Elastic Parton-Parton Scattering from AdS/CFT

Mannque Rho\textsuperscript{a,b1}, Sang-Jin Sin\textsuperscript{a,c2} and Ismail Zahed\textsuperscript{a,d3}

\textsuperscript{a} School of Physics, Korea Institute for Advanced Study, Seoul 130-012, Korea
\textsuperscript{b} Service de Physique Théorique, CE Saclay, 91191 Gif-sur-Yvette, France
\textsuperscript{c} Department of Physics, Hanyang University, Seoul 133-791, Korea
\textsuperscript{d} Department of Physics and Astronomy, SUNY-Stony-Brook, NY 11794

Abstract

Using the AdS/CFT correspondence and the eikonal approximation, we evaluate the elastic parton-parton scattering amplitude at large $N$ and strong coupling $g^2 N$ in $N=4$ SYM. We obtain a scattering amplitude with a Regge behavior that unitarizes at large $\sqrt{s}$. For partons with masses $m$ in the range $\sqrt{s} > m > \sqrt{-t}$, we suggest that our results may be of relevance to QCD in four dimensions.
1. Elastic quark-quark and gluon-gluon scattering at large $s$ and fixed $t$ pertains to the domain of non-perturbative QCD. Theoretical procedures based on resumming large classes of perturbative contributions have been proposed [1, 2], partially accounting for the reggeized form of the scattering amplitude and the phenomenological success of pomeron/odderon exchange models [2] (and references therein). In an inspiring approach, Nachtmann [3] and others [4, 5] suggested to use non-perturbative techniques for the elastic scattering amplitude. In the eikonal approximation and to leading order in $t/s$ the quark-quark amplitude at large $s$ was reduced to a correlation function of two light-like Wilson-lines. The latter was assessed in the stochastic vacuum model [3].

Recently, Maldacena [6] has made the remarkable conjecture that the large $N$ behavior of some gauge theories may be related to the behavior of string theory in a non-trivial geometry. This conjecture, made more precise by Gubser, Klebanov and Polyakov [7] and by Witten [8] and extended to non-supersymmetric case by Witten [9], provides an interesting and nonperturbative avenue (via AdS/CFT correspondence) for studying gauge theories at large $N$ and strong coupling $g^2N$. In particular the heavy quark potential was found to be Coulombic for supersymmetric theories and linear for non-supersymmetric theories [8, 13].

In this letter we suggest to use the AdS/CFT approach to analyze the elastic parton-parton scattering amplitude in the eikonal approximation for $N = 4$ SYM. In section 2 we briefly discuss the salient features of the parton-parton reduced elastic amplitude at large $\sqrt{s}$. The scattering amplitude in eikonal approximation is reduced to the Fourier transform of the connected part of correlators of Wilson lines. In section 3 we use the AdS/CFT approach to calculate the correlator. Following [10, 11], we propose that it is given by the minimum (regularized) area of world sheet whose boundary is the trajectories in the boundary and give a simple variational method that allows for a closed-form expression for the minimal area in the presence of a finite proper time cutoff. In section 4 we suggest that for partons with masses $\sqrt{s} > m > \sqrt{-t}$, the present results may be applicable to QCD in four dimensions at large $N$ and strong coupling $g^2N$. In section 5 we summarize and conclude.

2. First consider the quark propagator in an external non-Abelian gauge field. In the first quantized theory, the propagator from $x$ to $y$ in Euclidean space reads

$$S(x, y; A) = \langle x| \frac{1}{\nabla + im - 0} |y\rangle = \int_0^{\infty} dTe^{-mT} \langle x| e^{-\nabla T} |y\rangle$$  \hspace{1cm} (1)
that is [15]

$$S(x, y; A) = \int dT e^{-mT} \int_x^y d[x] \delta(1 - \dot{x}^2) \mathbf{P}_c e^{ig \int ds A_\mu \frac{1}{2} \mathbf{P}_s e^{-\frac{i}{2} \int ds \sigma_{\mu\nu} \dot{x}_\mu \dot{x}_\nu}}$$

