CHIRAL ANOMALIES IN FIELD THEORIES\textsuperscript{1}

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Abstract

The role of the contribution from the fermion mass term in the axial vector Ward identity in generating the U(1) axial anomaly, both local and global, is elucidated. Gauge invariance requires the fermion to decouple from the gauge field if it is very heavy. This identifies the Adler-Bell-Jackiw (ABJ) anomaly with the asymptotic limit of the sign reversed mass term. In an instanton background, the chiral limit \((m = 0)\) of the mass term does not vanish but consists of contributions from fermion zero modes. Space time integral of these zero mode contributions exactly cancels, thanks to the Atiyah-Singer index theorem, the integral of the ABJ anomaly and suggests that the Jacobian for global U(1) chiral transformation is trivial even in an instanton background. This can be realised in the representation of the fermion partition function in a Weyl basis. The resolution of the strong CP problem is thus achieved in an axionless physical world.

In chiral gauge theories the fermion partition function admits of a gauge invariant representation but only at the cost of locality. Implementation of fermion averaging of the gauge current with the invariant partition function yields the current whose covariant derivative is the covariant anomaly. With the covariant current as input one can derive an integrable current whose covariant derivative is the minimal consistent anomaly obeying the Wess-Zumino consistency condition. The distinction between the two currents disappears if either the covariant or the consistent anomaly vanishes. This is realised only if the fermion belongs to an anomaly-free representation of the gauge group.

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1 Introduction

In classical field theories there is a correspondence between a global symmetry of the action and a conserved Noether current. Presence of short distance singularities which need to be regularised for mathematical consistency complicates matters in quantum field theory (QFT). It may so happen that a regularisation scheme with mandatory attributes, like gauge invariance in a gauge theory of fermions, and at the same time consistent with the global symmetry cannot be formulated or simply does not exist. Traces of violation of the global symmetry in the form of non-conservation of the Noether current may survive as the regulator is removed at the end of calculation. This is the genesis of anomalies and anomalous Ward identities in QFT.

The topic of anomaly, in particular, axial anomaly came on the centre stage of particle physics research through the studies of neutral pion decay into two photons. The decay rate $1.2 \times 10^{16}$ per sec. was explained satisfactorily by Steinberger\(^1\) in 1949 in terms of triangle diagrams (Fig.1) with proton circulating in the fermion loop. The linear divergence of the amplitude was regulated by the Pauli–Villars method. Problem arose sixteen years later\(^2\), when decay rates obtained within the framework of current algebra and partial conservation of axial vector current (PCAC) were invariably smaller than the data by three orders of magnitude.

A popular working hypothesis, PCAC derives its dynamical basis in gauge theories of fermion like quantum chromodynamics (QCD) from the ‘naive’ operator relation (or, equivalently, naive Ward identity)

$$\partial_\mu (\bar{q} \gamma_5 \gamma_\mu \tau_3 q) = 2m (\bar{q} \gamma_5 \tau_3 q)$$ ...

(1.1)

which follows from field equations, with $q$ the quark doublet ($u, d$) and $\tau_3$ the isospin. One recognises in the left hand side the Noether current corresponding to the chiral symmetry $u \rightarrow e^{i\alpha\gamma_5}u, d \rightarrow e^{-i\alpha\gamma_5}d$, which should be conserved at the classical level in the chiral limit $m = 0$ of QCD. PCAC is just the statement that the mass term on the right hand side of (1.1) can be replaced by the neutral pion field

$$\partial_\mu (\bar{q} \gamma_5 \gamma_\mu \tau_3 q) = F_\pi m_\pi^2 \pi^0$$ ...

(1.2)

where $F_\pi$ is the pion decay constant, $m_\pi$ the pion mass, and $\pi^0$ the pion field. This is an unexceptionable step since the mass term has the right quantum numbers of a neutral pion and, therefore, can be regarded as the definition of the pion field in terms of quark constituents.

Problem with PCAC in $\pi^0 \rightarrow 2\gamma$ stemmed from the Sutherland–Veltman\(^3\) theorem which states
that substitution of the divergence of isospin axial current for the neutral pion field in the matrix element yields a null result for the decay rate. Coupled with the positive result of Steinberger, the unambiguous conclusion that emerges from the theorem is that the inadequacy of the PCAC relation stems really from the naive relation (1.1) which is flawed if quarks participate in electromagnetic interactions. The missing element was diagnosed as an anomaly, the Adler-Bell-Jackiw (ABJ) anomaly in the Noether current for chiral symmetry

$$\partial_\mu (\bar{q}\gamma_5 \gamma_\mu \tau_3 q) = 2m (\bar{q}\gamma_5 \tau_3 q) - \left( \frac{N_c}{3} \right) \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

... (1.3)

where \(N_c\) is the colour degree of freedom of quarks and \(F_{\mu\nu}\) the electromagnetic field tensor.

The ABJ anomaly, therefore, modifies the ‘naive’ PCAC relation (1.2) to

$$F_\pi m_\pi^2 \pi^0 = \partial_\mu (\bar{q}\gamma_5 \gamma_\mu \tau_3 q) + \left( \frac{N_c}{3} \right) \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

The decay rate now calculated by substituting the anomaly term for the pion field in the matrix element for \(\pi^0 \to 2\gamma\) is given by

$$\Gamma (\pi^0 \to 2\gamma) = \left( \frac{N_c}{3} \right)^2 \times 1.11 \times 10^{16} \text{ sec}^{-1}$$

... (1.4)

Depending on how one looks at the result (1.4), it may be regarded as either a signal success of the diagnosis of the problem in \(\pi^0 \to 2\gamma\) as due to anomaly, or in the light of later developments, a prediction of the number of colour degrees of freedom \(N_c = 3\) in QCD.

Success in \(\pi^0 \to 2\gamma\) problem brought into limelight the scenario of breaking symmetries at the classical level through anomalies in quantum field theories. Gauge theories become inconsistent if gauge symmetry is violated through anomaly. Cancellation of anomalies, therefore, constitutes an important constraint in building models for physical gauge theories with chiral coupling to fermions. Global chiral anomaly seems to play a key role in discussing physical effects associated with topologically nontrivial gauge field configurations.

