44

REGULARIZED PRODUCTS AND DETERMINANTS

Georg ILLIES

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Avril 1999

IHES/M/99/29

Regularized Products and Determinants

Georg Illies

Abstract

Zeta-regularized products are used to define determinants of operators in infinite dimensional spaces. This article provides a general theory of regularized products and determinants which delivers a better approach to their existence and explicit determination.

1 INTRODUCTION

The **zeta-regularized product** of a sequence $a_k \in \mathbb{C}^*$ is defined by

$$\prod_{k=1}^{\infty} a_k := \exp \left(-\frac{d}{ds} \sum_{k=1}^{\infty} a_k^{-s} \Big|_{s=0} \right), \quad (1.1)$$

provided that the Dirichlet series converges absolutely in a half plane and can be meromorphically continued to the left of $\Re(s) = 0$; the evaluation at $s = 0$ means the constant term of the Laurent expansion. This obviously generalizes the ordinary finite product.

Zeta-regularization was first used to define analytic torsion [RS] and since then has played a role in global analysis, the theory of dynamical zeta functions, and Arakelov theory. Theoretical physicists use zeta-regularization as a method for renormalization in quantum field theories [EORBZ] and various papers (e.g. [Ef], [Ko1], [Ko2], [Ko3], [Sa]) have calculated the regularized determinant of Laplacians. Zeta-regularization also appeared in a conjectural cohomological approach to motivic L -functions ([De1], [De2], [De3], [Ma]). In that context the question appeared as to which meromorphic functions of finite order (e.g. motivic L -functions) are zeta-regularized, i.e. can be represented as

$$f(z) = \prod (z - \rho)^{\pm 1} \quad (1.2)$$

where the product is over all zeroes and poles ρ of $f(z)$ with multiplicities and the sign of the exponent being positive for zeroes and negative for poles. This turns out to be the basic problem of zeta regularized determinants and it was the starting point of the following investigation which, we hope, gives a satisfying answer to the question.

Regularization entails several technical problems because of the meromorphic continuation of the Dirichlet series. For example, the regularized product of all primes does not exist as $\sum p^{-s}$ has the natural boundary $\Re(s) = 0$ ([LW]). The aim of this paper is to give a better approach to regularized products improving the formalism in [Vo], [CV], [QHS] and [JL1] (compare section 6 below) which is based on the representation of the Dirichlet series as the Mellin transform of the series

$$\theta(t) := \sum_{k=1}^{\infty} e^{a_k t}. \quad (1.3)$$

In many applications $\arg(a_k)$ varies in such a way that this series does not converge for any t , for example in the product (1.2). This problem can be solved by instead using a kind of Hankel integral of the Dirichlet series (see section 7). To treat the product (1.2) one has by definition to regard the function $\zeta(s, z) := \sum \pm (z - \rho)^{-s}$. This paper is also thought as an examination of the analytic and asymptotic properties of this generalized Hurwitz zeta function which should be interesting for its own sake.

Before giving a short overview we introduce the notion of a divisor which is basic for all what follows. A **divisor** D is given by a function $m_D : \mathbb{C} \rightarrow \mathbb{Z}$ such that there is a $\beta > 0$ with

$$\sum_{\rho \in \mathbb{C}} \frac{|m_D(\rho)|}{|\rho|^\beta} < \infty. \quad (1.4)$$

Condition (1.4) reflects that the Dirichlet series in the definition (1.1) must converge absolutely in a half plane. We recall a fundamental fact from the theory of entire functions of finite order (compare [Ti] for the proof): A function $m_D : \mathbb{C} \rightarrow \mathbb{Z}$ gives rise to a divisor if and only if there is a *meromorphic function* $f(z)$ of *finite order* (i.e. $f(z)$ is the quotient of two entire functions of finite order) such that

$$m_D(\rho) = \text{ord}_{z=\rho} f(z), \quad \rho \in \mathbb{C},$$

thus D is the divisor of $f(z)$ in the usual sense. And this function $f(z)$ is determined by D up to an exponential polynomial, i.e. a function $g(z)$ is meromorphic of finite order with divisor D if and only if there is a polynomial $P(z)$ with $g(z) = e^{P(z)} f(z)$.

After introducing some notations (section 2) we define a general class of regularized products in section 3, zeta-regularization being just an example; the rhs of (1.2) in case of its existence is called **regularized determinant** and denoted by

$$\Delta_D(z) := \prod_{\rho \in \mathbb{C}} (z - \rho)^{m_D(\rho)}. \quad (1.5)$$

We prove that $\Delta_D(z)$ is a meromorphic function of finite order with divisor D , thus equals $e^{P(z)}f(z)$ for a certain polynomial $P(z)$. *Regularization means finding this polynomial.* Section 4, a sort of theoretical excursion, discusses axiomatic generalizations of the regularization process and shows that a theory of regularization should deal with quasi-directed divisors (defined in sec. 2).

If the Dirichlet series in the definition of regularized products also satisfies certain exponential estimates and does not have too many poles we speak of *bounded regularizability* (section 5). In that case one can apply certain integral transformations, especially the Mellin transform, the mentioned Hankel integral and the Laplace transform, to get Theorems 3, 4 and 6 of sections 6 and 7 and 9. They give the equivalence of bounded regularizability with certain asymptotics for $\theta_D(t)$, $\zeta_D(s, z)$ and for the function $\theta_D(t, s)$ which is defined as Laplace transform of $\zeta_D(s, z)$. As a corollary of Theorem 4 one gets Theorem 5 in section 7 which is the **fundamental theorem** of the theory of regularization. It states that D is bounded regularizable if and only if for some $0 < \psi_i < \pi$, $i = 1, 2$ and $\varepsilon > 0$ an asymptotic

$$\log f(z) = \sum_i z^{\alpha_i} \log^{n_i} z + o(|z|^{-\varepsilon}) \quad |z| \rightarrow \infty \quad (1.6)$$

with finite sum, $\alpha_i \in \mathbb{C}$ and $n_i \in \mathbb{N}_0$, is valid for $-\psi_2 < \arg(z) < \psi_1$. $\Delta_D(z)$ exists in that case and also the polynomial $P(z)$ can be determined in terms of this asymptotic of $\log f(z)$ which is very intrinsic.

These results deliver a satisfying theory of regularization and apply to a large class of examples. In section 8.1 for instance it is shown that every meromorphic function of finite order representable by a Dirichlet series is regularized; this improves results of Jorgenson and Lang ([JL1], [JL2]) who had to assume that it also satisfies a functional equation. In 8.2 we regularize higher Γ -functions. Thus section 8 is applicable to various kinds of zeta and L -functions.

The function $\theta_D(t, s)$, introduced in section 9, is a type of multivalued theta function and plays a central role in [II2]. There is also an alternative approach to regularization via renormalizing certain divergent integrals (section 10). The following three sections contain technical proofs which were postponed.

The article reproduces the main results of chapter 2 of my thesis [II1] in a more special context. In some cases we only give sketches of the proofs, for complete proofs, generalizations and further results the reader is referred to [II1]. In [II2] it is shown how to apply the theory of regularization to generalize results of Cramér ([Cr]) and Guinand ([Gui]) thus improving results of [JL2].

Acknowledgements: I would like to thank C. Deninger for supervising my Ph.D.-thesis as well as M. Schröter, I. Vardi, C. Bree, C. Soulé, A. Voros, J.B. Bost and J. Jorgenson for helpful discussions and improvements. Parts of the article were written during a visit at the IHES.

2 NOTATIONS

In the sequel $f(z)$ denotes a meromorphic function of finite order and D its divisor. We define two important parameters: The **exponent** r of f and D is the infimum of all $\beta > 0$ satisfying (1.4); the **genus** g of f and D is the smallest $n \in \mathbf{N}_0$ such that (1.4) is satisfied for $\beta = n + 1$; note $g + 1 \geq r \geq g$.

We will say that D lies in a set $M \subset \mathbb{C}^*$ if $m_D(\rho) \neq 0$ implies that $\rho \in M$. Let $0 < \varphi_i < \pi$, $i = 1, 2$, then we define open connected sets

$$Wr_{\varphi_1, \varphi_2} := \{z \in \mathbb{C}^* \mid -\varphi_2 < \arg(z) < \varphi_1\}, \quad Wl_{\varphi_1, \varphi_2} := \mathbb{C}^* \setminus \overline{Wr_{\varphi_1, \varphi_2}}$$

and a contour C_{φ_1, φ_2} consisting of the ray from $e^{-\varphi_2 i} \infty$ to 0 and the ray from 0 to $e^{\varphi_1 i} \infty$; thus $\mathcal{C} = Wl_{\varphi_1, \varphi_2} \cup C_{\varphi_1, \varphi_2} \cup Wr_{\varphi_1, \varphi_2}$ is a disjoint union.

A divisor D is called **directed** if it lies in a $Wl_{\varphi_1, \varphi_2}$. It is called **quasi-directed** if it is directed with the exception of finitely many ρ , and it is called **strictly directed** if it lies in a $Wl_{\varphi_1, \varphi_2}$ with $\varphi_1 > \frac{\pi}{2}$ and $\varphi_2 > \frac{\pi}{2}$. We will also write $\rho \in D$ instead of $m_D(\rho) \neq 0$ and use the following notation

$$\sum_{\rho \in D} \varphi(\rho) := \sum_{\rho \in \mathcal{C}} m_D(\rho) \varphi(\rho).$$

3 XI FUNCTIONS AND REGULARIZATION

Definition 3.1 If D is a directed divisor, $U_D := \{z \in \mathbb{C} \mid |z| < |\rho|, \rho \in D\}$ and the argument is chosen so that $-\pi < \arg(z - \rho) < \pi$ then

$$\xi_D(s, z) := \sum_{\rho \in D} \frac{\Gamma(s)}{(z - \rho)^s}, \quad \Re(s) > r, \quad z \in U_D, \quad (3.1)$$

is called the **Hurwitz xi function** of D ; $\xi_D(s) := \xi_D(s, 0)$ is called the **xi function** of D .

Convergence is absolute and $\xi_D(s, z)$ is holomorphic in both variables.

Proposition 3.2 $\xi_D(s, z)$ satisfies the following differential equation:

$$\frac{d}{dz}\xi_D(s, z) = -\xi_D(s+1, z). \quad (3.2)$$

A function $f(z)$ is meromorphic of finite order with divisor D if and only if for some $l \geq g$:

$$\left(\frac{d}{dz}\right)^{l+1} \log f(z) = (-1)^l \xi_D(l+1, z). \quad (3.3)$$

Proof: Equation (3.2) follows by taking the term by term derivative. For (3.3) check that $\Delta_{D,a}^{\text{Wei},l}(z)$ defined in (3.6) below satisfies (3.3) and observe that the operation $(\frac{d}{dz})^{l+1} \log$ exactly kills exponential polynomials of degree $\leq l$. $\square$

Proposition 3.3 For $z \in U_D$ the following absolutely convergent Taylor series expansion is valid:

$$\xi_D(s, z) = \sum_{m=0}^{\infty} (-1)^m \xi_D(s+m) \frac{z^m}{m!}. \quad (3.4)$$

If $\xi_D(s)$ is meromorphic for $\Re(s) > -p$ then $\xi_D(s, z)$ is also meromorphic for $\Re(s) > -p$ and holomorphic for $z \in U$ for any simply connected $U \subset \mathbb{C}$ with $U_D \subset U$ and $\rho \notin U$ for all $\rho \in D$.