(2)

where $\mathbf{P}_{c,s}$ are orderings over color and spin matrices, and $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2$. The first exponent in the path integral is an arbitrary Wilson line in the fundamental representation of SU(N)$_c$, and the second exponent is a string of infinitesimal Thomas-precessions along the Wilson line. The integration is over all paths with $x(0) = x$ and $x(T) = y$. Notice that from the proper time integral, the dominant contributions come from $T \leq 1/m$. Heavy quarks travel shorter in proper time than light quarks and the mass gives an effective cutoff of the proper time range. This observation will be important below.

A quark with large momentum $p$ travels on a straight line with 4-velocity $\dot{x} = v = p/m$ (for $m \neq 0$) and $v^2 = 1$. Throughout, we will distinguish between the 4-velocity $v$ and the instantaneous velocity $V = dx/dt = p/E \leq 1$. For a straight trajectory, the 4-acceleration $\ddot{x} = a = 0$ and the spin factor drops. This is the eikonal approximation for $S$ in which an ordinary quark transmutes to a scalar quark. The present argument applies to any charged particle in a background gluon field, irrespective of its spin or helicity. The only amendments are: for antiquarks the 4-velocity $v$ is reversed in the Wilson line, while for gluons the Wilson lines are in the adjoint representation. With this in mind, quark-quark scattering can be also extended to quark-antiquark, gluon-gluon or scalar-scalar scattering. We note that for quark-antiquark scattering the elastic amplitude dominates at large $\sqrt{s}$ since the annihilation part is down by $\sqrt{-t/s}$.

Generically, we will refer to elastic parton-parton scattering as

$$Q_A(p_1) + Q_B(p_2) \rightarrow Q_C(k_1) + Q_D(k_2)$$

(3)

with $s = (p_1 + p_2)^2$, $t = (p_1 - k_1)^2$, $s + t + u = 4m^2$. We denote by $AB$ and $CD$ respectively, the incoming and outgoing color and spin of the quarks (polarization for gluons). Using the eikonal form (1) and LSZ reduction, the scattering amplitude $T$ may be reduced to [3, 4, 5]

$$T_{AB,CD}(s, t) \approx -2is \int d^2 b \ e^{i\mathbf{q}_\perp \cdot \mathbf{b}} \left\langle \left( \mathbf{W}_1(b) - 1 \right)_{AC} \left( \mathbf{W}_2(0) - 1 \right)_{BD} \right\rangle$$

(4)

where

$$\mathbf{W}_{1,2}(z) = \mathbf{P}_c \exp \left(iq \int_{-\infty}^{+\infty} d\tau A(b + v_{1,2} \tau) \cdot v_{1,2} \right).$$

(5)

The 2-dimensional integral in (4) is over the impact parameter $b$ with $t = -q^2_\perp$. In the CM frame $p_1 \approx (E, E, 0_\perp)$, $p_2 \approx (E, -E, 0_\perp)$, $q = p_1 - k_1 \approx (0, 0, q_\perp)$ and $s \approx 4E^2$. The
averaging in (4) is over the gauge configurations using the QCD action. The total cross section for $\sqrt{s} >> -t > 0$ follows from (4) in the form $\sigma = \Im T/s < \ln^2 s$, where the last inequality is just the Froissart bound.

The amplitude (4) allows for two gauge-invariant decompositions (repeated indices are summed over)

$$T_1 = T_{AB,AB} \quad T_2 = T_{AB,BA}$$

assuming that the gluon-gauge fields are periodic at the end-points. This can be achieved strongly by requiring periodicity on the hypersurface of constant proper time $\pm \infty$ or weakly by identifying the end-points, e.g. $x_A = (-\infty, -\infty, b) = x_B = (+\infty, +\infty, b)$. These conditions introduce a natural compactification of $M_4$. For gluon-gluon scattering the lines are doubled in color space (i.e., adjoint representation) and several gauge-invariant contractions are possible. For quark-quark scattering the singlet exchange in t-channel is $0^+$ (pomeron) while for quark-antiquark it is $0^-$ (odderon) as the two differ by charge conjugation.

