Improved Calculations of Quark Distributions in Hadrons: the case of pion.

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Abstract

The earlier introduced method of calculation of quark distributions in hadrons, based on QCD sum rules, is improved. The imaginary part of the virtual photon forward scattering amplitude on some hadronic current is considered in the case, when initial and final virtualities of the current $p_1^2, p_2^2$ are different, $p_1^2 \neq p_2^2$. The operator product expansion (OPE) in $p_1^2, p_2^2$ is performed. The sum rule for quark distribution is obtained using double dispersion representation of the amplitude on one side in terms of calculated in QCD OPE and on the other side in terms of physical states contributions. Double Borel transformation in $p_1^2, p_2^2$ is applied to the sum rule, killing background non-diagonal transition terms, which deteriorated the accuracy in previous calculations. The case of valence quark distribution in pion is considered, which was impossible to treat by the previous method. OPE up to dimension 6 operators is performed and leading order perturbative corrections are accounted. Valence $u$-quark distribution in $\pi^+$ was found at intermediate $x$, $0.15 < x < 0.7$ and normalization point $Q^2 = 2 \text{ GeV}^2$. These results may be used as input for evolution equations.

1 Introduction

The QCD sum rule approach, invented by Shifman, Vainstein and Zakharov in 1979 [1] is now well known as a powerful method, which make possible to calculate in QCD in non-model way and with a good accuracy various hadron characteristic like masses, decay widths, formfactors etc. The method is based on the operator product expansion (OPE), extended to the nonperturbative region. These results were obtained from consideration of 2 and 3-point correlators (for review see [2]). A bit later the structure functions – quark distribution in photon and hadrons were investigated in the QCD sum rule framework. The second moment of photon structure function was considered in [3], and for pion and nucleon – in [4, 5], but unfortunately it was difficult to extend this approach for calculating higher moments. The general method how to calculate hadron structure functions in the region of intermediate $x$ was suggested in [6] and developed in [7]. The method is based on the consideration of 4-point correlator, corresponding to forward scattering of
two currents, one of which has quantum number of hadron of interest, and the other is electromagnetic (or weak). In the first order of OPE, in the case, when the hadron is a meson, this corresponds to box diagrams like shown in Fig.1, where \( p \) is momentum of hadron current and \( q \) is momentum of photon. The problem of such diagrams is that even if \( p^2, q^2 \) are large and negative, in the case of forward scattering the singularity in \( t \)-channel for massless quarks is at \( t = 0 \), i.e. large distances in \( t \)-channel are of importance. However, as was shown in [6, 7] the situation changes drastically when the imaginary part of the scattering amplitude – the object of interest in case of structure functions – is considered. The imaginary part in \( s \)-channel (\( s = (q + p)^2 \)) of the forward scattering amplitude is dominated by small distances contributions at large (negative) \( p^2 \) and intermediate \( x \). (Here the standard notation is used: \( x \) is the Bjorken scaling variable, \( x = -q^2/2\nu, \nu = pq \)). The proof of this statement, given in [7], is based on the fact, that for the imaginary part of the forward amplitude the position of closest to zero singularity in momentum transfer is determined by the boundary of the Mandelstam spectral function and given by the equation

\[
t = -4 \frac{x}{1-x} p^2
\]  

(it is assumed, that \( |q^2| \gg |p^2| \)). Therefore, even at \( t = 0 \), but not small \( x \) and large \( p^2 \) the virtualities of intermediate states in the \( t \)-channel are large enough for OPE be available.

The further procedure is common for QCD sum rule (with some special nuances we will discuss later), i.e. dispersion representation on \( p^2 \) is saturated by physical states and the contribution of the lowest particle state is extracted using Borel transformation. In [7] the structure function of nucleon was calculated. Somewhat later, structure function of photon has also been calculated [8]. But one should note, that sum rule for \( d \)-quark distribution in proton obtained in [7], is applicable within rather narrow range of \( x \) (0.2 < \( x < 0.45 \)) and agreement with experiment is not good enough. Moreover, it was found to be impossible to calculate structure functions of \( \pi \)- and \( \rho \)-mesons in this way (that’s why the authors of [8] was forced to use special trick, based on VDM, to calculate \( \rho \)-meson structure function). The reason for this is that the sum rules, in form used in [7], have a serious drawback.

To understand, what kind of problem it is, let us shortly review the main points of the method. Let us consider 4-point correlator with two electromagnetic currents and two currents with quantum number of some hadron (for clarity the axial current, corresponding to charged pions, will be considered but conclusion is independent of the choice of current):

\[
\Pi_{\mu\nu\lambda\rho}(p_1, p_2; q_1, q_2) = -\int e^{ip_1x+iq_1y-ipp_2z}d^4xd^4yd^4zd^4z\langle 0 \mid T \{ j_{5\lambda}(x) j_{em}^{\mu\nu}(y) j_{em}^{\rho\lambda}(0) j_{5\rho}(z) \} \mid 0 \rangle
\]

\[
j_{5\lambda} = \bar{u}\gamma_5\gamma_\lambda d
\]  

By considering of forward scattering amplitude in accord with [7], put \( p_1 = p_2 \) at the very beginning. Among various tensor structures of \( \Pi_{\mu\nu\lambda\rho} \) it is convenient to consider the structure \( (p_{\mu}p_{\nu}p_{\lambda}p_{\rho}/\nu) \cdot \Pi(p^2, q^2, x) \), and the imaginary part \( Im \Pi(p^2, q^2, x) \) in \( s \)-channel is
related to pion structure function $F_{2\pi}(x)^1$. Let us write dispersion relation representation of $\text{Im} \tilde{\Pi}(p^2, q^2, x)$ in the $p^2$ variable. As was shown in [7] (see also [9],[10]) the correct form of dispersion representation is double dispersion relation

$$\text{Im} \tilde{\Pi} = a(x) + \int_0^\infty \frac{\varphi(x, u)du}{u - p^2} + \int_0^\infty \int_0^\infty \frac{dud'\rho(u, u', x)}{(u - p^2)(u' - p^2)}$$

(3)

