. There no restriction was found for \( \nu \) but it is known [22] that it must be in the range \( 0 < \nu < 1 \).
It is worth noting that the upper constraint $\nu < 1$ in Eq.(57) is consistent with the unitarity bound.

For $\nu > 1$ we found a boundary operator whose conformal dimension is the unitarity bound $d - \frac{2}{\nu}$. Whereas we have already found such a conformal dimension in the massless Neumann case Eq.(37) we have here a different multiplicative coefficient for the boundary two-point function. We note that the behavior of the scalar field for small $x_0$ is as it should be.

Now we consider the case $\nu = 0$, that is $m^2 = -\frac{d^2}{4}$, keeping still $\alpha = -\frac{2}{d}$. We then find

$$I_M^{\beta=0,\nu=0} = -\frac{d^2 \Gamma \left( \frac{d}{2} \right)}{16\pi^{d/2}} \int d^d x_0 \int d^d y \, \psi_\epsilon(\vec{x}) \, \psi_\epsilon(\vec{y}) \, \frac{\epsilon^{-d}}{|\vec{x} - \vec{y}|^d} + \cdots. \quad (59)$$

Taking the limit

$$\lim_{\epsilon \to 0} \epsilon^{-\frac{d}{2}} \psi_\epsilon(\vec{x}) = \psi_0(\vec{x}), \quad (60)$$

and making use of the AdS/CFT equivalence Eq.(56) we get the following boundary two-point function

$$\langle O_{M}^{\beta=0,\nu=0}(\vec{x}) O_{M}^{\beta=0,\nu=0}(\vec{y}) \rangle = \frac{d^2 \Gamma \left( \frac{d}{2} \right)}{8\pi^{d/2}} \frac{1}{|\vec{x} - \vec{y}|^d}. \quad (61)$$

Then the conformal operator $O_{M}^{\beta=0,\nu=0}$ on the boundary CFT has conformal dimension $\frac{d}{2}$. Now the field $\phi$ goes to the border as $x_{0}^{d/2} \psi_0(\vec{x})$ and no logarithm is present. We find again that the multiplicative coefficient of the two-point function is different from the corresponding ones for the Dirichlet and Neumann cases.

Let us now consider the case when $\beta \neq 0$. We consider first the case $\nu \neq 0$. Again the cases $\nu$ not integer and $\nu$ integer but non-zero must be considered separately. We first consider the case $\nu$ not integer. Up to contact terms or higher order terms in $\epsilon$ we find

$$\frac{d}{2} + \nu - ke \frac{K_{\nu+1}(ke)}{K_{\nu}(ke)} = \left( \frac{d}{2} - \nu \right) \left[ 1 - \frac{2^{1-2\nu} \Gamma(1-\nu)}{\Gamma(\nu)} (ke)^{2\nu} + \cdots \right], \quad (62)$$

and

$$\frac{1}{\beta(\alpha, \nu) + 2\nu} = \frac{1}{\beta(\alpha, \nu)} \left[ 1 + \frac{2^{1-2\nu} \alpha \Gamma(1-\nu)}{\beta(\alpha, \nu) \Gamma(\nu)} (ke)^{2\nu} + \cdots \right]. \quad (63)$$
Substituting in Eq. (47) we get
\[ I_{\beta, \nu}^{\neq 0} = -\frac{\nu}{\pi^2} \frac{1}{\beta^2(\alpha, \nu)} \frac{1}{\Gamma(\nu)} \times \int d^d x \int d^d y \psi_{\nu}(\vec{x}) \psi_{\nu}(\vec{y}) e^{2\nu-d} \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(\vec{x}-\vec{y})} k^{2\nu} + \cdots. \]  
(64)

Integration over \( \vec{k} \) yields
\[ I_{\beta, \nu}^{\neq 0} = -\frac{\nu}{\pi^2} \frac{1}{\beta^2(\alpha, \nu)} \frac{1}{\Gamma(\nu)} \int d^d x \int d^d y \psi_{\nu}(\vec{x}) \psi_{\nu}(\vec{y}) e^{2\nu-d} \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(\vec{x}-\vec{y})} k^{2\nu} + \cdots. \]  
(65)

