CM-P00058640

Ref.TH.1208-CERN

OPERATOR PRODUCT EXPANSIONS NEAR THE LIGHT CONE

Richard A. Brandt ^{*)}
CERN - Geneva

and
Department of Physics and Astronomy
University of Maryland, College Park

and
Giuliano Preparata ^{**)}
Universita di Roma, Roma

A B S T R A C T

The behaviour of products $A(x)B(o)$ of local field operators near the light cone $x^2=0$ is studied in renormalized perturbation theory. Operator product expansions are shown to exist and describe this behaviour in each order. This result is an extensive generalization of known results about short distance ($x \rightarrow 0$) behaviour in quantum field theory. For example, the local scalar current $j(x) \equiv : \varphi(x) \varphi(x) :$ of dimension two in φ^4 theory, ignoring logs and c numbers, has the short distance behaviour

$$j(x)j(o) \xrightarrow{x \rightarrow 0} c_0 \frac{1}{x^2} j(o) + c_1 \frac{1}{x^2} x^\mu : \varphi \partial_\mu \varphi :$$

It is shown that the light cone behaviour is

$$j(x)j(o) \xrightarrow{x^2 \rightarrow 0} \frac{1}{x^2} \sum_{n=0}^{\infty} x^{\alpha_1} \dots x^{\alpha_n} O_{\alpha_1 \dots \alpha_n}^{(n)}(o)$$

for some necessarily infinite set $\{O_{\alpha_1 \dots \alpha_n}^{(n)}\}$ of local field operators satisfying $\dim O_{\alpha_1 \dots \alpha_n}^{(n)} = n+2$. One must have $O_{(0)}^{(0)} = c_0 j$ and $O_{\alpha_1}^{(1)} = c_1 : \varphi \partial_{\alpha_1} \varphi :$, and the $O_{\alpha_1 \dots \alpha_n}^{(n)}$ for $n \geq 2$ do not contribute for $x \rightarrow 0$ but they are necessary to describe the $x^2 \rightarrow 0$ limit. Similar expansions are derived for the product $j_\mu(x)j_\nu(o)$ of vector currents and for other interesting products in φ^4 theory, the gluon model, all other renormalizable models, and insoluble field theoretic models. Properties of the expansions are discussed in detail.

The usefulness of such expansions arises from the fact that they describe, for example, the configuration space limit corresponding to the physical momentum space limit in which a very massive current interacts with a hadronic system in a high energy inelastic collision. The operator expansions predict the strength of the light cone singularities and thereby provide a means of measuring the dimensions of interacting fields, they determine properties of amplitudes in several variables, and they provide relations between form factors describing different experiments corresponding to different matrix elements of the current products. Several such applications are described and it is concluded that the present experimental results are in good agreement with the naïve field dimensions and canonical singularity structure of renormalizable field theories.

^{*)} A.P. Sloan Foundation Fellow. Address after 1 October 70: Department of Physics, New York University, N.Y.

^{**)} Address after 1 January 70: Department of Physics, New York University, N.Y.

1. INTRODUCTION

The behaviour of products $A(x)B(o)$ of (renormalized) local field operators at short distances $x^A \rightarrow 0$ has, in recent years, been understood in renormalized perturbation theory ¹⁾⁻³⁾ and in soluble field theoretic models ⁴⁾. One obtains operator expansions of the form

$$A(x)B(o) \xrightarrow{x \rightarrow 0} \sum_{i=0}^N F_i(x) O_i(o), \quad (1.1)$$

where $O_1, \dots, O_N$ is a finite set of local field operators and the $F_i(x)$ are functions with singularities $(1/x)^{d_i - d_A - d_B}$ (apart from logs), where the d 's are the dimensions of the fields ⁵⁾. The momentum space limit corresponding to (1.1) is, however, unphysical and so, although there are applications ⁶⁾⁻⁹⁾, the usefulness of expansions like (1.1) is somewhat limited. Of much greater physical interest is the behaviour of products like $A(x)B(o)$ near the light cone (LC) $x^2 \rightarrow 0$. The purpose of this paper is to study this LC behaviour in renormalized perturbation theory.

We show that operator product expansions near the LC exist and have the form

$$A(x)B(o) \xrightarrow{x^2 \rightarrow 0} \sum_{i=0}^M \sum_{m=0}^{\infty} F_{in}(x) x^{\alpha_1} \dots x^{\alpha_m} O_{i, \alpha_1, \dots, \alpha_m}^{(m)}(o), \quad (1.2)$$

where $\{O_i^{(n)} \alpha_1 \dots \alpha_n\}$ is an infinite set of local field operators satisfying

$$\dim O_{i, \alpha_1, \dots, \alpha_m}^{(m)} = d_i + m$$

and $F_{in}(x) \sim F_i(x)$ for $x^2 \rightarrow 0$. Thus, rather than a finite number of fields as occurs in the short distance case (1.1), an infinite number of fields occur in the LC expansion (1.2). Each term

$$T_{in}(x) \equiv x^{\alpha_1} \dots x^{\alpha_n} O_{i, \alpha_1, \dots, \alpha_n}^{(n)}(o)$$

has the same dimension d_i and carries the same LC singularity $\sim F_i(x) \sim x^{d_A + d_B - d_i}$. The short distance behaviour of $T_{in}(x)$ is, however, $x^{d_A + d_B - d_i + n}$ and hence it only contributes a short distance singularity if $n \leq d_i - d_A - d_B$.

Before delving into the derivation of expansions like (1.2), let us exhibit a specific example in φ^4 theory. For notational simplicity, we shall here ignore all logarithmic factors even though they actually occur in perturbation theory. We want to discuss the renormalized scalar current operator $j(x) \equiv : \varphi(x) \varphi(x) :$. Here and elsewhere we use the colon notation $:A(o)B(o):$ to denote a generalized Wick product of renormalized fields obtained from the ordinary product $A(x)B(o)$ by first subtracting off the singular expansion (1.1) (or a trivial modification of it) and then taking the limit $x \rightarrow 0$. The resulting quantity can be shown to be a finite local field operator having the same quantum numbers as the free field ordinary Wick product $:A(o)B(o):$ ¹⁾⁻³⁾. The relevant dimensions in the theory under consideration are $\dim I = 0$, $\dim \varphi = 1$, $\dim \partial_\alpha \varphi = 2$, $\dim : \varphi \varphi : = 2$, etc. Here I is the unit operator - the trivial local field.

We first exhibit the short distance behaviour ¹⁰⁾:

$$j(x)j(o) \xrightarrow{x \rightarrow 0} c_1 \left(\frac{1}{x^2}\right)^2 I + c_2 \left(\frac{1}{x^2}\right) j(o) + c_3 \left(\frac{1}{x^2}\right) x^\alpha : \varphi \partial_\alpha \varphi : (o) . \quad (1.3)$$

Here and elsewhere x^2 means $x^2 - i \epsilon x_0$.

Note that, for $x \rightarrow 0$, $(1/x^2)$ is a power more singular than $(1/x^2)x^\alpha$. Near the LC, however, each function has the same singularity and, in fact, an infinite number of terms with this singularity occurs in the LC expansion. The result is ¹⁰⁾

$$j(x)j(o) \xrightarrow{x^2 \rightarrow 0} c_1 \left(\frac{1}{x^2}\right)^2 I + \frac{1}{x^2} \sum_{n=0}^{\infty} x^{\alpha_1} \dots x^{\alpha_n} O_{\alpha_1 \dots \alpha_n}^{(n)}(o) , \quad (1.4)$$

where $\dim O^{(n)} = n+2$. Thus, each term in the sum has dimension two and carries a LC singularity $1/x^2$. For consistency with (1.3), we must have

$$O^{(0)} = c_2 j \quad \text{and} \quad O_{\alpha_1}^{(1)} = c_3 : \varphi \partial_{\alpha_1} \varphi : .$$

The other terms in (1.4) do not contribute to the short distance limit (1.3). They will be explicitly exhibited in Section 2.

We can now calculate the LC behaviour of, for example, the expectation value of $j(x)j(o)$ in the one-particle state of momentum p . We can write

$$\langle P | O_{\alpha_1 \dots \alpha_n}^{(n)}(o) | P \rangle = a_n p_{\alpha_1} \dots p_{\alpha_n} + b_n g_{\alpha_1 \alpha_2} p_{\alpha_3} \dots p_{\alpha_n} + \dots, \quad (1.5)$$

where the omitted terms each involve at least one $g_{\alpha\beta}$. Only the first term in (1.5) therefore contributes to the leading LC singularity of (1.4). Thus, defining

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad (1.6)$$

we obtain ¹⁰⁾

$$\langle P | j(x)j(o) | P \rangle_c \xrightarrow{x^2 \rightarrow 0} \frac{1}{x^2} f(x \cdot p) \quad (1.7)$$

as the leading LC singularity of the connected matrix element.

Expansions of the form (1.2) exist and describe the LC behaviour of the product of any local field operators in each order of renormalized perturbation theory and, more generally, in any theory in which expansions of the form (1.1) exist for all local field products at short distances. They might therefore be abstracted from these models and assumed to be true in the real world. The usefulness of such expansions stems from the fact that they describe the configuration space limit corresponding to the physical momentum space limit in which a very massive current interacts with a hadronic systems in a high energy inelastic collision. Knowledge of the LC expansion for the relevant current product determines the behaviour of the appropriate inelastic cross-sections in the specified limits. The relevance of the LC behaviour of the proton-proton matrix element of the product of electromagnetic currents to deep inelastic electron-proton scattering has already been noted ¹¹⁾. The advantages provided by the operator expansions are, however, numerous. For example, they predict (by dimensional analysis) the strength of the LC singularity and thereby provide a means of measuring the dimensions of interacting fields,

they determine, by their form, properties of inelastic form factors in several variables, and they provide relations between form factors describing different experiments corresponding to different matrix elements of the current products. Several such applications of LC operator expansions will be described in Section 5. The main conclusion we reach is the following: the present experimental results are in good agreement with the canonical singularity structure of renormalizable field theories - they provide no evidence for unrenormalizable theories and/or non-canonical dimensionality.