(We consider lowest twist contributions, the terms of order $p^2/q^2$ are neglected.) In order to derive (3) it is convenient to consider first the case, when $p_1^2 \neq p_2^2$ and go to the limit $p_1^2 \to p_2^2 = p^2$. Then the form (3) is evident. The last term in the right hand side (rhs) of (3) represents the properly double dispersion contribution, the second may be considered as subtraction term in variables $p_1^2$ or $p_2^2$ and the first term arises as subtraction from the second. The interesting for us contribution arises from the pion poles in both variables – $u$ and $u'$ in the last term in (3). This term corresponds to the diagram of Fig.2, where the axial current creates the pion, then the process of deep inelastic scattering of virtual photon on pion proceeds and finally pion is absorbed by axial current. Evidently this term is proportional to the pion structure function. All others in (3) may be considered as background. Accept a model of hadronic spectrum, in which $\rho, \varphi$ can be represented by contribution of resonance ($\pi$-meson) and continuum ($s_0$ is continuum threshold)

$$\rho(u, u', x) = f(x)\delta(u - m_\pi^2)\delta(u' - m_\pi^2) + \rho^0(x)\theta(u - s_0)\theta(u' - s_0)$$

$$\varphi(x, u) = \varphi_1(x)\delta(u - m_\pi^2) + \varphi_2(x)\theta(u - s_0)$$

(4)

where $f(x)$ is proportional to resonance ($\pi$-meson) structure function of interest,

$$f(x) \sim 2\pi F_2(x)$$

(5)

and $\varphi_{1,2}$ are some unknown functions, corresponding to non-diagonal transitions.

The substitution of (4), into (3) gives

$$\text{Im} \tilde{\Pi} = \frac{f(x)}{(p^2 - m_\pi^2)^2} + a(x) + \int_{s_0}^\infty \int_{s_0}^\infty \frac{\rho^0(x, u, u')dud'u'}{(u - p^2)(u' - p^2)} + \int_{s_0}^\infty \frac{\varphi_2(x, u)du}{(u - p^2)} + \frac{\varphi_1(x, m_\pi^2)}{(p^2 - m_\pi^2)}$$

(6)

The last term in (6) corresponds to Fig.3, where axial current creates a pion, deep inelastic scattering proceeds, but the final state is not a pion like in Fig.2, but some excited state with pion quantum numbers, which is absorbed by axial current. In order to separate the term proportional to the pion structure function – the first term in the rhs of (6), the Borel transformation in $p^2$ is applied to (6), which suppresses continuum contributions to (6). (The Borel parameter $M^2$ is chosen such that $e^{-s_0/M^2} \ll 1$). After Borel transformation we get:

$$\mathcal{B}_{M^2} \text{Im} \tilde{\Pi}(p^2, x) = f(x)\frac{1}{M^2} e^{-m_\pi^2/M^2} - \varphi_1(x) e^{-m_\pi^2/M^2} +$$

\footnote{As was mentioned in [7], the results are more reliable, if invariant amplitude at kinematical structure with maximal dimension is used.}
\[ + \int_{s_0}^{\infty} du \int_{s_0}^{\infty} du' \rho(x, u, u') e^{-(u+u')/M^2} + \int_{s_0}^{\infty} \varphi(x, u) e^{-\frac{u}{M^2}} \tag{7} \]

For the last two terms in rhs of (7), we can assume that \( \rho(x, u, u') \) and \( \varphi(x, u) \) are given by contribution of bare loop – Fig.1. Because of Borel suppression \( \sim \exp(-s_0/M^2) \ll 1 \) these terms are small and such an approximation does not introduce an essential error in the final result. However, the second term in rhs of (7) is not exponentially suppressed in comparison with the first. The only way to kill it is to differentiate both sides of (7) (multiplied by \( \exp(m_\pi^2/M^2) \)) over \( 1/M^2 \). Just this procedure was used in [6, 7] to determine nucleon structure functions. But, as is well known, the differentiation of approximate relation may seriously deteriorate the accuracy of the results. In QCD sum rules such procedure increases contribution of nonperturbative corrections and continuum contributions, sum rules become much worse or even fails (as for \( \rho \)-meson). For \( \pi \)-meson situation is even worse, because direct calculations show, that bare loop contribution corresponds only to non-diagonal transitions.

In this work we suggest the modified method of calculation of the hadron structure function, which is free from this problems and is completely based on QCD sum rules. We will illustrate it on an example on the \( \pi \)-meson structure function calculation, which usually is much "dangerous" case.

### 2 The idea of the method.

The idea of the method is to consider at the beginning non-equal \( p_1^2 \neq p_2^2 \) in (2) and perform all calculations for this case. Instead (3) dispersion representation takes the form

\[ \text{Im} \tilde{\Pi}(p_1^2, p_2^2, x) = a(x) + \int_{0}^{\infty} \frac{\varphi(x, u)}{u - p_1^2} du + \int_{0}^{\infty} \frac{\varphi(x, u)}{u - p_2^2} du + \int_{0}^{\infty} du_1 \int_{0}^{\infty} du_2 \frac{\rho(x, u_1, u_2)}{(u_1 - p_1^2)(u_2 - p_2^2)} \tag{8} \]

Apply to (8) double Borel transformation in \( p_1^2, p_2^2 \). This transformation kills three first terms in rhs of (7) and we have

\[ B_{M_1^2} B_{M_2^2} \text{Im} \tilde{\Pi}(p_1^2, p_2^2, x) = \int_{0}^{\infty} du_1 \int_{0}^{\infty} du_2 \rho(x, u_1, u_2) \exp\left[ -\frac{u_1}{M_1^2} - \frac{u_2}{M_2^2} \right] \tag{9} \]

One can divide the integration region over \( u_{1,2} \) into 4 areas (Fig.4):

- I \( u_1 < s_0; u_2 < s_0 \);
- II \( u_1 < s_0; u_2 > s_0 \);
- III \( u_2 < s_0; u_1 > s_0 \);
- IV \( u_{1,2} > s_0 \)

Using the standard QCD sum rule model of hadronic spectrum and the hypothesis of quark-hadronic duality, i.e. the model with one lower resonance plus continuum, one can easily notice\(^2\), that area (I) corresponds to resonance region. Spectral density can be written in this area as

\(^2\)We restrict ourselves by simplest model, because higher resonance contribution in any case will be suppressed after double borelization.
\[ \rho(u_1, u_2, x) = f_\pi^2 \cdot 2\pi F_2(x) \delta(u_1 - m_\pi^2) \delta(u_2 - m_\pi^2) \]  

where \( f_\pi \) is defined as

\[ \langle 0 \mid j_{\lambda 5} \mid \pi \rangle = f_\pi p_\lambda \]

\( f_\pi = 131 \) MeV

In area (IV), where both variables \( u_1, u_2 \) are far from resonance region, the non-perturbative effects may be neglected, and, as usual in sum rules, spectral function of hadron state is described by the bare loop spectral function \( \rho^0 \) in the same region

\[ \rho(u_1, u_2, x) = \rho^0(u_1, u_2, x) \]  

In the areas (II),(III) one of variables is far from resonance region, but other is in the resonance region, and spectral function in this region is some unknown function \( \psi(u_1, u_2, x) \), which corresponds to transitions like \( \pi \to \text{continuum} \), as shown in Fig.3.