3. In the eikonal approximation the parton-parton scattering amplitude is related to an appropriate correlator of two light-like Wilson lines. The typical duration in proper time of these light-like lines is $T \sim 1/m$ as we noted above. We now suggest to analyze the gauge-invariant correlators using the AdS/CFT [6, 8] approach for N=4 SYM. This correlator in large $N$ and strong coupling $g^2 N$ regime can be obtained from the minimal surface in the five-dimensional AdS space, [6, 11, 8] tracing the light-like Wilson lines at its boundaries as shown in Fig. 1. The classical action for a string world-sheet is

$$S = \frac{1}{2\pi \alpha'} \int d\xi d\sigma \sqrt{\det(G_{MN}\partial_\alpha X^M \partial_\beta X^N)}$$

The AdS metric $G_{MN}$ in Poincare coordinate is given by

$$ds^2 = R^2 \frac{-dt^2 + dx^2 + dy^2 + dw^2 + dz^2}{z^2}.$$  \hspace{1cm} (8)

where $R = (2g_{YM}^2 N\alpha')^{1/4}$ with $2\pi g_s = g_{YM}^2$ is the radius of the AdS space. The AdS space has a boundary in Minkowski space $M_4$ at $z = 0$. The boundary condition on the string world-sheet is given by the two time-like trajectories

$$x = V t = v \tau, y = 0, z = 0; \quad \text{and} \quad x = -V t, y = b, z = 0$$

with $V = v/\gamma$ and $t$ is the real time. The minimal surface associated to (7-9) leads to a set of coupled partial differential equations, which we have not yet managed to solve exactly. Instead, we provide a variational estimate as we now explain.
Figure 1: Wilson lines for moving quarks are described by the minimal surface in $AdS_5$ having two trajectories as its boundary.
First, we divide the string world-sheet by constant time slices, each containing a string connected to two boundary points. Then we minimize the string length and integrate over it to get an approximate minimal area. Specifically, we choose an orthogonal coordinate system on the world-sheet, and define the area $A$ through
\begin{equation}
A = \int d\xi d\sigma \sqrt{(\partial_\xi X^\mu)^2} \sqrt{\frac{(\partial_\sigma X^\mu)^2}{z^4}}.
\end{equation}

Let
\begin{equation}
l(t) := \int_{L(t)}^0 d\sigma \sqrt{\frac{dx^2 + dy^2 + dw^2 + dz^2}{z^4}}
\end{equation}
be the 'length' of the string ending on the two receding quarks at the boundary with separation $L(t)$. $l(t)$ depends on $t$ only through $L(t)(= L(\tau))$, so that $l(t)$ and $l(\tau)$ represent the same quantity. First, we minimize this 'length' and then form the Area by adding up the areas of strips between the hyper-planes of time $t$ and $t + dt$. The height of the strip at the boundary is $dt \sqrt{1 - V^2} = d\tau$ where $\tau$ is the proper time at the boundary. The height of strip at the central point of the string is given by $dt \sqrt{1 - z_0(t)^2}$ where $z_0(t)$ is the maximum value of $z$ at the time slice $t$. Therefore by the trapezoidal rule,
\begin{equation}
A_{\text{min}} \simeq \frac{1}{2} \int d\tau (1 + \gamma \sqrt{1 - \dot{z}_0^2}) l_{\text{min}}(t).
\end{equation}

where $\gamma = 1/\sqrt{1 - V^2}$ and $l_{\text{min}}(t)$ is the minimal length for a fixed time slice. Summarizing, we replaced $d\xi \sqrt{(\partial_\xi X^\mu)^2}$ by $d\tau (1 + \gamma \sqrt{1 - \dot{z}_0^2})/2$, which is $\sigma$ independent.