2 Axial Anomaly and Fermion Decoupling

In a gauge theory of fermion there is a contradiction at the quantum level between chiral invariance and gauge symmetry. The ABJ anomaly, or in the more general context of non-Abelian gauge theories of fermion, the anomaly in the U(1) axial vector current, arises because gauge invariance is to be preserved for consistency of the theory. The contradiction is transparent in
the condition for decoupling of the fermion from the background gauge field when it is very heavy. For the divergence of the U(1) axial vector current the decoupling condition assumes the form of an anomalous Ward identity

$$\langle \partial_\lambda \left( \bar{\psi}(x) \gamma_5 \gamma_\lambda \psi(x) \right) \rangle = 2m \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle - \lim_{m \to \infty} \left[ 2m \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle \right]$$

(2.1)

where $\langle \rangle$ denotes that the fermion degrees of freedom are integrated out. As we shall see below, (2.1) follows directly from gauge invariance and is known as Adler condition in QED. Eq.(2.1) will still be compatible with chiral symmetry and a conserved U(1) axial vector current would emerge in the chiral limit $m = 0$ if the second term on the right hand side were to vanish. But this is not to be. The asymptotics in field theory gives in the infinite mass limit the ABJ anomaly

$$\lim_{m \to \infty} \left[ 2m \langle \bar{\psi}(x) \gamma_5 \psi(x) \rangle \right] = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} tr F_{\mu\nu} F_{\lambda\rho}$$

(2.2)

where $F_{\mu\nu} = F_{\mu\nu}^a t_a$ are the field tensors with $t_a$ the generators of the gauge group.

To motivate the decoupling condition (2.1) we consider in QED the amplitude for creating two photons with momenta and polarisation $(p, \mu)$ and $(k, \nu)$ by the axial vector current $J_{\lambda\mu}(x) = \bar{\psi}(x) \gamma_5 \gamma_\lambda \gamma_\mu \psi(x)$

$$\langle p, \mu; k, \nu | J_{\lambda\mu}(0) | 0 \rangle = \epsilon_\mu(p) \epsilon_\nu(k) M_{\lambda\mu\nu}(p, k, m)$$

The key to the analysis is the Rosenberg tensor decomposition (see Fig.1)

$$M_{\lambda\mu\nu} = \epsilon_{\lambda\mu\nu\alpha} k_\alpha A(p, k, m) + \epsilon_{\lambda\mu\nu\alpha\beta} p_\alpha k_\beta B(p, k, m) + \epsilon_{\lambda\mu\nu} C(p, k, m) + [(k, \nu) \leftrightarrow (p, \mu)]$$

(2.3)
which follows from parity and Lorentz invariance.

Not all the form factors are independent. The form factor $A$ which gives the divergence of the axial vector current

$$(p + k)_\lambda M_{\lambda \mu \nu} = \epsilon_{\mu \nu \alpha \beta} p_\alpha k_\beta \left[ A(p, k, m) + A(k, p, m) \right] \quad \ldots (2.4)$$

is determined through gauge invariance by the form factors $B$ and $C$

$$A(p, k, m) = p^2 B(p, k, m) + p.k C(p, k, m) \quad \ldots (2.5)$$

The form factors $B$ and $C$ are of dimensions $[\text{mass}]^{-2}$, and, therefore, in perturbation theory they are represented by highly convergent amplitudes which vanish as $m^{-2}$ for large fermion mass

$$\lim_{m \to \infty} B(p, k, m) = \lim_{m \to \infty} C(p, k, m) = 0$$

Thus gauge invariance (2.5) guarantees that the divergence of the amplitude for the axial vector current given in (2.4) vanishes in the asymptotic $m \to \infty$ limit

$$\lim_{m \to \infty} (p + k)_\lambda M_{\lambda \mu \nu} = 0 \quad \ldots (2.6)$$

In perturbation theory the amplitude $M_{\lambda \mu \nu}$ for the triangle diagram is linearly divergent. The leading divergence, however, drops out due to symmetric integration of loop momentum leaving a potential logarithmic divergence, which can appear only in the form factor $A$ in (2.3). Gauge invariance (2.5) rules out even this residual logarithmic divergence.

The above observations suggest that the potential anomaly represented by the second term in (2.1) must be finite and independent of regularisation scheme. To verify this we start by calculating

$$\lim_{m \to \infty} \left[ 2m \langle \bar{\psi}(x) \gamma_5 \gamma(x) \rangle \right] = \lim_{m \to \infty} \left[ 2m \langle x | Tr \gamma_5 (i \slashed{D} + m)^{-1} | x \rangle \right], \quad \ldots (2.7)$$

where, to conform to our discussions in the subsequent sections, we work in Euclidean metric and write for the hermitian Dirac operator

$$\slashed{D} = \gamma_\mu (i \partial_\mu - g A_\mu) \quad \ldots (2.8)$$

with $A_\mu \equiv A_\mu^a t_a$, the gauge potential.
Our strategy is to develop the Green function appearing in (2.7) in a perturbative series

$$(i\slashed{D} + m)^{-1} = (-i\slashed{D} + m)G$$

with

$$G = (\slashed{D}^2 + m^2)^{-1} = G_0 - gG_0VG_0 + g^2G_0V^2G_0V + \ldots \quad \ldots (2.9)$$

The ‘free’ part is $G_0 = (p^2 + m^2)^{-1}$ with $p_\mu = i\partial_\mu$ the ‘momentum’ operator. The ‘potential’ $gV$ has two pieces

$$gV = gV_0 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$$

with $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. The first piece $gV_0$ is at most linear in $p$ and independent of $\gamma$–matrices. The trace with $\gamma_5$ in (2.7) starts to be nonvanishing only from terms of order $g^2$ onwards in the perturbative expansion (2.9) and one obtains (2.2)

$$\lim_{m \to \infty} [2m\langle \bar{\psi}(x)\gamma_5\psi(x) \rangle] = \lim_{m \to \infty} [2m^2\langle x| Tr(\gamma_5G) |x \rangle] = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}trF_{\mu\nu}F_{\lambda\rho}(x)$$

Note that the final result is local. All nonlocalities as well as contributions from higher order terms in the perturbative series (2.9) drop out in the infinite mass limit $m \to \infty$.

In the decoupling condition (2.1) one can set the mass term on the right hand side to zero in the chiral limit $m = 0$. The anomalous Ward identity thus obtained

$$\langle \partial_\lambda (\bar{\psi}\gamma_5\gamma_\lambda\psi) \rangle_{m=0} = -\frac{g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}trF_{\mu\nu}F_{\lambda\rho} \quad \ldots (2.10)$$

shows that the U(1) axial vector current, i.e. the Noether current corresponding to global chiral symmetry

$$\psi \rightarrow e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{i\alpha\gamma_5}, \quad \ldots (2.11)$$

of the massless Dirac operator (2.8), is not conserved. The divergence of the current is just the ABJ anomaly which is responsible for the two photon decay of neutral pion discussed in the preceding section.