After double Borel transformation the total answer for physical part can be written as

\[ \hat{B}_1 \hat{B}_2[Im \Pi] = 2\pi F_2(x) \cdot f_\pi^2 e^{-\frac{u_1^2}{M_1^2} + \frac{1}{M_2^2}} + \int_0^s du_1 \int_0^\infty du_2 \psi(u_1, u_2, x) e^{-\frac{u_1^2 + u_2^2}{M_1^2} + \frac{u_2^2}{M_2^2}} + \int_0^\infty du_1 \int_0^\infty du_2 \rho^0(u_1, u_2, x) e^{-\frac{u_1^2 + u_2^2}{M_1^2} + \frac{u_2^2}{M_2^2}} \]  

In what follows we put for simplicity \( M_1^2 = M_2^2 = 2M^2 \). The one of advantages of this method is that after double Borel transformation unknown contribution of (II), (III) areas (second and third term in (12)) are exponentially suppressed. Using duality arguments we estimate the contribution of all non-resonance region (i.e areas II,III,IV) as contribution of bare loop in the same region and demand their value to be small (less than 30%). So, equating physical and QCD representation of \( \Pi \), and taking in account cancellation of appropriate parts in left and right sides one can write the following sum rules (we omit all terms, which are suppressed after Borel transformation)

\[ Im \Pi_{QCD}^0 + \text{Power correction} = 2\pi F_2(x) f_\pi^2 \]

\[ Im \Pi_{QCD}^0 = \int_0^s \int_0^s \rho^0(u_1, u_2, x) e^{-\frac{u_1 + u_2}{2M^2}}; \]  

(The pion mass is neglected.) It can be shown (see Appendix), that for box diagram \( \psi(u_1, u_2, x) \sim \delta(u_1 - u_2) \) and, as a consequence, the second and third terms in (12) are zero in our model of hadronic spectrum.
3 Calculation of box diagram.

The diagrams, corresponding to unit operator contribution, are shown in Fig.1a,b. Note, that crossing diagram, shown in Fig.1c does not contribute, their contribution found to be 0 in leading twist. (This is a sequence of kinematics, so such crossing diagrams also are zero for higher dimension corrections in the leading twist.)

It is enough for us to calculate the distribution of valence $u$-quarks in pion, since $\bar{d}(x) = u(x)$. For this reason restrict ourselves by calculation of $Im\tilde{\Pi}$ for the diagram Fig.1a.

Consider first the case $p_1 = p_2$ and demonstrate, as was announced in Sec.2, that in this case the contribution of box diagram attributes only to non-diagonal transitions, like in Fig.3 and refers to background terms in (7). Diagram Fig.1a contribution is equal

$$Im\Pi_{\mu\nu\lambda\sigma} = -\frac{3}{(2\pi)^2} \frac{1}{2} \int \frac{d^4k}{k^4} \delta[(k + q)^2] \delta[(p - k)^2] \times$$
$$\times Tr[\gamma_\lambda \tilde{k} \gamma_\mu (\tilde{k} + \tilde{q}) \gamma_\nu \tilde{k} \gamma_\sigma (\tilde{k} - \tilde{p})] \quad (14)$$

Calculate the trace and omit the terms, which cannot contribute to the interesting for us structure $\sim p_\mu p_\nu p_\lambda p_\sigma / \nu$. We get

$$Im\Pi_{\mu\nu\lambda\sigma} = -\frac{12}{\pi^2} \int \frac{d^4k}{k^4} k_\mu k_\nu k_\lambda (k_\sigma - p_\sigma) \delta[(k + q)^2] \delta[(p - k)^2] \quad (15)$$

Calculation of the integral leads to:

$$Im\Pi_{\mu\nu\lambda\sigma} = -\frac{3}{\pi} p_\mu p_\nu p_\lambda p_\sigma \frac{1}{p^2} x^2 (1 - x) \quad (16)$$

(only the terms $\sim p_\mu p_\nu p_\lambda p_\sigma$ are kept) and

$$Im\tilde{\Pi}(p^2, x) = -\frac{3}{\pi} \frac{1}{p^2} x^2 (1 - x) \quad (17)$$

Substitute (17) into (6) and perform Borel transformation. We get:

$$\frac{3}{\pi} x^2 (1 - x) (1 - e^{-s_0/M^2}) = 2\pi f^2 x u_\pi(x) \frac{1}{M^2} + \varphi_1(x), \quad (18)$$

where $u_\pi(x)$ is the distribution of valence $u$ quarks in pion (pion mass is neglected). Looking at $M^2$ dependence in (18) it becomes evident, that in this approach the attempt to separate the pion contribution from the background by studying $M^2$ dependence (e.g. differentiation over $1/M^2$) is useless – up to small correction $\sim e^{-s_0/M^2}$ the box diagram contributes to the background only.

