Accelerated motion and special relativity transformations

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Abstract.
Transformation rules for coordinates, velocities and accelerations in accelerated reference frames are derived. Generalized approach of the special relativity is taken for a basis. A 7-dimensional space including projections of a velocity vector as three additional coordinates to time and geometric coordinates is studied. Turns in pseudoplane (dt, dv) of this 7-space describe accelerated motion of frame. In addition to the velocity of light c, the transformation formulas contain a fundamental constant A which has a meaning of maximal acceleration. It is demonstrated that if a source of light moves with acceleration with respect to some reference frame, the velocity of light is smaller than c and depends on the acceleration in this frame. The special relativity relation between energy, impulse, and mass gets changed for a particle in accelerated motion. A generalized wave operator being invariant to the above transformations is introduced. The components of electromagnetic field tensor and electromagnetic field potential get mixed in transformation relations for an accelerated frame.

PACS numbers: 03.30.+p

1. Introduction

At the moment the relativistic approach reducing to Rindler transformations (see, e.g., [1, 2]) is widely used for the analysis of phenomena in accelerated reference frames. However the definition of acceleration is in contradiction with invariance of derivation rules with respect to derivative degree in the specified approach. Really, a first derivative $v = dx/dt$, a second derivative $a = dv/dt$, and, in a common case, a derivative of any degree must be defined by uniform derivation rules. The additional algebraic relation between velocity $v$ and turn angle $\psi$ in a pseudoplane $(x, t)$ is introduced in the special relativity. In particular, this relation allows to receive the relativistic rule of composition of velocities. Yet when acceleration is defined a similar algebraic relation is not considered. For this reason generalized relativistic relations for acceleration can escape our understanding and the rule of composition of accelerations being similar to the special relativity rule of composition of velocities can not be formulated.

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Further it will be shown the mentioned contradiction in description of acceleration is eliminated, if it is assumed that light has additional kinematic properties apart from known property to move with the constant velocity $c$. Namely,

(i) light can move with acceleration;
(ii) there is the maximal acceleration $A$ of light;
(iii) velocity $v$ and acceleration $a$ of light follow to the relation
\[ c^2 - v^2 - \frac{c^2}{A^2} a^2 = 0. \]
(iv) the specified relation and the fundamental constants entered into it, the velocity $c$ and the acceleration $A$, do not depend on velocity and acceleration of a reference frame.

In 1981, the proposal of the existence of the maximal acceleration was first made by Caianiello [3] in the context of a quantization model formulated in an eight-dimensional geometric phase space, with coordinates $x^a = \{x^i, (\hbar/mc) u^i\}$, where $x^i$ is the position four-vectors, and $u^i = dx^i/ds$ is relativistic four-velocity ($i = 1, \ldots, 4$).

The objective of this work is to find the special relativity generalization to accelerated motions through derivation rules for acceleration as well as for velocity.

2. Formulation of problem

We shall consider a body $B$ in linear motion with respect to a frame $K$ with velocity $v$ and acceleration $a$. Let a frame $K'$ moves linear with respect to $K$ with velocity $\nu$ and acceleration $\alpha$. Let the motion of body $B$ is characterized by velocity $\nu'$ and acceleration $\alpha'$ with respect to $K'$. In Figure 1 frames explaining the considered situation are shown.

In the Newtonian mechanics, the kinematics of a linear uniform accelerated motion is described by the system of differential equations

\[
\begin{align*}
\frac{dt}{d'} &= dt', \\
\frac{dx}{d'} &= v \, dt' + dx', \\
\frac{dv}{d'} &= a \, dt' + dv', \\
\frac{da}{d'} &= da' = 0.
\end{align*}
\]
We assume that the kinematics of linear uniform accelerated motion generalizing the special relativity theory is described by the system of differential equations
\[
\begin{bmatrix}
\frac{dt}{dx} \\
\frac{dx}{dv} \\
da = da'
\end{bmatrix} = F 
\begin{bmatrix}
\frac{dt'}{dx'} \\
\frac{dx'}{dv'}
\end{bmatrix},
\]
where the elements of transformation matrix \( F \) are functions of velocity and acceleration.

For linear uniform motion of the frame \( K' \) the specified system of the differential equations should be reduced to Lorentz transformations. Therefore
\[
F(v, 0) = \begin{pmatrix}
\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & \frac{v}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} & 0 \\
\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Moreover for Newtonian limit it should be fulfilled
\[
F(v, a) = \begin{pmatrix}
1 & 0 & 0 \\
\frac{v}{a} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Also (2) should reduce to Rindler transformations
\[
t = \frac{c}{a} \sinh \left( \frac{at'}{c} \right), \quad x = \frac{c^2}{a} \cosh \left( \frac{at'}{c} \right)
\]
in a special case.

The desired generalization of the special relativity must meet to the specified requirements.

A uniform mathematical approach for the necessary generalizations is proposed in Section 3. An accelerated motion kinematics, including rules of composition of velocities and accelerations and transformations of kinematic variables, is considered in Section 4. In particular, the reduction to Rindler transformations is produced. A generalization of dynamic variables (energy, impulse, force) is produced in Section 5 where wave equation is also modified to describe accelerated motion of a wave. Section 6 contains transformations of electromagnetic field potential and tensor for an accelerated frame. The conclusions are presented in Section 7.