Consider now the case \( \nu \) integer and non-zero. We find
\[ \frac{d}{2} + \nu - k \epsilon K_{\nu+1}(k\epsilon) \frac{K_{\nu}(k\epsilon)}{K_{\nu}(k\epsilon)} = (\frac{d}{2} - \nu) \left[ 1 - (-1)^\nu \frac{2^{2-2\nu}}{\frac{d}{2} - \nu} \frac{1}{\Gamma^2(\nu)} (k\epsilon)^{2\nu} \ln k + \cdots \right], \]  
(66)

and
\[ \frac{1}{\beta(\alpha, \nu) + 2\alpha \nu - \alpha k \epsilon K_{\nu+1}(k\epsilon) \frac{K_{\nu}(k\epsilon)}{K_{\nu}(k\epsilon)}} = \frac{1}{\beta(\alpha, \nu)} \left[ 1 + (-1)^\nu \frac{2^{2-2\nu}\alpha}{\beta(\alpha, \nu)} \frac{1}{\Gamma^2(\nu)} (k\epsilon)^{2\nu} \ln k + \cdots \right]. \]  
(67)

Substituting in Eq. (47) we get
\[ I_{\beta, \nu}^{\neq 0} = -(-1)^\nu 2^{1-2\nu} \frac{1}{\beta^2(\alpha, \nu)} \frac{1}{\Gamma^2(\nu)} \times \int d^d x \int d^d y \psi_{\nu}(\vec{x}) \psi_{\nu}(\vec{y}) e^{2\nu-d} \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot(\vec{x}-\vec{y})} k^{2\nu} \ln k + \cdots. \]  
(68)

Making use of Eq. (95) we get Eq. (65) again. So both cases \( \nu \) integer and \( \nu \) not integer yield the same result.

Now taking the limit
\[ \lim_{\epsilon \to 0} \epsilon^{\nu-\frac{d}{2}} \psi_{\nu}(\vec{x}) = \psi_0(x), \]  
(69)

and making use of the AdS/CFT correspondence Eq. (56) we find the following boundary two-point function
\[ \langle O_{M}^{\beta, \nu} \rangle = \frac{2\nu}{\pi^2} \frac{1}{\beta^2(\alpha, \nu)} \frac{\Gamma(\frac{d}{2} + \nu)}{\Gamma(\nu)} \frac{1}{|\vec{x} - \vec{y}|^{2(\frac{d}{2} + \nu)}}, \]  
(70)
so that the operator $O^{\beta \neq 0, \nu \neq 0, \frac{d}{2}}_M$ has conformal dimension $\frac{d}{2} + \nu$. From Eq.(69) we find that the behavior of $\phi$ for small $x_0$ is as expected. Comparing Eqs.(22,40,70) we conclude that the multiplicative coefficients of the boundary two-point functions corresponding to the massive $\nu \neq 0$ Dirichlet, Neumann and $\beta \neq 0$ mixed cases are different.

We now consider the case $\nu = 0$ that is $m^2 = -\frac{d^2}{4}$. We find

$$ I^{\beta \neq 0, \nu = 0}_M = -\frac{1}{\beta^2(\alpha, 0)} \frac{\Gamma\left(\frac{d}{2}\right)}{4\pi^2} \int d^d x \ d^d y \ \psi_\epsilon(x) \ \psi_\epsilon(y) \ \frac{\epsilon^{-d}}{\ln^2 \epsilon} \ \frac{1}{|x - y|^d} + \cdots. \quad (71) $$

Taking the limit

$$ \lim_{\epsilon \to 0} (\epsilon^{\frac{d}{2}} \ln \epsilon)^{-1} \psi_\epsilon(x) = \psi_0(x), \quad (72) $$

and making use of the AdS/CFT correspondence Eq.(56) we find the following boundary two-point function

$$ \langle O^{\beta \neq 0, \nu = 0}_M(x) O^{\beta \neq 0, \nu = 0}_M(y) \rangle = \frac{1}{\beta^2(\alpha, 0)} \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^2} \ \frac{1}{|x - y|^d}, \quad (73) $$

so that the conformal operator $O^{\beta \neq 0, \nu = 0}_M$ on the boundary CFT has conformal dimension $\frac{d}{2}$. For small $x_0$ we find a logarithmic behavior $x_0^{d/2} \ln x_0 \ \psi_0(x)$. Again the multiplicative coefficient of the two-point function is different when compared to the corresponding ones for the Dirichlet, Neumann and $\beta = 0$ mixed cases.