Consider now the more promisable approach, $p_1^2 \neq p_2^2$. Since nonequality of $p_1^2, p_2^2$ is important for us only for Borel transformation, i.e. in the denominators of dispersion representation (8), in the calculation of numerator, resulting in kinematical structure $p_\mu p_\nu p_\lambda p_\sigma$ we can put $p_1 = p_2 = p$. Therefore, in order to understand the essential features of corresponding integrals in case of non-equal $p_1^2, p_2^2$, it is sufficient to study insread of (14) a more simple integral
\[ \text{Im} \ T(p_1^2, p_2^2, q^2, \nu) = \int d^4k \frac{1}{k^2} \frac{1}{(k + p_2 - p_1)^2} \delta[(k + q)^2] \delta[(p_1 - k)^2] \]  

(19)

The direct calculation of the integral in rhs of (19) (see Appendix) shows, that it may be represented in the form

\[ \text{Im} \ T(p_1^2, p_2^2, q^2, \nu) = \frac{\pi}{4\nu x} \int_0^{2\nu/x} \frac{1}{u - p_1^2} \frac{1}{u - p_2^2} \, du \]  

(20)

(Higher order terms in \( p_1^2/q^2, p_2^2/q^2 \) are neglected.) At \( p_1^2 = p^2 \) it gives

\[ \text{Im} \ T(p^2, q^2, \nu) = \frac{\pi}{4\nu xp^2} \]  

(21)

as it should be. (20) may be rewritten in the form of double dispersion representation (8) with \( a(x) = \varphi(x) = 0 \) and \( \rho(u, u', x) \) proportional to \( \delta(u - u') \)

\[ \nu \text{Im} \ T(p_1^2, p_2^2, x) = -\frac{\pi}{4x} \int_0^\infty \frac{\delta(u - u')}{(u - p_1^2)(u' - p_2^2)} \, du \, du' \]  

(22)

(Higher twist terms are omitted). From this consideration it becomes clear, that in order to go from the case of \( p_1^2 = p_2^2 = p^2 \) in the calculation of the box diagram Fig.1a (14) to \( p_1^2 \neq p_2^2 \), it is enough to substitute in the final result the factor \( 1/p^2 \) by\(^3\)

\[ \frac{1}{p^2} \rightarrow -\int_0^\infty \int_0^\infty \frac{\delta(u - u')}{(u - p_1^2)(u' - p_2^2)} \, du \, du' \]  

(23)

Therefore instead of (17) we get

\[ \tilde{\Pi}(p_1^2, p_2^2, x) = \frac{3}{\pi} x^2 (1 - x) \int_0^\infty \int_0^\infty \frac{\delta(u - u')}{(u - p_1^2)(u' - p_2^2)} \, du \, du' \]  

(24)

Perform double Borel transformation in \( p_1^2, p_2^2 \). It kills nondesirable depending on one variable subtraction terms in (8) and we have the sum rule for valence \( u \)-quark distribution in pion

\[ u_\pi(x) = \frac{3}{2\pi^2} \frac{M^2}{f_\pi^2} x(1 - x)(1 - e^{-s_0/M^2}) \]  

(25)

where it was put \( M_1^2 = M_2^2 = 2M^2 \). (As is known [11] the characteristic values of Borel parameters \( M_1^2, M_2^2 \) in double Borel transformation are about twice of Borel parameters in ordinary Borel transformation, used in mass calculations.)

Before going to more accurate consideration with account of higher dimension operators and leading order (LO) perturbative corrections, let discuss the unit operator contribution in order to estimate, if it is reasonable. The calculation of the pion decay constant \( f_\pi \), performed in [1], in the same approximation results in

\(^3\)It must be mentioned, that such substitution is valid only for box diagram, it does not take place for more complicated diagrams, considered in next Section.
Substitution (26) into (25) gives

\[ u_\pi(x) = 6x(1 - x) \]  

(27)

One can note, that

\[ \int_0^1 u_\pi(x)dx = 1 \]  

(28)

in agreement with the fact, that in the quark-parton model there is one valence quark in pion. Also, it can be is easily verified, that

\[ \int_0^1 xu_\pi(x)dx = 1/2 \]  

(29)

what corresponds to naive quark model, where no sea quarks exist. So one can say, that formally unit operator contribution corresponds to naive parton model.

Of course, eq.(28) has only formal sense, because, as was discussed in Introduction, our approach is correct only in some intermediate region of \( x \). The boundaries of \( x \), where this approach is correct, will be found, if one takes into account nonperturbative power corrections. In the next section we will discuss them. At the end of this section let us discuss perturbative corrections. We take into account only LO terms, proportional to \( \ln(Q^2/M^2) \), and choose \( Q^2 = Q^2_0 \simeq 2 GeV^2 \) – for the point we calculate our sum rules. Finally, the result for bare loop has the form (the second term in square brackets corresponds to perturbative correction, taken into account).

\[ u_\pi(x) = \frac{3M^2}{2\pi^2 f^2_\pi}(1-x) \left[ 1 + \frac{\alpha_s(\mu^2)\ln(Q^2/\mu^2)}{3\pi} \left( \frac{1}{x} + 4\ln(1-x) - \frac{2(1-2x)\ln x}{1-x} \right) \right] \cdot \left( 1 - e^{-s_0/M^2} \right) \]  

(30)

In the calculation we choose the normalization point \( M^2 = \mu^2 \). The fact that we take into account \( \alpha_s \)-correction at the point \( Q^2 = 2 GeV^2 \) means that our final results for the structure function (we write it in the next section) can be used as an input for evolution, starting from this value of \( Q^2_0 \).

4 Calculations of higher order terms in OPE.

In this section we discuss the power correction contribution to sum rules. The power correction with lower dimension is proportional to gluon condensate \( \langle G_{\mu\nu}G^{\mu\nu} \rangle \) with \( d = 4 \). As was discussed above, only s-channel diagrams (Fig.1a) exists in the case of double borelization. \( \langle G_{\mu\nu}G^{\mu\nu} \rangle \) correction was calculated in a standard way in the Fock-Schwinger gauge \( x_\mu A_\mu = 0 \) [12].
The quark propagator $iS(x, y) = \langle \psi(x)\bar{\psi}(y) \rangle$ in the external field $A_\mu$ has the well known form [13] (our sign of $g$ is opposite to that of [13]):

$$iS(x, y) = iS^0(x - y) - g \int d^4z iS^0(x - z) \cdot i\hat{A}(z) iS^0(z - y) + \frac{g^2}{\pi} \int d^4zd^4z' iS^0(x - z) i\hat{A}(z) iS^0(z - z') \cdot i\hat{A}(z') iS^0(z' - z) + \ldots$$  \hspace{1cm} (31)

Here $S^0$ is free quark propagator; $\hat{A} = (1/2)\dot{\lambda}^a\gamma_\mu A^a_\mu$ and

$$A^a_\mu(x) = \left( \frac{1}{2} \right) x_\rho G^a_\rho_\mu + \left( \frac{1}{3} \right) x_\alpha x_\rho [D_\alpha G^a_\rho_\mu(0)] + \left( \frac{1}{8} \right) x_\alpha x_\rho x_\beta [D_\alpha D_\beta G^a_\rho_\mu]$$  \hspace{1cm} (32)

When calculating one should take into account quark propagator expansion up to the third term and only the first term in the expansion of the external field $A_\mu$ (Fig.5).