Proof: The Taylor series follows from (3.2); for the meromorphy in s observe that shifting the coefficients does not change the convergence radius. The continuation in z is obtained by treating finitely many $\rho \in D$ separately. $\square$

Definition 3.4 A regularization sequence δ is a sequence of complex numbers $\delta_0, \delta_1, \dots$ with $\delta_0 = 1$. Formally let

$$\delta(s) := \delta_0 + \delta_1 s + \delta_2 s^2 + \dots$$

A directed divisor D is called **regularizable** if $\xi_D(s)$ is meromorphic in a half plane $\Re(s) > -\varepsilon$ with $\varepsilon > 0$. For $z \in U$

$$\Delta_D(z) := \exp(-CT_{s=0}(\delta(s)\xi_D(s, z))) \quad (3.5)$$

is called the δ -regularized determinant of D . One calls $\Delta_D(0)$ the δ -regularized product of D .

Note that $CT_{s=0}$ means the constant term in the Laurent expansion at $s = 0$. If $\delta(s)$ is a divergent series, then one has to develop $\xi_D(s, z)$ in a Laurent series and multiply it formally with the formal series for $\delta(s)$. In the sequel there will often appear formulas which must be interpreted in this formal sense.

Examples: 1) **xi-regularized** determinant (Jorgenson, Lang): $\delta(s) = 1$

2) **zeta-regularized** determinant: $\delta(s) = \Gamma^{-1}(s+1)$

3) **zero-renormalized** determinant: $\delta(s) = \Gamma(1-s)$

Remark: The factor $\delta(s)$ is introduced because of several reasons. First one wants to handle "scaled" products $\prod a(z-\rho)$ (compare [De1], [De2], [De3]). It also turned out that the canonical way of renormalization (see Theorem 7 in section 10) differs from zeta-regularization. A further reason is that in [JL1] xi-regularization was used which is technically the simplest regularization. While zeta-regularization as well as zero-renormalization generalize the ordinary finite product (as every regularization with $\delta_1 = \gamma$ does, γ the Euler-Mascheroni constant), xi-regularization does not. Zeta-regularization satisfies the product rule $\prod \rho^n = (\prod \rho)^n$ so comes closest to what one would expect for a product.

Fix $a \in \mathbb{C}$ with $m_D(a) = 0$ and $\ell \geq g$, then we define the absolutely convergent **Weierstrass product**

$$\Delta_{D,a}^{\text{Wei},\ell}(z) := \prod_{\rho \in \mathcal{C}} \left\{ \left(1 - \frac{z-a}{\rho-a}\right) \exp\left(\sum_{k=0}^{\ell} \frac{1}{k} \left(\frac{z-a}{\rho-a}\right)^k\right) \right\}^{m_D(\rho)}, \quad (3.6)$$

which is a meromorphic function of finite order with divisor D . For $a = 0$ and $g = l$ one has the usual **canonical Weierstrass product** (compare [Ti]).

Theorem 1 $\Delta_D(z)$ is a meromorphic function of finite order with divisor D . The explicit relation to Weierstrass products is given by

$$\Delta_D(z) = e^{P(z)} \Delta_{D,a}^{\text{Wei},\ell}(z) \quad (3.7)$$

where with a suitable branch of the logarithm

$$\begin{aligned} P(z) &= \sum_{m=0}^{\ell} \frac{(z-a)^m}{m!} \log^{(m)} \Delta_D(a) \\ \log^{(m)} \Delta_D(z) &= (-1)^{m+1} CT_{s=0}(\delta(s) \xi_D(s+m, z)) \quad m = 0, 1, \dots \end{aligned} \quad (3.8)$$

In the sequel a meromorphic function of finite order $f(z)$ with divisor D will be called **δ -regularized** if it equals $\Delta_D(z)$.

Proof of Theorem 1: We have

$$\left(\frac{d}{dz}\right)^{m+1} \log \Delta_D(z) = (-1)^m CT_{s=0}(\delta(s) \xi_D(s+m+1, z)) \quad (3.9)$$

$$= (-1)^m \xi_D(m+1, z) \quad \text{for } m \geq g, \quad (3.10)$$

the first equation holds because of (3.2) and is valid for all $m \in \mathbb{N}_0$. It is also true that $\xi_D(s, z)$ is holomorphic for $s = m+1$ if $m \geq g$ by the definition of g . Comparision with (3.3) proves the first assertion. One also easily checks

$$\left(\frac{d}{dz}\right)^{m+1} \log \Delta_{D,a}^{\text{Wei},\ell}(z)|_{z=a} = \begin{cases} 0 & \text{for } m = -1, 0, \dots, \ell-1 \\ (-1)^m \xi_D(m+1, a) & \text{for } m \geq \ell. \end{cases}$$

Using this as well as (3.9) and (3.10) the explicit relation to Weierstrass products follows by subtracting the Taylor series expansion around $s = a$ for $\log \Delta_{D,a}^{\text{Wei},\ell}(z)$ from that for $\log \Delta_D(z)$. $\square$

4 DETERMINANT SYSTEMS

We will call a function $f(z)$ **associated to a divisor** D if there is a polynomial $P(z)$ such that

$$f(z) = e^{P(z)} \Delta_{D,a}^{\text{Wei},g}(z) \quad \text{and} \quad \deg P \leq g$$

or equivalently that (3.3) is satisfied for $l = g$ (compare the proof of (3.3)). Observe that in this definition we have set $\ell = g$ in (3.6). Note also that if $\Delta_{D,a}^{\text{Wei},g}(z)$ is, in addition, entire then its order is exactly r and no entire function with divisor D can have a smaller order (compare [Ti]). So associated functions have minimal order because of $g \leq r$.

By Theorem 1 regularization means picking out a certain associated function to a divisor. Now we ask for extensions of this process to non-regularizable divisors. For $\alpha \in \mathbb{C}$ we define the translated divisor $D^{|\alpha}$ by $m_{D^{|\alpha}}(z + \alpha) := m_D(z)$ and the sum $D_1 + D_2$ by $m_{D_1+D_2}(z) := m_{D_1}(z) + m_{D_2}(z)$. Let $\mathfrak{D}^{\text{fin}}$ be the $\mathbb{Z}$ -module of all finite divisors, $\mathfrak{D}^{\text{breg}}$ that of all bounded regularizable quasi-directed divisors (comp. sec. 5), $\mathfrak{D}^{\text{reg}}$ that of all regularizable quasi-directed divisors, i.e. those which are regularizable directed after eliminating finitely many ρ , $\mathfrak{D}^{\text{qd}}$ that of all quasi-directed divisors, and $\mathfrak{D}$ that of all divisors. These are all translation-invariant with proper inclusions $\mathfrak{D}^{\text{fin}} \subset \mathfrak{D}^{\text{breg}} \subset \mathfrak{D}^{\text{reg}} \subset \mathfrak{D}^{\text{qd}} \subset \mathfrak{D}$. The following definition arises from the demand for generalizations of the characteristic polynomial to the infinite dimensional case.

Definition 4.1 *Let $\mathfrak{D}' \subseteq \mathfrak{D}$ be a translation-invariant $\mathbb{Z}$ -module. A determinant Δ on $\mathfrak{D}'$ attaches to every $D \in \mathfrak{D}'$ an associated function $\Delta_D(z)$, such that:*

- i) $\Delta_{D^{|\alpha}}(z + \alpha) = \Delta_D(z) \quad (\text{translation-invariance})$
- ii) $\Delta_{D_1+D_2}(z) = \Delta_{D_1}(z)\Delta_{D_2}(z) \quad (\text{linearity})$

for $D, D_1, D_2 \in \mathfrak{D}'$, $\alpha \in \mathbb{C}$. $(\mathfrak{D}', \Delta)$ is called a **determinant system**.

Examples: 1) $(\mathfrak{D}^{\text{fin}}, \Delta)$ with characteristic "polynomial" which is a rational function defined by $\Delta_D(z) := \prod_{\rho \in \mathbb{C}} (z - \rho)^{m_D(\rho)}$.

2) $(\mathfrak{D}^{\text{reg}}, \Delta)$ with the δ -regularized determinant Δ . For a $D \in \mathfrak{D}^{\text{reg}}$ which is not directed, $\Delta_D(z)$ can be defined by translation-invariance.

Theorem 2 answers the question of how large determinant systems can be.

Theorem 2 a) *There is no determinant system $(\mathfrak{D}, \Delta)$.*

b) *For every regularization sequence δ there is a determinant system $(\mathfrak{D}^{\text{qd}}, \Delta)$ which is an extension of the δ -regularized determinant system $(\mathfrak{D}^{\text{reg}}, \Delta)$.*

Proof: a) Let D be the divisor that consists only of zeroes of order one lying at the lattice points $\rho = m + ni$, $m, n \in \mathbb{Z}$. From translation invariance one gets $\Delta_D(z+1) = \Delta_D(z)$ and $\Delta_D(z+i) = \Delta_D(z)$. Hence $\Delta_D(z)$ must be a doubly-periodic entire non-constant function which is impossible by Liouville's theorem.

b) (Idea) One has to choose the exponential polynomials consistently with linearity and translation-invariance. This leads to a system of infinitely many linear equations with infinitely many variables which can be reduced to finite systems by Zorn's lemma. (See section 11 for a complete proof.) $\square$

Remark: Not every determinant system is extendable, so b) is an aesthetic property of regularization. The proof is non-constructive and its extensions are not uniquely determined. The meaning of regularization is that it gives large constructively defined determinant systems.

5 BOUNDED REGULARIZABILITY, SINGULARITIES AND ASYMPTOTICS

In this section we introduce the special case of *bounded regularizability* of divisors and give all the necessary technical definitions to formulate the results of sections 6, 7 and 9 which state its equivalence to various asymptotics.

Definition 5.1 Let D be a directed divisor, $0 < \sigma_i < \pi$ for $i = 1, 2$ and $p \in \mathbb{R} \cup \{\infty\}$, then D resp. $\xi_D(s)$ are called (σ_1, σ_2) -**bounded p -regular** if:

- i) $\xi_D(s)$ is meromorphic for $\Re(s) > -p$.
- ii) $\xi_D(s)$ has only finitely many poles in the strip $\alpha_1 < \Re(s) < \alpha_2$ for any $-p < \alpha_1 < \alpha_2$.
- iii) For all $-p < \alpha_1 < \alpha_2$ and $\sigma'_1 < \sigma_1$, $\sigma'_2 < \sigma_2$

$$\xi_D(s) = \begin{cases} O(e^{(\frac{\pi}{2} - \sigma'_2)\Im(s)}) & \text{for } \Im(s) \rightarrow \infty \\ O(e^{-(\frac{\pi}{2} - \sigma'_1)\Im(s)}) & \text{for } \Im(s) \rightarrow -\infty. \end{cases}$$

in the strip $\alpha_1 < \Re(s) < \alpha_2$.

We simply say **bounded p -regular**, if there are $0 < \sigma_i < \pi$ such that (σ_1, σ_2) -bounded p -regularity is valid. We have **bounded regularizability** if $p > 0$.

Note that every directed divisor D in $Wl_{\varphi_1, \varphi_2}$ is (φ_1, φ_2) -bounded $(-r)$ -regular as follows from Stirling's formula.

Definition 5.2 A **pB-System** consists of:

1. A finite or infinite sequence of pairs

$$(p_n, B_n(z))_{n=0,1,2,\dots}$$

with complex numbers p_n satisfying

$$\Re(p_0) \leq \Re(p_1) \leq \dots \leq \Re(p_n) \leq \dots$$

and polynomials $B_n(z) \in \mathbb{C}[z]$. $B_n(z) = \sum_k b_{n,k} z^k$.

