CM-P00058476

Ref.TH.1562-CERN

HARD PHOTON CORRECTIONS FOR THE PROCESS $e^+e^- \rightarrow \mu^\pm \mu^\mp$

F.A. Berends

CERN - Geneva

and

Instituut-Lorentz, University of Leiden, Leiden *)

K.J.F. Gaemers

Instituut-Lorentz, University of Leiden, Leiden

and

R. Gastmans **)

CERN - Geneva

A B S T R A C T

The differential cross-section for $e^+e^- \rightarrow \mu^\pm \mu^\mp$ is calculated to order α^3 for the case of e^+e^- colliding beam experiments in which the charge of the muons is not detected. Special attention is given to hard photon corrections. For a specific experimental set-up, detailed numerical results are presented. It is found that the radiative corrections lie in the range of $\pm 8\%$, and that approximate calculations cannot be trusted if an accuracy of a few percent is required.

*) Permanent address.

**) Aangesteld navorser, N.F.W.O., Belgium.

1. INTRODUCTION

Quantum electrodynamics has so far yielded a set of predictions which were found to be in remarkable agreement with experiment ^{1),2)}. The high precision measurements of static quantities, like the anomalous magnetic moment of the muon or the electron, test the theory to several orders in the fine structure constant, α , whereas scattering experiments probe the theory to smaller distances.

With the advent of e^+e^- colliding beam facilities, it has become possible to improve on the latter type of tests. Some successful ones have already been reported in the last couple of years ^{3),4)}. They deal with the reactions

$$e^+ e^- \rightarrow e^+ e^- , \quad (1)$$

$$e^+ e^- \rightarrow \mu^+ \mu^- , \quad (2)$$

$$e^+ e^- \rightarrow \gamma \gamma . \quad (3)$$

Integrated differential cross-sections were compared with theory up to energies of 1.2 GeV per beam. The measurements had a typical accuracy of 5% or larger. With these errors, the theoretical cross-sections can only be tested to lowest order.

With the new facilities at CEA, DESY, Orsay and SLAC, it will become possible to test reactions at higher energies and with a better precision. When differential cross-sections are measured with an accuracy of a few percent, it becomes necessary to know the theoretical cross-sections to order α^3 . This not only means that one has to calculate the virtual radiative corrections to the processes (1)-(3), but one also has to know the cross-sections for the processes where an extra photon is emitted, e.g.,

$$e^+ e^- \rightarrow \mu^+ \mu^- \gamma . \quad (4)$$

This reaction will in many experimental arrangements simulate events of reaction (2), since a photon may go undetected even if it is energetic. In order to evaluate this effect, one has to integrate the cross-section for reaction (4) over a phase space determined by the experimental set-up.

It is the purpose of this paper to study the reaction (2) up to order α^3 . At present we restrict ourselves to experiments where the charge of the muons is not detected. In that case the virtual corrections are well established. The problem then is to evaluate the effect of process (4) on the measurement of reaction (2). This requires the knowledge of the cross-section and a study of the phase space. Although the latter is treated quite generally, it is in the end applied to specific experimental situations. We are then able to evaluate numerically the complete radiative corrections up to order α^3 .

As the cross-section for process (4) is rather complicated, one often wants to use a simplified expression for it, e.g., the cross-section for soft photon emission. We therefore compare our exact results with such an approximate calculation. It turns out that for experiments with a few percent accuracy, the complete hard photon calculation has to be carried out. Recently, a similar conclusion was drawn from experimental information on reaction (1) ⁵⁾.

The outline of the paper is as follows. In Section 2, the basic lowest order cross-section is given, together with some general comments. In Section 3, the virtual radiative corrections are discussed. In Section 4, the cross-section for process (4) is given, both in the exact form and in the approximation where the emitted photon is treated as soft. Section 5 presents phase space considerations, and applies them to specific experimental arrangements. Numerical results are commented upon in Section 6, and, finally, conclusions are given in Section 7.