Now, $l_{\text{min}}(t)$ can be found by choosing a coordinate system such that the two quarks are located at $x = L(\tau)/2$ and $x = -L(\tau)/2$. Then
\begin{equation}
l(t) = \int_{-L(t)/2}^{L(t)/2} dx \sqrt{\frac{1 + z^2}{z^4}}.
\end{equation}

where $L(t) = \sqrt{b^2 + 4v^2 \tau^2}$, with $v\tau = Vt$ is the separation between the two time-like receding partons at the boundary $z = 0$. So, the problem of finding a minimal $l(t)$ is almost identical to the problem of finding a static $q\bar{q}$ potential [10, 11]. The result for a properly regularized length is
\begin{equation}
l(\tau) = -\frac{c_0}{L(\tau)}
\end{equation}

with $c_0 = (2\pi)^3/\Gamma(\frac{1}{3})^4$ and $z_0(t)$ is found to be
\begin{equation}
z_0(t) = c_1 L(t).
\end{equation}
where \( c_1 = 1/\sqrt{c_0} \approx 0.834 \). Hence

\[
A_{\text{min}} = -\frac{1}{2} \int^{-T}_{-T} d\tau (1 + \gamma \sqrt{1 - \frac{c_1^2}{c_0}}) - \frac{c_0}{\sqrt{b^2 + 4v^2\tau^2}}
\approx - \left(1 + \gamma \sqrt{1 - (2c_1V)^2}\right) \frac{c_0}{2v} \sin^{-1}(\frac{2vT}{b}).
\]

The result for the Wilson-line correlator in N=4 SYM at large \( N \) and fixed \( g^2N \) is

\[
\langle (W - 1)(W - 1) \rangle_{\text{connected}} \approx \exp(-S) = \exp\left(-\frac{R^2}{2\pi\alpha'}A_{\text{min}}\right)
= \exp \left[ c_0 \sqrt{2g^2N} \left(1 + \sqrt{\frac{1 - (2c_1V)^2}{1 - V^2}}\right) \frac{1}{2v} \ln(\frac{4vT}{b})\right]
\]

(17)

Hence

\[
\langle (W - 1)(W - 1) \rangle_{\text{connected}} \approx \left(\frac{2\sqrt{s}}{m^2b}\right) \left(c_0 \sqrt{2g^2N} \left(1 + \sqrt{\frac{1 - (2c_1V)^2}{1 - V^2}}\right) \frac{1}{2v} \ln(\frac{4vT}{b})\right)^{\frac{1}{2}}
\]

(18)

where we have made the substitution \( v = \sqrt{s}/2m, T \sim 1/m \). Notice that \( \sqrt{\frac{1 - (2c_1V)^2}{1 - V^2}} \) is always smaller than 1 as far as it is real and is an imaginary number for \( V > 0.6 \). Therefore the \( 1/\sqrt{s} \) behavior of the real part of the exponent is unavoidable. This result can be contrasted with the one-gluon exchange contribution to the connected correlator

\[
\langle (W - 1)(W - 1) \rangle_{\text{1c}} \sim \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 \frac{v_1 \cdot v_2}{((v_1 \tau_1 - v_2 \tau_2)^2 + b^2)^{\frac{3}{2}}} \sim 2\pi \frac{v_1 \cdot v_2}{\sqrt{1 - (v_1 \cdot v_2)^2}} \ln(1/\mu b)
\]

(19)

where \( \mu \) is an infrared regulator \( (T = 1/\mu) \). \(^1\) Equation (18) is our main result. It is important to realize that had we used time instead of proper time, the outcome would have been independent of \( s \). The proper time description and its ensuing cutoff is pressed upon us by both the underlying string dynamics and the eikonal representation following from (1-2). Since the elastic amplitude involves typically momenta of order \( \sqrt{-t} \), we expect the impact parameter in (18) to be \( b \sim 1/\sqrt{-t} \). As a result, the crossing singularity of the two Wilson loops at \( b = 0 \) is dynamically regulated.