### 3. Transformations in the special relativity

In this section the special relativity relations [4] will be derived, the applied approach being important for further analysis.

A square of linear element being used in the special relativity can be represented as
\[
(ds)^2 = c^2 (dt)^2 - (dx)^2.
\]
The variables is changed conveniently by:
\[
dx^1 = dx, \quad dx^4 = c\, dt.
\]
Using the new variables we have the square of linear element in form
\[(ds)^2 = (dx^4)^2 - (dx^1)^2.\]

The linear transformation preserving the square of linear element
\[||dx|| = V ||dx'||\]
is turn in \((dx^1, dx^4)\) pseudoeuclidean plane [5]. The transformation matrix is
\[
V = \begin{pmatrix}
\cosh \Psi & \sinh \Psi \\
\sinh \Psi & \cosh \Psi
\end{pmatrix},
\]
and
\[
\begin{align*}
dx^4 &= \cosh \Psi (dx^4)' + \sinh \Psi (dx^1)', \\
dx^1 &= \sinh \Psi (dx^4)' + \cosh \Psi (dx^1)',
\end{align*}
\]
where \(\Psi\) is turn angle.

To find a relation between the angle \(\Psi\) and the velocity \(v = dx/dt\) we shall consider the variation of coordinate differentials \(||dx||\) as function of the angle \(\Psi\) variation:
\[
\delta ||dx|| = \delta V ||dx'||.
\]
Taking into account that
\[
||dx'|| = V^{-1} ||dx||
\]
we can set
\[
\delta ||dx|| = (\delta V V^{-1}) ||dx||.
\]
In our case
\[
\delta V V^{-1} = \delta \Psi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and
\[
\begin{align*}
\delta dx^4 &= \delta \Psi dx^1, \\
\delta dx^1 &= \delta \Psi dx^4.
\end{align*}
\]

Now let us consider the differential
\[
\frac{\delta dx^1}{dx^4} = \frac{\delta dx^1}{dx^4} - dx^1 \frac{\delta dx^4}{(dx^4)^2}.
\]
Using (4) we receive
\[
\delta \frac{dx^1}{dx^4} = \delta \Psi \left[ 1 - \left( \frac{dx^1}{dx^4} \right)^2 \right].
\]
In (3), (6) going over to variables \(x\) and \(t\), taking into account that
\[
\frac{dx^1}{dx^4} = \frac{1}{c} \frac{dx}{dt} = \frac{v}{c},
\]
we receive
\[ \frac{dt}{c} = \cosh \Psi \, dt' + \frac{1}{c} \sinh \Psi \, dx', \]
\[ dx = c \sinh \Psi \, dt' + \cosh \Psi \, dx', \]
and
\[ \frac{\delta v}{c} = \delta \Psi \left( 1 - \frac{v^2}{c^2} \right). \]

These relations describe Lorentz transformations and the special relativity rule for composition of velocities.

The principal articles of the mathematical approach used above must be emphasized:

(i) The vector space is built on the differentials of \( x(t) \) function and argument \( t \).

(ii) The linear transformations preserving the square of linear element of this space are considered.

(iii) A correspondence exists between the said linear transformations and the derivative \( v = \frac{dx}{dt} \).

(iv) The rule of composition of the linear transformations corresponds to the composition rule in the space of derivatives (i.e. the velocity composition rule).

We shall assume that this approach has invariant meaning and can be applied to the function \( v(t) \). This point of view allows to obtain transformation rules for an accelerated frame and the rule of composition of accelerations in according with the special relativity principles.

4. Transformations for accelerated frames

In addition to the function \( x(t) \) we shall consider the function \( v(t) = \frac{dx}{dt} \). Correspondingly to the previous section consideration we shall construct a vector space on the differentials \( dx, dt, dv \) and set the square of linear element in the following form
\[ (ds)^2 = c^2 (dt)^2 - (dx)^2 - T^2 (dv)^2. \]

Here the constant \( T \) adjusts the dimensionality of \( dv \) to the dimensionality of linear element and has the dimensionality of time. Setting the constant \( L = cT \)

with the dimensionality of length and dividing the square of linear element by \( L^2 \) we obtain
\[ (d\sigma)^2 = \frac{(ds)^2}{L^2} = \frac{(dt)^2}{T^2} \left( \frac{dv}{L^2} \right)^2 - \frac{(dx)^2}{L^2} \left( \frac{dv}{c} \right)^2. \]

In this expression all components are dimensionless. Let us change variables
\[ x^1 = \frac{x}{L}, \quad x^4 = \frac{t}{T}, \quad x^1 = \frac{v}{c} = \frac{dx^1}{dx^4}. \]
Then the square of linear element is

$$\left( d\sigma \right)^2 = (dx^4)^2 - (dx^1)^2 - (dx_1^4)^2. \quad (7)$$

Further we shall make considerations about a sign of velocity differential square in the square of linear element. One can define dimensionless square of linear element through covariant and contravariant vector coordinates as

$$-(d\sigma)^2 = dx^4 dx_4 + dx^1 dx_1 + dx_1^4 dx^4.$$ 

Here all addends on the right have a positive sign. We shall use by relations

$$dx_1 = g_{11} dx^1 = dx^1, \quad dx_4 = g_{44} dx^4 = -dx^4, \quad dx_1^4 = dx^1_4,$$

where it is discounted that metric tensor components

$$g_{11} = 1, \quad g_{44} = g_{44}^* = -1,$$

and the turn matrix in pseudoeuclidean plane does not change a sign by transposition. In result we shall get (7). Thus coordinate $x^1_4$ is spacelike.