We note that in renormalisable theories in perturbative framework the decoupling condition (2.1), which is a special example of the decoupling theorem of Appelquist and Carazzone, is correct to all orders of perturbation, just as the Adler–Bardeen theorem assures us that the anomalous axial vector Ward identity (2.10) is not affected by radiative corrections in QED.

It should be remarked that setting the mass term to zero in (2.1), as was done in obtaining (2.10), may not always be legitimate in the chiral limit $m = 0$ if the gauge field is treated
nonperturbatively. The Euclidean Dirac operator (2.8) has zero modes if the background gauge field has a nontrivial topology. In this scenario the chiral limit of the mass term does not vanish and, as we shall see in the next section, consists precisely of the zero modes of the Dirac operator.

3 Path Integral Approach to Anomaly

In a seminal work Fujikawa\textsuperscript{9} interpreted the ABJ anomaly within the path integral framework as arising from the nontrivial Jacobian of the fermion measure under chiral transformation. In Euclidean metric, considered by Fujikawa, the partition function which generates fermion Green functions in a background gauge field configuration may be written as

\[ Z_f[A] \equiv \int d\mu \exp \left[ \int \bar{\psi} (i\slashed{D} + m) \psi d^4x \right] \]  

where \( d\mu \) is the integration measure for fermion and \( \slashed{D} \) is the Euclidean Dirac operator defined in (2.8). Fermion Green functions which are normalised expectation values of any product \( O \) of fermion fields are obtained from the functional integral (3.1)

\[ \langle O \rangle = \frac{1}{Z_f[A]} \int d\mu O \exp \left[ \int \bar{\psi} (i\slashed{D} + m) \psi d^4x \right] \]  

With \( \gamma \)-matrices chosen hermitian, the Dirac operator (2.8) is also hermitian with real eigenvalues \( \lambda_n \) and orthonormal eigenfunctions \( \phi_n(x) \)

\[ \slashed{D}\phi_n(x) = \lambda_n \phi_n(x), \quad \int \phi_m^+(x) \phi_n(x) d^4x = \delta_{mn} \]  

Each nonzero eigenvalue \( \lambda_n \) has its chirally conjugate partner \( -\lambda_n \) with eigenfunctions \( \phi_{-n} \)

\[ \slashed{D}\phi_{-n} = -\lambda_n \phi_{-n}, \quad \phi_{-n} = \gamma_5 \phi_n \]  

In perturbative field theories one is interested in gauge field configurations with only trivial topology. For such configurations the kernel space of the Euclidean Dirac operator (2.8) is of dimension zero. This means that the set \( \{ \phi_n(x) \} \) with nonzero eigenvalues constitute a complete basis in function space. The Dirac field \( \psi(x) \) can be expanded in this basis as

\[ \psi(x) = \sum (a_n + a_{-n} \gamma_5) \phi_n(x) \]  

where \( a_{\pm n} \) are complex-valued Grassmann generators. The four degrees of freedom corresponding to each mode of the Dirac field is accounted for if we split \( a_{\pm n} \) as

\[ a_{\pm n} = a_{\pm n} + i\beta_{\pm n} \]
with $\alpha, \beta$ real valued.

There are ambiguities\textsuperscript{10,11,12} on the issue whether in Euclidean metric $\bar{\psi}$ should be treated as independent of $\psi$. For the present, we follow the popular ansatz\textsuperscript{10} and expand $\bar{\psi}(x)$ with an independent set of Grassmann generators $\{\bar{b}_{\pm n}\}$,

$$\bar{\psi}(x) = \sum \phi_{+}^{n}(x) \left( \bar{b}_{n} + \bar{b}_{-n} \gamma_{5} \right)$$ \hspace{1cm} \ldots (3.6)

The representations for $\psi(x)$ and $\bar{\psi}(x)$ together with orthonormality of the eigenfunctions (3.3) yield for the fermion action

$$S_{f}(m) = \int \bar{\psi}(x) (i\mathcal{D} + m) \psi(x) d^{4}x$$
$$= \sum \left[ (i\lambda_{n} + m) \bar{b}_{n} a_{n} + (-i\lambda_{n} + m) \bar{b}_{-n} a_{-n} \right]$$ \hspace{1cm} \ldots (3.7)

The integration measure for the fermion fields in the basis $\{\phi_{n}(x)\}$ is

$$d\mu \equiv \Pi_{n} d\bar{b}_{n} d a_{n} d\bar{b}_{-n} d a_{-n}$$ \hspace{1cm} \ldots (3.8)

The standard rules of integration of Grassmann generators now yield for the partition function (3.1) the desired result

$$Z_{f}[A] = \Pi_{n} (\lambda_{n}^{2} + m^{2})$$
$$= \det (i\mathcal{D} + m)$$ \hspace{1cm} \ldots (3.9)

This confirms the correctness of the choice of the measure (3.8).

Ward identities, whether normal or anomalous, are obtained in path integral framework from the requirement that the partition function is invariant under infinitesimal symmetry transformations of the variables of integration. To derive the Ward identity corresponding to chiral symmetry one implements a ‘local’ chiral transformation of the variables of integration $\psi(x), \bar{\psi}(x)$ in the partition function (3.1)

$$\psi(x) \rightarrow \psi'(x) = (1 + i\alpha(x) \gamma_{5}) \psi(x)$$
$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) (1 + i\alpha(x) \gamma_{5})$$ \hspace{1cm} \ldots (3.10)

The fermion measure (3.8) changes and the new measure corresponding to the transformed variables of integration is given by $d\mu'$

$$d\mu' = \Pi_{n} d\bar{b}_{n}' d a_{n}' d\bar{b}_{-n}' d a_{-n}'$$
$$= d\mu \ J[\alpha],$$ \hspace{1cm} \ldots (3.11)

where $a_{n}'(\bar{b}_{n}')$ are the new set of Grassmann generators in the expansion of $\psi'(x) \left( \bar{\psi}'(x) \right)$ in the basis $\{\phi_{n}(x)\}$. The Jacobian $J[\alpha]$ can be calculated following standard procedure

$$J[\alpha] = \exp \left[ -2i \int d^{4}x \alpha(x) \sum_{n} \left( \phi_{+}^{n}(x) \gamma_{5} \phi_{n}(x) + \phi_{-}^{n}(x) \gamma_{5} \phi_{-n}(x) \right) \right]$$ \hspace{1cm} \ldots (3.12)
The fermion action also changes and the new action is given by

\[ S_f(m) \rightarrow S'_f(m) = \int d^4x \left[ \bar{\psi}(i\slashed{D} + m)\psi + i\alpha(x) \left( -\partial_{\mu} \left( \bar{\psi}\gamma_{5}\gamma_{\mu}\psi \right) + 2m\bar{\psi}\gamma_{5}\psi \right) \right] \] ... (3.13)

