Non-commutative Euclidean structures in compact spaces

B.-D. Dörfel
Institut für Physik, Humboldt-Universität zu Berlin
Invalidenstraße 110, D-10115 Berlin, Germany

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Abstract

Based on results for real deformation parameter $q$ we introduce a compact non-commutative structure covariant under the quantum group $SO_q(3)$ for $q$ being a root of unity. In a representation where $X^2$ is diagonal $P^2$ has been calculated. To manifest some typical properties an example of a one-dimensional $q$-deformed Heisenberg algebra is also considered and compared with non-compact case.

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1 Introduction

In paper [1] it was shown, how the $q$-deformation of the well-known group $SO(3)$ to quantum group $SO_q(3)$ can be used to define a non-commutative quantum space as a comodule of the quantum group. It is very natural to exploit the $R$ matrix as the main tool. Its decomposition into projectors generates a non-commutative (three-dimensional) Euclidean space of coordinates.

In all papers known to us the non-commutative structure has been defined for real $q$ only. The value of $q$ becomes important when we demand hermiticity for coordinates (and later on for momenta). For general complex $q$ the $R$ matrix looses its hermiticity which requires a new definition of conjugation for the coordinate operators. On the other hand their are at least two reasons why one should investigate the case of complex $q$.
First, real \( q \) implies always a non-compact coordinate space, while for a compact space we have to admit complex values of \( q \). In context with the fact, that non-commutative geometry [2] is considered to be the result of some deep dynamical principle which may be found e.g. in string theory the case of compactified dimensions is of special interest. We start here the consideration of an example with only compactified coordinates. The more interesting case with compact and non-compact dimensions (which seems to require different \( q \)) is due to further work. Second, we know the quantum group \( SO_q(3) \) for generic \( q \) and especially the case \( q \) being a root of unity, where it demonstrates some peculiarities [3,4]. It is therefore interesting how a non-commutative quantum space can be constructed in that special case. This will be the main aim of our paper.

As we have already mentioned, the key point is the definition of a conjugation for coordinates and momenta, which are later required to be self-adjoint with respect to that conjugation. Different conjugations result in different spaces and hence different physics. The conjugation we will propose below is of course equivalent to ordinary conjugation for real \( q \). We know two ways which are both consistent with \( SO_q(3) \). The choice that fits best with our problem is the one, where \( q \) is left untouched during conjugation. Thus if \( \bar{X} \) is the conjugate of an operator \( X \), the conjugate of \( qX \) is \( q\bar{X} \). This choice has been used already before, e.g. in [3]. The other way, one may find i.e. in [4], seems to work better in case if one deals with non-hermitean operators having only real eigenvalues, which will not be the case here.

At the first moment our definition looks rather unnatural but in Chap. 2 we shall describe how it works and mention the consequences. The most important one of them is that self-adjoint operators will have (instead of real ones) eigenvalues which are real functions of the parameter \( q \). But this is just what we need, because the scaling operator and its commutation properties force coordinates and momenta to have eigenvalues proportional to powers of \( q \).

The paper is organized as follows. In Chap. 2 we recall the basic formulae for the quantum space of \( SO_q(3) \) and state the modifications for our \( q \). In Chap. 3 we consider a one-dimensional example of a \( q \)-deformed Heisenberg algebra and demonstrate how it works for \( q \) being a root of unity. It is rather helpful to compare our results with earlier ones for real \( q \) with the same example. In our main Chap. 4 the non-commutative space covariant under \( SO_q(3) \) is considered and matrix elements of coordinates and momenta are calculated. The results are presented explicitly and do not contain any divergencies which usually occur if one simply replaces \( q \) in formulae derived earlier for real \( q \) only.
2 Euclidean phase space for q being a root of unity

First we have to recall some basic formulae of the non-commutative space from paper [1] which do not depend on the nature of $q$. The $R$ matrix of $SO_q(3)$ is decomposed like

$$\hat{R} = P_5 - \frac{1}{q^4}P_3 + \frac{1}{q^6}P_1 \quad (2.1)$$

We shall not give the projectors $P_i$ here, because we need only $P_3$. The non-commutative Euclidean space is defined by:

$$P_3XX = 0 \quad (2.2)$$

In the common basis (2.2) looks like:

$$X^3X^+ = q^2X^+X^3$$
$$X^3X^- = q^{-2}X^-X^3$$
$$X^-X^+ = X^+X^- + \lambda X^3X^3 \quad (2.3)$$