These diagrams have been calculated using the program of analytical calculation REDUCE. Surprisingly, in the case of the double borelization the sum of all diagrams Fig.5 was found to be 0. So, the gluon condensate contribution to the sum rule is absent.

Before we discuss the $d = 6$ contribution, let us make the following remark. Due to the fact that we are interested only in the intermediate values of $x$, we should take into account only loop diagrams. Really, one can easily see, that the diagrams with no loops (like those in Fig.6) are proportional to $\delta(1 - x)$ and is out of the region of the method applicability. There are a large number of loop diagrams, corresponding to $d = 6$ corrections. First of all, there are diagrams which correspond to interaction only with gluon vacuum field, i.e. only with external soft gluon lines (see Fig.7). Such diagrams may appear, if we take:

a) all possible combinations, which appear when expansion of quark propagator (31) is taken into account up to the fourth term and in expansion of the external gluon field (32) only the first term is kept. For example, it is the fourth term of the expansion for one quark propagator and the first term (free propagator) for other three (Fig.7a), the second term of the expansion for three quark propagator and one propagator is free (Fig.7b), the third term of the expansion for one quark propagator and the second term for other (Fig.7c) etc.

b) all possible combination, when the second and the third terms of expansion of gluon field (32) is taken into account, like those, shown in Fig.8.

The diagrams of Fig.7 are, obviously, proportional to $\langle g^3 f^{abc} G^a_\mu G^b_\nu G^c_\rho \rangle$ and when calculating it is convenient to use representation of this tensor structure given in [14]

$$\langle 0 \mid g^3 f^{abc} G^a_\mu G^b_\nu G^c_\rho \mid 0 \rangle = (1/24) \langle 0 \mid f^{abc} G^a_\nu G^b_\nu G^c_\nu \mid 0 \rangle \cdot (g_\mu g_\nu g_\rho +$$

$$+g_\mu g_\alpha g_\nu + g_\alpha g_\mu g_\rho + g_\rho g_\mu g_\alpha -$$

$$-g_\mu g_\alpha g_\rho - g_\mu g_\nu g_\alpha - g_\alpha g_\rho g_\nu - g_\rho g_\nu g_\alpha \rangle$$  \hspace{1cm} (33)

The diagrams of Fig.8 are proportional to $\langle 0 \mid D_\mu G^a_\nu D_\nu G^a_\rho \mid 0 \rangle$ and $\langle 0 \mid G^a_\mu D_\rho D_\sigma G^a_\nu \mid 0 \rangle$. Using the equation of motion it was found in [14]
\begin{equation}
- \langle 0 \mid D_\rho G_{\mu\nu} D_\sigma G_{\alpha\beta}^a \mid 0 \rangle = \langle 0 \mid D_\rho G_{\mu\nu}^a D_\rho D_\sigma G_{\alpha\beta}^a \mid 0 \rangle =
\end{equation}

\begin{align*}
&= 2O^- \left[ g_{\rho\sigma} (g_{\mu\beta} g_{\alpha\nu} - g_{\mu\alpha} g_{\nu\beta}) + \frac{1}{2} (g_{\mu\beta} g_{\alpha\sigma} g_{\rho\nu} + g_{\alpha\nu} g_{\mu\rho} g_{\beta\sigma} - g_{\alpha\sigma} g_{\mu\rho} g_{\nu\beta} - g_{\rho\nu} g_{\mu\alpha} g_{\beta\sigma}) \right] + \\
&+ \langle 0 \mid g_{\nu\rho} g_{\alpha\sigma} g_{\mu\beta} + g_{\nu\rho} g_{\alpha\beta} g_{\mu\sigma} - g_{\nu\beta} g_{\alpha\rho} g_{\mu\sigma} - g_{\nu\beta} g_{\alpha\rho} g_{\mu\sigma} \rangle;
\end{align*}

\begin{equation}
O^\pm = \frac{1}{72} \langle 0 \mid g^2 f_{\mu\nu}^a \mid 0 \rangle \pm \frac{1}{48} \langle 0 \mid g f^{abc} G_{\mu\nu}^a G_{\nu\lambda}^b G_{\lambda\mu}^c \mid 0 \rangle
\end{equation}

where \( j^a_\mu = \sum_i \tilde{\psi}_i \gamma^a \gamma_\mu \frac{\lambda_i}{2} \psi_i \).

From (33), (34) one may note, that these tensor structure are proportional to two different vacuum averages:

\begin{align*}
\langle 0 \mid g^2 j^2_\mu \mid 0 \rangle \quad \text{and} \quad \langle 0 \mid g^3 G_{\mu\nu}^a G_{\nu\rho}^b G_{\rho\mu}^c f^{abc} \mid 0 \rangle
\end{align*}

First of them \( \langle 0 \mid g^2 j^2_\mu \mid 0 \rangle \) by use of the factorization hypothesis easily reduces to \( \langle g\bar{\psi}\psi \rangle^2 \), which is well known.

\begin{equation}
\langle 0 \mid g^2 j^2_\mu \mid 0 \rangle = -\frac{4}{3} \langle 0 \mid g\bar{\psi}\psi \mid 0 \rangle \rangle^2
\end{equation}

But \( \langle 0 \mid g^3 G_{\mu\nu}^a G_{\nu\rho}^b G_{\rho\mu}^c f^{abc} \mid 0 \rangle \) is not well known, there are only some estimations based on the instanton model [15], [16]. Fortunately, in the sum of all diagrams of this two types all terms proportional to this dimension 6 gluonic condensate are exactly cancelled, and the sum of diagrams of Fig.7 and Fig.8 is proportional only to \( \langle g\bar{\psi}\psi \rangle^2 \).

We consider now an another type of diagrams which also give contribution to \( d = 6 \) power corrections. Such diagrams appear when the external quark field is taken into account, i.e., one should take into consideration the expansion of quark field:

\begin{equation}
\psi(x) = \psi(0) + x_{\alpha_1} [\nabla_{\alpha_1} \psi(0)] + \frac{1}{2} x_{\alpha_1} x_{\alpha_2} [\nabla_{\alpha_1} \nabla_{\alpha_2} \psi(0)] + \ldots
\end{equation}

where \( \nabla \) is covariant derivative.