Invariance of the partition function (3.1) under the infinitesimal local chiral transformation (3.10) now gives the anomalous axial Ward identity

\[ \langle \partial_{\mu} \left( \bar{\psi}\gamma_{5}\gamma_{\mu}\psi \right) \rangle = 2m\langle \bar{\psi}\gamma_{5}\psi \rangle - 2\sum_{n} \left( \phi_{n}^{+}\gamma_{5}\phi_{n} + \phi_{n}^{-}\gamma_{5}\phi_{n} \right) \] ... (3.14)

It is easy to recognise (3.14) as the decoupling condition (2.1) of the preceding section. Indeed

\[ \lim_{m \rightarrow \infty} \left[ 2m\langle \bar{\psi}\gamma_{5}\psi \rangle \right] = \lim_{m \rightarrow \infty} \left[ 2m\sum_{n} \left( \phi_{n}^{+}\gamma_{5}\phi_{n} + \phi_{n}^{-}\gamma_{5}\phi_{n} \right) \right] \]

\[ = 2\sum_{n} \left( \phi_{n}^{+}\gamma_{5}\phi_{n} + \phi_{n}^{-}\gamma_{5}\phi_{n} \right) \] ... (3.15)

where the infinite sum on the right hand side of (3.14) is to be cut off gauge invariantly, \(|\lambda_n| \ll M\) for \(M\) large. Fujikawa used the gauge invariant cut off exp \((-D/2)/M^2\) with large \(M\) to evaluate the infinite sum

\[ 2\sum_{n} \left( \phi_{n}^{+}(x)\gamma_{5}\phi_{n}(x) + \phi_{n}^{-}(x)\gamma_{5}\phi_{-n}(x) \right) = \lim_{M \rightarrow \infty} \langle x|Tr \left( \gamma_{5}e^{-\slashed{D}/M^2} \right) |x \rangle \]

\[ = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}trF_{\mu\nu}(x)F_{\lambda\rho}(x) \] ... (3.16)

It is clear that the left hand side of (3.16) should be augmented by zero modes if the Dirac operator admits of them. Zero modes always appear with definite chiralities, \(\epsilon_i = \pm 1\)

\[ \slashed{D}\phi_{0i} = 0, \quad \gamma_{5}\phi_{0i} = \epsilon_i\phi_{0i} \] ... (3.17)

This is because, in its kernel space the Dirac operator commutes with \(\gamma_{5}\). In the presence of zero modes the left hand side of (3.16) needs to be augmented by their contributions, i.e.,

\[ \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}trF_{\mu\nu}(x)F_{\lambda\rho}(x) = 2 \left[ \sum_{i} \epsilon_i\phi_{0i}^{+}\phi_{0i} + \sum_{n} \left( \phi_{n}^{+}\gamma_{5}\phi_{n} + \phi_{n}^{-}\gamma_{5}\phi_{-n} \right) \right] \] ... (3.18)

Space time integral of (3.18) gives the Atiyah–Singer index theorem

\[ \nu \equiv \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\lambda\rho} \int d^4xtrF_{\mu\nu}(x)F_{\lambda\rho}(x) = n_+ - n_- \] ... (3.19)

where \(\nu\) is the winding number (Pontryagin index) of the gauge field and \(n_+ (n_-)\) is the number of positive (negative) chirality zero modes. Eigenmodes corresponding to nonzero eigenvalues do not contribute to the space time integral (3.19) because \(\phi_n\) is orthogonal to \(\gamma_5\phi_n\). Note that nontrivial winding number, \(\nu \neq 0\), is realised through instanton–like configuration of the gauge field.
The presence of zero modes has profound impact on the chiral limit of the fermion mass term on the right hand side of the anomalous axial Ward identity

\[
\langle \partial_\mu (\bar{\psi} \gamma_5 \gamma_5 \psi) \rangle = 2m \langle \bar{\psi} \gamma_5 \psi \rangle - \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} tr F_{\mu\nu} F_{\lambda\rho} \quad \ldots (3.20)
\]

The zero modes can be isolated from the mass term

\[
2m \langle \bar{\psi} \gamma_5 \psi \rangle = 2m \langle \bar{\psi} \gamma_5 \psi \rangle' + 2 \sum \epsilon_i \phi_i^+ \phi_0^i \quad \ldots (3.21)
\]

where the first term on the right hand side is bereft of the zero modes and vanishes in the chiral limit

\[
2m \langle \bar{\psi} \gamma_5 \psi \rangle' = 4 \sum_{\lambda_n > 0} \frac{m^2}{m^2 + \lambda_n^2} \phi_n^+ \gamma_5 \phi_n \quad \ldots (3.22)
\]

The mass term, therefore, has now a nontrivial chiral limit consisting precisely of the zero modes, and the chiral limit of the axial Ward identity is not exactly what was obtained in the perturbative framework of the preceding section

\[
\langle \partial_\mu (\bar{\psi} \gamma_5 \gamma_5 \psi) \rangle_{m=0} = 2 \sum \epsilon_i \phi_i^+ \phi_0^i - \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} tr F_{\mu\nu} F_{\lambda\rho} \quad \ldots (3.23)
\]

The zero modes in the extra piece appearing on the right hand side arise from instanton–like configuration of the gauge field and, therefore, could not have been accessed in a perturbative framework.