For clarity, we first establish the existence of LC operator product expansions in φ^4 theory. We treat the simplest case of the field product $\varphi(x)\varphi(o)$ in Section 2. We deduce the LC expansion from a suitable representation of the short distance expansions of the products $\partial_{\alpha_1} \dots \partial_{\alpha_k} \varphi(x)\varphi(o)$ for all k . We then discuss some properties of the expansion including the nature of the basis fields $0, \alpha_1, \dots, \alpha_n$ and consistency with causality and translational invariance. We consider generalizations of these results to other examples in perturbation theory in Section 3. We remain in φ^4 theory in Section 3.A and derive the expansions for $j(x)j(o)$ and for the product $j_\mu(x)j_\nu(o)$ of vector currents. The restrictions from current conservation are given. In Section 3.B we consider the gluon model. We discuss the symmetries and partial symmetries of the perturbative expansion and construct a convenient set of basis fields. We then exhibit some specific LC expansions. Further perturbative models are briefly discussed in Section 3.C.

Our derivations immediately imply the validity of LC expansions in the soluble field theoretic models. We discuss this in Section 4 and give a more precise treatment of the dimensionality concept in order to include the Thirring model and its "non-canonical" dimensionality.

We describe some applications of our results in Section 5. We show in Section 5.A that the LC of configuration space describes the Bjorken scaling limit of a variety of physical amplitudes. It is shown in Section 5.B that the scaling behaviour observed in inelastic electroproduction can be derived from the LC singularity structure of renormalizable field theories with canonical dimensionality and rules out some significant deviations from such dimensionality. Further applications and conclusions are mentioned in Section 5.C.

2. FIELD PRODUCT IN φ^4 THEORY

In order to avoid irrelevant complications, we begin our discussion of light cone expansions by considering some examples in φ^4 theory, the simplest renormalizable perturbation theory. Our results will be valid to any finite order in this theory. We shall afterwards consider generalization both in perturbation theory and beyond it. In this section we consider only the simplest product $\varphi(x)\varphi(0)$ of the renormalized field operators.

Our procedure will be to deduce the LC expansion from a suitable set of short distance expansions rather than directly. This fact will form the basis of our non-perturbative discussion in Section 4. In addition to assuming the existence of these short distance expansions, we shall make several additional quite reasonable assumptions concerning certain properties of these expansions. The existence of these expansions in the general case, as well as the validity of the additional assumptions, have, however, been directly established in perturbation theory by W. Zimmermann¹²⁾. We shall also make several reasonable Taylor series expansions of fields and interchanges of limits without discussing their domains of validity. Our final result has, however, been directly verified in perturbation theory by W. Zimmermann¹²⁾. We therefore make no attempt here to justify these manipulations.

We begin our detailed discussion of φ^4 theory by choosing a suitable basis for the set of all local fields in the theory with the quantum numbers of $\varphi\varphi$. We shall classify these local fields according to their dimension. It is very convenient to also classify them according to the number of Lorentz indices they carry. These Lorentz indices can always be chosen such that a field has n indices if and only if it has dimension $n+2$. We leave the trivial proof of this statement as an exercise. Writing $\partial_\alpha \varphi(x) \equiv \varphi_\alpha(x)$, etc., a basis for the fields with dimension ≤ 5 is:

$$\begin{array}{ll} \dim 0 & \mathbb{I} \\ \dim 2 & :\varphi\varphi: \\ \dim 3 & :\varphi_\alpha\varphi: \\ \dim 4 & :\varphi_{\alpha\beta}\varphi:, \quad :\varphi_\alpha\varphi_\beta:, \quad g_{\alpha\beta}:\varphi^2: \\ \dim 5 & :\varphi_{\alpha\beta\gamma}\varphi:, \quad :\varphi_{\alpha\beta}\varphi_\gamma:, \quad g_{\alpha\beta}:\varphi_\gamma\varphi^3: \end{array} .$$

The complete basis $\overline{\mathcal{B}}$ is simply a continuation of this list. In choosing this independent basis, we have been guided by the following rules:

- i) the symmetry of the theory under $\varphi \rightarrow -\varphi$ precludes the occurrence of odd numbers of φ 's;
- ii) fields differing from the listed ones by permutations of the Lorentz indices have not been included because all these indices must be eventually contracted with c number coefficients and these coefficients can accomplish the permutations - in any case, the coefficients of interest to us will be totally symmetric;
- iii) generalized Wick products differing by permutations of the factors have not been included since these can be related to the given products - the products can, in fact, be chosen to be symmetric in the factors;
- iv) derivatives of Wick products have not been included since these can be related to Wick products of derivatives.

We have not precisely defined the generalized Wick products occurring in our basis. They are known to exist in all orders of perturbation theory, to be relatively local, and to have the indicated transformation properties ^{1)-3), 12)}. However, as is

the case with any choice of a basis in a linear vector space, there is a certain freedom in their definition. Our specific choice can be uniquely defined by specifying suitably many properties of the operators. Some such properties will be given in the following but others will remain unspecified.

We may now introduce a convenient notation and write the i^{th} field of dimension $n+2$ in the basis as $O_i \alpha_1 \dots \alpha_n^{(0)}$. Thus $O = : \varphi \varphi :$, $O_{\alpha_1} = : \varphi_{\alpha_1} \varphi :$, $O_{1\alpha_1\alpha_2} = : \varphi_{\alpha_1\alpha_2} \varphi :$, $O_{2\alpha_1\alpha_2} = : \varphi_{\alpha_1} \varphi_{\alpha_2} :$, $O_{3\alpha_1\alpha_2} = \mathcal{G}_{\alpha_1\alpha_2} : \varphi^4 :$, etc. Repeated i 's, as well as α 's, will be summed over.

We can use our basis to describe short distance behaviours in φ^4 theory. In any order of perturbation theory, there exist functions $F \dots(x)$ such that

$$\varphi(x)\varphi(o) \xrightarrow{x \rightarrow o} F_c(x)I + F^{(0)}(x)O(o), \quad (2.1)$$

$$\varphi_{\alpha}(x)\varphi(o) \xrightarrow{x \rightarrow o} F_{c\alpha}(x)I + F_{\alpha}^{(0)}(x)O(o) + F_{\alpha\gamma}^{(1)}(x)O^{\gamma}(o), \quad (2.2)$$

$$\varphi_{\alpha\beta}(x)\varphi(o) \xrightarrow{x \rightarrow o} F_{c\alpha\beta}(x)I + F_{\alpha\beta}^{(0)}(x)O(o) + F_{\alpha\beta\gamma}^{(1)}(x)O^{\gamma}(o) + F_{i\alpha\beta\gamma\delta}^{(2)}(x)O_i^{\gamma\delta}(o). \quad (2.3)$$

The fields occurring on the right are those in the basis with dimension less than or equal to that of the left and the short distance singularity of each $F(x)$ is determined, apart from logarithmic factors, by the difference of these dimensions. Thus, for example, $F^{(0)}(x) \sim (\log x^2)^a$ and $F_{\alpha}^{(0)}(x) \sim (\log x^2)^{a'} x_{\alpha}/x^2$, for some integers a and a' , in any order.

The above $F(x)$'s are, however, not independent. One has, for example, $F_{\alpha}^{(0)}(x) = \partial_{\alpha} F^{(0)}(x)$. This type of behaviour is expected since the $F_{\alpha}^{(0)}(x)O(o)$ term in $\partial_{\alpha} \varphi(x)\varphi(o)$ should be the derivative of the $F^{(0)}(x)O(o)$ term in $\varphi(x)\varphi(o)$. Consider next $F_{\alpha\beta}^{(0)}(x)$, which can be written as $Q_{\alpha\beta}^{(0)}(x) + L_{\alpha\beta}^{(0)}(x)$, where

$Q^{(0)}$ is quadratically divergent and $L^{(0)}$ is logarithmically divergent for $x \rightarrow 0$. Now, one has $Q_{\alpha\beta}^{(0)}(x) = \partial_\alpha \partial_\beta F^{(0)}(x)$, as is expected from comparison of (2.2) and (2.3). These properties of the F 's are true in perturbative theory¹²⁾ and will be assumed here. For notational simplicity, we shall make a second assumption about the F 's. We shall afterwards show why this assumption is not really necessary, although it is also correct in perturbation theory¹²⁾. We assume that, for example, each $F_{\alpha\dots}$ in (2.2) is a total derivative $\partial_\alpha F_{\dots}$ and each $F_{\alpha\beta\dots}$ in (2.3) is a double total derivative $\partial_\alpha \partial_\beta F_{\dots}$. This property is expected in view of the fact that it is shared by the left sides of (2.2) and (2.3). Thus $F_{\alpha\gamma}^{(1)}(x)$ has the form $\partial_\alpha F_\gamma^{(1)}(x)$ and $L_{\alpha\beta}^{(0)}(x)$ has the form $\partial_\alpha \partial_\beta L^{(0)}(x)$. At this point, we can simplify our notation considerably by, for example, replacing $F^{(0)}(x)$ in (2.1) by $F^{(0)}(x) + L^{(0)}(x)$. This is possible because $L^{(0)}(x)$ vanishes, apart from logs, like x^2 for $x \rightarrow 0$. Then we can write simply $F_{\alpha\beta}^{(0)}(x) = \partial_\alpha \partial_\beta F^{(0)}(x)$. We can now introduce our final notational simplification. Since, for example, $F_\gamma^{(1)}(x)$ vanishes like x for $x \rightarrow 0$, we can add $F_\gamma^{(1)}(x) O_\gamma^{(0)}$ to (2.1). Then (2.2) is obtained simply by differentiating (2.1).