here $\lambda = q - \frac{1}{q}$. It is natural to define a metric $g_{AB}$ and an invariant product $X \circ Y$

$$X \circ Y = g_{AB}X^AY^B \quad (2.4)$$

$$g_{+-} = -q, \quad g_{--} = -1/q, \quad g_{33} = 1$$

which let $X \circ X$ commute with $X^A$. $P_3$ can be expressed through a generalized $\epsilon$-tensor

$$P_{3}^{AB}_{CD} = \frac{1}{1+q^2} \epsilon^{FAB}_{FDC} \quad (2.5)$$

where its indices are moved according to formulae like

$$\epsilon_{ABC} = g_{CD}\epsilon_{AB}^{D} \quad (2.6)$$

$$\epsilon_{+-}^{3} = q, \quad \epsilon_{--}^{3} = -q, \quad \epsilon_{33}^{3} = 1 - q^2, \quad \epsilon_{+3}^{+} = 1, \quad \epsilon_{3+}^{+} = -q^2,$$
$$\epsilon_{-3}^{3} = -q^2, \quad \epsilon_{33}^{-} = 1 \quad (2.7)$$

Eq. (2.3) is then equivalent to

$$X^C X^B \epsilon_{BC}^A = 0 \quad (2.8)$$

and the $R$ matrix can be expressed in the form

$$\hat{R}_{CD}^{AB} = \delta_{C}^{A}\delta_{D}^{B} - q^{-4}\epsilon^{FAB}_{FDC} - q^{-4}(q^2 - 1)g^{AB}_{CD} \quad (2.9)$$

Now we come to the definition of conjugation. We still choose

$$\overline{X^{A}} = g_{AB}X^{B} \equiv X_{A} \quad (2.10)$$
like in paper [1]. But for generic complex $q$ this is consistent with eqns. (2.3) only if we define $\tilde{q} = q$ which means $q$ is unchanged under conjugation. This forces us to distinguish between $q$ (and its functions) and constant complex numbers which are to be conjugated as usual. (We mean e.g. the $i$ in the Heisenberg relation, s. b.)

As a consequence of our definition self-adjoint operators like $(X \circ X)$ still have orthogonal eigenstates for different eigenvalues (under the scalar product induced by conjugation), but their eigenvalues are no longer real. They are rather real functions of the parameter $q$. Examples will follow below. Another consequence is that the scalar product of a vector by itself is no longer of definite sign and can even vanish for special values of $q$. Similar modifications occur for unitary operators being discussed below. Based on eq (2.10) we can now proceed as in [1] and define a derivative, momentum, angular momentum and the scaling operator $\Lambda$ in the same way. For the components of the momentum we have the analog of (2.8), while for angular momentum

$$L^C L^B \epsilon_{BC}^A = -1/q^2 W L^A$$

and

$$q^A(q^2 - 1)^2 L \circ L = W^2 - 1$$

$$L^A W = W L^A$$

The scaling operator $\Lambda$ is introduced in the same way with the properties

$$\Lambda^{1/2} X^A = q^2 X^A \Lambda^{1/2}$$

$$\Lambda^{1/2} P^A = q^{-2} P^A \Lambda^{1/2}$$

$$\Lambda^{1/2} L^A = L^A \Lambda^{1/2}$$

$$\Lambda^{1/2} W = W \Lambda^{1/2}$$

Conjugation of vector values is analogous to eq. (2.10), $W$ is self-adjoint and $\Lambda$ is unitary up to normalization:

$$\Lambda^{1/2} = q^{-6} \Lambda^{-1/2}$$

Eqns. (2.12) lead to the standard $SO_q(3)$ algebra. The generalized Heisenberg relations are

$$P^A X^B - \hat{R}^{-1AB}_{CD} X^C P^D = -i/2 \Lambda^{-1/2} \{(1 + q^{-6})g^{AB} W - (1 - q^{-4}) \epsilon^{ABC} L_C\}$$

Now we have to study representations of this algebra. For $q$ being a root of unity the physical relevant representations become finite dimensional while for real $q$ they have infinite dimension. Thus there is no difference here between self-adjoint, essentially self-adjoint and hermitean operators.