2. An abscissa $p \in \mathbb{R} \cup \{\infty\}$ such that $p > \Re(p_n)$ for all n , and in addition for infinite sequences: $p = \lim_{n \rightarrow \infty} \Re(p_n)$.

pB -systems capture the simultaneous information about the occurring singular part distributions and the occurring asymptotics.

Example: If the divisor D is (σ_1, σ_2) -bounded p -regular, then there is a pB -system $(p_n, B_n(z))_{n=0,1,2,\dots}$ with abscissa p such that the poles of $\xi_D(s)$ in the half plane $\Re(s) > -p$ lie exactly at the values $s = -p_n$ and the Laurent expansions have the singular parts

$$B_n(\partial_s) \left[\frac{1}{s + p_n} \right] = \sum_k b_{n,k} \frac{(-1)^k k!}{(s + p_n)^{k+1}}.$$

$$(-p_n, B_n(\partial_s) \left[\frac{1}{s + p_n} \right])_{n=0,1,\dots}$$

is called the **singular part distribution** of $\xi_D(s)$ in that case.

Definition 5.3 Let $(p_n, B_n(z))_{n=0,1,\dots}$ be a pB -system with abscissa p as above and $0 < \sigma_i \leq \varphi_i < \pi$, $i = 1, 2$. A function $\theta : \mathcal{Wr}_{\varphi_1, \varphi_2} \rightarrow \mathbb{C}$ satisfies the **Cramér asymptotic with abscissa p in $\mathcal{Wr}_{\sigma_1, \sigma_2}$**

$$\theta(t) \sim \sum_{n=0}^{\infty} t^{p_n} B_n(\log t) \quad \text{for } |t| \rightarrow 0$$

if the estimate for $t \in \mathcal{Wr}_{\sigma'_1, \sigma'_2}$

$$\theta(t) - \sum_{\Re(p_n) < q} t^{p_n} B_n(\log t) = O(|t|^{q'}) \quad \text{for } |t| \rightarrow 0$$

is valid for any fixed $q, q' \in \mathbb{R}$ with $q' < q < p$ and $0 < \sigma'_i < \sigma_i$, $i = 1, 2$.

Analogously, for a function $g : \mathcal{Wr}_{\varphi_1, \varphi_2} \rightarrow \mathbb{C}$ one defines the meaning of the following **Stirling asymptotic with abscissa p in $\mathcal{Wr}_{\sigma_1, \sigma_2}$** :

$$g(z) \sim \sum_{n=0}^{\infty} z^{-p_n} B_n(\log z) \quad \text{for } |z| \rightarrow \infty.$$

In the definition we choose of course $\arg(t), \arg(z) \in]-\varphi_2, \varphi_1[$.

6 MELLIN TRANSFORM AND CRAMÉR ASYMPTOTICS

A typical method for showing that a Dirichlet series has a meromorphic continuation is using the Mellin transform.

Definition 6.1 For a strictly directed divisor D in $\mathcal{W}l_{\varphi_1, \varphi_2}$, $(\frac{\pi}{2} < \varphi_i < \pi)$, one defines the **partition function**

$$\theta_D(t) := \sum_{\rho \in D} e^{\rho t}, \quad t \in \mathcal{W}r_{(\varphi_2 - \frac{\pi}{2}), (\varphi_1 - \frac{\pi}{2})}. \quad (6.1)$$

The series converges absolutely and is holomorphic in t .

Proposition 6.2 One has the Mellin transform representation for $\Re(s) > r$

$$\xi_D(s) = \int_0^\infty \theta_D(t) t^{s-1} dt \quad (6.2)$$

and its inverse for $t \in \mathcal{W}r_{(\varphi_2 - \frac{\pi}{2}), (\varphi_1 - \frac{\pi}{2})}$ and $c > r$

$$\theta_D(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \xi_D(s) t^{-s} ds \quad (6.3)$$

with absolute convergence of the integrals.

Proof: Because of the theorem about Mellin inversion it suffices to prove (6.3) and by majorized convergence, this is reduced to the case of a one-point-divisor. In that case (6.3) is the inverse formula for Euler's Mellin integral for $\Gamma(s)$. $\square$

This approach is only possible for strictly directed divisors as otherwise the defining series for $\theta_D(t)$ does not converge for any t .

Theorem 3 Let $\frac{\pi}{2} < \sigma_i \leq \varphi_i < \pi$ for $i = 1, 2$ and D be strictly directed in $\mathcal{W}l_{\varphi_1, \varphi_2}$, and let $(p_n, B_n(z))_{n=0,1,\dots}$ be a pB -system with abscissa p . Then the following statements are equivalent:

A) $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular with singular part distribution

$$(-p_n, B_n(\partial_s) \left[\frac{1}{s + p_n} \right])_{n=0,1,\dots}$$

C) $\theta_D(t)$ satisfies a Cramér asymptotic with abscissa p of the form

$$\theta_D(t) \sim \sum_{n=0}^{\infty} t^{p_n} B_n(\log t) \quad \text{for } |t| \rightarrow 0$$

in $\mathcal{W}r_{(\sigma_2 - \frac{\pi}{2}), (\sigma_1 - \frac{\pi}{2})}$.

Proof: (sketch) **C)** $\Rightarrow$ **A)** is shown by (6.2): The poles and singular parts of $\xi_D(s)$ arise by integrating the terms of the Cramér asymptotic, and the exponential estimation for $\xi_D(s)$ in vertical strips can be shown by rotating the ray of integration in (6.2) in $\mathcal{W}r_{(\sigma_2 - \frac{\pi}{2}), (\sigma_1 - \frac{\pi}{2})}$.

A) $\Rightarrow$ **C)** follows from (6.3) by replacing the abscissa c of the line of integration by a smaller $c' > p$ and applying the residue theorem. The residues of the integrand produce the terms of the Cramér asymptotic. $\square$

Remark 1: This theorem is well known in the context of Mellin and Laplace transform (e.g. [Do] II, chap. 5, where a complete proof can be found). Using it one can decide whether a strictly directed divisor is bounded regularizable or not, by checking for the existence of a suitable Cramér asymptotic for the partition function. For example the regularized product of the eigenvalues of Laplacians on manifolds exists because of Cramér asymptotics arising from heat kernel expansions (comp. [Ef], [Ko1], [Ko2], [Ko3], [Sa]).

Remark 2: In the case of strictly directed divisors also the implication **A**) $\Rightarrow$ **B**') of Theorem 4 and, in particular, the asymptotic (7.8) can be obtained by the Mellin integral

$$\xi_D(s, z) = \int_0^\infty \theta_D(t) e^{-zt} t^{s-1} dt$$

using

$$\int_0^\infty t^{p_n} B_n(\log t) e^{-zt} t^{s-1} dt = B_n(\partial_s) \left[\frac{\Gamma(s + p_n)}{z^{s+p_n}} \right].$$

The Mellin transform approach to regularized products and determinants (the details can be found in section 2.4 in [III1]) was also extensively studied by Jorgenson and Lang ([JL1]).

7 HANKEL INTEGRALS AND STIRLING ASYMPTOTICS

The Mellin integral method has two shortcomings: It is possible only for strictly directed divisors and the partition function is a non-intrinsic construction, one wants criteria in terms of associated functions. In this section we solve these problems postponing the technical proofs until section 12. For powers a^s and $\log(a)$ we always use $-\pi < \arg(a) < \pi$.

Proposition 7.1 *Let D be a directed divisor in $\mathcal{W}l_{\varphi_1, \varphi_2}$ ($0 < \varphi_i < \pi$).*
a)

$$\xi_D(s, z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-s')}{z^{s-s'}} \xi_D(s') ds' \quad (7.1)$$

for $z \in \mathcal{W}r_{\varphi_1, \varphi_2}$ and $\Re(s) > c > r$ with absolute convergence of the integral.

b) Let $0 < \sigma_i < \varphi_i$ for $i = 1, 2$, $C = C_{\sigma_1, \sigma_2}$ and $z \in \mathcal{W}r_{\sigma_1, \sigma_2}$. Then for $\Re(s) > r$ and $\Re(s_0) > r$ one has the absolutely convergent integral representation

$$\xi_D(s, z) = \frac{1}{2\pi i} \int_C \frac{\Gamma(s-s_0+1)}{(z-w)^{s-s_0+1}} \xi_D(s_0, w) dw. \quad (7.2)$$

In the sequel the representations (7.1) and (7.2) play a similar role as (6.2) and (6.3) in section 6. For the explicit description of the Stirling asymptotics we need the following definition.

Definition 7.2 Let δ be a fixed regularization sequence. Then for any $q \in \mathbb{C}$ we define the linear map

$$\begin{aligned} [q] : \mathbb{C}[z] &\longrightarrow \mathbb{C}[z] \\ B(z) &\longmapsto B^{[q]}(z), \end{aligned}$$

by

$$CT_{s=0} \left(\delta(s) B(\partial_s) \left[\frac{\Gamma(s+q)}{z^{s+q}} \right] \right) = z^{-q} B^{[q]}(\log z). \quad (7.3)$$

For $P_k(z) := z^k$ we get:

$$P_k^{[q]}(z) = \sum_{j=0}^k (-1)^j \binom{k}{j} \Gamma^{(k-j)}(q) z^j \quad (7.4)$$

in the case that $q \neq -n$ for all $n \in \mathbf{N}_0$, while for $q = -n$

$$\begin{aligned} P_k^{[q]}(z) &= \sum_{j=0}^k (-1)^j \binom{k}{j} CT_{s=q}(\Gamma^{(k-j)}(s)) z^j \\ &\quad + \frac{(-1)^n}{n!} \left(\frac{(-1)^{k+1}}{k+1} z^{k+1} + (-1)^k k! \delta_{k+1} \right). \end{aligned} \quad (7.5)$$

Special case $B(z) = b_0$: Easy calculations using the fact that $\Gamma(z)$ is holomorphic for $(-z) \notin \mathbf{N}_0$ as well as the expansion $\Gamma(s) = \frac{1}{s} - \gamma + \dots$ (γ the Euler-Mascheroni constant) and $\Gamma(s-n) = \Gamma(s)((s-1)(s-2)\dots(s-n))^{-1}$ deliver

$$B^{[q]}(z) = \begin{cases} b_0 \Gamma(q) & \text{for } q \neq -n \\ b_0 \frac{(-1)^{n+1}}{n!} \left(z + \gamma - \delta_1 - \sum_{j=1}^n \frac{1}{j} \right) & \text{for } q = -n, \end{cases} \quad (7.6)$$

(for zeta-regularization as well as for zero-renormalization one has $\delta_1 = \gamma$.) The following basic properties of $[q]$ are clear by (7.4), (7.5) and the definition.

Proposition 7.3 a) $[q]$ is bijective for $q \neq 0, -1, -2, \dots$ with $\deg B^{[q]} = \deg B$.
b) $[q]$ is injective for $q = 0, -1, -2, \dots$ with $\deg B^{[q]} = \deg B + 1$ and with $\dim \text{Coker}([q]) = 1$
c)

$$\frac{d}{dz} [z^{-q} B^{[q]}(\log z)] = -z^{-(q+1)} B^{[q+1]}(\log z). \quad (7.7)$$

Remark 1: In particular, every Stirling asymptotic with abscissa p can be represented as linear combination of terms of the form $z^{-q} B^{[q]}$ up to a polynomial

$P(z)$ with¹ $P(z)z^p \rightarrow 0$ for $|z| \rightarrow 0$, and this polynomial is uniquely determined. This shows that the asymptotics in **B')** of Theorem 4 and **B)** of Theorem 5 are general Stirling asymptotics which are written in a special manner. And this also means that the Stirling asymptotic (7.8) is an *effective method* to determine the regularized determinant among all associated functions for D .