It is of great interest to note that the zero mode terms which appear explicitly on the right hand side of (3.23) are exactly cancelled by similar terms contained now (see (3.18)) in the ABJ anomaly. Thus, irrespective of whether or not the gauge field configuration gives rise to zero modes, the chiral limit of the local axial anomaly comprises of only nonzero eigenmodes of the Dirac operator

\[
\langle \partial_\mu (\bar{\psi} \gamma_5 \gamma_\mu \psi) \rangle_{m=0} = -2 \sum_{|\lambda_n| \neq 0} \left( \phi_n^+ \gamma_5 \phi_n + \gamma_{-n}^+ \gamma_5 \phi_{-n} \right) \quad \ldots (3.24)
\]

We, therefore, conclude that the space-time integral of the chiral limit of the divergence of the axial vector current always vanishes. This follows from the orthogonality of \(\phi_n\) and \(\gamma_5 \phi_n\) if one uses (3.24) or from the Atiyah–Singer index theorem if instead one uses (3.23)

\[
\int \langle \partial_\mu (\bar{\psi} \gamma_5 \gamma_5 \psi) \rangle_{m=0} d^4x = 0 \quad \ldots (3.25)
\]

This, as we shall see later, has a profound impact on issues of physics related to global chiral anomaly.
4 Chiral Gauge Theories and the Covariant and Consistent Anomalies

The ABJ anomaly in the U(1) axial vector Ward identity constitutes an unambiguous evidence of a fundamental incompatibility of chiral invariance and gauge symmetry in regularisation scheme in perturbative framework of quantum field theory. In a vector-like gauge theory, such as QCD, chiral invariance is an expendable attribute and the ABJ anomaly results from strict adherence to gauge symmetry. In chiral gauge theories where gauge fields are coupled chirally to fermions in the Dirac operator

\[ D = \gamma_\mu \left( i \partial_\mu + t_a A_\mu^a (1 - \gamma_5) \right) = \left( i \partial + A_\mu^a (1 - \gamma_5) \right), \]

loss of chiral invariance jeopardises gauge symmetry and hence the consistency of the theory.

The fermion action

\[ S_F = \int \bar{\psi} D \psi d^4x \]

is invariant under the local chiral gauge transformations

\[ \psi(x) \rightarrow e^{i\alpha(x) \frac{1-\gamma_5}{2}} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\alpha(x) \frac{1+\gamma_5}{2}} \]

\[ A_\mu(x) \rightarrow e^{i\alpha(x)} \left( A_\mu(x) + \frac{1}{i} \partial_\mu \right) e^{-i\alpha(x)} \]

with \( \alpha(x) = t_a \alpha_a(x) \) the gauge function. Dimensional regularisation, popular in perturbative gauge theories, has serious problem with \( \gamma_5 = \frac{1}{4!} \epsilon_{\mu\nu\lambda\rho} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho \). The totally antisymmetric tensor of rank four \( \epsilon_{\mu\nu\lambda\rho} \) does not admit of suitable generalisation to arbitrary space-time dimensions. Thus, one is yet to find a consistent and systematic scheme for regulating divergences in chiral gauge theories in weak coupling perturbation in the continuum.

On lattice, the finite spacing \( a \) between lattice sites provides a built-in regularisation of all short distance singularities in field theories. Here too, the prospects for a consistent formulation of chiral gauge theory are not really bright. The major problem on lattice is the species doublers of fermion and their removal. The doublers appear as unwanted zeros of the Fourier transform of the ‘free’ Dirac operator on lattice, over and above the zero at the origin of momentum space which correspond to the physical fermion. In the ‘naive’ Dirac operator \( (\gamma_\mu \sin(p_\mu a) / a) \) the doublers are located at the edges of the Brillouin zone \( -(\pi/a) \leq p_\mu \leq (\pi/a) \). The doublers are not specific for the naive Dirac operator. According to the celebrated theorem of Nielsen and Ninomiya\(^{14} \) these are generic and can be avoided only at a price, by breaking explicitly
locality and/or chiral symmetry in the Dirac operator. The most popular model for lattice fermion, the Wilson model\textsuperscript{15}, removes the doublers by giving them masses of the order of the lattice cut-off $O(1/a)$

$$D_W(p) = \gamma_\mu \sin(p_\mu a)/a + ir (1 - \cos(p_\mu a))/a \quad \ldots (4.3)$$

Gauge invariance is implemented simply through link variables as in all lattice models. But the explicit breaking of chiral symmetry for nonzero ‘$r$’ makes the model patently inappropriate for chiral gauge theories. Current spurt in interest in the subject stems mainly from the realisation that for lattice Dirac operators $D$ obeying the Ginsparg–Wilson\textsuperscript{16} relation

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D, \quad \ldots (4.4)$$

chiral symmetry is restored and species doublers are removed in the continuum limit\textsuperscript{17}. The issue of nonlocality implied in the Ginsparg–Wilson relation, particularly in the context of chiral gauge theories, is yet to be resolved\textsuperscript{18}.

**Covariant Anomaly\textsuperscript{19}**: Apart from the absence of a consistent and systematic regularisation scheme, chiral gauge theories are, in general, afflicted with anomalies in the gauge current. The Dirac operator (4.1) in chiral gauge theory is non-hermitian. A fallout of this is that Fujikawa’s\textsuperscript{9} recipe for constructing a gauge invariant partition function, which assumes a hermitian Dirac operator, needs to be modified. The Dirac operator $\mathcal{D}$ in (4.1) maps $\psi$ into the space of spinors in the domain of $\mathcal{D}^+$. The eigenvalue equations (3.3) are, therefore replaced by

$$\mathcal{D} \phi_n = \lambda_n \chi_n, \quad \mathcal{D}^+ \chi_n = \lambda_n \phi_n, \quad \ldots (4.5)$$

where $\lambda_n^2$ are real, nonnegative and constitute the eigenvalue spectrum of $\mathcal{D}\mathcal{D}^+$ and $\mathcal{D}^+\mathcal{D}$. The sets of eigen functions $\{\phi_n\}$ and $\{\chi_n\}$ of $\mathcal{D}^+\mathcal{D}$ and $\mathcal{D}\mathcal{D}^+$ respectively constitute an orthonormal basis for expanding $\psi$ and $\bar{\psi}$

$$\psi = \sum_n a_n \phi_n, \quad \bar{\psi} = \sum_n \bar{b}_n \chi_n^+ \quad \ldots (4.6)$$

in terms of the Grassmann generators $a_n, \bar{b}_n$. The fermion measure defined as

$$d\mu[A] = \Pi_n d\bar{b}_n da_n \quad \ldots (4.7)$$

is a gauge invariant functional of $A_\mu$ and yields the partition function\textsuperscript{19}

$$Z_{inv}[A] \equiv \int d\mu[A] \exp[\int \bar{\psi} \mathcal{D} \psi d^4x] = (\det \mathcal{D}^+\mathcal{D})^{1/2} = (\det \mathcal{D}\mathcal{D}^+)^{1/2} \quad \ldots (4.8).$$
Both $D^+D$ and $DD^+$ change by a similarity transformation under gauge transformation. The representation (4.8) is thus formally gauge invariant.