Making use of the above properties of the expansion coefficients and employing the ensuing notational simplifications, we arrive at the general expansion

$$q_{\alpha_1 \dots \alpha_k}(x) q^{(0)} \xrightarrow{x \rightarrow 0} \partial_{\alpha_1} \dots \partial_{\alpha_k} F(x) I + \sum_{n=0}^{\infty} \partial_{\alpha_1} \dots \partial_{\alpha_k} F_{i\beta_1 \dots \beta_n}(x) O_i^{\beta_1 \dots \beta_n}(0) \quad (2.4)$$

The F 's are inductively defined as above so that (2.4) gives the correct singularity structure for each k . Since $\partial_{\alpha_1} \dots \partial_{\alpha_k} F_{i\beta_1 \dots \beta_k}(x)$ is logarithmically divergent for $x \rightarrow 0$, $F_{i\alpha_1 \dots \alpha_n}(x) \sim |x|^n$ for $x \rightarrow 0$ and only the first k terms in the sum in (2.4) contribute singularities for $x \rightarrow 0$. Note that (2.4) follows from taking k derivatives of the relation

$$q(x) q^{(0)} \xrightarrow{x \rightarrow 0} F(x) I + \sum_{n=0}^{\infty} F_{i\beta_1 \dots \beta_n}(x) O_i^{\beta_1 \dots \beta_n}(0) \quad (2.5)$$

We are not claiming that (2.5) represents $\varphi(x)\varphi(0)$ in powers of x , but only use (2.5) to summarize (2.4). We shall see below, however, that (2.5) does correctly describe the LC limit $x^2 \rightarrow 0$.

To proceed from the short distance behaviours (2.4) to the LC, we introduce a lightlike vector η^μ ($\eta^2 = 0$) and consider the Taylor series

$$\varphi(x+\eta)\varphi(0) = \sum_{k=0}^{\infty} \frac{1}{k!} (\eta \cdot \partial)^k \varphi(x)\varphi(0), \quad (2.6)$$

which is well defined on a suitable domain for $x \neq 0$ and $x^2 \neq 0$.

In the limit $x \rightarrow 0$, $(x+\eta)^2 \rightarrow 0$ and so the left side of (2.6) approaches the LC product of interest. The behaviour of each term on the right side is, however, known in the limit from (2.4) and so, assuming this limit can be interchanged with the sum, we obtain

$$\varphi(x+\eta)\varphi(0) \xrightarrow{x \rightarrow 0} \sum_{k=0}^{\infty} \frac{1}{k!} \eta^{\alpha_1} \dots \eta^{\alpha_k} [\partial_{\alpha_1} \dots \partial_{\alpha_k} F_c(x) I + \sum_{n=0}^{\infty} \partial_{\alpha_1} \dots \partial_{\alpha_k} F_{i\beta_1 \dots \beta_n}(x) O_i^{\beta_1 \dots \beta_n}(0)]. \quad (2.7)$$

We next interchange summations to arrive at

$$\varphi(x+\eta)\varphi(0) \xrightarrow{x \rightarrow 0} F_c(x+\eta) I + \sum_{n=0}^{\infty} F_{i\beta_1 \dots \beta_n}(x+\eta) O_i^{\beta_1 \dots \beta_n}(0), \quad (2.8)$$

which can be written as

$$\varphi(x)\varphi(0) \xrightarrow{x^2 \rightarrow 0} F_c(x) I + \sum_{n=0}^{\infty} F_{i\beta_1 \dots \beta_n}(x) O_i^{\beta_1 \dots \beta_n}(0). \quad (2.9)$$

This is our desired result.

Equation (2.9) can be made more explicit as follows. The known structure of the F 's, in any order of perturbation theory, is

$$F_{i\beta_1 \dots \beta_n}(x) = b_{in} x_{\beta_1} \dots x_{\beta_n} (\log x^2)^{a_{in}} + \bar{b}_{in} g_{\beta_1 \beta_2} x^2 x_{\beta_3} \dots x_{\beta_n} (\log x^2)^{\bar{a}_{in}} + \dots, \quad (2.10)$$

where the omitted terms each involve at least one factor x^2 . Thus, only the first term in (2.10) contributes to the leading LC singularity and we obtain

$$\varphi(x)\varphi(0) \xrightarrow{x^2 \rightarrow 0} F_E(x)I + \sum_{n=0}^{\infty} b_{in}(\log x^2)^{d_{in}} x^{\alpha_1} \dots x^{\alpha_n} O_{i\alpha_1 \dots \alpha_n}(0). \quad (2.11)$$

This has the form (1.2) discussed in the Introduction. We remind the reader not impressed with the rigour of our derivation that (2.11) has been directly confirmed in perturbation theory by W. Zimmermann¹²⁾.

Let us now indicate why our assumption that each coefficient function in (2.4) is the indicated derivative can be avoided. Without using this property (2.4) would have the form

$$\begin{aligned} \varphi_{\alpha_1 \dots \alpha_k}(x)\varphi(0) &\xrightarrow{x \rightarrow 0} \partial_{\alpha_1} \dots \partial_{\alpha_k} F_E(x)I + \partial_{\alpha_1} \dots \partial_{\alpha_k} F(x)O(0) \\ &+ \partial_{\alpha_2} \dots \partial_{\alpha_k} F_{\alpha_1 \beta_1}(x)O^{\beta_1}(0) + \dots + \partial_{\alpha_k} F_{\alpha_1 \dots \alpha_{k-1} \beta_1 \dots \beta_{k-1}}(x)O^{\beta_1 \dots \beta_{k-1}}(0) \\ &+ F_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_k}(x)O^{\beta_1 \dots \beta_k}(0). \end{aligned} \quad (2.12)$$

Introducing the lightlike vector η^μ and proceeding as above, we now obtain

$$\varphi(x+\eta)\varphi(0) \xrightarrow{x \rightarrow 0} \sum_n G_{\beta_1 \dots \beta_n}(x, \eta) O^{\beta_1 \dots \beta_n}(0), \quad (2.13)$$

with

$$G_{\beta_1 \dots \beta_n}(x, \eta) = \int_0^\eta d\xi_1^{\alpha_1} \int_0^{\xi_1} d\xi_2^{\alpha_2} \dots \int_0^{\xi_{n-1}} d\xi_n^{\alpha_n} F_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_n}(x + \xi_n), \quad (2.14)$$

where, in the integrand, $\xi_i^\alpha = \eta^\alpha$ for $\alpha \neq \alpha_i$. Knowledge of the short distance behaviour of $F_{\alpha_1 \dots \alpha_n}$ now determines the behaviour of $G_{\beta_1 \dots \beta_n}(x, \eta)$ for $x \sim 0$ and we again obtain a result of the form (2.11).

As a special case of the LC expansion (2.9), we may consider the short distance limit. The short distance behaviour (2.4) for each k clearly follows from (2.9). It is important to notice, however, that the short distance behaviours (2.4) for $k \geq 2$ do not follow from (2.11). This is because the terms in (2.10) omitted from (2.11) may contain short distance singularities as important as those kept. The leading LC behaviour of $\varphi_{\alpha_1 \dots \alpha_k}(x) \varphi(o)$, on the contrary, does follow from (2.11) as well as from (2.9) provided $\alpha_{in} \neq 0$.

In the remainder of this section we shall consider some properties of our result (2.11). We first note that the basis of local fields which contribute to the leading operator LC singularity in (2.11) is actually much smaller than that indicated above. Namely, the fields in the basis involving $g_{\alpha\beta}$'s give rise to x^2 's in (2.11) and therefore to weaker LC singularities. Thus the basis $\mathcal{B}$ which actually contributes singularities to (2.11) is the following:

$$\begin{array}{ll}
 \text{dim } 0 & I \\
 \text{dim } 2 & :\varphi\varphi: \\
 \text{dim } 3 & :\varphi_{\alpha}\varphi: \\
 \text{dim } 4 & :\varphi_{\alpha\beta}\varphi: , :\varphi_{\alpha}\varphi_{\beta}: \\
 & \vdots \\
 \text{dim } 2r & :\varphi_{\alpha_1 \dots \alpha_{2r-2}}\varphi: , \dots , :\varphi_{\alpha_1 \dots \alpha_{r-1}}\varphi_{\alpha_1 \dots \alpha_{r-1}}: \\
 \text{dim } 2r+1 & :\varphi_{\alpha_1 \dots \alpha_{2r-1}}\varphi: , \dots , :\varphi_{\alpha_1 \dots \alpha_r}\varphi_{\alpha_1 \dots \alpha_{r-1}}: \\
 & \vdots
 \end{array}$$

Note that, although (2.11) with $\mathcal{B}$ gives the correct LC behaviour, it will not, in general, give the correct short distance behaviour of $\varphi_{\alpha_1 \dots \alpha_k}(x) \varphi(o)$.

We could, of course, have guessed that $\mathcal{B}$ was sufficient to describe the LC behaviour of $\varphi(x) \varphi(o)$ from the start. From the set of fields of a given dimension, the subset containing the greatest number of symmetric Lorentz indices not coming from $g_{\alpha\beta}$'s is the subset which gives the leading LC behaviour. This

is because such indices can be contracted with x^μ 's to decrease the dimension and therefore to increase the LC singularity. For example, if the local fields $\mathcal{P}_{\alpha_1 \dots \alpha_r}$ and $\mathcal{P}_{\alpha_1 \dots \alpha_r \alpha_{r+1} \dots \alpha_{r+2s}}$ have the same dimension $r+2$, then the first can contribute a LC singularity which is at worse $x^{\alpha_1} \dots x^{\alpha_r} / x^2$, whereas the second can give $x^{\alpha_1} \dots x^{\alpha_{r+2s}} / (x^2)^{s+1}$. This maximally indexed subset generates precisely $\mathcal{B}$. To change an element of $\mathcal{B}$ (without changing its dimension) by including more than two ϕ 's must decrease the number of its Lorentz indices and hence exclude it.