Remark 2: Part c) of the proposition together with (3.2) shows that the Stirling asymptotics in **B')** and **B)** can be differentiated term by term.

Theorem 4 Let $0 < \sigma_i \leq \varphi_i < \pi$ for $i = 1, 2$ and D be a directed divisor in $\mathcal{Wl}_{\varphi_1, \varphi_2}$, let $p' \in \mathbb{R} \cup \{\infty\}$ and $\xi_D(s)$ be meromorphic $\Re(s) > -p'$ (compare Proposition 3.3). Then for any regularization sequence δ , s_0 with $\Re(s_0) > -p'$ and a pB -system $(p_n, B_n(z))_{n=0,1,\dots}$ with abscissa $p \in \mathbb{R} \cup \{\infty\}$ the following statements are equivalent:

A) $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular with the singular part distribution

$$(-p_n, B_n(\partial_s) \left[\frac{1}{s + p_n} \right])_{n=0,1,\dots}$$

B') There is a polynomial $P_{s_0}(z)$ with $P_{s_0}(z)z^{p+s_0} \rightarrow 0$ for $|z| \rightarrow 0$ and such that the Stirling asymptotic with abscissa $p + \Re(s_0)$

$$CT_{s=0}(\delta(s)\xi_D(s+s_0, z)) \sim P_{s_0}(z) + \sum_{n=0}^{\infty} z^{-(p_n+s_0)} B_n^{[p_n+s_0]}(\log z) \quad \text{for } |z| \rightarrow \infty$$

is valid in $\mathcal{Wr}_{\sigma_1, \sigma_2}$.

The polynomial in **B')** is then uniquely determined: $P_{s_0}(z) = 0$.

The idea of the proof given in section 12 is rather similar to the proof of Theorem 3. To get the Stirling asymptotic **B')** from the singular part distribution **A)** one uses (7.1), shifts the line of integration and applies the residue theorem. The other direction is a little bit more difficult but the basic idea is of course to use (7.2) and integrate the Stirling asymptotic term by term. Some technical difficulties arise because (7.2) is not valid for $z = 0$.

Using equations (3.2), (3.3) and (3.5) one obtains the following theorem as an easy corollary of Theorem 4.

Theorem 5 Let $0 < \sigma_i \leq \varphi_i < \pi$ for $i = 1, 2$ and D be a directed divisor in $\mathcal{Wl}_{\varphi_1, \varphi_2}$; let $f(z)$ be a meromorphic function of finite order with divisor D . Then for a regularization sequence δ , $m \in \mathbf{N}_0$ and a pB -system $(p_n, B_n(z))_{n=0,1,\dots}$ with abscissa $p \in \mathbb{R} \cup \{\infty\}$ the following statements are equivalent:

A) $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular with singular part distribution

$$(-p_n, B_n(\partial_s) \left[\frac{1}{s + p_n} \right])_{n=0,1,\dots}$$

¹Observe: Terms z^{-q} with $q \geq p$ make no sense in Stirling asymptotics with abscissa p .

B) There is a polynomial $P_f(z)$ with $P_f(z)z^p \rightarrow 0$ for $|z| \rightarrow 0$, such that the Stirling asymptotic with abscissa $(p+m)$

$$\log^{(m)} f(z) \sim P_f^{(m)}(z) + (-1)^{m-1} \sum_{n=0}^{\infty} z^{-(p_n+m)} B_n^{[p_n+m]}(\log z) \quad \text{for } |z| \rightarrow \infty.$$

is valid in Wr_{σ_1, σ_2} .

$P_f(z)$ in B) can then be chosen independent of m , it is (up to the choice of the logarithm) uniquely determined.

If, in addition, $p > 0$, so D is bounded regularizable, then for the δ -regularized determinant one has $P_{\Delta_D} = 0$, i.e. the following Stirling asymptotic with abscissa p in Wr_{σ_1, σ_2} is valid:

$$\log \Delta_D(z) \sim - \sum_{n=0}^{\infty} z^{-p_n} B_n^{[p_n]}(\log z) \quad \text{for } |z| \rightarrow +\infty \quad (7.3)$$

Theorem 5 which can be regarded as the fundamental theorem about bounded regularizability by Remark 1 states that whenever a $\log^{(m)} f(z)$ satisfies any Stirling asymptotic with abscissa greater zero, then $f(z)$ and its divisor D are bounded regularizable, and (7.8) allows to determine its regularized determinant, i.e. the polynomial $P(z)$ mentioned in the introduction.

The triple equivalence **A**) $\Rightarrow$ **B**) $(\Rightarrow$ **C**)) given by Theorems 3 and 5 where the latter equivalence is valid only for strictly directed divisors will be generalized in section 9 (Theorems 4 and 6) to an equivalence **A**) $\Rightarrow$ **B'**) $\Rightarrow$ **C'**) valid for all directed divisors which summarizes all informations about singular part distributions and asymptotics of $\xi_D(s, z)$ and $\theta_D(t, s)$.

8 EXAMPLES

8.1 DIRICHLET SERIES

Corollary 8.1 *If $f(z)$ is meromorphic of finite order and has an absolutely convergent Dirichlet series representation*

$$f(z) = 1 + \sum_{n=0}^{\infty} \frac{\beta_n}{\alpha_n^z}, \quad \Re(z) > \sigma_0,$$

with $\beta_n \in \mathbb{C}$ and $\alpha_n \in \mathbb{R}^{>1}$ with $\lim_{n \rightarrow \infty} \alpha_n = \infty$, then $f(z)$ is δ -regularized for every regularization sequence δ , i.e. $f(z) = \Delta_D(z)$, in particular, $f(z)$ is associated to its divisor (compare section 4).

Proof: By the Taylor series expansion for $\log(1+x)$ it is clear that the *trivial* Stirling asymptotic $\log f(z) \sim 0$ as $|z| \rightarrow 0$ with abscissa $+\infty$ is valid in $Wr_{(\frac{\pi}{2}-\varepsilon), (\frac{\pi}{2}-\varepsilon)}$, so by Theorem 5 the assertion is clear. $\square$

Remark 1: Using only the Mellin integral method one needs to assume that $f(z)$ also satisfies a *functional equation* and examines

$$\theta_{D_+}(t) := \sum_{\rho \in D, \Im(\rho) > 0} e^{\rho t}, \quad \theta_{D_-}(t) := \sum_{\rho \in D, \Im(\rho) \leq 0} e^{\rho t}$$

separately. For $f(z) = \zeta(z)$ the Riemann zeta function a classical result of Cramér ([Cr]) delivers the Cramér asymptotics for $\theta_{D_+}(t)$ and $\theta_{D_-}(t)$ (with logarithmic terms in contrast to the examples from the spectra of Laplacians mentioned in section 6) and thus regularizability of $\zeta(z)$ ([So], [ScSo]).

In [JL2] Cramér's result was generalized to a class of $f(z)$ as in the above corollary which in addition satisfy a functional equation, and their result implies regularizability of all these functions. Corollary 8.1 shows regularizability for a much larger class and moreover one no longer needs Cramér's result. The methods of this section of [II2] also apply to the "polynomial Bessel fundamental class" introduced in [JL3]. Nevertheless a functional equation is necessary if one wants to get information about $\theta_{D_+}(t)$ (compare [II2]).

Remark 2: Theorem 5 gives satisfying criteria for deciding whether a function is bounded regularizable or not. For example, consider the function

$$f(z) = \frac{1}{\sqrt{2\pi}}(z^2 + 1)(1 + e^{-z})\Gamma(z) + e^{-z^2}\Gamma'(z).$$

It can be immediately seen that it is zeta-regularized: $(\sqrt{2\pi})^{-1}\Gamma(z)$ is zeta-regularized because of (8.5) below, and this is true for $(z^2 + 1)$ because it is a characteristic polynomial, and this holds for $(1 + e^{-z})$ because it is a Dirichlet series; the second summand is small (in an angular domain) compared to the first and does not change the Stirling asymptotic of $\log f(z)$.

8.2 Γ -FUNCTIONS

In §2.8 of [Vi] the functions $\Gamma_n(z)$ were defined which appear in the functional equations of Selberg zeta functions and which are special cases of the general higher Γ -functions introduced by Barnes ([Bar]). They are simple examples for regularization with non-trivial Stirling asymptotics and their zeta-regularization can already be found in [Va] and [Ma] §3.3. We give the following definition which is equivalent to that of Vigneras.

Definition 8.2 *The sequence $(\Gamma_n(z))_{n=0,1,\dots}$ of Γ -functions of order n is defined by the following conditions:*

- 0) $\Gamma_0(z) = \frac{1}{z}$
- 1) $\Gamma_n^{-1}(z)$, $n \in \mathbf{N}$, is an entire function of finite order and the divisor D_n of $\Gamma_n(z)$ consists exactly of the $\rho = -k$, $k \in \mathbf{N}_0$ with multiplicity $-\binom{n+k-1}{n-1}$.
- 2) $\Gamma_n(1) = 1$ for all $n \in \mathbf{N}_0$.

3) For all $n \in \mathbf{N}_0$ the following functional equation is valid

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}.$$

Using higher Bernoulli polynomials ([No]) one has for $n \in \mathbf{N}_0$

$$\begin{aligned} \theta_{D_n}(t) &= -(-\theta_{D_1}(t))^n = -\frac{1}{(1-e^{-t})^n}, \\ &= -\sum_{\nu=0}^{\infty} \frac{(-1)^\nu B_\nu^n(0)}{\nu!} t^{\nu-n} \quad \text{for } 0 < |t| < 2\pi \end{aligned} \quad (8.1)$$

thus by Theorem 3 the D_n are bounded regularizable. Applying (7.6) and (7.8) one gets the Stirling asymptotic with abscissa $+\infty$ for the δ -regularized determinant

$$\begin{aligned} \log \Delta_{D_n}(z) &\sim (-1)^{n+1} \sum_{k=0}^n \frac{B_{n-k}^n(0)}{(n-k)!k!} (\log z + \gamma - \delta_1 - \sum_{j=1}^k \frac{1}{j}) z^k \\ &+ \sum_{k=1}^{\infty} (-1)^{n+k} \frac{(k-1)! B_{n+k}^n(0)}{(n+k)!} z^{-k} \quad \text{for } |z| \rightarrow \infty \end{aligned} \quad (8.2)$$

in $\mathcal{W}r_{(\pi-\varepsilon),(\pi-\varepsilon)}$.