The chiral gauge current

$$\left[\bar{\psi}t_a\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\psi\right]$$

transforms covariantly under gauge transformation (4.3). Fermion averaging of the current with the gauge invariant measure (4.7) yields

$$J_a^\mu(x) \equiv \int d\mu[A] \left[\bar{\psi}t_a\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\psi\right] \exp\left[\int_\mathcal{D}\bar{\psi}\psi d^4x\right]$$

$$= \sum_n \frac{1}{\lambda_n} \chi_n^+t_a\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\phi_n.$$  

Gauge invariant regularisation can be implemented by suppressing large eigenvalues and the current thus obtained transforms covariantly and is called the covariant current.

Formal application of field equations suggest that the gauge current should be covariantly conserved. This, however, may not be true for the fermion averaged current $J_a^\mu(x)$ if it is anomalous,

$$G^a(x) \equiv \partial_\mu J_a^\mu(x) - f^{abc}A_b^\mu(x)J_c^\mu(x)$$

$$= \sum_n \left\{\chi_n^+t_a\frac{1}{2}(1+\gamma_5)\chi_n - \phi_n^+t_a\frac{1}{2}(1-\gamma_5)\phi_n\right\}.$$  

Following Fujikawa’s\textsuperscript{9} recipe for gauge invariant regularisation one obtains the covariant anomaly

$$G^a(x) = \lim_{M \to \infty} \int \frac{d^4k}{(2\pi)^4} Tr t_a \left[\frac{1}{2}(1+\gamma_5)e^{ik.x}e^{-\frac{\pi\tau}{M^2}}e^{-ik.x}\frac{1}{2}(1+\gamma_5)e^{ik.x}e^{-\frac{\pi\tau}{M^2}}e^{-ik.x}\right]$$

$$= -\frac{1}{32\pi^2}\epsilon_{\mu\nu\lambda\rho} tr [t_aF_{\mu\nu}F_{\lambda\rho}].$$  

where $F_{\mu\nu} = t_aF_{a\mu\nu}$ are the field tensors.

**Consistent Anomaly\textsuperscript{10,20}:** In perturbative treatment of chiral gauge theories the fermion measure in the partition function is independent of the gauge field. A fallout of this is that, unlike $Z_{inv}[A]$ in (4.8), the perturbative partition function

$$Z_{pert}[A] \equiv e^{W[A]} = \int d\mu \exp\left[\int_\mathcal{D}\bar{\psi}\psi d^4x\right],$$  

and hence the effective action $W[A]$ need not be gauge invariant. The gauge current with fermion averaging implemented through this perturbative partition function

$$J_{W^\mu}^a(x) \equiv \frac{\delta}{\delta A_{\mu}^a(x)}W[A],$$

$$= \left\langle\bar{\psi}t_a\gamma_{\mu}\frac{1}{2}(1-\gamma_5)\psi\right\rangle_W.$$  

13
will, in general, not transform covariantly. However, it must obey the integrability condition

\[ \frac{\delta J^a_{W\mu}(x)}{\delta A^b_{\mu}(x')} - \frac{\delta J^b_{W\nu}(x')}{\delta A^a_{\nu}(x)} = 0, \]  

(4.15)

since it is defined in (4.14) through the functional derivative of the effective action \( W[A] \). The current \( J^a_{W\mu}(x) \) is called the consistent current and its covariant derivative

\[ G^a(x) \equiv \partial_\mu J^a_{W\mu}(x) - f^{abc} A^b_{\mu} J^c_{W\mu}(x) \]  

(4.16)

is the consistent anomaly.

Gauge transformation properties of an arbitrary functional of gauge fields are best discussed with the help of the generators

\[ L^a(x) = \partial_\mu \frac{\delta}{\delta A^a_{\mu}(x)} - f^{abc} A^b_{\mu} \frac{\delta}{\delta A^c_{\mu}(x)} \]  

(4.17)

Thus the consistent anomaly \( G^a_W(x) \), representing as it does the gauge variation of the effective action \( W[A] \), is given by

\[ G^a_W(x) = L^a(x)W[A] \]  

(4.18)

The algebra of the generators

\[ [L^a(x), L^b(x')] = f^{abc} \delta^a(x-x')L^c(x) \]  

(4.19)

shows that the consistent anomaly must obey the Wess-Zumino\(^{21} \) consistency condition

\[ L^a(x)G^b_W(x') - L^b(x')G^a_W(x) = f^{abc} \delta^4(x-x')G^c_W(x) \]  

(4.20)

On the other hand, the anomaly \( G^a_W(x) \) is a measure of the non-covariance of the consistent current \( J^a_{W\mu}(x) \)

\[ L^b(x')J^a_{\mu}(x) = -f^{abc} \delta^4(x-x') + \frac{\delta G^b_W(x')}{\delta A^a_{\mu}(x)} \]  

(4.21)

As for the covariant anomaly (4.12), one finds, as expected, an incompatibility with the Wess-Zumino consistency condition

\[ L^a(x)G^b(x') - L^b(x')G^a(x) = 2f^{abc} \delta^4(x-x')G^c(x), \]  

(4.22)

where the factor 2 on the right hand side spoils consistency. Thus, the anomaly itself is a measure of the ‘inconsistency’. The origin of the ‘inconsistency’ may be traced to the fermion measure \( d\mu[A] \) given by (4.7) for averaging of the gauge current in the definition (4.10) of
the covariant current $\mathcal{J}_a^\mu(x)$. A nontrivial covariant anomaly $G^a(x)$ corresponds to a nontrivial
dependence of the measure $d\mu[A]$ on the gauge field. This is suggested also from the observation
that the definition
$$\mathcal{J}_a^\mu(x) \equiv \frac{\delta}{\delta A^a_\mu(x)} \ln Z_{inv}[A] \quad \ldots (4.23)$$
where $Z_{inv}[A]$ is the gauge invariant partition function (4.8), has all the attributes, it is covari-
ant, consistent and anomaly free. The price that one pays for this ‘perfect’ current is a high
degree of nonlinearity.