Let us introduce the following notations for dimensionless motion parameters: $x^1, x^4, x_1^4, x_{44}$ as coordinate, time, velocity and acceleration of the body $B$ with respect to the frame $K$; $(x^1)', (x^4)', (x_1^4)', (x_{44})'$ as identical quantities with respect to the frame $K'$; and $x^1, x^4, x_1^4, x_{44}$ are coordinate, time, velocity and acceleration of the frame $K'$ with respect to the frame $K$.

A turn in the space of differentials

$$||dx|| = U ||dx'||$$

preserves the square of linear element. In particular the turn in a pseudoplane $(dx^1, dx^4)$

$$V = \begin{pmatrix} \cosh \Psi & \sinh \Psi & 0 \\ \sinh \Psi & \cosh \Psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

describes the uniform velocity motion of frame. The turn in a pseudoplane $(dx^4, dx_1^4)$

$$A = \begin{pmatrix} \cosh \Phi & 0 & \sinh \Phi \\ 0 & 1 & 0 \\ \sinh \Phi & 0 & \cosh \Phi \end{pmatrix}$$

describes the accelerated motion of frame.

Since turns are in general non-commuting, a motion with $U = V A$ must be distinguished from a motion with $U = A V$.

If the parameters of motion (velocity and acceleration) of the frame $K_1$ are measured with respect to the frame $K$, we shall call the frame $K_1$ being embedded in the frame $K$. Let us introduce an ordering relation on the embedded frames given by the sequence of determination of parameters of motion. Figures 2(a,b) show the embedded frames describing $V A$- and $A V$-motion respectively.
4.1. \( V A \)-motion

We shall consider turns described by a matrix

\[
U = VA = \begin{pmatrix}
\cosh \Psi \cosh \Phi & \sinh \Psi & \cosh \Psi \sinh \Phi \\
\sinh \Phi & 0 & \cosh \Phi \\
0 & 0 & 0
\end{pmatrix}.
\]

Then

\[
\begin{align*}
\delta d^4 x &= \cosh \Psi \cosh \Phi (\delta d^4 x)' + \sinh \Psi (\delta d^1 x)' + \cosh \Psi \sinh \Phi (\delta d_4^1)' , \\
\delta d^1 x &= \sinh \Psi \cosh \Phi (\delta d^4 x)' + \cosh \Psi (\delta d^1 x)' + \sinh \Psi \sinh \Phi (\delta d_4^1)' , \\
\delta d^1_4 x &= \sinh \Phi (\delta d^4 x)' + \cosh \Phi (\delta d_4^1)' .
\end{align*}
\]

(8)

To find the relation connecting the angles \( \Psi \) and \( \Phi \) occurring the turn matrix with velocity \( v = dx/dt \) and acceleration \( a = dv/dt \) of body \( B \) we shall study the variation of coordinate differentials \( ||dx|| \) as function of the angles \( \Psi \) and \( \Phi \) variations:

\[
\delta ||dx|| = \delta U ||dx'|| .
\]

Taking into account that

\[
||dx'|| = U^{-1} ||dx||
\]

we shall express

\[
\delta ||dx|| = (\delta U U^{-1}) ||dx|| .
\]

(9)

In our case \( U = VA \) and

\[
\delta U U^{-1} = \delta V V^{-1} + V (\delta A A^{-1}) V^{-1}
\]

\[
= \delta \Psi \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \delta \Phi \begin{pmatrix}
0 & 0 & \cosh \Psi \\
0 & 0 & \sinh \Psi \\
\cosh \Phi & -\sinh \Phi & 0
\end{pmatrix} .
\]

Substituting last matrix in (9) we obtain

\[
\begin{align*}
\delta d^4 x &= \delta \Psi \delta d^4 x + \cosh \Psi \delta \Phi \delta d_4^1 , \\
\delta d^1 x &= \delta \Psi \delta d^4 x + \sinh \Psi \delta \Phi \delta d_4^1 , \\
\delta d^1_4 x &= \cosh \Psi \delta \Phi \delta d^4 x - \sinh \Psi \delta \Phi \delta d^1 x .
\end{align*}
\]

(10)
Accelerated motion and special relativity transformations

To find a relation connecting the angles $\Psi$ and $\Phi$ with the velocity of body $B$ we shall consider the differential $\delta x_4^1$. Using (5) and (10) we obtain

$$\delta x_4^1 = \left[1 - (x_4^1)^2\right] \delta \Psi + (\sinh \Psi - x_4^1 \cosh \Psi) x_4^1 \delta \Phi,$$

where notation $x_4^1 = \frac{dx_4^1}{dx^4}$ was introduced.