Proposition 8.3 *The functions $\Gamma_n(z)$ are well defined; one has*

$$\Gamma_n(z) = e^{-P_n(z)} \Delta_{D_n}(z)$$

with polynomials $P_n(z)$ of degree $\leq n$ which are determined (e.g. using Lagrange interpolation) by the relations

$$P_n(j) = \sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} \log \Delta_{D_{n-i}}(1), \quad j = 1, \dots, n+1. \quad (8.3)$$

The values $\log \Delta_{D_n}(1) := -CT_{s=0}(\delta(s)\xi_{D_n}(s, 1))$ can be expressed in terms of the Riemann zeta function:

$$\log \Delta_{D_n}(1) = \sum_{l=0}^{n-1} \tau_{n,l} \zeta'(-l) + (\delta_1 - \gamma) \sum_{l=0}^{n-1} \tau_{n,l} \zeta(-l) \quad (8.4)$$

for $n \geq 1$ and $\log \Delta_{D_0}(1) = \delta_1 - \gamma$, with the Euler-Mascheroni constant γ and $\tau_{n,l}$ from the development

$$\binom{n+x-1}{n-1} = \sum_{l=0}^{n-1} \tau_{n,l} (x+1)^l.$$

Proof: One shows that there exists exactly one choice of polynomials $P_n(z)$ with

$$\begin{aligned} P_{n+1}(z+1) + P_n(z) - P_{n+1}(z) &= 0 \\ P_n(1) &= \log \Delta_{D_n}(1) \end{aligned}$$

for $n \in \mathbf{N}_0$, with $\deg P_0 = 0$. (Because of $\deg P_0 = 0$ and the first equation one gets $\deg P_n \leq n$, by induction it is easy to prove that the $P_n(1)$ are given as in the proposition, and in the other direction, that the uniquely determined $P_n(z)$ with $\deg P_n \leq n$ and with these $P_n(1)$ satisfy the two equations.) The expression for $\log \Delta_{D_n}(1)$ follows from $\delta(s)\Gamma(s) = \frac{1}{s} + (\delta_1 - \gamma) + \dots$ and

$$\xi_{D_n}(s, 1) = -\Gamma(s) \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (k+1)^{-s} = -\Gamma(s) \sum_{l=0}^{n-1} \tau_{n,l} \zeta(s-l). \quad \square$$

$\Gamma_1(z)$ is the usual Γ -function, $\Gamma_2^{-1}(z) = G(z)$ is known as Barnes' G -function. For these two functions we will give the result more explicitly. It is well known that $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\log \sqrt{2\pi}$ and $\zeta(-1) = -\frac{1}{12}$. With the Kinkelin-Glaisher constant A one can express $\zeta'(-1) = \frac{1}{12} - \log A$ (compare [Vo] pp. 461-464, [Al], p. 357), but we use just $\zeta'(-1)$.

Corollary 8.4 *For $n = 1, 2$ one has*

$$\Delta_{D_1}(z) = \frac{\Gamma_1(z)}{\sqrt{2\pi}} e^{-(\delta_1 - \gamma)(z - \frac{1}{2})} \quad (8.5)$$

$$\Delta_{D_2}(z) = \frac{\Gamma_2(z)}{\sqrt{2\pi}} e^{\zeta'(-1) + z \log \sqrt{2\pi} + \frac{\delta_1 - \gamma}{2}((z-1)^2 - \frac{1}{6})} \quad (8.6)$$

By combining (8.5) and (8.2) one gets the usual Stirling formula for $\Gamma(z)$.

9 THE FUNCTION $\tilde{\theta}_D(t, s)$

We now define and examine the function $\tilde{\theta}_D(t, s)$ for a directed divisor. This function is a Laplace transform of $\xi_D(s, z)$ for the variable z which turns out to be a "mixture" of $\theta_D(t)$ and $\xi_D(s)$ and is an essential tool in [II2]. We give without proof a sort of generalization of Theorem 3 to directed divisors in terms of this function.

In the sequel $\mathcal{W}l_{\varphi_1, \varphi_2}$ is regarded as the subset of all those $t \in \widetilde{\mathcal{C}}^*$ with $\varphi_1 < \arg(t) < -\varphi_2 + 2\pi$ (and use these arguments for $\log t$). Then $\mathcal{W}l_{\varphi_1, \varphi_2}$ is also defined for $\varphi_i \leq 0$, which is needed in what follows.

Definition 9.1 *For $\rho \in \mathcal{C} \setminus \mathbb{R}^{\geq 0}$, $s \in \mathcal{C}$ with $(-s) \notin \mathbf{N}_0$ and $t \in \widetilde{\mathcal{C}}^*$ we define*

$$\widetilde{\exp}(\rho, t, s) := \frac{e^{-\pi i(s-1)}}{2\pi i} \Gamma(s) \Gamma(1-s, \rho t) \cdot e^{\rho t}. \quad (9.1)$$

For a directed divisor D in $\mathcal{W}l_{\varphi_1, \varphi_2}$ (with $0 < \varphi_i < \pi$) we define

$$\tilde{\theta}_D(t, s) := \sum_{\rho \in D} \widetilde{\exp}(\rho, t, s) \quad \text{for } t \in \mathcal{W}l_{(\frac{\pi}{2}-\varphi_1), (\frac{\pi}{2}-\varphi_2)}, \Re(s) > r. \quad (9.2)$$

In the definition of $\widetilde{\exp}(\rho, t, s)$, a type of multivalued exponential function, the *incomplete Gamma function* (obviously holomorphic in α and z)

$$\Gamma(\alpha, z) := \int_z^\infty e^{-\tau} \tau^{\alpha-1} d\tau, \quad \alpha \in \mathbb{C}, z \in \widetilde{\mathbb{C}}^*$$

is used. Properties of $\Gamma(\alpha, z)$ are well known (e.g. [EMOT] II, chap. 9). We state the needed properties of $\widetilde{\exp}(\rho, t, s)$ in Lemma 13.1 and give a selfcontained proof. In particular, by the lemma one can see that the defining sum for $\tilde{\theta}_D(t, s)$ converges absolutely and is holomorphic in the given domains.

Proposition 9.2 *With D as in the above definition, $t \in \mathcal{W}l_{(\frac{\pi}{2}-\varphi_1), (\frac{\pi}{2}-\varphi_2)}$ and $\Re(s) > r$ one has*

$$\tilde{\theta}_D(t, s) = \frac{t^{-(s-1)}}{2\pi i} \int_0^{e^{i\alpha}\infty} e^{wt} \xi_D(s, w) dw \quad (9.3)$$

for every $\alpha \in]-\varphi_2, \varphi_1[$ satisfying $\Re(e^{i\alpha}t) < 0$, and the integral converges absolutely. $\tilde{\theta}_D(t, s)$ satisfies the following functional equations:

$$\tilde{\theta}_D(t, s+1) - \tilde{\theta}_D(t, s) = \frac{t^{-s}}{2\pi i} \cdot \xi_D(s) \quad (9.4)$$

and if D is strictly directed in $\mathcal{W}l_{\varphi_1, \varphi_2}$ with $\frac{\pi}{2} < \varphi_i < \pi$,

$$\tilde{\theta}_D(t, s) - e^{2\pi i(s-1)} \tilde{\theta}_D(\exp(2\pi i)t, s) = \theta_D(t), \quad t \in \mathcal{W}r_{(\varphi_2-\frac{\pi}{2}), (\varphi_1-\frac{\pi}{2})}, \quad (9.5)$$

where the overlap in $\mathcal{W}l_{(\frac{\pi}{2}-\varphi_1), (\frac{\pi}{2}-\varphi_2)}$, the domain of definition for $\tilde{\theta}_D(t, s)$, is identified with $\mathcal{W}r_{(\varphi_2-\frac{\pi}{2}), (\varphi_1-\frac{\pi}{2})}$ which is the domain of definition for the partition function $\theta_D(t)$ (compare Definition 6.1).

Proof: By majorized convergence using Lemma 12.1 the Laplace integral representation is obtained from (13.4). The functional equations follow from the corresponding ones for $\widetilde{\exp}(\rho, t, s)$ given in Lemma 13.1. $\square$

Remark 1: The proposition shows that $\tilde{\theta}_D(t, s)$ behaves like $\xi_D(s)$ in the variable s and like $\theta_D(t)$ in the variable t . In particular, $\tilde{\theta}_D(t, s)$ is meromorphic for $\Re(s) > -p$ if and only if this is true for $\xi_D(s)$.

For $q \in \mathbb{C}$, a regularization sequence δ and $B(z) \in \mathbb{C}[z]$ we define the polynomial $B^{[q]}(z) \in \mathbb{C}[z]$ by

$$-\frac{1}{2\pi i} C T_{s=0} \left(\delta(s) B(\partial_s) \left[\frac{\pi(-z)^{s+q}}{\sin(\pi(s+q))} \right] \right) = B^{[q]}(\log z) z^q,$$

with $\arg(-z) := \arg(z) - \pi$.

Theorem 6 Let $0 < \sigma_i \leq \varphi_i < \pi$ for $i = 1, 2$ and D be a directed divisor in $\mathcal{W}_{\varphi_1, \varphi_2}$ such that $\xi_D(s)$ is meromorphic for $\Re(s_0) > -p'$ (comp. Remark 1). For $s_0 \in \mathbb{C}$ with $\Re(s_0) > -p'$, a regularization sequence δ and a pB -system $(p_n, B_n(z))_{n=0,1,\dots}$ with abscissa $p < \infty$ the following statements are equivalent:
A) $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular with singular part distribution

$$(-p_n, B_n(\partial_s) \left[\frac{1}{s + p_n} \right])_{n=0,1,\dots}$$

C') There exists a polynomial $\tilde{P}_{s_0}(t, t^{-1})$ with $\tilde{P}_{s_0}(t, t^{-1})t^{-(p+s_0-1)} \rightarrow 0$ for $|t| \rightarrow \infty$ and such that the Cramér asymptotic with abscissa $p + \Re(s_0) - 1$

$$CT_{s=0} \left(\delta(s)t^{s+s_0-1} \tilde{\theta}_D(t, s+s_0) \right) \sim \tilde{P}_{s_0}(t, t^{-1}) + \sum_{n=0}^{\infty} t^{p_n+s_0-1} B_n^{[[p_n+s_0-1]]}(\log t) \quad (9.6)$$

for $|t| \rightarrow 0$ is valid in $\mathcal{W}_{(\frac{\pi}{2}-\sigma_1), (\frac{\pi}{2}-\sigma_2)}$.

The polynomial in **C')** is then uniquely determined:

$$\tilde{P}_{s_0}(t, t^{-1}) = \frac{1}{2\pi i} \sum_{k=0}^{n-1} CT_{s=0}(\delta(s)\xi_D(s+s_0-k-1))t^k$$

with n such that $n-1 < p + \Re(s_0) - 1 \leq n$.

As this theorem has no direct application to regularization we omit the analogue of Proposition 7.3 for $[[q]]$ which shows that the Cramér asymptotic in **C')** is a general one written in a special manner, and we give only the idea of the proof of Theorem 6.

Proof: (idea) By Theorem 4 it suffices to prove **B')** $\Leftrightarrow$ **C')**. This equivalence can be shown using (9.3) and its inversion by a Hankel integral ((2.6.11) in [II2]) integrating the asymptics term by term. For details see section 2.6.1 in [II2]. $\square$

Remark 2:

$$B^{[[q]]}(\log t)t^q - B^{[[q]]}(\log(\exp(2\pi i)t))(\exp(2\pi i)t)^q = B(\log t)t^q$$

and (9.5) lead one to rediscover the implication **A**) $\Rightarrow$ **C**) in Theorem 3 but now one has with Theorem 4 the general equivalence **A**) $\Leftrightarrow$ **B')** $\Leftrightarrow$ **C')** for all directed divisors already mentioned in section 7. Because of (9.3) **C')** is an explicit determination of the Cramér asymptotic of the Laplace transform of $\xi_D(s, z)$ which is the basic meaning of Theorem 6.

10 RENORMALIZED DETERMINANTS

The following is a generalization of ideas from §5 of [Vo]. Because of (3.2) and (3.3) every meromorphic function $\Delta_D(z)$ of finite order with divisor D has

representations of the form

$$\log \Delta_D(z) = \int_{a_{\ell+1}}^z \int_{a_{\ell}}^{\lambda_{\ell}} \dots \int_{a_1}^{\lambda_1} (-1)^{\ell} \xi_D(\ell+1, \lambda_0) d\lambda_0 d\lambda_1 \dots d\lambda_{\ell} \quad (10.1)$$

for certain $\ell \geq g$ and $a_i \in \mathbb{C}$, e.g. $\Delta_{D,a}^{\text{Wei},\ell}(z)$ defined by equation (3.6) for $a_i = a$. Easy considerations show that one must have $|a_i| = \infty$ in order to get a determinant (compare section 4) by this. But then (10.1) is divergent, so one has to *renormalize* the divergent integral.