We next want to discuss the consistency of (2.11) with causality. We begin by putting $-x^2 + i\epsilon x_0$ for x^2 in (2.11) and introducing the perturbative behaviour for $F_c(x)$:

$$\begin{aligned} \phi(x) \phi(0) \xrightarrow{x^2 \rightarrow 0} & b_c [\log(-x^2 + i\epsilon x_0)]^{2c} (x^2 - i\epsilon x_0)^{-1} \\ & + \sum_n b_{in} [\log(-x^2 + i\epsilon x_0)]^{2in} x^{\alpha_1} \dots x^{\alpha_n} O_{i\alpha_1 \dots \alpha_n}(0) . \end{aligned} \quad (2.15)$$

We shall also choose the O 's to be Hermitian. This is most simply accomplished by choosing the generalized Wick products in $\mathcal{B}$ to be symmetric under exchange of their "factors" and by adjoining a factor of i if necessary. The causality requirement on (2.15) that $[\phi(x), \phi(0)]$ vanish for $x^2 < 0$ is then easily seen to be equivalent to the reality of the b 's. We shall henceforth assume this is the case:

$$O_{i\alpha_1 \dots \alpha_n}(0)^* = O_{i\alpha_1 \dots \alpha_n}(0), \quad b_c = \bar{b}_c, \quad b_{in} = \bar{b}_{in} . \quad (2.16)$$

The requirement of translational invariance places interesting restrictions on the O 's occurring in (2.15). To deduce some of these restrictions we translate Eq. (2.15) by $-x$, take its Hermitian conjugate, and then change variables from x to $-x$. We obtain

$$\begin{aligned} \phi(x) \phi(0) \xrightarrow{x^2 \rightarrow 0} & b_c [\log(-x^2 + i\epsilon x_0)]^{2c} (x^2 - i\epsilon x_0)^{-1} \\ & + \sum_n b_{in} [\log(-x^2 + i\epsilon x_0)]^{2in} (-1)^n x^{\alpha_1} \dots x^{\alpha_n} O_{i\alpha_1 \dots \alpha_n}(x) . \end{aligned} \quad (2.17)$$

Since on the left side we have arrived back at (2.15), we can expand the $O_{i\alpha_1 \dots \alpha_n}(x)$ on the right side in a Taylor series about $x=0$ and equate coefficients of $x^{\alpha_1 \dots \alpha_n}$ to obtain the relations

$$(d) \sum_{s=0}^r (-1)^s \frac{1}{(r-s)!} b_{is} [\log(-x^2 + i\epsilon x_0)]^{2is} O_{i\alpha_1 \dots \alpha_s \alpha_{s+1} \dots \alpha_r}(0) \\ = b_{ir} [\log(-x^2 + i\epsilon x_0)]^{2ir} O_{i\alpha_1 \dots \alpha_r}(0). \quad (2.18)$$

Here we have written $2\alpha_0 \beta \equiv 0_{\beta, \alpha}$ etc., and (d) means the equation must be symmetrized in the α 's.

As an example of the restrictions imposed by (2.18), we consider the simplest case $r=1$. Then (2.18) becomes

$$2b_1 [\log(-x^2)]^{2i_1} O_{\alpha_1}(0) = b_0 [\log(-x^2)]^{2i_0} O_{\alpha_1}(0). \quad (2.19)$$

This requires that

$$2_1 = 2_0 \quad \text{and} \quad 2b_1 O_{\alpha_1}(0) = b_0 O_{\alpha_1}(0). \quad (2.20)$$

These relations can be directly verified in perturbation theory.

The discussion of the behaviour of matrix elements of $\varphi(x)\varphi(0)$ near the LC proceeds as in Section 1 [Eqs. (1.5)-(1.7)]. In the decomposition of the general matrix element

$$\langle p_1 \dots p_r | O_{\alpha_1 \dots \alpha_n}(0) | q_1 \dots q_s \rangle,$$

only the covariant monomials containing no $g_{\alpha\beta}$'s contribute to the leading LC singularity. If, for simplicity, we assume that all the a 's are equal ($a_{in} = a$), we obtain a behaviour of the form

$$\langle p_1 \dots p_r | \varphi(x)\varphi(0) | q_1 \dots q_s \rangle \xrightarrow{x^2 \rightarrow 0} [\log(-x^2)]^2 F(p_1, x, \dots, q_s, x). \quad (2.21)$$

We have not discussed the convergence properties of the infinite sums such as occur in (2.15). In perturbation theory, the corresponding power series obtained by taking matrix elements are everywhere convergent and define analytic functions of the variables $p_1 \cdot x$. The operator convergence properties of their sums are, however, somewhat delicate. This can be seen, for example, by commuting both sides of (2.15) with $\varphi(y)$ with y chosen such that $y^2 < 0$ and $(x-y)^2 > 0$. Then the left side is in general non-vanishing whereas, if we commute termwise on the right side, we obtain a vanishing result¹³⁾. We must therefore conclude that the commutation cannot be taken inside of the infinite sum. This behaviour is not surprising and is analogous to the distributions theoretic relation

$$\delta(s+t) = \sum_n \frac{1}{n!} (t \cdot \partial)^n \delta(s) . \quad (2.22)$$

In (2.22), the left side has support at $s = -t$ whereas each term on the right side has support at $s = 0$. Just as (2.22) is only valid for analytic testing functions, (2.15) is only supposed to be valid between (sharp momentum) scattering states. This shows, incidently, that an infinite number of terms must be present in (2.15).

3. PERTURBATION THEORETIC GENERALIZATIONS

By repeating essentially the same analysis used in Section 2, we can deduce the LC behaviour of any product of local field operators to any order in any renormalizable perturbation theory. Various interesting examples of this will be considered in the present section.

A. φ^4 theory

To maintain the simplicity provided by the absence of spin and internal indices, we begin by considering some further examples from φ^4 theory. We first discuss the behaviour of $j(x)j(0)$, where

$$j(x) \equiv :\varphi(x)\varphi(x): = O(x) \quad (3.1)$$

is a scalar current density of dimension two. The short distance behaviours have the form

$$\begin{aligned} \partial_{\alpha_1} \cdots \partial_{\alpha_k} j(x) j(0) \xrightarrow{x \rightarrow 0} \partial_{\alpha_1} \cdots \partial_{\alpha_k} G_c(x) I + \partial_{\alpha_1} \cdots \partial_{\alpha_k} G(x) O(0) \\ + \dots + \partial_{\alpha_1} \cdots \partial_{\alpha_k} G_{i\beta_1 \dots \beta_{k+2}}(x) O_{i\beta_1 \dots \beta_{k+2}}(0) . \end{aligned} \quad (3.2)$$

By the procedure of Section 2, we deduce from these the LC behaviour

$$j(x) j(0) \xrightarrow{x^2 \rightarrow 0} G_c(x) I + \sum_n G_{i\alpha_1 \dots \alpha_n}(x) O_{i\alpha_1 \dots \alpha_n}(0) . \quad (3.3)$$

Now, the G 's have the forms

$$G_c(x) = d_c \left(\frac{1}{x^2}\right)^2 (\log x^2)^{c_c} \quad (3.4)$$

$$G_{i\alpha_1 \dots \alpha_n}(x) = d_{in} x_{\alpha_1} \cdots x_{\alpha_n} \left(\frac{1}{x^2}\right) (\log x^2)^{c_{in}} + \dots , \quad (3.5)$$

where each of the omitted terms have at least one $x_\alpha x_\beta$ replaced by a $x^2 g_{\alpha\beta}$ and so have lesser LC singularities. Thus (3.3) becomes

$$j(x) j(0) \xrightarrow{x^2 \rightarrow 0} d_c \left(\frac{1}{x^2}\right)^2 (\log x^2)^{c_c} + \frac{1}{x^2} \sum_n d_{in} (\log x^2)^{c_{in}} x_{\alpha_1} \cdots x_{\alpha_n} O_{i\alpha_1 \dots \alpha_n}(0) . \quad (3.6)$$

By ignoring the logarithmic factors (i.e., putting $c_{in} = 0$), and defining

$$O_{\alpha_1 \dots \alpha_n}(0) = \sum_i d_{in} O_{i\alpha_1 \dots \alpha_n}(0) , \quad (3.7)$$

we obtain the simplified expression (1.4) of Section 1. In any finite order of perturbation theory, the c_{in} will be bounded by some finite integer c ($c_{in} \leq c$) and so, defining

$$\tilde{d}_{in} = \begin{cases} d_{in} & \text{if } c_{in} = c \\ 0 & \text{if } c_{in} < c \end{cases} , \quad (3.8)$$

(3.6) can be simplified to

$$j(x)j(0) \xrightarrow{x^2 \rightarrow 0} d_c \left(\frac{1}{x^2}\right)^2 (\log x^2)^{c_c} + \left(\frac{1}{x^2}\right) (\log x^2)^{c_c} \sum_n \tilde{d}_{in} x_{\alpha_1} \cdots x_{\alpha_n} O_i^{\alpha_1 \cdots \alpha_n}(0). \quad (3.9)$$

$$= d_c \left(\frac{1}{x^2}\right)^2 (\log x^2)^{c_c} + \left(\frac{1}{x^2}\right) (\log x^2)^{c_c} \sum_n x_{\alpha_1} \cdots x_{\alpha_n} \tilde{O}^{\alpha_1 \cdots \alpha_n}(0), \quad (3.10)$$

where

$$\tilde{O}^{\alpha_1 \cdots \alpha_n} = \sum_i \tilde{d}_{in} O_i^{\alpha_1 \cdots \alpha_n}. \quad (3.11)$$

Note that (3.10) may only give correctly the leading LC singularity. Note also that, for consistency with causality, an infinite number of terms must be present in (3.10).