In the high energy limit \( (V \approx 1) \), the gauge-invariant combinations of the parton-parton scattering amplitude (6) reads

\[
\mathcal{T}(s,t) \approx c_1(s) \left(\frac{2is}{t}\right) \left(2\sqrt{-st}\right) \left[c_0 \sqrt{2g^2N} \left(\frac{m^2}{2s} + 0.67i\right)\right]
\]

(20)

\(^1\) For parallel moving quarks, the result is \( 1/b \) instead of \( \ln b \). Coulomb’s law is 2-dimensional for non-parallel moving light-like quarks and 3-dimensional for heavy or parallel moving light-like quarks.

7
for large $N$ and fixed $g^2 N$. Here $c_1(s) = 2\pi \int_0^\infty db' b'^{1-2\alpha(s)} J_0(b')$ and

$$\alpha(s) = \frac{1}{2} c_0 \sqrt{2g^2 N} \left( \frac{m}{\sqrt{s}} + 0.67i \right)$$  \hspace{1cm} (21)

This result is remarkable. The amplitude is nonperturbative in $g^2 N$ and it is different from the BFKL result [1, 2] especially in the presence of $1/\sqrt{s}$ in the exponent. Also, it shows that the elastic parton-parton scattering amplitude in $N=4$ SYM with a finite proper time cutoff $T \sim 1/m$ reggeizes and unitarizes at large $s$. One should notice that in large $s$ limit, the exponent has a finite piece since $\gamma/v = 1/V \rightarrow 1$. However, this piece is imaginary in high energy as indicated in eq.(20), which assures the unitarity. The Regge intercept is $\alpha(s)$. The appearance of the mass $m$ is the natural infrared cutoff in the problem. For quarks and gluons, it is simply their constituent mass.

The Regge behavior signals the occurrence of log $s$-dependence in the elastic amplitude, a behavior usually expected in asymptotically free and non-conformal gauge theories. How could this property be present for $N=4$ SYM with conformal symmetry? In fact the appearance of the string picture in the $N=4$ SYM was a much appreciated surprise in Maldacena’s result [6]. In our case, the Regge behavior follows from the the stringy behavior of $N=4$ SYM in the large $N$ limit. Also, conformal symmetry is upset in the presence of a finite proper time cutoff $T \sim 1/m$. The receding partons with momenta $\sqrt{s}$ defines a range in rapidity space of the order of $\ln s$. Powers of $\ln s$ count the number of ‘gluons’ exchanged in the t-channel. Since (20) can be written as power series of $\ln s$, it contains terms of infinite number of gluon exchange. A similar observation was also made by Verlinde and Verlinde [4] in the process of mapping high-energy scattering onto a conformally symmetric two-dimensional sigma model.

4. We now suggest that for intermediate quark masses $m$ such that $\sqrt{s} > m > \sqrt{-t}$ the result (??) maybe of relevance to QCD in four dimensions. Indeed, the gauge-invariant forms $T_{1,2}$ suggest a compactification of $M_4$ in the time-direction (weak-form). On the surface of constant $\pm T$ the gluons (gluinos) satisfy periodic (antiperiodic) boundary conditions. For finite $T$ these conditions discriminate between particles and their supersymmetric partners. For fixed but large $T \sim 1/m$ (arbitrarily small $m$), we expect the breaking of supersymmetry to be small, and it can be treated by a small perturbation from the AdS metric, possibly in the form of a light black-hole in the AdS space. To the leading order the calculation can be done with the AdS metric. And this is result of the previous section. For finite $T$, similar calculations can be done by changing the background metric from the
AdS to the a AdS black-hole:

\[ ds^2 = R^2 - f dt^2 + dx^2 + dy^2 + dw^2 + \left( \frac{1}{f} \right) dz^2. \]  