In this case there appear diagrams like those in Figs.9-11, where quark (and antiquark) line is expanded and the first and the second terms of the expansion (36) are taken. The expansion of the external gluon field (32) is also accounted up to the second term. For the diagrams of Fig.10 gluon propagator in the external field is also accounted (we discuss it a bit later).

All these diagrams can be divided into two types with quite a different physical sense. The first type of diagrams – like those in Fig.9 – corresponds to the case, where all interactions with vacuum proceeds out of the loop. Such diagrams correspond to logarithm corrections (evolution) to the corresponding non-loop diagrams (without hard gluon line). Since, as was discussed in Sec.3, we will not take into account these non-loop diagrams, then it seems reasonable, that at the same level of accuracy we do not take into account their evolution. So, all the diagrams of this type should be omitted. The problem of correct calculation of non-loop diagrams and their leading logarithmic correction is a
special problem, which will not be discussed here. In any case, estimations and physical reason show, that their contribution would be significant at large $x$ and negligible in intermediate region. We shall see at the end of the paper, that sum rules themselves indicate region of $x$ where effects of the non-loop diagrams and their evolution may be neglected.

So, according to the previous discussion, we should bear in mind only those diagrams, where interaction with vacuum takes place inside the loop. (Figs.10,11). Such diagrams cannot be treated as evolution of any non-loop diagrams and are pure power correction of dimension 6. All these diagrams are, obviously, proportional to

$$\langle 0 \mid \bar{\psi}_a^d \psi_{\beta b} (D_{\sigma} G^{\mu\nu}_{\mu\nu})^n \mid 0 \rangle; \quad \langle 0 \mid \bar{\psi}_a^d (\nabla_{\tau} \psi_{\beta b}) G^{\mu\nu}_{\mu\nu} \mid 0 \rangle;$$

$$\langle 0 \mid (\nabla_{\tau} \bar{\psi}_a^d) \psi_{\beta b} G^{\mu\nu}_{\mu\nu} \mid 0 \rangle$$

These tensor structures were considered in [9] where using the equation of motion the following results have been obtained

$$\langle 0 \mid \bar{\psi}_a^d \psi_{\beta b} (D_{\sigma} G^{\mu\nu}_{\mu\nu})^n \mid 0 \rangle = \frac{g}{3^3 \cdot 2^5} \left( \frac{g_{\sigma\nu}\gamma^\mu - g^{\sigma\mu}\gamma_\nu}{\beta_\alpha} \right) (\lambda^n)^{bd}$$

$$\langle 0 \mid \bar{\psi}_a^d (\nabla_{\tau} \psi_{\beta b}) G^{\mu\nu}_{\mu\nu} \mid 0 \rangle = \frac{g}{3^3 \cdot 2^6} \left[ g^{\sigma\mu}\gamma_\nu - g^{\sigma\nu}\gamma_\mu - i\varepsilon^{\sigma\nu\lambda}\gamma_5 \gamma_\lambda \right]_{\beta_\alpha} (\lambda^n)^{bd}$$

(37)

$$\langle 0 \mid (\nabla_{\tau} \bar{\psi}_a^d) \psi_{\beta b} G^{\mu\nu}_{\mu\nu} \mid 0 \rangle$$

can be easily calculated using the results of [9]

$$\langle 0 \mid (\nabla_{\tau} \bar{\psi}_a^d) \psi_{\beta b} G^{\mu\nu}_{\mu\nu} \mid 0 \rangle = \frac{g}{3^3 \cdot 2^6} \left[ g^{\sigma\mu}\gamma_\nu - g^{\sigma\nu}\gamma_\mu + i\varepsilon^{\sigma\mu\lambda}\gamma_5 \gamma_\lambda \right]_{\beta_\alpha} (\lambda^n)^{bd}$$

(38)

For diagrams in Fig.10 we use the following expansion of gluon propagator

$$S_{\nu\mu}(x - y, y) = -\frac{i}{(2\pi)^4} g J^{\nu\mu Kl} \int \frac{dk}{k^4} e^{-iku}. \left\{ -i k_\lambda y_\alpha G_{\alpha\lambda}^{\mu\nu} - \frac{2}{3} (y_\mu y_\nu) - \frac{iy_\beta}{k^2} (k^2 \delta_{\alpha\lambda} - 2k_\alpha k_\lambda) (D_\alpha G_{\beta\lambda})^l + \frac{1}{3} (y_\alpha + \frac{2ik_\alpha}{k^2}) (D_\lambda G_{\alpha\lambda})^l \right\}$$

$$\delta_{\nu\mu} + \frac{2}{3} \left( G_{\nu\mu}^l + 2 \frac{k_\alpha}{k^2} (D_\alpha G_{\nu\mu})^l \right)$$

(39)

This expression can be found using the a method of calculation of gluon propagator in external vacuum gluon field, suggested in [12]. The same result up to $\sim G$ term is explicitly written in [13] (see also [17]). The total number of $d = 6$ diagrams is enormous - about 500. All of them were calculated by use of REDUCE program. The final result for $d=6$ correction after double Borel transformation have the form

$$Im \Pi^{d=6} = -\frac{1}{(2\pi)^7} g^2 (ga)^2 \frac{M^4}{3^3 \cdot 2^5} \left[ (-5784x^4 - 1140x^3 - 20196x^2 + 20628x - 8292) ln(2) + 
+ 4740x^4 + 8847x^3 + 2066x^2 - 2553x + 1416 \right] \frac{1}{x(1-x)^2}$$