If D is quasi-directed and bounded regularizable according to Theorem 4 one has a Stirling asymptotic for $\xi_D(\ell+1, z)$, and (10.1) with $a_i = \infty$ can be renormalized if one has a renormalization for every integral of the form $\int_{\infty}^z \lambda^{-q} B(\log \lambda) d\lambda$, $q \in \mathbb{C}$, $B(z) \in \mathcal{U}[z]$ (taking of course the value of the integral in case of absolute convergence). In the sequel for $B(z) \in \mathcal{U}[z]$ and $q \in \mathbb{C}$ we define $B^{(q-1)}(z) \in \mathcal{U}[z]$ by

$$\begin{aligned} \frac{d}{dz} \left(z^{-(q-1)} B^{(q-1)}(\log z) \right) &= z^{-q} B(\log z) \\ B^{(0)}(0) &= 0. \end{aligned}$$

Thus the $z^{-(q-1)} B^{(q-1)}(\log z)$ are just those primitives of the $z^{-q} B(\log z)$ whose constant terms are zero.

Definition 10.1 *A renormalization sequence ω is a sequence $(\omega_n)_{n=0,1,\dots}$ of complex numbers. For such a renormalization sequence ω and $D \in \mathfrak{D}^{\text{breg}}$, i.e. D is a quasi-directed bounded regularizable divisor, the ω -renormalized determinant $\Delta_D(z)$ of D is defined by*

$$\log \Delta_D(z) = \int_{\infty}^z \int_{\infty}^{\lambda_{\ell}} \dots \int_{\infty}^{\lambda_1} (-1)^{\ell} \xi_D(\ell+1, \lambda_0) d\lambda_0 d\lambda_1 \dots d\lambda_{\ell}$$

for $\ell \geq g$, integrating $(\ell+1)$ times using the Stirling asymptotic for $\xi_D(\ell+1, z)$ from Theorem 4 and following the renormalization rule

$$\int_{\infty}^z \lambda^{-q} B(\log \lambda) d\lambda := \begin{cases} z^{-(q-1)} B^{(q-1)}(\log z) & \text{for } q \neq 1 \\ B^{(0)}(\log z) + \omega(B) & \text{for } q = 1 \end{cases} \quad (10.2)$$

with $\omega(B) := \sum_k \omega_k^0 b_k$ for $B(z) = \sum_k b_k z^k$.

One can easily prove that the definition of $\Delta_D(z)$ is independent of $\ell \geq g$ and of the lines of integration and that it delivers indeed a determinant system on $\mathfrak{D}^{\text{breg}}$. The Stirling asymptotic that determines $\log \Delta_D(z)$ in the same way as (7.8) for the δ -regularized determinant is derived by integrating the Stirling asymptotic for $\xi_D(\ell+1, z)$ term by term following (10.2).

The next Theorem shows that renormalization and regularization in fact are essentially the same.

Theorem 7 *There is a bijection between the set of regularization sequences δ and the set of renormalization sequences ω such that δ -regularized determinant and the ω -renormalized determinant deliver the same determinant system on $\mathfrak{D}^{breg}$.*

The ω^0 -renormalized determinant with $\omega_n^0 := 0$ for all $n \in \mathbf{N}_0$ delivers the zero-renormalization as defined in example 3 in section 3.

Proof: By Theorem 5 and the properties of the map $[q]$, in particular, (7.7) and the fact that Stirling asymptotics for $\log^{(m)} f(z)$ can be differentiated term by term (Remark 2 in section 7) one easily sees that it is sufficient to observe the following:

1. $\delta(s) = \Gamma(1-s)$ is a regularization sequence with $B^{[0]}(0) = 0$ for all $B(z) \in \mathcal{C}[z]$. This follows from (7.5) as then one has $CT_{s=0}(\Gamma^{(k)}(s)) = (-1)^{k+1} k! \delta_k$ for all $k \in \mathbf{N}_0$.
2. Let Ω_1 be the $\mathcal{C}$ -vector space of all renormalization sequences and Ω_2 that of all regularization sequences. Define Λ to be the $\mathcal{C}$ -vector space of all $\mathcal{C}$ -linear maps from $\mathcal{C}[z]$ to $\mathcal{C}$. Then regard the maps

$$\begin{aligned} \alpha_1 : \Omega_1 &\longrightarrow \Lambda \\ \omega &\longmapsto (B \mapsto \omega(B)) \\ \alpha_2 : \Omega_2 &\longrightarrow \Lambda \\ \delta &\longmapsto (B \mapsto (B^{[0]\delta} - B^{[0]0})) \end{aligned}$$

where in the latter definition $[q]_\delta$ means the map $[q]$ for the regularization sequence δ while $[q]_0$ means $[q]$ for the special regularization sequence of zero renormalization ($\delta(s) = \Gamma(1-s)$). These maps are obviously isomorphisms and $\alpha_1^{-1} \circ \alpha_2$ is the demanded isomorphism between Ω_2 and Ω_1 . $\square$

11 PROOF OF THEOREM 2 b)

Given a system of relations

$$\sum_k D_1^{|\alpha_{1,k}^{(i)}|} + \dots + \sum_k D_n^{|\alpha_{n,k}^{(i)}|} = D^{(i)}, \quad i \in I, \quad (11.1)$$

with $D^{(i)} \in \mathfrak{D}^{\text{reg}}$, $D_1, \dots, D_n \in \mathfrak{D}^{\text{qd}} \setminus \mathfrak{D}^{\text{reg}}$, $\alpha_{m,k}^{(i)} \in \mathcal{C}$ for $i \in I$, $m = 1, \dots, n$ and with finite sums over the index k . We first regard logarithms of associated functions for large real z . We choose the logarithms of the regularized determinants $\log \Delta_{D^{(i)}}(z) := -CT_{s=0}(\delta(s)\xi_{D^{(i)}}(s, z))$ and logarithms $\log \Delta_{D_m}^{\text{in}}(z)$ of certain associated functions $\Delta_{D_m}^{\text{in}}(z)$. We search for polynomials

$$P_m(z) = \sum_{l=0}^{g_m} (-1)^{l+1} \frac{x_{m,l}}{l!} z^l, \quad m = 1, \dots, n$$

such that

$$\log \Delta_{D_m}(z) = P_m(z) + \log \Delta_{D_m}^{\text{in}}(z)$$

is consistent with *i*) and *ii*) of Definition 4.1 under (11.1). With the polynomials

$$P_{(i)}(z) := \log \Delta_{D^{(i)}}(z) - \sum_k \log \Delta_{D_1}^{\text{in}}(z - \alpha_{1,k}^{(i)}) - \dots - \sum_k \log \Delta_{D_n}^{\text{in}}(z - \alpha_{n,k}^{(i)}) \quad (11.2)$$

this is equivalent to

$$P_{(i)}(z) = \sum_k P_1(z - \alpha_{1,k}^{(i)}) + \dots + \sum_k P_n(z - \alpha_{n,k}^{(i)}), \quad i \in I, \quad (11.3)$$

and this is equivalent to a system of linear equations for the $x_{m,l}$.

Now by Zorn's lemma a system of linear equations has a solution if every finite subsystem has one. Thus it suffices to prove that there is always a solution if $|I| < \infty$. So wlog we may assume that $\delta(s)$ is a polynomial (as the finitely many $\log \Delta_{D^{(i)}}(z)$ depend only on finitely many δ_n). And after a trivial translation we also assume that all divisors are directed and that there is an $\alpha > 0$ such that $|z| < \alpha$ implies $|z - \alpha_{m,k}^{(i)}| < |\rho|$ for all ρ that occur in the $D^{(i)}, D_m$; we always assume $|z| < \alpha$. If we choose $\log \Delta_m^{\text{in}}(z) = \log \Delta_{D_m,0}^{\text{Wei},g_m}(z)$ (compare equation (3.6)), then we have with the coefficients given in the proof of Theorem 1 b) and using Proposition 3.3:

$$\begin{aligned} \log \Delta_{D_m}^{\text{in}}(z - \alpha_{m,k}^{(i)}) = \\ -CT_{s=0} \left(\delta(s) (\xi_{D_m}(s, z - \alpha_{m,k}^{(i)}) - \sum_{l=0}^{g_m} (-1)^l \frac{(z - \alpha_{m,k}^{(i)})^l}{l!} \xi_{D_m}(s+l)) \right) \end{aligned}$$

and thus by equation (11.2)

$$\begin{aligned} P_{(i)}(z) = & -CT_{s=0} \left(\delta(s) \left(\sum_{l=0}^{g_1} \sum_k (-1)^l \frac{(z - \alpha_{1,k}^{(i)})^l}{l!} \xi_{D_1}(s+l) + \right. \right. \\ & \left. \left. \dots + \sum_{l=0}^{g_n} \sum_k (-1)^l \frac{(z - \alpha_{n,k}^{(i)})^l}{l!} \xi_{D_n}(s+l) \right) \right), \quad i \in I \quad (11.4) \end{aligned}$$

(as $\sum_k \xi_{D_1}(s, z - \alpha_{1,k}^{(i)}) + \dots + \sum_k \xi_{D_n}(s, z - \alpha_{n,k}^{(i)}) = \xi_{D^{(i)}}(s, z)$). Comparison of (11.4) and (11.3) leads to introduce the functions

$$\begin{aligned} x_{m,l}(s) &:= \delta(s) \xi_{D_m}(s+l) \\ P_m(s, z) &:= \sum_{l=0}^{g_m} (-1)^{l+1} \frac{x_{m,l}(s)}{l!} z^l \\ P_{(i)}(s, z) &:= \sum_k P_1(s, z - \alpha_{1,k}^{(i)}) + \dots + \sum_k P_n(s, z - \alpha_{n,k}^{(i)}) \quad (11.5) \end{aligned}$$

where the $x_{m,l}(s)$ and the $P_m(s, z)$ are all holomorphic for $\Re(s) > r_{\max} := \max r_m$ while $P_{(i)}(s, z)$ is meromorphic for $\Re(s) > 0$ with

$$CT_{s=0}(P_{(i)}(s, z)) = P_{(i)}(z), \quad i \in I, \quad (11.6)$$

for $|z| < \alpha$ as is seen from equation (11.4). We now expand (11.5) and (11.3) by powers of z :

$$P_{(i)}(s, z) = \sum_{l=0}^{g_{\max}} p_{(i),l}(s) z^l$$

and $P_{(i)}(z) = \sum_{l=0}^{g_{\max}} p_{(i),l} z^l$. Regard the $p_{(i),l}(s)$ and correspondingly the $p_{(i),l}$ as the components of vectors $\vec{p}(s), \vec{p} \in \mathbb{C}^M$, and regard the $x_{m,l}(s)$ and $x_{m,l}$ as the components of vectors $\vec{x}_{m,l}(s), \vec{x}_{m,l} \in \mathbb{C}^N$. The expansion of (11.3) and (11.5) by powers of z delivers a matrix $B \in \text{Mat}(N \times M, \mathbb{C})$ such that

$$\vec{p}(s) = B \cdot \vec{x}(s) \quad \text{for } \Re(s) > r_{\max} \quad (11.7)$$

and it has to be shown that there is a $\vec{x} \in \mathbb{C}^N$ such that $\vec{p} = B \cdot \vec{x}$. But there is a matrix $\hat{B} \in \text{Mat}(M \times N', \mathbb{C})$ such that a solution exists if and only if $\hat{B} \cdot \vec{p} = 0$. With this $\hat{B}$ one has

$$\hat{B} \cdot \vec{p} = \hat{B} \cdot CT_{s=0}(\vec{p}(s)) = CT_{s=0}(\hat{B} \cdot \vec{p}(s)) = 0$$

where the first equality is obtained from (11.6) and the last from (11.7). $\square$

Remark: Observe that in the proof the operation $CT_{s=0}$ is applied to functions $f(s) = f_1(s) + f_2(s)$ with $f_1(s)$ being meromorphic around $s = 0$ and $f_2(s)$ defined only for $\Re(s) > 0$ but continuous at $s = 0$.