The representations will be studied in detail in Chap. 4.
3 Representations of a one-dimensional $q$-deformed Heisenberg algebra

We consider now a one-dimensional example of a $q$-deformed Heisenberg algebra. That is neither a projection of the Euclidean space nor based on the deformation of any symmetry group. It is even not non-commutative in the sense of space coordinates because there is only one. Nevertheless it is based on a modified Leibniz rule and has been studied for real $q$ in great detail \[5,6\]. It reflects very nicely the deep role which is played by the scaling operator $\Lambda$ that one has to introduce in a general non-commutative structure of coordinates and momenta. The algebra looks as follows:

\[
\frac{1}{\sqrt{q}} PX - \sqrt{q} XP = -iU
\]

\[
UP = qPU
\]

\[
UX = \frac{1}{q} XU
\]  

(3.1)

Conjugation is given by

\[
\bar{P} = P
\]

\[
\bar{X} = X
\]

\[
\bar{U} = U^{-1}
\]  

(3.2)

There is no problem for real $q$, for complex $q$ we demand $\bar{q} = q$ with the same consequences as in Chap. 2. Then (3.2) is consistent with (3.1). Now we put $q$ a root of unity, $q = e^{i\pi/r}$, $q^r = -1$. The integer $r$ is taken larger than 2. We shall consider a representation of the algebra (3.1) based on eigenvectors of $P$. From the second equation it follows that applying $U$ to such an eigenstate we obtain another one with eigenvalue multiplied by $q^{-1}$. Therefore we have

\[
P \mid n > \pi_0 = \pi_0 q^n \mid n > \pi_0
\]  

(3.3)

where $n$ is integer, $0 \leq n \leq 2r - 1$, and it is sufficient to have $\pi_0 > 0$. Further

\[
U \mid n > \pi_0 = \mid n - 1 > \pi_0
\]  

(3.4)

and we can introduce the scalar product

\[
\pi_0 < n \mid m > \pi_0 = \delta_{nm}
\]  

(3.5)

Now we have an example that the self-adjoint operator $P$ has eigenvalues being real numbers multiplied by powers of $q$. Those powers are a consequence of the properties of $U$. For our $q$ choosen we can see that the eigenstate $U \mid 0 > \pi_0$ has the same eigenvalue as $\mid 2r - 1 > \pi_0$. Disregarding the case of degeneration we have

\[
U \mid 0 > \pi_0 = C(\pi_0) \mid 2r - 1 > \pi_0
\]  

(3.6)
where \( C \) is a phase factor and different \( C \) label different representations. From eqns. (3.4) and (3.6) we have \( U^{2r} = C \) for any state in our representation. Now it is straightforward to define another unitary operator \( U' \) by

\[
U' = U e^{-i\alpha}
\]

(3.7)

where we have put \( C = e^{i\alpha} \). Then \( U'^{2r} = 1 \) and it is more convenient to work with a new system \( | n >' \)

\[
U' | n >' = | n - 1 >'
\]

(3.8)

The new eigenstates are just multiplied by phase factors. For shortness we have omitted the upper index \( \pi_0 \). From the first equation of (3.1) and its conjugate one can deduce

\[
X P = \frac{i}{\lambda} \left( \sqrt{qU} - \frac{1}{\sqrt{q}U^{-1}} \right)
\]

(3.9)

which shows the action of \( X \) on the states \( | n >' \) states:

\[
X | n >' = \frac{1}{q^n \lambda \pi_0} \left( \sqrt{q} e^{i\pi n} | n - 1 >' - \frac{1}{\sqrt{q}} e^{-i\pi} | n + 1 >' \right)
\]

(3.10)

This system of \( 2r \) equations can be solved and the eigenvalues and eigenstates of \( X \) can be found. But it is easier to exploit the eigenstates of \( U' \), as we shall demonstrate below. We start with

\[
| \phi_0 > = \sum_{n=0}^{2r-1} | n >'
\]

(3.11)

\[
| \phi_k > = (\pi_0)^{-k} P^k | \phi_0 > = \sum_{n=0}^{2r-1} q^{kn} | n >'
\]

and integer \( 0 \leq k \leq 2r - 1 \). Obviously

\[
U' | \phi_k > = q^k | \phi_k >
\]

(3.12)

We mention that for real \( q \) those states are non-normalizable which is not the case here.