It can be shown\(^{19}\) that the consistent current coincides with the covariant current if the func-
tional curl of the latter vanishes
$$J^a_{W\mu}(x) = J^a_\mu(x) + \int_0^1 dg \int d^4x' A^b_\nu(x') \left\{ \frac{\delta J^b_{\nu}(x')}{\delta A^a_\mu(x)} - \frac{\delta J^a_{\nu}(x)}{\delta A^b_\mu(x')} \right\} \quad \ldots (4.24)$$
where $J^a_\mu(x)$ is the covariant current corresponding to the Dirac operator $D^a = (i\partial + gA^1_\mu(\begin{array}{c} 1 \\ \gamma^5 \end{array}))$
with coupling constant $g$. One can obtain from (4.24) an explicit representation of the consistent
anomaly using the expression (4.12) for the covariant anomaly\(^{19}\)
$$G^a_W(x) = f_0^1 dg G^{ag} + \frac{1}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \int_0^1 dgg(1-g) tr \left( [t_a, A_\mu] \left( F^{a\nu}_{\lambda\rho} A_\nu + A_\nu F^{a\rho}_{\lambda\nu} \right) \right) \quad \ldots (4.25)$$
The above analysis shows that the distinction between covariant and consistent currents disap-
ppears if and only if the anomaly in either current vanishes. The fundamental requirement that
the chiral gauge theory is free of either anomaly imposes the unique constraint on the group
generators of the chiral fermions
$$tr \left( t_a \{ t_b, t_c \} \right) = 0 \quad \ldots (4.26)$$
which is symmetric in all the indices. An interesting application in the Standard Model is
to take $t_a = Q$, the matrix of electric charge, and $t_b, t_c$ the isospin matrices. The constraint
$trQ = 0$ is obeyed in the Standard Model since each generation of quark doublet of three colours
is paired with a lepton doublet.

5 Global Chiral Anomaly and the Strong CP Problem.

Global U(1) axial anomaly is the sine qua non for the strong CP problem. The problem consists
in the gross disagreement in the experimental data for the CP violating electric dipole moment
of neutron (EDMN) which are consistent with a null result and theoretical estimates that
invariably give a large value. Strong CP problem provides the unique arena where the concept of a global chiral anomaly is confronted with direct experimental data.

The two possible sources for CP violation in QCD action

\[
S_{QCD} = S_G + \int \bar{q}(i\gamma_\mu q) d^4x + m \int \bar{q}e^{2i\alpha_{ew}\gamma_5} q d^4x + \theta_{QCD}\Delta S \quad \ldots (5.1)
\]

are the chiral phase \(\alpha_{ew}\) in the quark mass which arises from the electroweak sector of the Standard Model, and the QCD vacuum term with parameter \(\theta_{QCD}\)

\[
\theta_{QCD}\Delta S = \theta_{QCD} \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\lambda\rho} \int tr F_{\mu\nu} F_{\lambda\rho} d^4x \quad \ldots (5.2)
\]

In (5.1) \(S_G\) represents the contributions from the gauge fields. For gauge fields with nontrivial topology the coefficient of \(\theta_{QCD}\) in (5.2) gives precisely the winding number \(\nu \neq 0\),

\[
\nu = \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\lambda\rho} \int tr F_{\mu\nu} F_{\lambda\rho} d^4x \quad \ldots (5.3)
\]

The chiral phase in the mass term in (5.1) can be transformed away by relabelling the quark fields

\[
q \rightarrow e^{-i\alpha_{ew}\gamma_5} q, \quad \bar{q} \rightarrow \bar{q}e^{-i\alpha_{ew}\gamma_5} \quad \ldots (5.4)
\]

There relabelling, however, introduces a Jacobian

\[
J(\alpha_{ew}) = \exp \left[ -i\alpha_{ew} \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \int tr F_{\mu\nu} F_{\lambda\rho} d^4x \right] \quad \ldots (5.5)
\]

where the coefficient of \(\alpha_{ew}\) in the exponent is \(2\nu\), i.e. twice the winding number of the gauge field configuration, which is nontrivial precisely in sectors where instantons live. The relabelling, therefore, merely shifts \(\alpha_{ew}\) to \(\theta_{QCD}\) giving an effective \(\tilde{\theta}\)

\[
\tilde{\theta} = \theta_{QCD} - 2N_f\alpha_{ew} \quad \ldots (5.6)
\]

where \(N_f\) is the number of quark flavours. All physical quantities in this scenario, therefore, depends on \(\tilde{\theta}\) and not on \(\theta_{QCD}\) or \(\alpha_{ew}\) individually. Theoretical estimates\(^{22}\) for CP-violating EDMN are all in the range

\[
d_n^{th} \approx \tilde{\theta} \times 10^{-15\pm1} \text{e.cm} \quad \ldots (5.7)
\]

Experimental data \(d_n^{ex} \leq 10^{-26}\text{e.cm}\), therefore, suggests \(\tilde{\theta} < 10^{-9}\). Such a small value requires near cancellation of two parameters \(\theta_{QCD}\) and \(\alpha_{ew}\) as in (5.6), which arise from completely different sectors of the Standard Model. This is the strong CP problem, which is essentially a problem of fine tuning.
Attempts to remedy the strong CP problem by invoking a spontaneously broken global chiral U(1) symmetry, the Peccei-Quinn symmetry, have been pursued vigorously\textsuperscript{22}. The idea essentially is that the effective $\bar{\theta}$ becomes a dynamical variable in this scenario involving the field of the pseudoscalar Goldstone boson associated with the broken Peccei-Quinn symmetry. The dynamical $\bar{\theta}$ could then settle down to a minimum consistent with the conservation of P and CP. The axion has been virtually ruled out by experiments and the strong CP problem in its original formulation is no closer to a resolution now than it was at the time of its conception\textsuperscript{22}.

**Question of Global Chiral Anomaly**: In view of the prevailing impasse, with axion window virtually closed, it is worthwhile to reexamine critically the basic premises that lead up to the strong CP problem. The question of a nontrivial global chiral anomaly clearly stands out as the most vulnerable among these basic premises.

The chiral limit of the axial vector Ward identity in a instanton-like background gauge field was given in (3.23)

\[
\langle \partial_{\mu} (\bar{\psi} \gamma_5 \gamma_\mu \psi) \rangle_{m=0} = 2 \sum \epsilon_i \phi_{0i} \phi_{0i} - \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} t^r F_{\mu\nu} F_{\lambda\rho} \quad \ldots (3.23)
\]

where the zero modes $\phi_{0i}(x)$ are a fallout of the nontrivial winding number $\nu$ of the gauge field. It is natural to identify the right hand side of (3.23) as the density of global chiral anomaly in an instanton-like background. Its space-time integral, the global chiral anomaly, vanishes by the Atiyah-Singer index theorem (3.19). This patently contradicts a nontrivial Jacobian as in (5.5), the cornerstone of the strong CP problem. The popular perception of a nontrivial global chiral anomaly and hence a nontrivial Jacobian (5.5) not only leads to the strong CP problem but is afflicted with contradictions in the chiral limit.