2. THE REACTION $e^+e^- \rightarrow \mu^+\mu^-$ IN LOWEST ORDER

The lowest order cross-section formula for the reaction

$$e^+(p_+) + e^-(p_-) \rightarrow \mu^+(q_+) + \mu^-(q_-) \quad (5)$$

is obtained from the Feynman diagram of Fig. 1. It reads:

$$\frac{d\sigma^0}{d\Omega_\mu} = \frac{\alpha^2}{2s} \left[\frac{s/4 - \mu^2}{s/4 - m^2} \right]^{\frac{1}{2}} \left[\frac{1}{2} + \frac{2(m^2 + \mu^2)}{s} - 2 \cos^2 \theta \left(-\frac{1}{4} + \frac{m^2 + \mu^2}{s} - \frac{4m^2\mu^2}{s^2} \right) \right], \quad (6)$$

where

$$s = (p_+ + p_-)^2 = (q_+ + q_-)^2, \quad (7)$$

and m , μ , and θ are the electron mass, the muon mass, and the angle between e^+ and μ^+ .

From Eq. (6), it is clear that to lowest order in α , $d\sigma(\theta)/d\Omega_\mu$ and $d\sigma(\pi-\theta)/d\Omega_\mu$ are equal. In higher order, where one and two photon exchange amplitudes may interfere, this is no longer the case. However, in experiments where the charge of the muons is not detected, one effectively measures the quantity

$$S = \frac{d\sigma(\theta)}{d\Omega_\mu} + \frac{d\sigma(\pi-\theta)}{d\Omega_\mu}. \quad (8)$$

But then, as first pointed out by Putzolu⁶⁾, the interference terms between the exchange of an odd number of photons and the exchange of an even number of photons cancel. On the other hand, in the difference

$$D = \frac{d\sigma(\theta)}{d\Omega_\mu} - \frac{d\sigma(\pi-\theta)}{d\Omega_\mu} \quad (9)$$

only these interference terms survive.

A similar argument can be applied to the process

$$e^+(p_+) + e^-(p_-) \rightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k), \quad (10)$$

for which the Feynman diagrams are given in Fig. 2. Now, the interference terms between diagrams 2a-b and 2c-d do not contribute.

Until now, the charge of the final state particles is not detected in e^+e^- colliding beam experiments. One is therefore justified to concentrate in the first instance on charge symmetric experiments. Of course, measurements of the quantity D [Eq. (9)] would be highly interesting, since this quantity directly tests the two-photon exchange diagrams.

3. VIRTUAL RADIATIVE CORRECTIONS

The α^2 virtual photon corrections to the amplitude arise from the vertex corrections (Figs. 3a-b), the vacuum polarization (Figs. 3c-d), and the two-photon exchange graphs (Figs. 3e-f). Since we are interested in the cross-section to order α^3 , we only have to take into account the interference terms of the diagrams of Fig. 3 with the basic diagram of Fig. 1. As explained in Section 2, the effect of the two photon exchange graphs can be omitted, since we restrict the calculation to a situation which is charge symmetric for the muons.

The expressions we need for the vertex corrections and the vacuum polarization are well-known⁷⁾. The renormalized correction to the electron vertex, γ_μ , takes the form

$$\gamma_\mu F_1(s, m^2) + [\gamma_\mu, \not{k}_+ + \not{k}_-] \frac{F_2(s, m^2)}{4m}, \quad (11)$$

where, up to first order in α ,

$$\begin{aligned} \text{Re } F_1(s, m^2) = & -\frac{\alpha}{\pi} \left\{ 1 + \frac{1+2a^2}{4a} \log b \right. \\ & + \frac{1+a^2}{2a} \left[-\text{Li}_2(b) - \frac{\pi^2}{3} + \frac{1}{4} (\log b)^2 - \log b \log(1-b) \right] \\ & \left. - \left(1 + \frac{1+a^2}{2a} \log b \right) \log \frac{m}{\lambda} \right\}, \end{aligned} \quad (12)$$

and

$$\text{Re } F_2(s, m^2) = \frac{\alpha}{\pi} \frac{1-a^2}{4a} \log b. \quad (13)$$