12 PROOF OF THEOREM 4

The following estimate is needed to apply majorized convergence to integrals over $\xi_D(s, z)$.

Lemma 12.1 *Let $0 < \varphi'_i < \varphi_i < \pi$, $i = 1, 2$ and let D be a directed divisor in $\mathcal{W}l_{\varphi_1, \varphi_2}$ and given $r' \geq r$ such that $c := \sum_{\rho \in D} |m_D(\rho) \rho_k^{-r'}| < \infty$. Then for $\Re(s_0) > r$*

$$\sum_{\rho \in D} \left| \frac{m_D(\rho)}{(z - \rho)^{s_0}} \right| = O(|z|^{r' - \Re(s_0)}) \quad (12.1)$$

for $z \in \mathcal{W}r_{\varphi'_1, \varphi'_2}$ and $|z| \rightarrow \infty$.

Proof: We split the series in $\sum_{|\rho| < \frac{1}{2}|z|}$ and $\sum_{|\rho| \geq \frac{1}{2}|z|}$ and treat these two series separately. For the first series we estimate $|z - \rho| > \frac{1}{2}|z|$ and use

$\sum_{|\rho| < x} |m_D(\rho)| \leq cx^{r'}$ for any $x > 0$ which follows immediately from the definition of c . This last inequality on the other hand implies

$$\sum_{x_1 \leq |\rho| < x_2} \left| \frac{m_D(\rho)}{\rho^\alpha} \right| \leq cx_1^{r'} \frac{1}{x_1^\alpha} + \int_{x_1}^{x_2} cr' y^{r'-1} \frac{1}{y^\alpha} dy \quad (12.2)$$

for all $\alpha \in \mathbb{R}^{>0}$ and $0 < x_1 < x_2$ (as the rhs obviously maximizes $\int_{x_1}^{x_2} y^{-\alpha} d\mu(y)$ under the condition $\int_{x_1}^x d\mu(y) \leq cx^{r'}$ for all $x \in [x_1, x_2]$). Observing that there is a $\beta > 0$ such that $|z - \rho| > \beta|\rho|$ for all $\rho \in D$ and $z \in \mathcal{W}r_{\varphi'_1, \varphi'_2}$ and using (12.2) we get the estimate also for the second series. $\square$

In the sequel we often tacitly use the following estimate: For $0 \leq \varphi < \frac{\pi}{2}$, $\alpha_1 < \alpha_2$ and $m \in \mathbb{N}_0$

$$\Gamma^{(m)}(s) = O(e^{-\varphi|\Im(s)|}), \quad |\Im(s)| \rightarrow \pm\infty \quad (12.3)$$

for $\alpha_1 < \Re(s) < \alpha_2$. For $m = 0$ this is part of the Stirling formula, for $m > 0$ it follows by applying Cauchy's inequalities.

Proof of Proposition 7.1: By majorized convergence (for b) apply the above Lemma) the two integral representations have to be proved only for one-point-divisors. In the sequel for expressions a^s we always use $\arg(a) \in]-\pi, \pi[$.

a) Let $\Re(s) > c > 0$ and $\Re(\rho) < 0$, $\Re(z) > 0$, then by Euler's Mellin integral for $\Gamma(s)$ and its inversion one has

$$\begin{aligned} \frac{\Gamma(s)}{(z - \rho)^s} &= \int_0^\infty e^{-zt} t^{s-1} e^{\rho t} dt \\ &= \int_0^\infty e^{-zt} t^{s-1} \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s')}{(-\rho)^{s'}} t^{-s'} ds' \right) dt \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s-s')}{z^{s-s'}} \frac{\Gamma(s')}{(-\rho)^{s'}} ds', \end{aligned}$$

the last equation by interchanging the integrations (Fubini). Using the identity theorem one gets this formula as needed for $\rho \in \mathcal{W}l_{\varphi_1, \varphi_2}$ and $z \in \mathcal{W}r_{\varphi_1, \varphi_2}$ because both sides are holomorphic in the variables ρ and z .

b) For $\rho \in \mathcal{W}l_{\varphi_1, \varphi_2}$, $z \in \mathcal{W}r_{\varphi_1, \varphi_2}$ and $\Re(s) > 0$ we will prove

$$\frac{\Gamma(s)}{(z - \rho)^s} = \frac{1}{2\pi i} \int_C \frac{\Gamma(s - s_0 + 1)}{(z - w)^{s-s_0+1}} \frac{\Gamma(s_0)}{(w - \rho)^{s_0}} dw. \quad (12.4)$$

For $z_0 \in \mathcal{W}r_{\varphi_1, \varphi_2}$ and $\Re(s_0) < 1$ one has

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{1}{(z_0 - w)^{s-s_0+1}} \frac{1}{w^{s_0}} dw &= z_0^{-s} \frac{e^{i\pi s_0} - e^{-i\pi s_0}}{2\pi i} \int_0^\infty \frac{v^{-s_0}}{(1+v)^{s-s_0+1}} dv \\ &= z_0^{-s} \frac{\sin \pi s_0}{\pi} \frac{\Gamma(1-s_0)\Gamma(s)}{\Gamma(s-s_0+1)} \\ &= \frac{\Gamma(s)}{z_0^s} \frac{1}{\Gamma(s_0)\Gamma(s-s_0+1)}, \end{aligned}$$

the first equation by substituting $w \mapsto -z_0 v$ and deforming the contour (residue theorem), the second because of the representation 1.5 (2) in [EMOT] for Euler's beta function $B(u, v) = \Gamma(u)\Gamma(v)\Gamma^{-1}(u+v)$ and the third because of the equation $\Gamma(1-s_0)\Gamma(s_0) = \pi \sin^{-1} \pi s_0$. Now for $\rho \in \mathcal{W}l_{\varphi_1, \varphi_2}$ such that $z = z_0 + \rho \in \mathcal{W}r_{\varphi_1, \varphi_2}$ replace the contour C by the shifted contour $C - \rho$. The value of this integral is independent of ρ (residue theorem) and (by majorized convergence for $\rho \rightarrow 0$) equals the value of the above integral. Applying the substitution $w \rightarrow (w - \rho)$ and the identity theorem yields (12.4) in the demanded generality. $\square$

Proof of Theorem 4: $\mathbf{A} \Rightarrow \mathbf{B}'$: Let $-p < -q < r < c$ with $\Re(p_n) \neq q$ for all n . Then by the residue theorem (7.1) for $\Re(s) > c$ and $z \in \mathcal{W}r_{\varphi_1, \varphi_2}$ becomes

$$\begin{aligned} \xi_D(s, z) &= \frac{1}{2\pi i} \int_{-q-i\infty}^{-q+i\infty} \frac{\Gamma(s-s')}{z^{s-s'}} \xi_D(s') ds' \\ &+ \sum_{\Re(p_n) < q} \underbrace{\text{Res}_{s'=-p_n} \left(\frac{\Gamma(s-s')}{z^{s-s'}} \xi_D(s') \right)}_{B_n(\partial_s) \left[\frac{\Gamma(s+p_n)}{z^{s+p_n}} \right]} \end{aligned}$$

and this is also valid for all $\Re(s) > -q$ as is seen by the identity theorem. From this $\mathbf{B}'$) easily follows for $\Re(s_0) > -p$. If $-p' < \Re(s_0) \leq -p$ then first take the Stirling asymptotic for $s'_0 = s_0 + k$ with $k \in \mathbf{N}$ such that $-p < \Re(s'_0) \leq -p+1$ and integrate k times.

$\mathbf{B}' \Rightarrow \mathbf{A}$: First note that we just need to show that $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular, but we do not need to determine the singular part distribution as then because of $\mathbf{A} \Rightarrow \mathbf{B}'$) and the properties of the map $[q]$ it must be the demanded one. Let now $0 < \sigma'_i < \sigma''_i < \sigma$. We have for $\Re(s) > r$ and $C = C_{\sigma'_1, \sigma'_2}$

$$\xi_D(s, z) = \frac{1}{2\pi i} \int_C CT_{s_1=0}(\delta(s_1) \xi_D(s_1 + s_0, w)) \frac{\Gamma(s-s_0+1)}{(z-w)^{s-s_0+1}} dw \quad (12.5)$$

which is obtained by applying partial integration to (7.2) using (3.2) where the necessary estimates for $|w| \rightarrow \infty$ are derived by integrating (12.1). Now as (12.5) is not valid for $z = 0$ one has to use a little trick: One deforms the contour C and uses a "shifted" Stirling asymptotics. Let $\varepsilon > 0$ then by Taylor series expansion one obtains a pB -systems $(\tilde{p}_n, \tilde{B}_n)$ with abscissa p such that

$$\begin{aligned} \tilde{R}_q(z) &:= CT_{s_1=0}(\delta(s_1) \xi_D(s_1 + s_0, z)) \\ &- \tilde{P}_{s_0}(z) - \sum_{\Re(\tilde{p}_n) < q} \frac{\tilde{B}_n(\log(z+\varepsilon))}{(z+\varepsilon)^{\tilde{p}_n+s_0}} \\ &= O(|z|^{-(q'+\Re(s_0))}) \quad \text{for } |z| \rightarrow \infty \end{aligned}$$

is valid for $z \in \mathcal{W}r_{\sigma'_1, \sigma'_2}$ for all fixed $q' < q < p$.

Now we choose a contour C' which equals C for large $|w|$ but is a little bit

deformed near $z = 0$ such that $z = 0$ lies to its right side and $(-\varepsilon)$ and all $\rho \in D$ lie to its left side. Then (12.5) is also valid for $z = 0$ and C' instead of C . Thus one gets for $\Re(s) > \max(r, -\Re(p_0), \deg \tilde{P}_{s_0})$:

$$\begin{aligned} \xi_D(s) &= \sum_{\Re(\tilde{p}_n) < q} \frac{1}{2\pi i} \int_{C'} \frac{\tilde{B}_n(\log(w + \varepsilon))}{(w + \varepsilon)^{\tilde{p}_n + s_0}} \frac{\Gamma(s - s_0 + 1)}{(-w)^{s - s_0 + 1}} dw \\ &\quad + \frac{1}{2\pi i} \int_{C'} \tilde{R}_q(w) \frac{\Gamma(s - s_0 + 1)}{(-w)^{s - s_0 + 1}} dw \\ &= \sum_{\Re(\tilde{p}_n) < q} \tilde{B}_n(\partial_y) \left[\varepsilon^{y-s} \frac{\Gamma(s - y)}{\Gamma(s_0 - y)} \right]_{|y = -\tilde{p}_n} \\ &\quad + \frac{1}{2\pi i} \int_{C'} \tilde{R}_q(w) \frac{\Gamma(s - s_0 + 1)}{(-w)^{s - s_0 + 1}} dw \end{aligned}$$

where the integral over $\tilde{P}_{s_0}$ vanishes because of the residue theorem and one applies a slight generalization of (12.4) (obtained by deforming the contour and using the identity theorem) to the integral

$$\int_{C'} \frac{\tilde{B}_n(\log(w + \varepsilon))}{(w + \varepsilon)^{\tilde{p}_n + s_0}} \frac{1}{(-w)^{s - s_0 + 1}} dw = \tilde{B}_n(\partial_y) \left[\int_{C'} \frac{(w + \varepsilon)^{y - s_0}}{(-w)^{s - s_0 + 1}} dw \right]_{|y = -\tilde{p}_n}$$