To find a relation connecting the angles $\Psi$ and $\Phi$ with the acceleration of body $B$ we shall consider the differential

$$\delta \frac{dx_4^1}{dx^4} = \frac{\delta dx_4^1}{dx^4} - dx_4^1 \frac{\delta dx^4}{(dx^4)^2}.$$

Using (10) we obtain

$$\delta x_4^1 = -x_4^1 x_4^1 \delta \Psi + \left\{ \cosh \Psi \left[1 - (x_4^1)^2\right] - \sinh \Psi x_4^1 \right\} \delta \Phi.$$

The relations (8), (11) and (12) describe transformations of coordinates generalized for the case of accelerated motion of the body $B$ and the frame $K'$ as well as the rules of composition of velocities and accelerations.

Further we shall consider a solution of equations (11) and (12), when the velocity of body $B$ is zero with respect to $K'$. In this case $v = v$. The frames explaining the rule of composition of accelerations are shown in Figure 3.

4.1.1. Rule of composition of accelerations

Because, for $V_A$-motion the acceleration $a$ of the frame $K'$ is determined with respect to the uniform velocity frame $K_1$ (see Figure 3), the rule of composition of accelerations in the frame $K$ depends on the velocity $v$ of the embedded frame $K_1$.

From (11) under initial condition ($\Psi = 0, x_4^1 = 0$) follows

$$x_4^1 = x_4^1 = \tanh \Psi.$$

If the velocity $x_4^1 = 0$ and $\Psi = 0$ then (12) gets reduced to

$$\delta x_4^1 = \left[1 - (x_4^1)^2\right] \delta \Phi.$$

Whence we have

$$x_4^1 = \tanh(\Phi + \phi') = \tanh \phi,$$
where $\phi'$ is integration constant and notation $\phi = \Phi + \phi'$ is used. Find integration constant $\phi'$ from the following considerations. We shall consider that $\phi' = 0$ corresponds $a' = 0$, and $\Phi = 0$ corresponds $a = 0$. Thus

$$x_{44}^1 = \tanh \Phi, \quad (x_{44}^1)' = \tanh \phi'.$$

Whence and from (14) the rule of composition of accelerations follows

$$x_{44}^1 = \frac{x_{44}^1 + (x_{44}^1)'}{1 + x_{44}^1 (x_{44}^1)'}.$$

Changing for dimensional acceleration in accordance with

$$x_{44}^1 = \frac{dx_4^1}{dx^4} = \frac{T}{c} \frac{dv}{dt} = \frac{a}{A},$$

where constant with dimensionality of acceleration

$$A = \frac{c}{T},$$

was introduced, we obtain

$$a = \frac{a + a'}{1 + \frac{a a'}{A^2}}.$$

Thus the resulting acceleration $a$ is always less than or equal to $A$. Note that the above rule of composition of accelerations was derived by Scarpetta within the framework of Caianiello’s model [6].

For non-zero velocity $x_4^1$ with using (13) the equation (12) under the initial condition $(x_4^1)' = 0$ yields

$$x_{44}^1 = \sqrt{1 - (x_4^1)^2} \tanh (\Phi + \phi'). \quad (15)$$

Because of correspondence between $\phi'$ and $a'$, $\Phi$ and $a$ we have

$$x_{44}^1 = \sqrt{1 - (x_4^1)^2} \tanh \Phi \quad (16)$$

and

$$(x_{44}^1)' = \tanh \phi'.$$

In last relation it is taken into account that $(x_4^1)' = 0$. Whence and from (15) the general rule of composition of accelerations follows

$$x_{44}^1 = \sqrt{1 - (x_4^1)^2} \frac{x_{44}^1 + \sqrt{1 - (x_4^1)^2} (x_{44}^1)'}{\sqrt{1 - (x_4^1)^2 + x_{44}^1 (x_{44}^1)'}}.$$

Changing for dimensional values we obtain

$$a = \sqrt{1 - \frac{v^2}{c^2}} \frac{a + \sqrt{1 - \frac{v^2}{c^2}} a'}{\sqrt{1 - \frac{v^2}{c^2} + \frac{a a'}{A^2}}}.$$
4.1.2. Transformation of differentials of coordinates

From (13) and (16) follows

\[
\cosh \Psi = \frac{1}{\sqrt{1 - (x_1^4)^2}}, \quad \sinh \Psi = \frac{x_1^4}{\sqrt{1 - (x_1^4)^2}},
\]

\[
\cosh \Phi = \frac{\sqrt{1 - (x_1^4)^2}}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}}, \quad \sinh \Phi = \frac{x_{44}}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}}.
\]