Before constructing the eigenstates of \( X \) we shortly comment on the eigenstates of \( U' \) and \( U \). Our definition of an adjoint operator in Chap. 2 and the induced scalar product lead to unitary (isometry) operators with respect to that product which will have properties differing from the usual ones, as we have already seen for self-adjoint operators. The eigenstates of our unitary operators may not be orthogonal and can contain zero norm states. This is due to the fact that usual argumentations breaks down, if the eigenvalues depend on \( q \) as in eq. (3.12). So explicitly

\[
< \phi_k | \phi_m > = \sum_{n=0}^{2r-1} q^{n(k+m)}
\]

(3.13)
what is non-zero for \( m = k = 0 \) or \( m + k = 2r \). Hence we have only two non-zero norm states for \( k = 0 \) and \( r \) and the eigenstates \( | \phi_k > \) and \( | \phi_{2r-k} > \) for \( k = 1, \ldots, r-1 \) are not orthogonal. The situation is improved if we remember orthogonal matrices in real space. Indeed \( U' \) can be considered as such a matrix. We define a new basis by combining the pairs of vectors with indices \( k \) and \( 2r - k \) to new ones

\[
| \tilde{\phi}_m > = \sum_{n=0}^{2r-1} (Re q^{kn} + Im q^{kn}) | n >' \\
| \tilde{\phi}_{2r-m} > = \sum_{n=0}^{2r-1} (Re q^{kn} - Im q^{kn}) | n >' \\
1 \leq m \leq r - 1
\] (3.14)

and adding \( | \phi_0 > \) and \( | \phi_r > \). The new basis is orthogonal and can be normalized. In that basis we have (after normalization) \( U'^T = U'^{-1} \) and \( U' \) contains boxes with \( \sin \frac{\pi m}{r} \) and \( \cos \frac{\pi m}{r} \) in the usual way of orthogonal matrices. The situation for \( U \) can be read off from eq. (3.7). In general the eigenvalues of our unitary matrices are combinations of phases and functions of \( q \). Keeping in mind all that we can still work with the states (3.11) as a basis to construct the \( X \) eigenstates.

From the algebra (3.1) follows

\[
X | \phi_k >= d_k | \phi_{k-1} >
\] (3.15)

for \( 1 \leq k \leq 2r - 1 \) and

\[
X | \phi_0 >= d_0 | \phi_{2r-1} >
\] (3.16)

Next we have to calculate \( d_k \). We apply the conjugate of eq. (3.9) to \( | \phi_k > \) and find

\[
d_k = \frac{i}{\lambda \pi_0} (e^{i \frac{\pi q^{k-\frac{1}{2}}}} - e^{-i \frac{\pi q^{-k+\frac{1}{2}}}})
\] (3.17)

This formula works for all \( 0 \leq k \leq 2r - 1 \). We construct the eigenstates the following way

\[
X | x_m > = x_m | x_m >
\]

\[
| x_m > = \sum_{k=0}^{2r-1} a_k | \phi_k >
\] (3.18)

yielding the recursion relation for the coefficients

\[
a_{k+1} = \frac{x_m}{d_{k+1}} a_k
\] (3.19)

Consistency requires

\[
a_0 = \frac{x_m}{d_0} a_{2r-1}
\] (3.20)
We can put $a_0 = 1$ and the solution of eqs. (3.19) and (3.20) are

$$a_k = (x_m)^k(\prod_{l=1}^{k} d_l)^{-1}$$

$$\left(x_m\right)^{2r} = \prod_{l=1}^{2r} d_l = \frac{i^{2r}}{\lambda^{2r}\pi_0^{2r}}4\cos^2\frac{\alpha}{2}$$

(3.21)

The r.h.s. of eq. (3.21) is a positive number. The $X$ eigenvalues are given by all roots of unity (or powers of $q$) multiplied by this number in power $\frac{1}{2r}$. The states (3.18) are orthogonal for different $m$. Thus we have obtained the same picture for the operator $X$ as in eq. (3.3) for the operator $P$, except just one point. For $\alpha = \pi$ all eigenvalues of $X$ vanish, moreover $X$ becomes nilpotent, $X^r = 0$. We know that ordinary hermitean operators cannot be nilpotent (except the trivial case). Looking closer one finds $d_0 = d_r = 0$ and it follows that the (non-orthonormal) system $|\phi_k>$ forms its canonical basis. Therefore the whole situation is connected with the existence of zero-norm states which are usually not allowed but cannot be forbidden here as explained in Chap. 2.