Our procedure can be generalized in a straightforward manner to take additional Lorentz indices into account. As an important example, we consider a vector current $j_\mu(x)$ of dimension three, for example $:\bar{\psi}(x) \gamma_\mu \psi(x):$. The short distance behaviour has the form

$$\begin{aligned} j_\mu(x)j_\nu(0) \xrightarrow{x^2 \rightarrow 0} & E_{\mu\nu}(x)I + E_{\mu\nu}(x)O(0) + E_{\mu\nu\alpha_1}(x)O^{\alpha_1}(0) \\ & + E_{\mu\nu\alpha_1\alpha_2}(x)O_i^{\alpha_1\alpha_2}(0) + E_{\mu\nu\alpha_1\alpha_2\alpha_3}(x)O_i^{\alpha_1\alpha_2\alpha_3}(0) \\ & + E_{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4}(x)O_i^{\alpha_1\alpha_2\alpha_3\alpha_4}(0). \end{aligned} \quad (3.12)$$

Thus the LC behaviour has the form

$$j_\mu(x)j_\nu(0) \xrightarrow{x^2 \rightarrow 0} E_{\mu\nu}(x)I + \sum_n E_{\mu\nu\alpha_1 \cdots \alpha_n}(x)O_i^{\alpha_1 \cdots \alpha_n}(0). \quad (3.13)$$

To simplify the notation, we shall ignore logarithmic factors in our discussion of the behaviour of the $E_{\mu\nu}$'s. Then the LC singularities are of the form

$$\begin{aligned}
 E_{i\mu\nu\alpha_1\cdots\alpha_n}(x) \longrightarrow & c_{1in}\left(\frac{1}{x^2}\right)^3 x_\mu x_\nu x_{\alpha_1} \cdots x_{\alpha_n} + c_{2in}\left(\frac{1}{x^2}\right)^2 g_{\mu\nu} x_{\alpha_1} \cdots x_{\alpha_n} \\
 & + c_{3in}^{(1)}\left(\frac{1}{x^2}\right)^2 g_{\mu\alpha_1} x_\nu x_{\alpha_2} \cdots x_{\alpha_n} + \text{perm.} \\
 & + c_{4in}^{(1)}\left(\frac{1}{x^2}\right)^2 g_{\nu\alpha_1} x_\mu x_{\alpha_2} \cdots x_{\alpha_n} + \text{perm.} \\
 & + c_{5in}^{(12)}\left(\frac{1}{x^2}\right) g_{\mu\alpha_1} g_{\nu\alpha_2} x_{\alpha_3} \cdots x_{\alpha_n} + \text{perm.} ,
 \end{aligned} \tag{3.14}$$

where "perm" means to add similar terms with the distinguished α 's taking on $\alpha_1 \cdots \alpha_n$. We define the new local field operators

$$P_{r\alpha_1\cdots\alpha_n}(0) \equiv \sum_{\text{perm}} \sum_i c_{rin}^{(i)} O_{i\alpha_1\cdots\alpha_n}(0) , \tag{3.15}$$

and obtain from (3.13) and (3.14) the LC expansion

$$\begin{aligned}
 j_\mu(x) j_\nu(0) \xrightarrow{x^2 \rightarrow 0} & E_{\mu\nu}(x) I + \sum_n \left\{ x^{-6} x_\mu x_\nu x_{\alpha_1} \cdots x_{\alpha_n} P_1^{\alpha_1\cdots\alpha_n}(0) \right. \\
 & + x^{-4} [g_{\mu\nu} x_{\alpha_1} \cdots x_{\alpha_n} P_2^{\alpha_1\cdots\alpha_n}(0) + x_\nu x_{\alpha_1} \cdots x_{\alpha_n} P_{3\mu}^{\alpha_1\cdots\alpha_n}(0) + x_\mu x_{\alpha_1} \cdots x_{\alpha_n} P_{4\nu}^{\alpha_1\cdots\alpha_n}(0)] \\
 & \left. + x^{-2} x_{\alpha_1} \cdots x_{\alpha_n} P_{5\mu\nu}^{\alpha_1\cdots\alpha_n}(0) \right\} ,
 \end{aligned} \tag{3.16}$$

where we have written $x^{-6} = (x^2)^{-3}$, etc.

If the current $j_\mu(x)$ is conserved, then the expansion (3.16) can be simplified further. The current conservation condition $\partial^\mu j_\mu(x) = 0$ places essentially two constraints on (3.16) and reduces the number of operator sequences from five to three. The final result can be conveniently written in the manifestly conserved form

$$\begin{aligned}
 j_\mu(x) j_\nu(0) \xrightarrow{x^2 \rightarrow 0} & (\partial_\mu \partial_\nu - g_{\mu\nu} \square) x^{-2} \sum_n x^{\alpha_1} \dots x^{\alpha_n} \mathcal{R}_{\alpha_1 \dots \alpha_n}(0) \\
 & + i \epsilon_{\mu\nu\alpha\beta} \partial^\alpha x^{-2} \sum_n x^{\alpha_1} \dots x^{\alpha_n} \mathcal{R}_{\alpha_1 \dots \alpha_n}^\beta(0) \\
 & + [g_{\mu\nu} \partial_\alpha \partial_\beta - g_{\mu\alpha} \partial_\beta \partial_\nu - g_{\mu\beta} \partial_\alpha \partial_\nu + g_{\mu\alpha} g_{\beta\nu} \square] (\log x^{-2}) \sum_n x^{\alpha_1} \dots x^{\alpha_n} \mathcal{R}_{\alpha_1 \dots \alpha_n}^{\alpha\beta}(0).
 \end{aligned}
 \tag{3.17}$$

Here the $(\log x^2)$ term does not violate our neglect of logs since the log goes away after it is differentiated. The relation between the $\mathcal{R}$'s and the $\mathcal{P}$'s can be found by explicitly performing the differentiations exhibited in (3.17) but will not be given here.

In a similar manner we can describe the LC behaviour of any product of local field operators in φ^4 theory. This includes products with more than two factors, such as $A(x)B(y)C(0)$ for x^2 , y^2 and $(x-y)^2$ all approaching zero.

B. Gluon Model

In this subsection we shall discuss the LC expansion of some operator products in the gluon model in perturbation theory. This physically interesting ¹⁴⁾ renormalizable field theory describes the interaction of $\text{spin } \frac{1}{2}$ quarks via a neutral vector meson coupled to the (conserved) baryon current. Our study will illustrate how internal quantum numbers can be incorporated into our expansions. Some specific applications of our results will be given in Section 5.

The formal Hamiltonian density is

$$\mathcal{H} = \psi^\dagger (-i \underline{\alpha} \cdot \underline{\nabla} + \beta M_0 + g_0 \beta \gamma^\mu B_\mu) \psi + \mathcal{H}_B,
 \tag{3.18}$$

where ψ is a $\text{spin } \frac{1}{2}$ quark field of dimension $\frac{3}{2}$, B_μ is a neutral vector meson field of dimension 1, $\mathcal{H}_B$ is the Hamiltonian of B_μ , and

$$M_0 = \alpha_0 \lambda^0 + \alpha_8 \lambda^8 \quad (3.19)$$

in the quark mass term. Thus, the chiral $SU(3) \times SU(3)$ symmetry breaking is formally due entirely to the quark mass term and $SU(2)$ is formally an exact symmetry. The Hamiltonian is also formally invariant to the local gauge transformations ¹⁵⁾

$$\Psi(x) \rightarrow e^{i g \Lambda(x)} \Psi(x), \quad B_\mu(x) \rightarrow B_\mu(x) + \partial_\mu \Lambda(x), \quad (3.20)$$

for arbitrary $\Lambda(x)$ satisfying

$$(\square + m_{B_0}^2) \Lambda(x) = 0. \quad (3.21)$$

Finally, (3.18) is formally invariant under the baryonic charge conjugation transformation.

We define the renormalized perturbation theory corresponding to (3.18) such that charge conjugation, $SU(2)$, and local gauge invariance ¹⁶⁾ are exact symmetries, $SU(3)$ is exact in the limit when the renormalized α_8 vanishes, and $SU(3) \times SU(3)$ is exact when the renormalized α_8 and α_0 (or M) vanish ^{17), 18)}.

In this theory we define renormalized vector, axial vector, scalar, and pseudoscalar current densities of dimension three:

$$V_\mu^a = : \bar{\Psi} \gamma_\mu \lambda^a \Psi : \quad (3.22a)$$

$$A_\mu^a = : \bar{\Psi} \gamma_\mu \gamma_5 \lambda^a \Psi : \quad (3.22b)$$

$$S^a = : \bar{\Psi} \lambda^a \Psi : \quad (3.22c)$$

$$P^a = : \bar{\Psi} \gamma_5 \lambda^a \Psi : \quad (3.22d)$$

The corresponding V and A charges satisfy $U(3) \times U(3)$ commutation relations and the S and P charges define a $(\bar{3}, 3) + (3, \bar{3})$ representation of $SU(3) \times SU(3)$. S and P are related to the divergences of V and A respectively according to

$$\partial^\mu V_\mu^a = i\alpha_g f^{abc} S^b, \quad (3.23)$$

and

$$\partial^\mu A_\mu^a = i\alpha_g d^{abc} P^b + i\alpha_g d^{abc} P^b, \quad a = 1, \dots, 8. \quad (3.24)$$

Let us consider now the LC behaviour of $S^a(x)P^b(0)$. We begin as in Section 2 by first defining the relevant basis of local field operators. For simplicity we shall only consider the fields contributing to the leading LC singularity. As discussed in Section 2, for a given dimension, these will be the fields carrying the greatest number of symmetric Lorentz indices not arising from the metric tensor. It is also convenient to choose the basis to have definite transformation properties with respect to the generalized charge conjugation transformation C .