The source of these afflictions is easily traced to the popular identification of the partition function with the determinant of the Dirac operator

\[
Z_f[A]_{\nu \neq 0} = \text{det}(i\slashed{D} + m) \quad \ldots (5.8)
\]

which is unphysical in the chiral limit because of zero modes. A key to the problem is provided by the theorem\textsuperscript{23} which states that there are no wrong chirality zero modes of the Dirac operator $\slashed{D}$, i.e., in the Atiyah-Singer index theorem (3.19) positive (negative) chirality zero modes $n_+(n_-)$ are associated with positive (negative) winding number $\nu$. Thus

\[
\dim \ker(D_R D_L) = 0, \quad \nu \geq 0 \\
\dim \ker(D_L D_R) = 0, \quad \nu \leq 0 
\quad \ldots (5.9)
\]
where $D_L, D_R = D_L^\dagger$ are the Weyl components of the Dirac operator $\slashed{D}$

$$\slashed{D} = \begin{pmatrix} 0 & D_L \\ D_R & 0 \end{pmatrix} \quad \ldots (5.10)$$

The theorem (5.9), therefore, assures that the partition functions defined as

$$Z_f[A]_{\nu \geq 0} = \det (D_R D_L + m^2)$$

$$Z_f[A]_{\nu \leq 0} = \det (D_L D_R + m^2) \quad \ldots (5.11)$$

in the respective gauge field sectors, are not afflicted, unlike (5.8), with zero modes and hence have smooth chiral limits. In the trivial sector $\nu = 0$ the two representations coincide.

The representations in (5.11) require that instead of the Dirac basis $\{\phi_n(x)\}$ of Sec.3 we use eigenfunction sets of Weyl operators $D_R D_L$ and $D_L D_R$ appropriate respectively for positive and negative $\nu$. Thus for $\nu \geq 0$, one writes

$$\phi_n(x) = \frac{1}{\sqrt{2}} \left( \frac{1}{\lambda_n} D_L \phi_{nL}(x) \right), \quad \phi_{-n}(x) = \frac{1}{\sqrt{2}} \left( \frac{1}{\lambda_n} D_L \phi_{nL}(x) - \phi_{nL}(x) \right) \quad \ldots (5.12)$$

where $\phi_{nL}(x)$ are orthonormal eigenfunctions of the positive definite hermitian operator $D_R D_L$

$$D_R D_L \phi_{nL}(x) = \lambda_n^2 \phi_{nL}(x) \quad \ldots (5.13)$$

The set $\{\phi_{nL}(x)\}$ with $\lambda_n^2 > 0$ provides a complete set of functions in the Weyl basis in $\nu \geq 0$ sector. In the resulting axial vector Ward identity

$$\langle \partial_\mu \left( \bar{\psi} \gamma_5 \gamma_\mu \psi \right) \rangle_{\nu > 0} = 2m \langle \bar{\psi} \gamma_5 \psi \rangle - \left\{ \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon^{\mu\nu} F_{\lambda\rho} - \sum \phi_{0i}^+ \phi_{0i} \right\} \quad \ldots (5.14)$$

the contribution from the mass term on the right hand side now vanishes smoothly in the chiral limit. The global chiral anomaly given by the space-time integral of the chiral limit of the four divergence of the axial vector current, therefore, vanishes and instead of (5.5), we now have

$$J(\alpha_{ew})_{\nu \neq 0} = \exp \left[ -i \alpha_{ew} \int \left\{ \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \epsilon^{\mu\nu} F_{\lambda\rho} - \sum \epsilon_i \phi_{0i}^+ \phi_{0i} \right\} d^4x \right] \quad \ldots (5.15)$$

The vanishing of the global chiral anomaly means that the chiral phase $\alpha_{ew}$ in the quark mass in (5.1) is unphysical and can be transformed away trivially by a global chiral rotation (5.4) without affecting in any way $\theta_{QCD}$. The vacuum parameter $\theta_{QCD}$ remains invariant. The crux of the strong CP problem, the problem of fine tuning, therefore, melts away. CP symmetry is ensured simply through the natural choice $\theta_{QCD} = 0$. 

18
6 Concluding Remarks

Ever since its conception in the context of the problem of neutral pion decay into two photons, chiral anomaly has been a topic of abiding interest and challenge in particle physics. The interest stems in a large measure from the need to couple fermions chirally to gauge fields in building models in particle physics. The challenge consists in formulating a consistent and systematic regularisation scheme in chiral gauge theories.

The paper highlights and elucidates the seminal role of the mass term in the axial vector Ward identity in generating the local ABJ anomaly and the global U(1) axial anomaly. Gauge invariance demands that the fermion gets decoupled from the divergence of the U(1) axial vector current if it is very heavy. This identifies the ABJ anomaly with the asymptotic limit of the fermion mass term with sign reversed. On the other hand, the chiral limit \((m = 0)\) of the same mass term does not vanish and consists of contributions from fermion zero modes when the background gauge field has a nontrivial topology \(\nu \neq 0\). The space time integral of the chiral limit cancels the integral of the ABJ anomaly, the (sign–reversed) asymptotic limit of the mass term, thanks to the Atiyah-Singer index theorem. This suggests, contrary to popular perception, that the Jacobian for global U(1) chiral transformation is trivial even in an instanton background. The triviality of the Jacobian is realised in a representation of the fermion partition function in the Weyl basis (5.11) which has a null kernel space. The point of interest in all this is that there is no strong CP problem in an axionless physical world.

Current interest in lattice formulation of chiral gauge theory centres around Dirac operators for lattice fermion which obey the Ginsparg-Wilson relation (4.4). Apart from redefining chiral symmetry on lattice, the Ginsparg-Wilson relation introduces nonlocality. It is interesting to note that in continuum formulation also it is possible to define a gauge invariant partition function (4.8) but only at the cost of locality. Fermion averaging of the gauge current implemented with this partition function yields the covariant current. The consistent current which obeys integrability, can be generated with the covariant current as input. The covariant derivative of the consistent current thus obtained yields the minimal anomaly which obeys the Wess-Zumino consistency condition. Both the anomalies, covariant and consistent, and the distinction between the two currents vanish if the fermion belongs to anomaly free representation (4.26).
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