(39)
\((ga)^2 \equiv 4\pi \alpha_s \cdot (2\pi)^4 \langle 0 \mid \bar{\psi}\psi \mid 0 \rangle^2\). Before we write the final result of the sum rules, let us make one note. One can see, that in contribution of \(d = 6\) operator (39) strong coupling constant \(g^2\) appears as factor, and again it appears in structures \((ga)^2\). The factor \(g^2\) corresponds to interaction with quark propagator (vertices of hard gluon line in diagrams in Fig.9,10, or vertices of external gluon in diagrams in Fig.6,7), and it is reasonable to take it at the renormalization point \(\mu^2 = Q_0^2\). On the other side, \(g^2\) in structure \((ga)^2\) appears as a consequence of use of equation of motion and its normalization point should be taken in such a way that the quantity \(\alpha_s(0 \mid \bar{\psi}\psi \mid 0)^2\) is renormalization group invariant. Finally, substituting results for bare loop (30) and power corrections (39), we can write the sum rule for quark distribution function in pion:

\[
x_u(x) = \frac{3}{2\pi^2} \frac{M^2}{f_\pi^2} x^2 (1-x) \left[ \left( 1 + \frac{\alpha_s(M^2) \cdot \ln(Q_0^2/M^2)}{3\pi} \right) \left( \frac{1+4x \ln(1-x)}{x} - 2(1-2x)\ln x \frac{1}{1-x} \right) \right] \\
\cdot (1 - e^{-s_0/M^2}) - \frac{4\pi \alpha_s(Q_0^2) \cdot 4\pi \alpha_s(M^2) a(M^2)}{(2\pi)^4 \cdot 3^7 \cdot 2^6 \cdot M^6} \cdot \frac{\omega(x)}{x^3 (1-x)^3}
\]

(40)

where \(\omega(x)\) is the expression in square brackets in (39). We choose the effective LO QCD parameter \(\Lambda_{QCD} = 200\) MeV, \(Q_0^2 = 2\) GeV\(^2\). The value of renorminvariant parameter is equal

\[\alpha_s a^2 = \alpha_s(M^2 = 1\text{ GeV}^2) \cdot (0.55\text{ GeV}^3)^2 = 0.13\text{ GeV}^6\]

The value of \(a\) was taken from the best fit [18] of the sum rule of nucleon masses (see [9], Appendix B). Continuum threshold was varied in the interval, \(0.8 < s_0 < 1.2\) GeV\(^2\) and it was found that the results only slightly depend on it. The analysis of the sum rule (40) shows, that they are fulfilled in the region \(0.15 < x < 0.7\); power corrections are less than 30%, and continuum contribution is small (<25%). Stability in Borel mass parameter \(M^2\) dependence in the region \(0.4\text{ GeV}^2 < M^2 < 0.6\text{ GeV}^2\) is good; especially in the region of \(x \leq 0.4\) the \(M^2\) dependence is almost constant (see Fig. 12).

The final result for \(u_\pi(x)\), (at \(M^2 = 0.45\) GeV\(^2\), \(S_0 = 0.8\) GeV\(^2\)), is shown in Fig.13 (thick solid line). On Fig.13 is also plotted the curve of \(u\)-quark distribution in pion, found in [19] by using evolution equation and the fit to the data on Drell-Yan process, performed in [20]). Bearing in mind, that NLO \(\alpha_s\)-corrections are not accounted and one may expect, that they would increase \(u_\pi(x)\) at low \(x\) and decrease at large \(x\), one may consider the agreement as good. We also show in the same figure pure bare loop contribution (line with squares) and contribution (30) of bare loop with perturbative correction (crossed line). One can see, that pure bare loop is not in a quite good agreement with experiment and both perturbative correction and power correction improve the agreement with experiment. Let us discuss, why stability became worse when \(x\) became larger (see Fig.12). From our point of view, it reflect the influence of non-loop diagrams (and their evolution), which were not accounted as it was discussed in sect.4. Indeed, the non-loop diagram which formally are proportional to \(\delta(1-x)\), of course really would correspond to some function with maximum close to \(x=1\) and fast decreasing when \(x\) decreased. That is why effects of such diagrams (and their evolution) are negligible at \(x \leq 0.4 - 0.5\), but may be more or less sensible at large \(x\), and deterioration of stability probably reflects
this fact. We repeat, finally that obtained valence $u$-quark distribution function $u_\pi(x)$ can be used as input for evolution equation (starting from point $Q_0^2 = 2 \text{GeV}^2$).

Let us now discuss at the end the estimations for the moments of quark distribution which can be found with the help of the results obtained.

To get the moments, one should make some suggestions about the region of small $x \lesssim 0.15$ and large $x \gtrsim 0.7 \pm 0.8$, where our method is inapplicable. Of course, in this case the estimation of moments are not model-independent and the accuracy of estimation of moments should be treated as lower than for the structure function (40) itself.

If we make a natural supposition, that at $x \lesssim 0.15$ $u_\pi(x) \sim 1/\sqrt{x}$ according to Regge behaviour, and at large $x \gtrsim 0.7$, $u_\pi(x) \sim (1-x)^2$ according to quark counting rules, then, matching these functions with our result (40), one may find, that

$$M_0 = \int_0^1 u_\pi(x)dx \approx 0.84$$

$$M_1 = \int_0^1 xu_\pi(x)dx \approx 0.21$$

at $M^2 = 0.4 \div 0.45\text{GeV}^2$. These results only slightly depend on the choice of the points of matching (not more than 5% when we vary lower matching point in the region $0.15 \div 0.2$ and the upper one in the region $0.65 \div 0.75$. One may note that $M_0$ which has the physical meaning of the number of quarks in pion (and should be $M_0 = 1$) is really close to 1 within our accuracy $\sim 10 \div 20\%$. $M_1$ has physical meaning of the part of momentum carried by a valence quark, and the value $M_1 \approx 0.21$ is in good agreement with well known fact, that two valence quarks in pion take about 40% of the total momentum.

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Appendix

In this Appendix the double dispersion representation (22) of integral (19) is proved. It is convenient to change variables in (19) and put $p - k = k'$. Then (19) takes the form (prime is omitted)

$$\text{Im } T(p_1^2, p_2^2, q^2, \nu) = \int d^4k \frac{1}{(p_1 - k)^2} \frac{1}{(p_2 - k)^2} \delta[(p + q - k)^2] \delta(k^2) \quad (A.1)$$