Now we can compare our results with those for real $q$ obtained in papers [5] and [6]. The main difference is that all our representations have finite dimensions which avoids the mathematical problems of the real case. Also we do not have the two sectors with respect to the sign of $\pi_0$. On the other hand we have to introduce an additional parameter $C$ (or $\alpha$) characterizing the representation. As soon as $\alpha \neq \pi$ the operators $X$ and $P$ are manifestly equivalent in our representation. As far as the degenerate case $\alpha = \pi$ is concerned we believe that it is the reflection of the fact that the existence of eigenstates of $P$ with vanishing eigenvalue cannot be excluded from the very beginning as we did with demanding $\pi_0 > 0$.

4 $SO_q(3)$ deformation in compact space

In this Chapter we give the representations of the $q$-deformed algebra (2.8), (2.11) - (2.15) for $q = e^{\frac{i\pi}{r}}$. We have not written the $L^A X^B$ and $L^A P^B$ relations which are the same as in [1]. We are also not going to repeat the derivations of papers [1] and [7] leading to the $T$-operators and explaining the appearance of the Clebsch-Gordon coefficients because on the algebraic level there are no changes. The changes start as soon as representations are considered, what shall be done now.

We choose $L \circ L$, $L^3$ and $X \circ X$ as a complete set of commuting variables. One can proceed as in the undeformed case and exploit eqs. (2.11) and (2.12). For the angular momentum the eigenvalues are

$$L \circ L \mid j, m, n > = \frac{q^{-6}}{(q^2 - q^{-2})^2} (q^{4j+2} + q^{-4j-2} - q^2 - q^{-2}) \mid j, m, n >$$
\[ L^3 | j, m, n > = - \frac{q^{-3}}{(q - q^{-1})} \left( q^{2m} - \frac{q^{2j+1} + q^{-2j-1}}{q + q^{-1}} \right) | j, m, n > \]  

(4.1)

where \( j \) and \( m \) are integers, \( |m| \leq j \) and \( 0 \leq j \leq j_{\text{max}} \). (Note that the sign of \( L^3 \) is opposite to the usual one, because we have kept the conventions of paper [1].) For \( q \) being a root of unity we must remember that there are two types of representations, called types I and II in paper [3]. We allow only type II representations for the construction of the non-commutative space. That the type I representations can be omitted consistently follows from paper [4]. The type II representations behave as for \( q = 1 \) (and general real \( q \)) except the fact \( j_{\text{max}} \leq \frac{r}{2} - 1 \). The states are fully determined by the quantum numbers \( j, m \) and \( n \). From the first eq. of (2.13) we read off

\[ X^2 | j, m, n > = l_0^2 q^{4n} | j, m, n > \]  

(4.2)

It is sufficient to choose the integer \( n \) as \( 0 \leq n \leq r - 1 \) and \( l_0 > 0 \). The parameter \( l_0 \) plays the same role as \( \pi_0 \) in the one-dimensional case.

All our representations are unitary and either irreducible or fully reducible [3]. Irreducible representations are labelled by the integer \( j \). Because of eq. (4.2) we deal with finite dimensional irreducible representations like in the one-dimensional case before. That and the existence of a \( j_{\text{max}} \) are the main differences with respect to real \( q \).

The states are normalized in the usual way (this defines our metric). The phase factors can be chosen to fulfill

\[ \Lambda_{\pm}^2 | j, m, n > = q^{-3} | j, m, n - 1 > \]

\[ \Lambda_{\pm}^{-2} | j, m, n > = q^{-3} | j, m, n + 1 > \]  

(4.3)

From eq. (2.11) the matrix elements of \( L^\pm \) can be obtained. We mention for further use

\[ W | j, m, n > = \frac{\{2j + 1\}}{\{1\}} | j, m, n > \]  

(4.4)

where we have introduced the abbreviations

\[ \{a\} = q^a + q^{-a} \]

\[ [a] = \frac{q^a - q^{-a}}{\lambda} \]  

(4.5)

In papers [1] and [7] one finds how the \( SO_q(3) \) structure can be used to define reduced matrix elements for \( X^A \) and \( P^A \). For the non-vanishing matrix elements we quote the results

\[ < j + 1, m + 1, n | X^+ | j, m, n > = q^{m-2j} \sqrt{[j + m + 1][j + m + 2]} < j + 1, n | X^- | j, n > \]

\[ < j - 1, m + 1, n | X^+ | j, m, n > = q^{m+2j+2} \sqrt{[j - m][j - m - 1]} < j - 1, n | X^- | j, n > \]

\[ < j + 1, m - 1, n | X^- | j, m, n > = q^m \sqrt{[j - m + 1][j - m + 2]} < j + 1, n | X^- | j, n > \]
\[
\begin{align*}
<j - 1, m - 1, n | X^- | j, m, n> &= q^m \sqrt{[j + m][j + m - 1]} <j - 1, n || X^- || j, n> \\
<j + 1, m, n | X^3 | j, m, n> &= q^{m-j+\frac{1}{4}} \sqrt{1 + q^2} \sqrt{[j - m + 1][j + m + 1]} <j + 1, n || X^- || j, n> \\
<j - 1, m, n | X^3 | j, m, n> &= -q^{m+j+\frac{1}{4}} \sqrt{1 + q^2} \sqrt{[j - m][j + m]} <j - 1, n || X^- || j, n>
\end{align*}
\]