One easily sees that this basis includes the operators ¹⁹⁾

$$\begin{aligned} O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c}(0) &= :(\Delta_{\alpha_1}^* \dots \Delta_{\alpha_n}^* \bar{\Psi}) \gamma_5 \gamma_\mu \lambda^c (\Delta_{\beta_1} \dots \Delta_{\beta_m} \Psi): \\ &\pm :(\Delta_{\beta_1}^* \dots \Delta_{\beta_m}^* \bar{\Psi}) \gamma_5 \gamma_\mu \lambda^c (\Delta_{\alpha_1} \dots \Delta_{\alpha_n} \Psi):, \end{aligned} \quad (3.25)$$

where we have introduced the covariant derivative

$$\Delta_\alpha = \partial_\alpha - ig B_\alpha. \quad (3.26)$$

Under C we have

$$C O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c} C^{-1} = \pm O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c}. \quad (3.27)$$

Also, as in φ^4 theory, we have

$$\dim O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c} = (n+m+1) + 2 = n+m+3. \quad (3.28)$$

The remainder of the basis consists of the fields

$$\tilde{H}_{\alpha\beta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}(0) = :G_{\alpha\gamma, \alpha_1 \dots \alpha_n} \tilde{G}_{\beta, \beta_1 \dots \beta_m}^{\gamma}:, \quad (3.29)$$

where

$$G_{\alpha\beta} = B_{\alpha, \beta} - B_{\beta, \alpha} \quad \tilde{G}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} G^{\gamma\delta}. \quad (3.30)$$

These fields are even under $\mathcal{C}$ and we again have

$$\dim \tilde{H}_{\alpha\beta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m} = (n+m+2) + 2 = n+m+4. \quad (3.31)$$

By repeating the procedure of Section 2, we thus obtain, in each order of perturbation theory, the expansion

$$\begin{aligned} S^2(x) P^b(0) \xrightarrow{x^2 \rightarrow 0} & x^{-4} \sum_{\substack{n, m \\ c \pm}} \beta_{nm}^{\pm abc} (\log x^2)^{\alpha_{nm}^{\pm abc}} x^\alpha x^{\alpha_1} \dots x^{\beta_m} O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c}(0) \\ & + x^{-4} \sum_{nm} \beta_{nm}^{ab} (\log x^2)^{\alpha_{nm}^{ab}} x^\alpha x^\beta x^{\alpha_1} \dots x^{\beta_m} \tilde{H}_{\alpha\beta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}(0). \end{aligned} \quad (3.32)$$

We can make this expansion more explicit by employing the fact that the only $SU(3) \times SU(3)$ symmetry breaking is in the mass term. Since the leading LC singularity is mass independent, it must therefore be $SU(3) \times SU(3)$, and hence $\mathcal{C}$, symmetric. Thus (3.32) must have the form

$$\begin{aligned} S^2(x) P^b(0) \xrightarrow{x^2 \rightarrow 0} & x^{-4} \sum_{\substack{n, m \\ \pm}} \beta_{nm}^{\pm} d_{\pm}^{abc} (\log x^2)^{\alpha_{nm}^{\pm}} x^\alpha x^{\alpha_1} \dots x^{\beta_m} O_{5\mu, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}^{\pm c}(0) \\ & + x^{-4} \delta^{ab} \sum_{nm} \beta_{nm} (\log x^2)^{\alpha_{nm}} x^\alpha x^\beta x^{\alpha_1} \dots x^{\beta_m} \tilde{H}_{\alpha\beta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m}(0), \end{aligned} \quad (3.33)$$

where $d_+^{abc} \equiv d^{abc}$ is the symmetric octet coupling matrix and $d_-^{abc} \equiv f^{abc}$ is the antisymmetric one.

As a second example from the gluon model, we consider the product $J_\mu(x)J_\nu(0)$ of electromagnetic currents. For notational simplicity, we again neglect logarithmic factors. The basis includes currents like (3.25) but without the γ_5 and only those even under charge conjugation. We call these fields $O^c_\alpha, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. They have dimension $(n+m+1)+2=n+m+3$, as in φ^4 theory, Ref. 20). Another set of possible fields in the basis is given by

$$H_{\alpha\beta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m} = : G_{\alpha\gamma, \alpha_1 \dots \alpha_n} G^{\gamma}_{\beta, \beta_1 \dots \beta_m} : \quad (3.34)$$

These have dimension $(n+m+2)+2$. The $O^c_\alpha, \dots$ and the $H_{\alpha\beta}, \dots$ contribute to the LC expansion as in (3.13) and (3.16) since the relation between Lorentz indices and dimension is the same as encountered in φ^4 theory. In particular, using current conservation, we obtain an expression of the form (3.17).

Another set of possible fields in the basis is given by

$$H_{\alpha\beta\gamma\delta, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_m} = : G_{\alpha\gamma, \alpha_1 \dots \alpha_n} G_{\beta\delta, \beta_1 \dots \beta_m} : \quad (3.35)$$

These have dimension $n+m+4$, which is the number of their Lorentz indices rather than that number plus two. One might therefore fear that they will carry stronger singularities than the previous fields. This is not the case, however, because of the antisymmetry properties of (3.35). The coefficient $E_{\mu\nu, \alpha\beta\gamma\delta, \alpha_1, \dots, \beta_m}$ of (3.35) in the expansion of $J_\mu(x)J_\nu(0)$ can be decomposed as in (3.14) but with singularities greater by a factor of $(1/x^2)$. Clearly only the weakest type singularity

$$\sim x^{-4} g_{\mu\alpha} g_{\nu\beta} x_\gamma x_\delta x_{\alpha_1} \dots x_{\beta_m}$$

gives a non-vanishing contribution. Likewise, (3.35) contributes to only the third form in (3.17), but with a singularity $1/x^2$ rather than $\log x^2$. After differentiation, the leading singularity generated again does not, of course, contribute and so the effective singularity

is the same as before. The final effect of (3.35) is to allow an additional possible manifestly conserved term in (3.17) of the form ²¹⁾

$$\partial^\nu \partial^\delta (\log -x^2) \sum_{nm} H_{\mu\nu\gamma\delta, \alpha_1 \dots \beta_m} . \quad (3.36)$$

The final possible fields of the basis are obtained from (3.34) and (3.35) by replacing the $G_{\alpha\beta}$'s by $\tilde{G}_{\alpha\beta}$'s. In terms of this basis, one can describe the LC behaviour of $J_\mu(x) J_\nu(0)$ with expressions similar to (3.13), etc. The LC cone expansion for any product of local fields in the gluon model can be found in the same way.

C. Other Models

It should be clear from the above that operator product expansions near the LC are valid in any renormalizable perturbation theory. This follows from the fact that short distance operator product expansions are valid in these theories. Included in these models are the super-renormalizable $g\varphi^3$ theory and the physically interesting σ models ²²⁾. The expansions for these models have essentially the same form as those exhibited above.

We shall not give further examples of expansions from other models since nothing essentially new arises. We only point out that in models like $g\varphi^3$ theory with dimensional coupling constants, it is convenient to put explicit factors of the coupling constants into the basis fields. Thus, in φ^3 theory, for example, there are three fields of dimension two, namely $\partial_\alpha \varphi$, $:\varphi\varphi:$, and $g\varphi$. In terms of such a basis, the leading short distance and LC singularities will be mass independent and, in lowest order, coupling constant independent.

4. NON-PERTURBATION THEORETIC GENERALIZATIONS

As we have already mentioned, an immediate consequence of our derivations is that operator product expansions near the LC will be valid in any theory in which operator product expansions at short distances are valid. In addition to the renormalizable perturbation theories, this class of theories includes all known exactly soluble field theoretic models²³⁾. More generally, or rather more phenomenologically, if, as seems to be indicated, short distance expansions are valid in the real world, then so are LC expansions.

Because of a phenomenon which occurs in the Thirring model, we are forced at this point to be more precise about the notion of dimensionality which we have been using. One says that a local field $\chi(x)$ has dimension d if there exists a one-parameter group $U(s)$ of unitary transformations such that

$$U(s) \chi(x) U^{-1}(s) = s^d \chi(sx) . \quad (4.1)$$

Examples are the free massless scalar field with $d=1$ and free spinor field with $d=\frac{3}{2}$. We shall refer to this notion of dimension as "dynamical" dimension. For the usual fields in free field theories, dynamical dimension coincides with naïve dimension. We shall say a field has canonical dimension if it has a dynamical dimension equal to that of the corresponding free field. In a theory in which all local fields have dimensions and short distance expansions such as (1.1) are valid, application of (4.1) to (1.1) implies that the $F_i(x)$ behave as stated like $(1/x)^{d_i - d_A - d_B}$, with no logarithmic factor. This is what happens in free field theories.

In any finite order of a renormalizable perturbation theory, because of the occurrence of logarithmic factors, the renormalized fields do not have well-defined dynamical dimensions. Nevertheless, the short distance behaviour of any Wightman function is, apart from logarithmic factors, the same as it would be if the fields did have canonical dynamical dimensions. Put differently, the short distance behaviour is determined, apart from logs, by the naïve

dimensions of the fields. In particular, the nature of short distance expansions, and, by our analysis, of LC expansions are so determined. We shall describe this situation by saying that the fields have effective canonical dimensions.

In any theory with effective canonical dimensionality and with short distance expansions, LC expansions very similar to those given in Sections 1-3 will exist. Included in such theories are free field models, renormalizable perturbation theories in any finite order, and most of the known exactly soluble models. In theories with effective non-canonical dynamical dimensionality and with short distance expansions, our derivations show that LC expansions will also exist. In these theories, the singular functions will, of course, be somewhat different from those encountered in Sections 1-3. The Thirring model is the only one we know of that exhibits non-canonical dynamical dimensionality ²⁴⁾.

A final point we should mention concerns the nature of the sum (if it exists) of the perturbative expansions of the renormalizable field theories. It is possible, and has been suggested, that the logarithmic factors occurring in each order sum up to a power and so change the dynamical dimensions of the fields. This is what happens in the Thirring model. We suspect that this Thirring model phenomenon arises because of the zero mass particle present and will not occur in realistic models with no massless particles. There is no evidence that the logs of renormalizable perturbation theories add up to a power ²⁵⁾. In the following section we shall point out the existence of empirical indications of the validity of effective canonical dimensionality.

5. APPLICATIONS

A. Probing the LC

In this Section we shall discuss some applications of the operator product expansions near the LC derived in the preceding sections. Further applications will be given elsewhere. We begin in this subsection by showing what physical quantities and in what region of momentum space the LC is relevant.