Assume that $q^2 = q'^2$, $t = (p_1 - p_2)^2 = 0$ and choose the Lorenz system, where 4-vector $P = (p_1 + p_2)/2$ has only $z$-component equal $P_z$. From

$$t = (p_1 - p_2)^2 = p_1^2 + p_2^2 + 2p_1p_2 = 0 \quad (A.2)$$

it follows

$$P^2 = -P_z^2 = (p_1^2 + p_2^2)/2 \quad (A.3)$$

Introduce 4-null-vector $r = p_1 - p_2, r^2 = 0$

$$rP = (p_1^2 - p_2^2)/2 \quad (A.4)$$

We have

$$p_1 = P + r/2, \quad p_2 = P - r/2 \quad (A.5)$$

Use the notation

$$qP = \nu = qp_1 = qp_2 \quad (A.6)$$

Then

$$q'p_1 = q'p_2 = \nu + (p_1^2 - p_2^2)/2, \quad qr = q'r = 0 \quad (A.7)$$

We can choose the coordinate system, where 4-vector $q$ has only time and $z$-components and

$$q_z = -\nu/\sqrt{-P^2}, \quad q_0 = \sqrt{\nu^2 - q^2P^2}/\sqrt{-P^2} \approx \nu/\sqrt{-P^2} = -q_z \quad (A.8)$$

The last equality corresponds to account of lower twist terms. From (A.4) and (A.7) for 4-vector $r$ with components $r = \{r_0, r_\perp, r_z\}$ it follows

$$r_0 = \frac{1}{2} \frac{p_1^2 - p_2^2}{\sqrt{-P^2}} \frac{1}{\sqrt{\nu^2 - q^2P^2}} \approx \frac{1}{2} \frac{p_1 - p_2}{\sqrt{-P^2}}, \quad r_z = \frac{1}{2} \frac{p_1^2 - p_2^2}{\sqrt{-P^2}} \approx r_0, \quad r_\perp = -\frac{1}{4} q^2 (p_1^2 - p_2^2)/\nu^2 - q^2P^2 \quad (A.9)$$

The components $r_0$ and $r_z$ are equal in the lowest twist approximation and of order $\sqrt{-P^2}$ if $p_1^2 \sim p_2^2$, while $r_\perp \sim (p_1^2 - p_2^2)/\nu^{1/2}$, i.e. of the next order in this approximation and may be neglected. The argument of the first $\delta$-function in (A.1) is equal
\[
s - 2 - \frac{1}{\sqrt{-P^2}} \left[ \nu + P^2 + \frac{1}{4}(p_1^2 - p_2^2) \right] k_z - \left[ 2\sqrt{\frac{\nu^2 - q^2 P^2}{-P^2}} + \frac{1}{2} P_1^2 + P_2^2 \right] k_0 + r_{\perp} k_{\perp} \cos \varphi = 0 \tag{A.10}
\]

where \( \varphi \) is the azimuthal angle between \( p_{\perp} \) and \( k_{\perp} \). The last term in (A.10) may be omitted – it is of the next order in \( p^2/\nu \): it may appear only squared because of integration over \( \varphi \) in (A.1)). (This fact can be proved by direct calculation.) From the inequality

\[
k_0^2 - k_z^2 \geq 0, \tag{A.11}
\]

the inequality follows, which defines the integration domain over \( k_z \) in the integral (A.1):

\[
k_z^2 P^2 - \sqrt{-P^2} [\nu + P^2 + \frac{1}{4}(p_1^2 - p_2^2)] k_z - \frac{1}{4} P^2 \geq 0 \tag{A.12}
\]

It is convenient to use the notation

\[
u = 2\sqrt{-P^2} k_z \tag{A.13}
\]

The integration domain is

\[-2\nu \leq \nu \leq -P^2(1 - x) \tag{A.14}\]

The denominators in (A.1) are calculated by using the relations:

\[
(p_1 - k)^2 = p_1^2 - 2 p_1 k = p_1^2 - 2 P k - r k \quad (p_2 - k)^2 = p_2^2 - 2 P k + r k \tag{A.15}
\]

\[
P k = -\sqrt{-P^2} k_z, \quad r k \approx r_0(k_0 + k_z) = r_0\sqrt{-P^2}(1 - x) = \frac{1}{2}(p_1^2 - p_2^2)(1 - x) \tag{A.16}
\]

(In the above equalities (A.10) and (A.9) were exploited). As a result we get (\( \delta \)-functions were eliminated by integration over \( k_{\perp}^2 \) and \( k_0 \)):

\[
Im T = \frac{\pi}{4 q_0 \sqrt{-P^2}} \int_{-2\nu}^{-P^2(1-x)} du \frac{1}{p_1^2 + v - (p_1^2 - p_2^2)(1 - x)/2} \times \frac{1}{p_2^2 + v + (p_1^2 - p_2^2)(1 - x)/2} \tag{A.17}
\]

Changing variables

\[
u = -P^2(1 - x) - u x \tag{A.18}
\]

gives the final answer

\[
Im T = \frac{\pi}{4 \nu x} \int_0^\infty \frac{1}{u - p_1^2} \frac{1}{u - p_2^2} du \tag{A.19}
\]

(The upper limit of integration was put as infinity, what is legitimate in the lowest twist approximation).
References


Figure 1: Diagrams, corresponding to unit operator contribution. Dashed lines with arrows correspond to photon, thick solid - to hadron current.
Figure 2: Diagram of forward photon-pion scattering.
Figure 3: Example of non-diagonal transition
Figure 4: Integration region in double dispersion representation.
Figure 5: Diagrams, corresponding to $d = 4$ operator contribution. Dashed lines with arrows correspond to photon, thick solid - hadron current, wave lines correspond to external gluon field.

+ all combinations
Figure 6: Examples of non-loop diagrams of dimension 4. Wave lines correspond to gluons, dot means derivative, other notations as in Fig.1.
Figure 7: Diagrams of dimension 6, see text. All notations as in Fig.6.

\[ a + b + c + \text{all combination} \]
Figure 8: Diagrams of dimension 6. External gluon line with dot corresponds to derivatives of gluon lines. All notations as in Fig.6
Figure 9: Dimension 6 diagrams with out the loop vacuum interaction. All notations as in Fig.8
Figure 10: Diagrams of dimension 6, corresponding to quark propogator expansion (eq.37). All notations as in Fig.8.
Figure 11: Diagrams of dimension 6, corresponding to quark and gluon propagator expansion (eq.37, 38). All notations as in Fig.8

\[\begin{align*} + & \quad + & \quad + & \quad + \\
+ & \text{all combinations} \end{align*}\]
Figure 12: Borel mass dependence of quark distribution function at various $x$
Figure 13: Quark distribution function in pion, noted "total". For comparison fit from [19], noted "GR", is shown. Also bare loop ("bare") and bare loop with perturbative corrections (noted "1"), are shown.