\hspace{1cm} (4.6)

The matrix elements on the r.h.s. are the reduced ones. Using conjugation properties (2.10) we have
\[
<j + 1, n || X^- || j, n> = -q^{2j+2} <j, n || X^- || j + 1, n>
\]

\hspace{1cm} (4.7)

Therefore only one reduced matrix element has to be determined what is easily obtained from the first eq. of (2.3) and (4.2). We fix the phase by setting
\[
<j + 1, n || X^- || j, n> = \frac{l_0 q^{j+2n}}{\sqrt{2[2j+1][2j+3]}}
\]

\hspace{1cm} (4.8)

By the way, the first eq. of (2.3) also tells us that \(<j, n || X^- || j, n> must vanish.

Now we come to the matrix elements of \(P^A\). Based on eqs. (4.6) and (4.7) they are calculable relying on the matrix elements of the values \(X \circ P\) and its conjugate \(P \circ X\).

The Heisenberg relation (2.15) with the help of the \(R\) matrix (2.9) yields after contraction
\[
P \circ X - q^6 X \circ P = -\frac{i}{2} \lambda^{-\frac{1}{2}} (1 + q^{-6}) (q^2 + 1 + q^{-2}) W
\]

\hspace{1cm} (4.9)

Together with its conjugation eq. (4.9) gives
\[
X \circ P = \frac{i (\Lambda^\frac{1}{2} - \Lambda^{-\frac{1}{2}}) W}{2 q^2 (q^2 - 1)}
\]

\hspace{1cm} (4.10)

\[P \circ X = \frac{i (q^{-6} \Lambda^{-\frac{1}{2}} - q^6 \Lambda^\frac{1}{2}) W}{2 q^2 (q^2 - 1)}\]

Therefore \(X \circ P\) has matrix elements only between neighbouring \(n\). We consider now
\[
<j, m, n | X \circ P | j, m, n + 1> = -q^2 \{[2j + 3][2j + 2]
\]

\hspace{1cm} (4.11)

\[
= -\frac{i}{2} \frac{W_j q^5 (q^2 - 1)}{q^2 (q^2 - 1)}
\]

where the reduced matrix elements of \(P^A\) are defined analogous to eqs. (4.6) including the fact that they are no longer diagonal in \(n\). Now it is straightforward to take
\[
<j, m, n + 1 | X \circ P | j, m, n> = -q^2 \{[2j + 3][2j + 2]
\]

\hspace{1cm} (4.12)

\[
= \frac{i}{2} \frac{W_j q}{q^2 - 1}
\]
We put in eqs. (4.7) and (4.6) and the conjugation relations
\begin{align*}
< j + 1, n \parallel P^- \parallel j, n + 1 > &= -q^{2j+2} < j, n + 1 \parallel P^- \parallel j, n > \\
< j + 1, n + 1 \parallel P^- \parallel j, n > &= -q^{2j+2} < j, n \parallel P^- \parallel j + 1, n + 1 > \quad (4.13)
\end{align*}

The system (4.11) and (4.12) can be rewritten as two recursion relations in $j$ for the two unknowns, the reduced matrix elements of $P$. An easy way to solve it, is to start with $j = 0$, read off the general formula and prove it by insertion. For clearness, we present all non-vanishing reduced matrix elements
\begin{align*}
< j + 1, n \parallel P^- \parallel j, n + 1 > &= -iq^{-j-6-2n}Z^{-1}, < j, n + 1 \parallel P^- \parallel j + 1, n > = -iq^{-3j-8-2n}Z^{-1} \\
< j + 1, n + 1 \parallel P^- \parallel j, n > &= iq^{3j-2-2n}Z^{-1}, < j, n \parallel P^- \parallel j + 1, n + 1 > = iq^{j-4-2n}Z^{-1} \quad (4.14)
\end{align*}

where the common denominator is
\[ Z = 2l_0\lambda \sqrt{2j + 1} \]

Neither eq. (4.8) nor eq. (4.14) contains any divergencies because of the condition $j_{\text{max}} \leq \frac{r}{2} - 1$. If $j + 1$ exceeds $j_{\text{max}}$ the matrix element simply vanishes as it does for $j - 1 = -1$.