Let us first consider the weak or electromagnetic scattering of a lepton off of a hadronic system α to produce an arbitrary final hadronic state. Calling q the momentum transferred to the lepton and p the total momentum of the system α , the total cross-section of interest has the form

$$\frac{d^2\sigma^r}{dq^2 d\nu} \propto \int d^4k \theta(k_0) \delta(k^2 - q^2) \delta(k \cdot p - \nu) \int d^4x e^{-ik \cdot x} \langle \alpha | J_\mu(x) J_\nu(0) | \alpha \rangle \varepsilon_r^\mu \varepsilon_r^\nu. \quad (5.1)$$

Here $\nu \equiv q \cdot p$ is the initial energy variable, r is the polarization of the leptons, and J_μ is the hadronic current to which the leptons couple. The matrix element in (5.1) is connected and spin averaged. Changing the orders of integration in (5.1) gives

$$\frac{d^2\sigma^r}{dq^2 d\nu} \propto \int d^4x \Delta_+(x, p, \nu, q^2) \langle \alpha | J_\mu(x) J_\nu(0) | \alpha \rangle \varepsilon_r^\mu \varepsilon_r^\nu, \quad (5.2)$$

where we have defined

$$\Delta_+(x, p, \nu, q^2) = \int d^4k e^{-ik \cdot x} \theta(k_0) \delta(k^2 - q^2) \delta(k \cdot p - \nu). \quad (5.3)$$

The form (5.2) is very useful for our configuration space purposes. The integral (5.3) can be simply evaluated in the frame $p = (1, \vec{0}) \equiv \mathcal{S}$ (we take $p^2 = 1$) to give

$$\Delta_+(x, \mathcal{S}, \nu, q^2) = \frac{\pi}{i\nu} [e^{i(\nu^2 + q^2)^{1/2} x} - \text{h.c.}] e^{-i\nu t}, \quad (5.4)$$

where we have written $r = |\underline{x}|$, $t = x_0$. We refer to the Bjorken ²⁶⁾ scaling limit as the A limit

$$v \rightarrow \infty, q^2 \rightarrow \infty, \omega \equiv \frac{q^2}{2v} \text{ fixed,} \quad (5.5)$$

and obtain

$$\Delta_+(x, s, v, q^2) \xrightarrow{A} \frac{\pi}{ir} \left[e^{iv(r-t) + i\omega r} - e^{-iv(r+t) - i\omega r} \right]. \quad (5.6)$$

Only the regions $r = \pm t$ in the (first/second) term are important and so we can deduce the covariant result

$$\Delta_+(x, p, v, q^2) \xrightarrow{A} \frac{\pi}{i\tau} \sin\left(\frac{v x^2}{2\tau} + \omega \tau\right) \quad (5.7)$$

in terms of the "transverse" variable

$$\tau^2 \equiv (p \cdot x)^2 - x^2. \quad (5.8)$$

It follows from (5.7) that, in the A limit, $\Delta_+(x, p, v, q^2)$ is highly oscillating outside of the region

$$\frac{x^2}{2\tau} \lesssim \frac{1}{v}, \quad \tau \lesssim \frac{1}{\omega} \quad \text{or} \quad x^2 \lesssim \frac{2}{q^2}$$

and, therefore, has effective support on the LC $x^2 \sim 0$. Referring back to (5.2), we see that the A limit of the cross-section can be obtained simply from the behaviour of $\langle \alpha | J_\mu(x) J_\nu(0) | \beta \rangle$ on the LC. This is precisely the behaviour we have studied in the previous sections.

For reactions in which a hadronic system produces a lepton pair and an arbitrary final hadronic state, the analysis becomes more complicated because of kinematic restrictions on the k space integration region in the analogue of (5.1). In the frame $p = (\sqrt{s}, 0)$, the physical k space region will have the form $\{ |\underline{k}| < \mathcal{K}(q^2, s) \}$ for some functions $\mathcal{K}(q^2, s)$. Proceeding as above it can be shown that the relevant configuration space region is given by

$$|x^2 - \mu^2| \lesssim \frac{1}{q^2} . \quad (5.9)$$

So, for large q^2 and μ , the LC is again the dominant region. The extra condition that $\mu(q^2, s)$ be large can be satisfied in many cases of interest.

Another important use of the LC is to determine the behaviour of amplitudes for large values of a single mass variable. We illustrate this by consideration of a scalar vertex function

$$A(q^2, k^2) = \int d^4x e^{-iq \cdot x} \langle 0 | T[A(x), B(0)] | p \rangle . \quad (5.10)$$

Here $A(x)$ and $B(x)$ are scalar currents and $|p\rangle$ is a state of one scalar particle of momentum $p = q - k$ and mass $p^2 = m^2$. We can write (5.10) as

$$A(q^2, k^2) = \frac{1}{4\pi} (\nu^2 - m^2 p^2)^{-\frac{1}{2}} \int d^4x \Delta_+(x, p, \nu, q^2) \langle 0 | T[A(x), B(0)] | p \rangle , \quad (5.11)$$

where $\nu = q \cdot p = \frac{1}{2}(q^2 + k^2 - m^2)$. Thus, the behaviour of $A(q^2, k^2)$ for $q^2 \rightarrow \infty$ and fixed $q^2/2\nu$, i.e., for fixed q^2/k^2 , is determined by the LC behaviour of $A(x)B(0)$. A special case of this limit is the old Bjorken²⁷⁾ limit $q^2/k^2 \rightarrow 0$, in which case only the equal time behaviour of $A(x)B(0)$ is relevant. Another special case is the limit $q^2 \rightarrow \infty$ with k^2 fixed so that $q^2/2\nu \rightarrow 1$. This limit is important because it determines the number of subtractions needed in fixed k^2 dispersion relations.

The analysis of this subsection can clearly be extended to many kinematically more complicated processes. Our operator expansions can thus be used to derive and relate these processes. Correspondingly, the observed nature of these processes can be used to learn about the existence and properties of the field theory describing the hadronic structure.

B. Deep Inelastic Electron Proton Scattering

Our purpose here is to use the expansion (3.17) to study the process $e + p \rightarrow e + \text{anything}$. The relevance of the LC behaviour of the matrix element $\langle p | [J_\mu(x), J_\nu(0)] | p \rangle$ to the A limit this reaction has been known for some time ¹¹⁾. Our use of the operator expansion will, however, enable us to deduce a number of new results from the observed scaling behaviour. We follow the notation of Ref. 11), where more details and references can be found. We shall first work with the result (3.17) of ignoring logarithmic factors and afterwards discuss the effect of these logs and of their possible role in changing the singularity structure ²⁸⁾.

The total cross-section (5.1) of interest can be written

$$\frac{d^2\sigma}{dq^2 d\nu} = \frac{\pi\alpha^2}{E^2 |q|^2 \sin^2 \frac{\Theta}{2}} \left[W_2(q^2, \nu) \cos^2 \frac{\Theta}{2} + 2W_1(q^2, \nu) \sin^2 \frac{\Theta}{2} \right], \quad (5.12)$$

where E is the initial electron energy and Θ the scattering angle and we have set the proton mass equal to unity: $p^2 = 1$. The structure functions are defined by

$$\frac{1}{2\pi} \int d^4x e^{iq \cdot x} \langle p | [J_\mu(x), J_\nu(0)] | p \rangle = (p_\mu - \rho q_\mu)(p_\nu - \rho q_\nu) W_2(q^2, \nu) - (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) W_1(q^2, \nu), \quad (5.13)$$

where $\rho \equiv -\nu/q^2 = (2\omega)^{-1}$, and the A limits are

$$\lim_A \nu W_2(q^2, \nu) = F_2(\rho), \quad (5.14)$$

$$\lim_A W_1(q^2, \nu) = F_1(\rho). \quad (5.15)$$

The transverse and longitudinal structure functions are

$$F_T = F_1, \quad F_L = \rho F_2 - F_1. \quad (5.16)$$

Experimentally ²⁹⁾, (5.14) is well satisfied in a non-trivial way
 $[F_2(\rho) \sim \text{const. for } \rho \geq 2]$ and F_L/F_T is small, as suggested by
 the gluon model ³⁰⁾.

It is convenient to introduce new structure functions by
 writing

$$\begin{aligned} \frac{1}{2\pi} \int d^4x e^{iq \cdot x} \langle P | [J_\mu(x), J_\nu(0)] | P \rangle &= [q^2 p_\mu p_\nu - \nu(p_\mu q_\nu + q_\mu p_\nu) + \nu^2 g_{\mu\nu}] V_2(q^2, \nu) \\ &\quad - (q^2 g_{\mu\nu} - q_\mu q_\nu) V_1(q^2, \nu). \end{aligned} \quad (5.17)$$

Equations (5.14)-(5.16) imply

$$\lim_A (-\nu^2) V_2(q^2, \nu) = \rho F_2(\rho) \quad (5.18)$$

$$\lim_A \nu V_1(q^2, \nu) = \rho F_L(\rho). \quad (5.19)$$

In configuration space (5.17) reads

$$\begin{aligned} \frac{1}{2\pi} \langle P | [J_\mu(x), J_\nu(0)] | P \rangle &= -[\Box p_\mu p_\nu - (p \cdot \partial)(p_\mu \partial_\nu + p_\nu \partial_\mu) + (p \cdot \partial)^2 g_{\mu\nu}] \hat{V}_2(x^2, x \cdot p) \\ &\quad - (\partial_\mu \partial_\nu - \Box g_{\mu\nu}) \hat{V}_1(x^2, x \cdot p), \end{aligned} \quad (5.20)$$

in terms of the Fourier transforms

$$V_i(q^2, \nu) = \int d^4x e^{iq \cdot x} \hat{V}_i(x^2, x \cdot p). \quad (5.21)$$

The A limit of the V_i 's is given by the LC behaviour
 of the $\hat{V}_i$'s and these can be determined from (3.17). We define as
 in (1.4)-(1.7) the matrix elements

$$\langle P | \sum_n x^{\alpha_1} \dots x^{\alpha_n} \mathcal{R}_{\alpha_1 \dots \alpha_n}(0) | P \rangle = f_0(x \cdot p) + O(x^2), \quad (5.22)$$

$$\langle P | \sum_n x^{\alpha_1} \dots x^{\alpha_n} \mathcal{R}_{2\alpha_1 \dots \alpha_n}^{\alpha\beta}(0) | P \rangle = g^{\alpha\beta} f(x \cdot p) + p^\alpha p^\beta f_2(x \cdot p) + O(x^2). \quad (5.23)$$