In the massless case we have $\nu = \frac{d}{2}$. For $d$ odd we make use of Eqs.(90,94), whereas for $d$ even we use Eqs.(91,95). In both cases we get

$$ I^{\nu = \frac{d}{2}}_M = -\frac{d}{2\pi^2} \ \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \int d^d x \ d^d y \ \psi_\epsilon(x) \ \psi_\epsilon(y) \ \frac{1}{|x - y|^{2d}} + \cdots. \quad (74) $$

Taking the limit

$$ \lim_{\epsilon \to 0} \psi_\epsilon(x) = \psi_0(x), \quad (75) $$

and making use of the AdS/CFT equivalence Eq.(56) we get the following boundary two-point function

$$ \langle O^{\nu = \frac{d}{2}}_M(x) O^{\nu = \frac{d}{2}}_M(y) \rangle = \frac{d}{\pi^2} \ \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \ \frac{1}{|x - y|^{2d}}. \quad (76) $$

Then the operator $O^{\nu = \frac{d}{2}}_M$ has conformal dimension $d$. The scalar field goes to the border as $\psi_0(x)$ as expected. Comparing Eqs.(23,76) we conclude that the boundary CFT’s corresponding to the massless Dirichlet and mixed cases are equal.
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7 Appendix A. Boundary Two-Point Functions

The coefficients $\nu$, $\alpha$ and $\beta(\alpha, \nu)$ are defined in Eqs.(6,13,45) respectively. Let us also define

$$\sigma(\nu) = \frac{d}{2} - \nu .$$  (77)

7.1 Dirichlet Boundary Condition

$$\langle O^{\nu=0}(\vec{x})O^{\nu=0}(\vec{y}) \rangle = \frac{2\nu}{\pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} + \nu)}{\Gamma(\nu)} \frac{1}{|\vec{x} - \vec{y}|^{2(\frac{d}{2}+\nu)}}$$  (78)

$$\langle O^{\nu=\frac{d}{2}}(\vec{x})O^{\nu=\frac{d}{2}}(\vec{y}) \rangle = \frac{d}{\pi^{\frac{d}{2}}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \frac{1}{|\vec{x} - \vec{y}|^{2d}}$$  (79)

$$\langle O^{\nu=0}(\vec{x})O^{\nu=0}(\vec{y}) \rangle = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^{d}}$$  (80)

7.2 Neumann Boundary Condition

$$\langle O^{\nu=\frac{d}{2}}(\vec{x})O^{\nu=\frac{d}{2}}(\vec{y}) \rangle = \frac{1}{\sigma^{2}(\nu)} \langle O^{\nu=0}(\vec{x})O^{\nu=0}(\vec{y}) \rangle$$  (81)

$$\langle O^{\nu=\frac{d}{2}}(\vec{x})O^{\nu=\frac{d}{2}}(\vec{y}) \rangle = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^{2\frac{d}{2}}}$$  (82)

$$\langle O^{\nu=0}(\vec{x})O^{\nu=0}(\vec{y}) \rangle = \frac{1}{\sigma^{2}(0)} \langle O^{\nu=0}(\vec{x})O^{\nu=0}(\vec{y}) \rangle$$  (83)
7.3 Mixed Boundary Condition

\[ \langle O_M^{\beta=0,0<\nu<1}(\vec{x}) O_M^{\beta=0,0<\nu<1}(\vec{y}) \rangle = \sigma^2(\nu) \frac{1}{2\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2} - \nu\right)}{\Gamma(1 - \nu)} \frac{1}{|\vec{x} - \vec{y}|^{2(\frac{d}{2} - \nu)}} \]  
\[ (84) \]

\[ \langle O_M^{\beta=0,\nu>1}(\vec{x}) O_M^{\beta=0,\nu>1}(\vec{y}) \rangle = \sigma^2(\nu)(\nu - 1) \frac{\Gamma\left(\frac{d-2}{2}\right)}{2\pi^{\frac{d}{2}}} \frac{1}{|\vec{x} - \vec{y}|^{2\frac{d-2}{2}}} \]  
\[ (85) \]