Substituting the above expressions in (8) we obtain transformations of differentials of coordinates:

\[
dx^4 = \frac{1}{\sqrt{1 - (x_1^4)^2}} (dx^4)' + \frac{x_1^4}{\sqrt{1 - (x_1^4)^2}} (dx^1)' + \frac{1}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}} (dx_4^1)',
\]

\[
dx^1 = \frac{x_1^4}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}} (dx^4)' + \frac{1}{\sqrt{1 - (x_1^4)^2}} (dx^1)' + \frac{x_1^4}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}} (dx_4^1)',
\]

\[
dx_4^1 = \frac{x_{44}}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}} (dx^4)' + \frac{\sqrt{1 - (x_1^4)^2}}{\sqrt{1 - (x_1^4)^2 - (x_{44})^2}} (dx_4^1)'.
\]

Thus we have received a concrete form of the transformations (2). By using dimensional values for \(v \ll c\) and \(a \ll A\) the transformations are reduced to

\[
dt = dt' + \frac{1}{c} v dx' + \frac{1}{c^2} a dv',
\]

\[
dx = v dt' + dx' + \frac{1}{c^2} a v dv',
\]

\[
dv = a dt' + dv'.
\]

For Newtonian limit (\(c \to \infty\) and \(A \to \infty\)) we obtain the system of the differential equations (1).

4.2. A V-motion

In this case turn can be described by a matrix

\[
U = A V = \begin{pmatrix}
\cosh \Phi & \cosh \Psi & \sinh \Phi \\
\sinh \Psi & \cosh \Phi & 0 \\
\sinh \Phi & \sinh \Psi & \cosh \Phi
\end{pmatrix}.
\]

The transformations of coordinates and the rules of composition of velocities and accelerations can be obtained, much as it was made in the previous Section. We shall give results for the case when \(a' = 0\). This case is illustrated on Figure 4.
The rule of composition of velocities, when acceleration \( x_{44} \neq 0 \), have form

\[
x_4 = \sqrt{1 - (x_{44})^2} x_4 + \sqrt{1 - (x_{44})^2} (x_4')' \frac{1}{\sqrt{1 - (x_{44})^2 + x_4^2}}.
\]

or in dimensional units

\[
v = \sqrt{1 - \frac{a^2}{A^2}} v + \sqrt{1 - \frac{a^2}{A^2}} v' \frac{1}{\sqrt{1 - \frac{a^2}{A^2} + \frac{v}{c^2}}}.
\]

Let the body \( B \) is light source, i.e. \( v' = c \). Then light speed is

\[
v = c \sqrt{1 - \frac{a^2}{A^2}}
\]

with respect to the frame \( K \). Therefore maximum velocity of accelerated motion is smaller than \( c \).

Transformation of differentials of coordinates have form

\[
\begin{align*}
dx^4 &= \frac{1}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} (dx^4)' + \frac{x_4^4}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} \frac{1}{\sqrt{1 - (x_{44})^2}} (dx^1)' + \frac{1}{\sqrt{1 - (x_{44})^2}} (dx_4)', \\
dx^1 &= \frac{x_4^4}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} (dx^4)' + \frac{x_4^4}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} \frac{1}{\sqrt{1 - (x_{44})^2}} (dx^1)', \\
dx_4^1 &= \frac{x_4^4}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} (dx^4)' + \frac{x_4^4}{\sqrt{1 - (x_4')^2 - (x_{44})^2}} \frac{1}{\sqrt{1 - (x_{44})^2}} (dx^1)' + \frac{1}{\sqrt{1 - (x_{44})^2}} (dx_4').
\end{align*}
\]

By using dimensional values for \( v \ll c \) and \( a \ll A \) the transformations are reduced to

\[
\begin{align*}
dt &= dt' + \frac{1}{c^2} v dx' + \frac{1}{A^2} a dv', \\
dx &= v dt' + dx', \\
dv &= a dt' + \frac{1}{c} v a dx' + dv'.
\end{align*}
\]

For Newtonian limit \( (c \to \infty \) and \( A \to \infty \) we obtain the system of the differential equations (1).

5. The special case: Rindler transformations

Let us derive Rindler transformations from \( \mathbf{V} \mathbf{A} \)-motion transformations for example. Consider the relations (8) when

\[
(dx^1)' = 0, \quad (dx_4)' = 0.
\]

In this case \( x_4^1 = x_4^4, \ x_{44}^1 = x_{44}^4 \) and

\[
\begin{align*}
dx^4 &= \cosh \Psi \cosh \Phi (dx^4)', \\
dx^1 &= \sinh \Psi \cosh \Phi (dx^4)', \\
dx_4^1 &= \sinh \Phi (dx^4)'.
\end{align*}
\]