Our next aim is to calculate the eigenvalues of $P^2 \equiv P \circ P$. We shall follow the lines of Chap. 3 and start with the definition of a unitary operator
\[ U = q^3 \Lambda^\frac{1}{2} \quad (4.15) \]

Going back to eq. (4.3) we have
\[ U \mid n > = \mid n - 1 > \quad (4.16) \]

where we have omitted all quantum numbers which are unchanged. After
\[ U \mid 0 > = e^{ia} \mid r - 1 > \quad (4.17) \]

we introduce
\[ U' = Ue^{-\frac{ia}{r}} \]
\[ U' \mid n >' = \mid n - 1 >' \quad (4.18) \]

The eigenstates of the operator $U'$ are given by
\begin{align*}
\mid \phi_k > &= \sum_{n=0}^{r-1} q^{2nk} \mid n >' \\
U' \mid \phi_k > &= q^{2k} \mid \phi_k > \quad (4.19)
\end{align*}
Note that the eigenstates for even \( k \) can be produced by the operator \( X^2/l_0^2 \) acting \( k/2 \) times on \( |\phi_0>\). From the algebra (2.13) follows

\[
P \circ P |\phi_k> = \tilde{d}_k |\phi_{k-2}>
\]

(4.20)

where we shall calculate \( \tilde{d}_k \) below. For the \( P \)-eigenstates we use the ansatz

\[
P \circ P |p_n> = p_n^2 |p_n>
\]

\[
|p_n> = \sum_{k=0}^{r-1} a_k |\phi_k>
\]

(4.21)

Eq. (4.20) yields the recursion relation

\[
a_{k+2} = \frac{p_m^2}{\tilde{d}_{k+2}} a_k
\]

(4.22)

Now it is necessary to distinguish between even and odd \( r \). In the first case we obtain two different solutions putting \( a_0 = 1, a_1 = 0 \) and vice versa. They contain either even or odd numbers of \( k \) in the sum (4.21). Consistency gives for the eigenvalues

\[
(p^2)^{r}_{+} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k}
\]

\[
(p^2)^{r}_{-} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k+1}
\]

(4.23)

For odd \( r \) the sum (4.20) contains all numbers and hence

\[
(p^2)^{r} = \prod_{k=0}^{r-1} \tilde{d}_{k}
\]

(4.24)

The coefficients \( \tilde{d}_k \) are calculated via the matrix elements of \( P^2 \) between the \( |j, n>\) states. We have the same structure as in the first parts of eqs. (4.11) and (4.12), e.g.

\[
< j, n + 2 | P^2 | j, n > = - q^2 \{ [2j + 3][2j + 2] \}
\]

\[
< j, n + 2 | P^- | j + 1, n + 1 > < j + 1, n + 1 | P^- | j, n >
\]

\[
+ [2j][2j - 1]
\]

\[
< j, n + 2 | P^- | j - 1, n + 1 > < j - 1, n + 1 | P^- | j, n >
\]

(4.25)

With the results of eq. (4.14) we get

\[
< j, n + 2 | P^2 | j, n > = - \frac{q^{4n-10}}{4l_0^2 \lambda^2}
\]

(4.26)

and the same way

\[
< j, n - 2 | P^2 | j, n > = - \frac{q^{4n-2}}{4l_0^2 \lambda^2}
\]

(4.27)

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A little bit more lengthy is the calculation of the diagonal element due to the doubling of terms connected with intermediate states having quantum numbers \( n \pm 1 \).

\[
<j, n \mid P^2 \mid j, n> = \frac{q^{-4n-6}}{4l_0^2 \lambda^2} \{4j + 2\} \tag{4.28}
\]

As soon as the quantum numbers of the r.h.s. ket vector are fixed there are no further non-vanishing matrix elements. Now we consider

\[
P^2 \mid n \rangle = n >' < n, n \mid P^2 \mid n >'
\]

\[
+ \mid n + 2 \rangle < n + 2, n \mid P^2 \mid n >' + \mid n - 2 \rangle < n - 2, n \mid P^2 \mid n >'
\]

\[
\sum_{n=0}^{r-1} q^{2nk} \mid n >' (q^{-4k} e^{-2im \tau} < n + 2 \mid P^2 \mid n > + q^{4k} e^{2im \tau} < n - 2 \mid P^2 \mid n >
\]