\[ \langle O_M^{\beta\neq0,\nu\neq0,\frac{d}{2}}(\vec{x}) O_M^{\beta\neq0,\nu\neq0,\frac{d}{2}}(\vec{y}) \rangle = \frac{1}{\beta^2(\alpha, \nu)} \langle O_D^{\nu\neq0}(\vec{x}) O_D^{\nu\neq0}(\vec{y}) \rangle \]  
\[ (86) \]

\[ \langle O_M^{\nu=\frac{d}{2}}(\vec{x}) O_M^{\nu=\frac{d}{2}}(\vec{y}) \rangle = \langle O_D^{\nu=\frac{d}{2}}(\vec{x}) O_D^{\nu=\frac{d}{2}}(\vec{y}) \rangle \]  
\[ (87) \]

\[ \langle O_M^{\beta\neq0,\nu=0}(\vec{x}) O_M^{\beta\neq0,\nu=0}(\vec{y}) \rangle = \frac{1}{\beta^2(\alpha, 0)} \langle O_D^{\nu=0}(\vec{x}) O_D^{\nu=0}(\vec{y}) \rangle \]  
\[ (88) \]

\[ \langle O_M^{\beta=0,\nu=0}(\vec{x}) O_M^{\beta=0,\nu=0}(\vec{y}) \rangle = \sigma^2(0) \langle O_D^{\nu=0}(\vec{x}) O_D^{\nu=0}(\vec{y}) \rangle \]  
\[ (89) \]

8 Appendix B. Some Useful Formulae

8.1 Series Expansions for the Modified Bessel Functions \( K_\nu \)

For \( \nu \) not integer

\[ K_\nu(z) = \frac{1}{2} \frac{\Gamma(\nu) \Gamma(1 - \nu)}{\Gamma(1 - \nu)} \left( \frac{z}{2} \right)^{-\nu} \left[ \sum_{n=0}^{\nu-1} \frac{\left( \frac{z}{2} \right)^{2n}}{n! \Gamma(n + 1 - \nu)} - \frac{\left( \frac{z}{2} \right)^{2\nu}}{\sum_{n=0}^{\nu-1} \frac{\left( \frac{z}{2} \right)^{2n}}{n! \Gamma(n + 1 + \nu)}} \right] . \]  
\[ (90) \]

For \( \nu \) integer and non-zero

\[ K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^{-\nu} \sum_{n=0}^{\nu-1} (-1)^n \frac{\Gamma(\nu - n)}{n!} \left( \frac{z}{2} \right)^{2n} \]
\[ - (-1)^{\nu} \left( \frac{z}{2} \right)^{\nu} \sum_{n\geq0} \ln \left( \frac{z}{2} \right) - \frac{\lambda(n + 1) + \lambda(\nu + n + 1)}{2} \right] \frac{\left( \frac{z}{2} \right)^{2n}}{n! \Gamma(n + 1 + \nu)} , \]  
\[ (91) \]

where

\[ \lambda(1) = -\gamma \quad \lambda(n) = -\gamma + \frac{1}{m} \quad (n \geq 2), \]  
\[ (92) \]
and $\gamma$ is the Euler constant.

For $\nu = 0$

$$K_0(z) = -\sum_{n \geq 0} \left[ \ln \left( \frac{z}{2} \right) - \lambda(n + 1) \right] \frac{\left( \frac{z}{2} \right)^{2n}}{n! \Gamma(n + 1)}.$$  \hspace{1cm} (93)

### 8.2 Integration over the Momenta

$$\int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot \vec{x}} k^\rho = C_\rho \frac{1}{|\vec{x}|^{d+\rho}} \quad \rho \neq -d,-d-2,...$$  \hspace{1cm} (94)

$$\int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot \vec{x}} k^\rho \ln k = \frac{dC_\rho}{d\rho} \frac{1}{|\vec{x}|^{d+\rho}} + C_\rho \frac{\ln |\vec{x}|}{|\vec{x}|^{d+\rho}} \quad \rho \neq -d,-d-2,...$$  \hspace{1cm} (95)

where

$$C_\rho = \frac{2^\rho}{\pi^{|\rho|}} \frac{\Gamma(\frac{d+\rho}{2})}{\Gamma(-\frac{\rho}{2})}.$$  \hspace{1cm} (96)

### References