Here angles \( \Psi \) and \( \Phi \) are related with velocity and acceleration of the frame \( K' \) by (13) and (16). Let these angles are small and

\[
\Psi \approx x_4^1, \quad \sinh \Phi \approx \tanh \Phi \approx x_{44}^1, \quad \cosh \Phi \approx 1.
\]
Then the transformations (20) are reduced to
\[ dx^4 = \cosh(x_4^4) (dx^4)', \]
\[ dx^1 = \sinh(x_4^4) (dx^4)', \]
\[ dx_4^4 = x_4^4 (dx^4)'. \]
Integrating these equations by condition of constancy of acceleration \( x_4^4 \) and assuming integration constants are zeros, we obtain
\[ x^4 = \frac{1}{x_4^4} \sinh[x_4^4 (x^4)'], \quad x^1 = \frac{1}{x_4^4} \cosh[x_4^4 (x^4)']. \]
Whence Rindler transformations follow:
\[ t = \frac{c}{a} \sinh \left( \frac{a t'}{c} \right), \quad x = \frac{c^2}{a} \cosh \left( \frac{a t'}{c} \right). \]
Note that in Rindler transformations the time \( t' \) is proper time \( \tau = s/c \). Really, from (19) follows
\[ (ds)^2 = c^2 (dt')^2. \]

6. Relativistic mechanics

Relativistic mechanics generalized to accelerated motion can be constructed by analogy with relativistic mechanics being invariant with respect to uniform velocity motion.

6.1. 7-dimensional velocity

Differentials included in the expression for the square of linear element
\[ (ds)^2 = c^2 (dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - T^2 (dv^1)^2 - T^2 (dv^2)^2 - T^2 (dv^3)^2 \]
can be considered as coordinates of a vector in 7-space. We have for contravariant coordinates of a vector
\[ dx^\alpha = \{ c \, dt, \, dx^1, \, dx^2, \, dx^3, \, T \, dv^1, \, T \, dv^2, \, T \, dv^3 \} \]
and for covariant coordinates of a vector
\[ dx_\alpha = \{ c \, dt, \, -dx_1, \, -dx_2, \, -dx_3, \, -T \, dv_1, \, -T \, dv_2, \, -T \, dv_3 \}. \]
Using the introduced coordinates we rewrite the square of linear element in form
\[ (ds)^2 = dx^\alpha dx_\alpha, \quad (\alpha = 1, \ldots, 7). \]
Expressing it as
\[ ds = c \, dt \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}} \]
we define 7-dimensional velocity as
\[ u^\alpha \equiv \frac{dx^\alpha}{ds} = \frac{\partial s}{\partial x_\alpha} = \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}, \, \frac{v^b}{c \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}, \, \frac{a^b}{A \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} \right\}. \]
in contravariant coordinates and
\[ u_\alpha \equiv \frac{dx_\alpha}{ds} = \frac{\partial s}{\partial x^\alpha} = \begin{cases} \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}, & -v_b, A \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}} \end{cases} \]
in covariant coordinates. Here the index \( b \) enumerating geometric coordinates ranges over values 1, 2, 3, and \( v^b = v_b, \ a^b = a_b, \ v^b v_b = v^2, \ a^b a_b = a^2 \). It is obvious that
\[ u^\alpha u_\alpha = 1. \] (21)

6.2. Free particle action

We shall define free particle action as
\[ S = -mc \int_{s_1}^{s_2} ds, \]
where the integration is over a line in 7-space, \( s_1, s_2 \) are points of the specified line, and \( m \) is the particle mass. The action can be expressed as an integral over time
\[ S = -mc^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}} dt. \]

6.3. Energy, impulse, force

We define 7-dimensional impulse as
\[ p^\alpha = -\frac{\partial S}{\partial x^\alpha} = mc u^\alpha \]
in contravariant coordinates and
\[ p_\alpha = -\frac{\partial S}{\partial x^\alpha} = mc u_\alpha \]
in covariant coordinates. Let us introduce relativistic energy
\[ E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}, \]
relativistic impulse
\[ p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}, \]
relativistic kinetic force
\[ f = \frac{ma}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}}. \]
Using the introduced quantities we can express the components of 7-impulse in form
\[ p^\alpha = \left\{ \frac{E}{c}, \ p^b, \ T f^b \right\} \quad \text{and} \quad p_\alpha = \left\{ \frac{E}{c}, \ -p_b, \ -T f_b \right\}. \]
From (21) follows
\[ p^\alpha p_\alpha = m^2 c^2. \]
It can be written as the relation for energy, impulse, kinetic force, and mass in relativistic mechanics generalized to accelerated motions:

$$\frac{E^2}{c^2} - p^2 - T^2 f^2 = m^2 c^2. \quad (22)$$

From it follows that a massless accelerated particle is like to a particle in uniform velocity motion with “effective” mass

$$\frac{1}{c} \sqrt{\frac{E^2}{c^2} - p^2} = \frac{f}{A}. \quad (23)$$

Transformation of 7-impulse components can be described by the formalism similar to that for transformation of coordinates differentials. For example, in the case of AV-motion of frame, the transformations (18) imply

$$E = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E') + \frac{v}{\sqrt{1 - \frac{a^2}{A^2}}} (p') + \frac{a T^2}{\sqrt{1 - \frac{a^2}{A^2}}} (f'), \quad (24)$$

$$p^1 = \frac{v}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E') + \frac{1}{\sqrt{1 - \frac{a^2}{A^2}}} (p')$$

$$f = \frac{a}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}}} (E') + \frac{v a}{c^2 \sqrt{1 - \frac{v^2}{c^2} - \frac{a^2}{A^2}} \sqrt{1 - \frac{a^2}{A^2}}} (p') + \frac{1}{\sqrt{1 - \frac{a^2}{A^2}}} (f').$$