From eq. (4.19) follows

\[
P^2 \mid \phi_k > = \sum_{n=0}^{r-1} q^{2nk} \mid n >' (q^{-4k} e^{-2im \tau} < n + 2 \mid P^2 \mid n > + q^{4k} e^{2im \tau} < n - 2 \mid P^2 \mid n >
\]

Substituting eqs. (4.26)-(4.28) we can read off

\[
\tilde{d}_k = \frac{q^{-6}}{4l_0^2 \lambda^2} \left( \{4j + 2\} - q^{-4k} e^{-2im \tau} - q^{4k} e^{2im \tau} \right)
\]

\[
= \frac{q^{-6}}{4l_0^2} \left[ 2k + 2j + 3 + \frac{\alpha}{\pi} \right] \left[ 2k - 2j + 1 + \frac{\alpha}{\pi} \right] \tag{4.31}
\]

The product in eqs. (4.23) and (4.24) can be taken easily. Finally we have for even \( r \)

\[
(p^2_\pm)^2 = \frac{-i \tau}{2r \lambda^2 l_0^2} 4 \cos^2 \frac{\alpha}{2} \tag{4.32}
\]

and for odd \( r \)

\[
(p^2)^r = \frac{i^{2r}}{2r \lambda^2 l_0^2} 4 \sin^2 \alpha \tag{4.33}
\]

While \( \tilde{d}_k \) depends on \( j \), \( p^2 \), of course, does not. It is remarkable that eq. (4.32) very much resembles eq. (3.21) derived for the one-dimensional model. The difference between (4.32) and (4.33) is due to the different role played by \( \alpha \) in both cases.

For odd \( r \) all eigenvalues of \( P^2 \) are given by the \( r \) different roots of unity (or powers of \( q \)) multiplied by the r.h.s. of eq. (4.33) in power \( 1/r \). For even \( r \) any eigenvalue is degenerated twice, disregarding the obvious degeneration with respect to \( j \) and \( m \). Because of the – sign in front (produced by the factor \( q^{-6} \) in (4.31)) \( p^2_\pm \) contains a phase
factor $e^{\frac{2\pi}{i}(2m+1)}$ where $m$ goes from zero to $\frac{r}{2} - 1$. All eigenvectors (4.21) are orthogonal and normalizable. (Note that this is not true for the $|\phi_k>$ states.)

The main difference to real $q$ is the finiteness in dimension for the eigenvector space.

In addition, $P^2$ becomes nilpotent (and non-diagonalizable) for special $\alpha$, namely $\alpha = \pi$ ($r$ even) and $\alpha = 0, \pi$ ($r$ odd). The reason is the same as mentioned before, the existence of special zero-norm states. It is interesting to look at the canonical basis for the cases above. This is done by finding the roots of the r.h.s. of eq. (4.31), which depend on angular momentum $j$.

First consider the case $r = 4s$ ($s$ integer). Every "angle" has two zeros, one for even and one for odd $k$. The canonical basis of $P^2$ consists of four series of length $s$, their first members depending on $j$. We have always $(P^2)^s = 0$.

The situation for $r = 4s + 2$ is a little bit more complicated. For $j = 0$ we have two series of length $s + 1$ and two of length $2s$. While the length of the first two grows the other two shrink for increasing $j$. At $j = s$ both zeros coincide. We are left with only two series of length $r/2$. For higher $j$ until $j_{max} = 2s$ the picture expresses mirror symmetry. Therefore $(P^2)^{s+1} = 0$ (for $j = 0$) varies until $(P^2)^{2s+1} = 0$ ($j = s$) and again $(P^2)^{s+1} = 0$ ($j = 2s$).

For odd $r$ the situation is symmetric with respect to $\alpha = 0$ and $\alpha = \pi$. We have always two series. For $j = 0$ they have lengths $r/2 + 1/2$ and $r/2 - 1/2$, respectively. For $j = j_{max} = r/2 - 3/2$ the longer one reaches $r - 1$, the smaller is just 1. Thus $(P^2)^{\frac{r}{2}+\frac{1}{2}} (j = 0)$ varies until $(P^2)^{r-1} = 0$ ($j = j_{max}$).

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References