Comparison with (5.20), using

$$\text{Im}(x^2 - i\epsilon x_0)^{-1} = \pi \epsilon(x_0) \delta(x^2), \quad \text{Im} \log(-x^2 + i\epsilon x_0) = \pi \epsilon(x_0) \Theta(x^2), \quad (5.24)$$

gives the results

$$\widehat{V}_2(x^2, x \cdot p) \xrightarrow{x^2 \rightarrow 0} -\epsilon(x_0) \Theta(x^2) f_2'(x \cdot p) \quad (5.25)$$

$$\widehat{V}_1(x^2, x \cdot p) \xrightarrow{x^2 \rightarrow 0} -\epsilon(x_0) \delta(x^2) f_0'(x \cdot p). \quad (5.26)$$

These are precisely the LC behaviours shown in Ref. 11) to be equivalent to the scaling laws (5.18) and (5.19) or (5.14) and (5.15). Indeed, direct substitution of (5.25) and (5.26) into (5.21) gives the results (5.18) and (5.19) with

$$\rho F_2(\rho) = -2\pi \int d\lambda e^{-i\lambda/2\rho} \lambda f_2'(\lambda), \quad (5.27)$$

$$\rho F_1(\rho) = -\frac{\pi i}{2} \int d\lambda e^{-i\lambda/2\rho} f_0'(\lambda). \quad (5.28)$$

We have thus derived the validity of the scaling laws (5.18) and (5.19) in the large class of theories in which (3.17) holds. Strictly speaking, because these theories really give extra logarithmic factors, we can only deduce that (5.18) and (5.19) are valid apart from powers of $\log q^2$. Indeed, it is a known fact that in low orders these theories give scaling apart from logs. This is satisfactory since the presence of such logarithmic factors could easily escape experimental detection at present. One might think that this result is trivial because we have built in scaling via the mass independence of (3.17). The point is, however, that, because of the possibility of non-canonical dimensions, mass independence is not equivalent to scaling. We can thus reach the strong conclusion that the observed scaling behaviour is consistent with canonical dimensionality but not with many types of non-canonical dimensionality. If a single operator with a non-vanishing proton-proton matrix element in (3.17) had a dimension significantly less than its canonical one, then the scaling limits (5.18)

and/or (5.19) would be divergent by the corresponding power of q^2 . If we further believe in the relevance of a perturbative model, then we can conclude that the logarithmic factors in the model do not add up to significantly change the singularity structure. It is therefore an interesting and non-trivial fact that non-trivial scaling is equivalent to the presence of the LC singularity structure required by canonical dimensionality.

We have thus seen how the existence of operator product expansions like (3.17) enables one to correlate experimental results with the nature of possible field theoretic models for the hadrons. In addition to providing evidence for essentially canonical dimensionality, the observed scaling strongly suggests the relevance of renormalizable field theories. Non-renormalizable models, made finite, say, by the introduction of infinitely many subtraction constants, possess much worse LC singularities. A further major advantage of our formalism is that, unlike the matrix element statements like (5.25) and (5.26), it enables one to compare and relate different processes since these processes simply involve different matrix elements of the same operators.

Before leaving electroproduction, we wish to comment on what happens if $F_L = 0$. It is clear from the above analysis that, neglecting the unlikely possibility that the proton-proton matrix element of each $\mathcal{R}_{\alpha_1 \dots \alpha_n}$ in (3.17) vanishes, $F_L = 0$ means that the leading allowed singularity $1/x^2$ in the first piece of (3.17) is not present. Assuming canonical dimensionality, this means that the $(1/x^2)$ must be replaced by $(\log x^2)$. This comes from both the non-leading contributions of the given operators satisfying $\dim \mathcal{R}_{\alpha_1 \dots \alpha_n} = n+2$ and from the leading contributions of additional operators satisfying $\dim \mathcal{R}_{\alpha_1 \dots \alpha_n} = n+4$. Calling the matrix element of the sum of these operators still $f_0(x \cdot p)$ as in (5.22), (5.26) becomes replaced by

$$\hat{V}_1(x^2, x \cdot p) \xrightarrow{x^2 \rightarrow 0} -\epsilon(x_0) \Theta(x^2) [f_0(x \cdot p) + 2f'(x \cdot p)]. \quad (5.29)$$

C. Further Applications

In this final subsection we shall briefly mention some further applications of the formalism we have developed. Detailed treatments of these applications have been given elsewhere ^{31),32)}.

An especially interesting application ³¹⁾ is to study the recent measurement ³³⁾ of the cross-section $d\sigma/dq^2$ for the reaction proton + proton $\rightarrow \mu$ pair + anything. Here the double-proton matrix element $\langle pp' | J_\mu(x) J_\nu(0) | pp' \rangle$ is involved and the relevant region is given by (5.9) with $\mathcal{K}^2 = [(s + q^2 - 4m^2)/4s] - q^2$ even though $s = (p + p')^2 \rightarrow \infty$ in the scaling limit. Most of the experimental points correspond to the LC $x^2 \sim 0$ and so the expansion (3.17) can be used. The matrix element of $R \equiv \sum x^{\alpha_1} \dots x^{\alpha_n} R_{\alpha_1 \dots \alpha_n}$ etc., is now more complicated, but Regge theory provides an enormous simplification in that it implies that, for example $\langle pp' | R | pp' \rangle \xrightarrow{s \rightarrow \infty} s^\alpha [\bar{h}(p \cdot x) + h(p' \cdot x)]$ ³¹⁾. Further use of Regge theory for the dimensionless structure functions above the value $\rho \sim 2$ suggested by SLAC enables us to express the cross-sections in terms of a two-parameter function which can be fit very nicely to the data which fall ~ 5 orders of magnitude for $2 \leq q^2 \leq 6 \text{ GeV}^2$.

Another class of applications involves the determination of the asymptotic behaviour of vertex functions when a mass becomes large, as described in subsection A ³²⁾. Knowledge of this asymptotic behaviour, plus the information, learned from the electroproduction and μ pair results, that asymptotia sets in at $q^2 \sim 2 \text{ GeV}^2$, enables us to write the analogue of finite energy sum rules in the mass variable and thus approximately calculate both the "infinite" mass contributions and the continuum contributions. These contributions give corrections to the result of simply saturating the mass dispersion relation with a low lying meson. In this way we can understand, for example, pion pole dominance of matrix elements of the divergence of the axial vector current and we can estimate corrections to vector meson dominance which are in good agreement with experiment. Consider, for example, the $\pi \rightarrow 2\gamma$ off-shell amplitude. The asymptotic behaviour in the photon-

mass variable is determined from the second piece of Eq. (3.17). This information enables us to obtain an experimentally correct determination of the rate and also to relate the amplitude to, say, the electroproduction ones via (3.17).

We feel that these successful applications support the correctness of our assumption that the hadrons can be described by a renormalizable field theory with effective canonical dimensionality. It is clearly desirable to explore additional applications in order to further test these ideas and to gain more familiarity with the concepts involved. It is possible that the LC will provide an increase in the usefulness of configuration space field theoretic methods in understanding some aspects of particle physics.

ACKNOWLEDGEMENT

We are deeply indebted to Wolfhart Zimmermann for useful discussions on the material in this paper.

R E F E R E N C E S

- 1) K. Wilson, unpublished.
- 2) R.A. Brandt, Ann.Phys.(N.Y.) 44, 221 (1967).
- 3) W. Zimmermann, Commun.Math.Phys. 6, 161 (1967).
- 4) See the references given in Ref. 2).
- 5) This dimensional concept is discussed in Refs. 1)-3) and below.
- 6) R.A. Brandt, Phys.Rev. 166, 1795 (1968).
- 7) W. Zimmermann, Commun.Math.Phys. 8, 66 (1968) and Springer Tracts in Modern Phys., Vol. 50, 143 (1969).
- 8) R.A. Brandt, Ann.Phys.(N.Y.) 52, 122 (1969).
- 9) K. Wilson, Phys.Rev. 129, 1499 (1969).
- 10) Recall that we are ignoring $\log x^2$ factors here.
- 11) R.A. Brandt, Phys.Rev.Letters 22, 1149 (1969);
ibid 23, 1260 (1969); and
Phys.Rev. (in press).
- 12) W. Zimmermann, private communication.
- 13) We thank R. Haag for raising this question.
- 14) For an application and references to earlier applications, see
R.A. Brandt and G. Preparata, Phys.Rev., in press.
- 15) V.I. Ogievetskii and I.V. Polubarinov, JETP 14, 179 (1962).
- 16) Local gauge invariance in renormalized perturbation theory is
discussed in Ref. 8) for quantum electrodynamics and in Ref. 7)
for massive neutral vector meson theory.
- 17) That such statements are meaningful is shown by the work of
K. Symanzik, to be published.
- 18) $U(3) \times U(3)$ does not, however, become exact in this limit because
of the Adler anomaly. See S.L. Adler, Phys.Rev. 177, 2426
(1969), and references therein.

- 19) These operators are, of course, not all independent.
- 20) By mass independence, only $c = 0, 3$ and 8 occur in the expansion.
- 21) In the spirit of our neglect of logs, these terms could be omitted since, without additional logarithmic factors, they do not have the correct analyticity structure in the zero mass limit.
- 22) M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960);
M. Lévy, Nuovo Cimento 52A, 23 (1967).
- 23) References are given in Ref. 2).
- 24) This property of the Thirring model was discovered by K. Wilson, to be published.
- 25) The most recent and elegant account can be found in K. Symanzik, to be published.
- 26) J.D. Bjorken, Phys.Rev. 179, 1547 (1969).
- 27) J.D. Bjorken, Phys.Rev. 148, 1469 (1966).
- 28) We also ignore a possible term of the form (3.36). Its presence would not change our conclusions.
- 29) M. Briedenback et al., Phys.Rev.Letters 23, 935 (1969).
- 30) C. Callan and D.J. Gross, Phys.Rev.Letters 22, 156 (1969).
- 31) G. Altarelli, R.A. Brandt and G. Preparata, to be published.
- 32) R.A. Brandt and G. Preparata, to be published.
- 33) J.H. Christenson, G.S. Hicks, L.M. Lederman, P.J. Limon, B.G. Pope and E. Zavattini, to be published.