6.4. Wave equation

7-dimensional derivative operators are given by

$$\frac{\partial}{\partial x^a} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x^b}, -\frac{1}{T} \frac{\partial}{\partial v^b} \right\} \quad \text{and} \quad \frac{\partial}{\partial x^a} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^5}, \frac{1}{T} \frac{\partial}{\partial v^b} \right\}.$$

Using these operators we get wave equation

$$\frac{\partial^2}{\partial x^a \partial x_a} \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^5} - \frac{1}{T^2} \frac{\partial^2}{\partial v^2} = 0.$$

This wave equation is different from the traditional one by the last addend allowing to describe accelerated and motion of wave. as well as the processes of wave initiation and disappearance.

Let $\phi(t, x, v)$ be an arbitrary function describing the wave field. We shall try for the solution of wave equation in form

$$\phi(t, x, v) = \phi_0 \exp \left[ i \left( \kappa_b x^b + \xi_b v^b - \omega t \right) \right], \quad (25)$$

where $\omega$ is the wave frequency, $\kappa_b$ are the coordinates of wave vector. By analogy we shall name $\xi_b$ coordinates of wave vector of velocity.

Substituting expression (23) in the wave equation we obtain a relation connecting $\omega, \kappa$ and $\xi$:

$$\frac{\omega^2}{c^2} - \kappa^2 - \frac{\xi^2}{T^2} = 0. \quad (24)$$
Multiplying this equation by Planck constant square and comparing the result to (22) within the framework of wave-corpuscle duality we obtain the set of relations:

\[ E = \hbar \omega, \quad p = \hbar \kappa, \quad f = \frac{\hbar}{T^2} \xi, \]

the first two are de Broglie relations, while the last one describes the relation between the wave vector of velocity \( \xi \) and the relativistic particle force.

Differentiating expression (24) by wave parameters we get

\[ \frac{\omega d\omega}{c^2} - \kappa^b d\kappa_b - \frac{\xi^b d\xi_b}{T^2} = 0. \]  

(25)

By analogy with the traditional definition of a wave velocity

\[ v^b = \frac{\partial \omega}{\partial \kappa_b} = \frac{\kappa^b}{\omega} c^2. \]

from (25) we define wave acceleration

\[ a^b = \frac{\partial \omega}{\partial \xi_b} = \frac{\xi^b}{\omega} A^2. \]

Substituting expressions for the wave velocity and the wave acceleration in (24) we receive the equation

\[ c^2 - v^2 - a^2 T^2 = 0, \]  

(26)

which we shall name equation of wave motion. Integration of the equation of motion (26) for wave which propagates in one direction reduces to the following two solutions for the wave velocity

\[ v_I = \pm c, \quad v_{II} = \pm c \cos(t/T + C_1), \]

where \( C_1 \) is integration constant. Therefore two types of wave motion are possible. In the first case a wave is in linear motion with constant velocity \( c \), and in the second case a wave oscillates along line with amplitude \( L \). At time moments \( t/T + C_1 = \pi n, \) where \( n \) is integer, wave velocity for oscillatory motion is equal \( \pm c \), and the change of type of wave motion is made possible. Then oscillatory motion of wave can be transformed into linear uniform velocity motion and vice versa. For two-dimensional case, the uniform circular motion is a particular solution of the equation (26). This solution is described relations:

for circle radius

\[ r = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}, \]

for centripetal acceleration

\[ a = A \sqrt{1 - \frac{v^2}{c^2}}. \]
The wave equation generalizing the Klein-Gordon equation for massive particles to accelerated motion can be written in the following form
\[
\frac{\partial^2 \psi}{\partial x^\alpha \partial x^\alpha} + \frac{m^2 c^2}{\hbar^2} \psi = 0.
\]
Substituting (23) in the wave equation we obtain a relation connecting \(\omega, \kappa\) and \(\xi\):
\[
\frac{\omega^2}{c^2} - \kappa^2 - \frac{\xi^2}{T^2} = \frac{m^2 c^2}{\hbar^2}.
\]
(27)

Let’s assume
\[
\omega = \frac{mc^2}{\hbar} + \omega',
\]
where \(\omega' \ll mc^2/\hbar\). For \(\omega'\) the equation (27) is reduced to following
\[
\frac{2m\omega'}{\hbar} - \kappa^2 - \frac{\xi^2}{T^2} = 0.
\]
(28)

Differentiating this expression by wave parameters we obtain
\[
\frac{m\, d\omega'}{\hbar} - \kappa^b \, d\kappa_b - \frac{\xi^b \, d\xi_b}{T^2} = 0.
\]
(29)

By analogy to the traditional definition of wave packet group velocity
\[
v^b_g = \frac{\partial \omega'}{\partial \kappa_b} = \frac{\hbar}{m} \kappa^b
\]
from (29) we define a wave packet group acceleration
\[
a^b_g = \frac{\partial \omega'}{\partial \xi_b} = \frac{\hbar}{m} \frac{\xi^b}{T^2}.
\]

Substituting expressions for group velocity and acceleration of wave packet in (28) we obtain expression
\[
\hbar \omega' = \frac{m \, v^2_g}{2} + \frac{m \, T^2 \, a^2_g}{2},
\]
which is a generalization of known de Broglie’s relation.