Using (12.3) and choosing C' suitably for the integral over $\tilde{R}_q$ one easily sees that $\xi_D(s)$ is (σ_1, σ_2) -bounded p -regular. $\square$

13 PROPERTIES OF $\widetilde{\exp}(\rho, t, s)$

Lemma 13.1 a) $\widetilde{\exp}(\rho, t, s)$ satisfies the functional equations

$$\widetilde{\exp}(\rho, t, s) - e^{2\pi i(s-1)} \widetilde{\exp}(\rho, \exp(2\pi i)t, s) = e^{\rho t} \quad (13.1)$$

and

$$\widetilde{\exp}(\rho, t, s + 1) - \widetilde{\exp}(\rho, t, s) = \frac{t^{-s}}{2\pi i} \cdot \Gamma(s)(-\rho)^{-s}. \quad (13.2)$$

b) Let $K \subset \{s \in \mathbb{C} \mid \Re(s) > 0\}$ be a compact set and $\varepsilon > 0$. Then there is a $c > 0$ such that for $\rho \in \mathbb{C} \setminus \mathbb{R}^{\geq 0}$, $s \in K$ and $t \in \widetilde{\mathbb{C}}^*$ with $\arg(-\rho) \in]-\pi, \pi[$ and $-\arg(-\rho) - \frac{\pi}{2} + \varepsilon < \arg(t) < -\arg(-\rho) + \frac{3\pi}{2} - \varepsilon$, one has the estimate

$$|\widetilde{\exp}(\rho, t, s)| < c|\rho t|^{-\Re(s)}. \quad (13.3)$$

Proof: For $\frac{\pi}{2} < \arg(t) < \frac{3\pi}{2}$ we have by definition

$$\widetilde{\exp}(\rho, t, s) = \frac{t^{-(s-1)}}{2\pi i} \Gamma(s) \int_0^\infty \frac{e^{wt}}{(w - \rho)^s} dw \quad (13.4)$$

with absolute convergence. (13.2) is obtained by partial integration. Rotating the ray of integration (residue theorem) one first gets for $\Re(s) < 1$ and $-\arg(-\rho) - \frac{\pi}{2} < \arg(t) < -\arg(-\rho) + \frac{\pi}{2}$ with $\arg(-\rho) = \alpha - \pi$ and a small $\delta > 0$

$$\begin{aligned} \widetilde{\text{exp}}(\rho, t, s) &= e^{2\pi i(s-1)} \widetilde{\text{exp}}(\rho, \exp(2\pi i)t, s) \\ &= \frac{t^{-(s-1)}}{2\pi i} \Gamma(s) e^{\rho t} \underbrace{\left(\int_0^{e^{i(\alpha+\delta)\infty}} - \int_0^{e^{i(\alpha-\delta)\infty}} \frac{e^{wt}}{w^s} dw \right)}_{(e^{-\pi i(s-1)} - e^{\pi i(s-1)}) t^{s-1} \Gamma(1-s)} \\ &= \frac{1}{2\pi i} \Gamma(s) \Gamma(1-s) 2i \sin(\pi s) e^{\rho t} = e^{\rho t}, \end{aligned}$$

thus (13.1), by the identity theorem also in general. It remains to prove (13.3). We assume $\arg(\rho t) \in]-\varepsilon', \varepsilon'[\mathbf{}$ for $0 < \varepsilon' < \frac{\pi}{2}$, the general case follows then by rotating the ray of integration.

$$\widetilde{\text{exp}}(\rho, t, s) = \frac{\Gamma(s)}{2\pi i} (-\rho t)^{-(s-1)} \int_0^\infty \frac{e^{(-\rho t)w}}{(w+1)^s} dw$$

immediately gives the estimate $|\widetilde{\text{exp}}(\rho, t, s)| < c_1 |\rho t|^{-(\Re(s)-1)}$ for $|\rho t| \geq 1$ for a suitable $c_1 > 0$ and thus (13.3) by (13.2). For $|\rho t| \leq 1$ on the other hand with $0 < \alpha := \Re(\rho t) \leq 1$ and the trivial estimate $e^{-x} \leq \frac{1}{x+\alpha}$ for $x \in \mathbb{R}^{\geq 0}$ one has

$$\int_0^\infty \frac{e^{-\Re(\rho t)w}}{|(w+1)^s|} dw = \alpha^{\Re(s)-1} \int_0^\infty \frac{e^{-x}}{(x+\alpha)^{\Re(s)}} dx \leq \alpha^{\Re(s)-1} \int_0^\infty \frac{dx}{(x+\alpha)^{\Re(s)+1}} dx,$$

and the assertion easily follows also for $|\rho t| \leq 1$. $\square$

14 MISCELLEANEA

In chapter 2 of [II1] the formalism of regularized determinants was developed more generally: Following Jorgenson and Lang ([JL1]) divisors with non-integer multiplicities $m_D : \mathcal{C} \rightarrow \mathcal{C}$ (instead of $\mathcal{C} \rightarrow \mathbb{Z}$) were regarded, then everything can be carried out with almost no difficulties, except that the associated functions become multivalued with the $\rho \in D$ as branch points. Also essential singularities for $\xi_D(s)$ were allowed. In that case it is necessary that the formal power series $\delta(s)$ is convergent near zero. With this assumption almost everything can be done in general although some not completely trivial convergence problems occur.

The maps $[q]$ and $[[q]]$ defined in sections 7 and 8 are special cases of the following construction: For $q \in \mathcal{C}$, a regularization sequence δ and a function $h(s)$, which is meromorphic in a neighborhood of q we define a linear map $[h, q] : \mathcal{C}[z] \rightarrow \mathcal{C}[z]$ (notation: $B(z) \mapsto B^{[h, q]}(z)$) by

$$CT_{s=0}(\delta(s)B(\partial_s)[h(s)z^{s+q}]) = B^{[h, q]}(\log z)z^q.$$

If $h_1(s)$ and $h_2(s)$ are two such function, then if $h_1(s)$ is, in addition, holomorphic at $s = q$ the composition law $[h_2, q] \circ [h_1, q] = [h_1 \cdot h_2, q]$ is easily checked. For example this implies $[[q]] = -\frac{1}{2\pi i} [1 - q] \circ [q]$ for $q \neq -n$, $n \in \mathbb{N}_0$. Also a sort of inverse of $[q]$ can be defined (compare Satz 2.3.6 in [II1]).

References

- [Al] G. Alnquist, *Asymptotic Formulas and Generalized Dedekind Sums*, Exp. Math. 7 (1998), 343-359
- [Bar] E.W. Barnes, *On the Theory of the Multiple Gamma Function*, Phil. Trans. of the Royal Soc. (A) 19 (1904), 374-439
- [Cr] H. Cramér, *Studien über die Nullstellen der Riemannschen Zeta-funktion*, Math. Zeitschrift 4 (1919), 104-130
- [CV] P. Cartier, A. Voros, *Une nouvelle interpretation de la formule des traces de Selberg*, in: The Grothendieck Festschrift, Vol. 2, Birkhäuser, 1991, 1-67
- [De1] C. Deninger, *Motivic L-functions and regularized determinants*, in: Motives, Proc. of Symp. Pure Math. 55/1, AMS, Providence, 1994, 707-743
- [De2] C. Deninger, *Motivic L-functions and regularized determinants II*, in: F. Catanese (Hrsg.) *Proc. Arithmetic Geometry*, Cortona, 1994
- [De3] C. Deninger, *Some Analogies between Number Theory and dynamical Systems on foliated Spaces*, Documenta Mathematica, extra vol. ICM 1998. I, Plenary Talks, 23-46
- [Do] G. Doetsch, *Handbuch der Laplacetransformation I/II*, Birkhäuser, Basel, 1950/1955
- [Ef] L. Efrat, *Determinants of Laplacians on surfaces of finite volume*, Comm. Math. Phys. 119 (1988), 443-451, *Erratum*, Comm. Math. Phys. 138 (1991), 607
- [EMOT] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher transcendental functions I, II, III*, McGraw-Hill, New York, 1953
- [EORBZ] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, S. Zerbini, *Zeta regularization techniques with applications*, World Scientific, Singapore, 1994
- [Gui] A.D. Guinand, *Fourier reciprocities and the Riemann zeta function*, Proc. London Math. Soc. (2) 51 (1950), 401-414

- [Il1] G. Illies, *Regularized products, trace formulas and Cramér functions*, Ph.D.-thesis (in German), Schriftenreihe des mathematischen Instituts der Universität Münster, 3. Serie, Heft 22, 1998
- [Il2] G. Illies, *Cramér functions and Guinand equations*, IHES-preprint 1999
- [JL1] J. Jorgenson, S. Lang, *Basic Analysis of regularized series and products*, LNM1564, Springer, Berlin, 1994
- [JL2] J. Jorgenson, S. Lang, *On Cramér's theorem for general Euler products with functional equation*, Math. Ann. 297/3 (1993), 383-416
- [JL3] J. Jorgenson, S. Lang, *Extension of analytic number theory and the theory of regularized harmonic series from Dirichlet series to Bessel series*, Math. Ann. 306 (1996), 75-124
- [Ko1] S.Y. Koyama, *Determinant expressions of Selberg zeta functions I*, Trans. AMS 324 (1991), 149-168
- [Ko2] S.Y. Koyama, *Determinant expressions of Selberg zeta functions II*, Trans. AMS 329 (1992), 755-772
- [Ko3] S.Y. Koyama, *Determinant expressions of Selberg zeta functions III*, Proc. AMS 113 (1991), 303-311
- [LW] E. Landau, A. Walfisz, *Über die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen*, Rend. di Palermo 44 (1919), 82-86
- [Ma] Y.I. Manin, *Lectures on zeta functions and motives*, preprint MPI Bonn, 1992
- [No] N.E. Norlund, *Memoire sur les polynomes de Bernoulli*, Acta Mathematica 43 (1920), 121-196
- [QHS] J.R. Quine, S.H. Heydari, R.Y. Song, *Zeta-regularized products*, Trans. of the AMS 338, 1 (1993), 213-231
- [RS] D. Ray, I. Singer, *Analytic torsion for analytic manifolds*, Ann. Math. 98 (1973), 154-177
- [Sa] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. 110 (1987), 113-120
- [ScSo] M. Schröter, C. Soulé *On a Result of Deninger Concerning Riemann's Zeta Function* in: Motives, Proc. of Symp. Pure Math. 55/1, AMS, Providence, 1994, 745-747

- [So] C. Soulé, letter to C. Deninger, 13.2.1991, as: M. Schröter. S. Soulé, *On a result of Deninger concerning Riemann's zeta function*, in: *Motives*, Proc. of Symp. Pure Math. 55/1, AMS, Providence, 1994, 745-747
- [Ti] E.C.Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, Oxford, 1939
- [Va] I. Vardi, *Determinants of Laplacians and multiple Gamma Functions*, Siam J. Math. Anal. 19, 1 (1988), 493-507
- [Vi] M.F. Vigneras, *L'équation fonctionnelle de la fonction zeta de Selberg du groupe modulaire $SL(2, \mathbb{Z})$* , Asterisque 61 (1979), 235-249
- [Vo] A. Voros, *Spectral Functions, Special Functions and the Selberg Zeta Function*, Comm. Math. Phys. 110 (1987), 439-465

Georg Illies
 IHES
 Le Bois-Marie
 35, Route de Chartres
 F-91440 Bures-sur-Yvette
 illies@ihes.fr