(22)

with \( f = 1 - (\mu z)^4 \). In our case, the scale \( \mu \) will be interpreted as a proper-time cutoff instead of temperature. By choosing periodic boundary conditions for the gluons, and antiperiodic boundary conditions for the gluinos, the latters acquire masses of order \( 1/T \sim m \sim \mu > \sqrt{-t} \) and decouple from t-channel exchange. If we assess the ‘length’ \( l(\tau) \) in the presence of the black hole as a function of the separation \( L(\tau) \), the outcome is more like a screened Coulomb potential [14, 12]. From (16), we note that the logarithmic contribution is coming from a large separation \( L(t) \) or large \( t \) (\( \tau \)) for finite impact parameter \( b \). However, for very small impact parameter \( b << \sqrt{s}/m^2 \), we get log dependence from both large and small time regions. Since we are integrating over \( 0 < b < \infty \), which of course contains the small \( b \) region, we still get the \( \log s \) dependence for very small impact parameters. Therefore we expect that the Regge behavior survives under SUSY breaking and the main feature of our result should also hold for the real QCD. More detailed analysis will be given elsewhere.

Finally, the present arguments also show that at large \( N \) and strong \( g^2 N \), the cross sections for quark-quark and quark-antiquark scattering are the same. The gluon-gluon scattering amplitude could be calculated similarly with the substitution \( N \rightarrow N^2 \), due to the adjoint representation of the gluon.

5. We have presented arguments for evaluating the elastic parton-parton scattering amplitude at large \( N \) and strong \( g^2 N \) in N=4 SYM. Although the latter is conformally invariant in \( M_4 \) the appearance of the string picture and the necessity to regulate the elastic contribution by a finite proper time \( T \sim 1/m \) has led to a Regge-like behavior that unitarizes at large \( \sqrt{s} \). The result cannot be arrived at by perturbation theory. The nature of the result depends sensitively on the string character of the underlying description and hence not applicable to Abelian-like theories such as QED.

Our main result follows from a physically motivated variational estimate of the minimal surface in the AdS space. The exact form of the extremal surface is too involved to be written down analytically. Although an exact result would of course be ideal, we do not expect our estimate of the parton-parton cross section to change appreciably. Indeed, the dominant contributions arise from scattering with large impact parameter \( b \sim 1/\sqrt{-t} \), for which our approximation for the extremal surface should be legitimate. For a large impact
parameter $b$, the extremal surface is smoothly twisted and eikonal approximation is also good.

We have suggested that for partons with masses $m$ in the range $\sqrt{s} < m < \sqrt{-t}$, the present results may be applicable to QCD at large $N$ and strong $g^2 N$. Another and perhaps complementary way to see this is to note that large $N$ QCD reduces to a master field description [16]. The latter may be regarded as the starting point in Maldacena [6] conjecture leading to the AdS metric. From a different point of view, Verlinde and Verlinde [4] have shown that at large $s$ the elastic amplitudes in QCD follows from a conformally symmetric two-dimensional sigma model, where the conformal symmetry is precisely broken when the light-like quark lines are regulated in the time-like directions. Could it be that in large $N$ the effective action derived by Verlinde and Verlinde [4] map onto an AdS type action?

Acknowledgments

We thank KIAS for the generous hospitality and support during the work. IZ is grateful to Maciek Nowak for discussions and Romuald Janik for comments. MR and SJS would like to thank Igor Klebanov, Taekoon Lee and Kyungsik Kang for helpful comments and discussions. The work of IZ was supported in part by US-DOE grant DE-FG-88ER40388 and that of SJS by the program BSRI-98-2441.
References