7. Transformations for electromagnetic field potential and tensor

Let us introduce an electromagnetic field potential \(A^i\). The component \(A^4 = \varphi\) is a scalar potential. Thereto let us introduce an electromagnetic field tensor
\[
F^i_k = \left\{ F^i_1, F^i_2, F^i_3, F^i_4, F^i_5, F^i_6 \right\} = \left\{ E^1, E^2, E^3, B^3, B^2, B^1 \right\},
\]
where \(E^b\) are electric intensity components, \(B^b\) are magnetic induction components.

Transformation formalism for potential and tensor of electromagnetic field follows from (17) and (18) with due regard that potential \(A^i\) gets transformed similar to \(dx^i\), tensor components \(F^i_k\) get transformed similar to a velocity \(dx^b_4\), and the sets of components \(\{F^i_4, F^i_5\} = \{-E^2, B^3\}, \{F^i_4, F^i_3\} = \{-E^3, -B^2\}\) get transformed similar to \(\{dx^4, dx^1\}\) components. Because the transformations (17) and (18)
do not affect on components \(dx^2\) and \(dx^3\) then potential components \(A^2\), \(A^3\) and electromagnetic field tensor components \(F^2_3 = B^1\) remains preserved.

In particular for \(A\) \(V\)-motion, the transformations have form
\[
\varphi = \frac{1}{\sqrt{1-(x_4')^2}} (\varphi)' + \frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (A^1)' + \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^1)',
\]
\[
A^1 = \frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (\varphi)' + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (A^1)',
\]
\[
E^1 = \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (\varphi)' + \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (A^1)' + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^1)',
\]
\[
B^3 = -\frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^2)' + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (B^3)',
\]
\[
E^2 = \frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^2)' - \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (B^3)',
\]
\[
B^2 = \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^2)' + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (B^3)',
\]
\[
E^3 = \frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^3)' + \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (B^3)'.
\]

Note that in these relations addends proportional to the derivatives of electromagnetic field tensor are omitted. Besides, the variables in (17) and (18) are dimensionless, therefore those in (30) are also dimensionless.

For zero velocity, we have
\[
\varphi = \frac{1}{\sqrt{1-(x_4')^2}} (\varphi)' + \frac{x_4}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^1)', \quad A^1 = (A^1)',
\]
\[
E^1 = \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (\varphi)' + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^1)', \quad B^2 = (B^2)', \quad B^3 = (B^3)',
\]
\[
E^2 = \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^2)', \quad E^3 = \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} (E^3)'.
\]

Let point charge \(e\) is immovable about a frame \(K'\). In the frame \(K'\), the electromagnetic potential and the electric intensity have form
\[
(\varphi)' = \frac{e}{r}, \quad (A^1)' = 0, \quad (E)' = \frac{e}{r^2}.
\]

Here \(r\) is the distance to a charge. Let the frame \(K'\) is in accelerated motion with respect to a frame \(K\). Then in the frame \(K\) at the direction of acceleration, electric intensity is determined by the following expression:
\[
E^1 = \frac{x_{14}}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} \frac{e}{r} + \frac{1}{\sqrt{1-(x_4')^2} \sqrt{1-(x_4')^2}} \frac{e}{r^2}.
\]

Thus for \(a \ll A\), electric intensity contains two addends, the first one is proportional to acceleration and varies inversely with a distance to a charge, while the second one does not depend from acceleration and is in inverse ratio with square of a distance to a charge. It is known result which can be obtained by using Lienart-Wiechert potentials (see, e.g., [7]).
8. Conclusions

The present work considers accelerated motions in the context which differs radically from Caianiello’s conception (see [8] and references therein). However our results are similar to those obtained in works of Caianiello and his colleagues [6, 9]. We assume that acceleration has to be considered as well as velocity in the special relativity. Namely, as turn angle in pseudoplane \((dx, dt)\) corresponds to velocity as turn angle in pseudoplane \((dv, dt)\) must correspond to acceleration. Because of this, the dimension of space-time should be increased through three coordinates of velocity instead of coordinates of four-velocity used in Caianiello’s model. In turn it explains difference between coordinate transformations, and between relations for kinematic and dynamic parameters in our and Scarpetta’s considerations [6].

The generalization of special relativity transformations to accelerated reference frames requires a new fundamental constant \(T\) of time dimensionality to be introduced. This constant breeds a set of secondary constants: fundamental length \(L = cT\), and fundamental acceleration \(A = c/T\). Within the framework of this paper the problem on quantitative values of these constants is not discussed.

Note that the introduced 7-space is pseudoeuclidean and the above considered transformations for accelerated frames preserve its metric.

Since in according to the special relativity, three velocity components and three space turn angles make full Lorentz group, the present work supposes natural generalization to uniform rotations of reference frames.

References