The quantities a , b and the dilogarithm are defined by

$$a = [1 - 4m^2/s]^{\frac{1}{2}},$$

$$b = \frac{1-a}{1+a} \quad (14)$$

$$Li_2(b) = - \int_0^b \frac{\log(1-x)}{x} dx.$$

The last term in Eq. (12) represents the usual infrared divergence, which was given a meaning through the introduction of a small fictitious photon mass, λ .

The renormalized photon propagator is effectively given by

$$-i \frac{g_{\mu\nu}}{s} \left[1 - \Pi(s, m^2) - \Pi(s, \mu^2) \right] \quad (15)$$

with, up to order α ,

$$\text{Re } \Pi(s, m^2) = \frac{\alpha}{\pi} \left[\frac{8}{9} - \frac{a^2}{3} + a \left(\frac{1}{2} - \frac{a^2}{6} \right) \log b \right]. \quad (16)$$

Only the real parts of the quantities, F_1 , F_2 and Π are required, since the interference term with the lowest order diagram is real.

Summing up all the contributions from the graphs of Figs. 1 and 3, one finds for the differential cross-section S_V [Eq. (8)]:

$$S_V = 2 \frac{d\sigma^0}{d\Omega_\mu} \left[1 + \delta_{Vc}(m^2) + \delta_{Vc}(\mu^2) + \delta_{VP}(m^2) + \delta_{VP}(\mu^2) \right] \quad (17)$$

$$+ 2 \frac{d\sigma^{mag}(m^2)}{d\Omega_\mu} + 2 \frac{d\sigma^{mag}(\mu^2)}{d\Omega_\mu}.$$

Here, $d\sigma^0/d\Omega_\mu$ is given by Eq. (6), and the other quantities are given by

$$\delta_{VC}(m^2) = 2 \operatorname{Re} F_1(s, m^2),$$

$$\delta_{VP}(m^2) = -2 \operatorname{Re} T(s, m^2),$$

$$\frac{d\sigma^{\text{mag}}(m^2)}{d\Omega_\mu} = \frac{\alpha^2}{s} \left[\frac{s/4 - \mu^2}{s/4 - m^2} \right]^{\frac{1}{2}} \left(1 + \frac{2\mu^2}{s} \right) \operatorname{Re} F_2(s, m^2), \quad (18)$$

$$\frac{d\sigma^{\text{mag}}(\mu^2)}{d\Omega_\mu} = \frac{\alpha^2}{s} \left[\frac{s/4 - \mu^2}{s/4 - m^2} \right]^{\frac{1}{2}} \left(1 + \frac{2m^2}{s} \right) \operatorname{Re} F_2(s, \mu^2).$$

In the relativistic limit, the expressions of Eqs. (17) and (18) reduce to those given by Bonneau and Martin⁸⁾. It should be stressed that the π^2 term in the relativistic limit of the quantity S_V differs from the often quoted result of Ref. 6). We also disagree with this π^2 term as given in Refs. 9)-12).

4. SOFT AND HARD PHOTON EMISSION

The differential cross-section for reaction (10) will be obtained from the expression

$$d^9\sigma^B = \frac{1}{(2\pi)^5} \delta(p_+ + p_- - q_+ - q_- - k) \frac{m^2 \mu^2}{[(p_+ p_-)^2 - m^4]^{\frac{1}{2}}} \quad (19)$$

$$\times \frac{e^6}{8} \sum_{\text{spins}} |T|^2 \frac{d\vec{q}_+}{q_{+0}} \frac{d\vec{q}_-}{q_{-0}} \frac{d\vec{k}}{k_0},$$

where

$$T = \sum_{i=1}^4 T_i = \sum_{i=1}^4 N_i / D_i. \quad (20)$$

The four terms in the T matrix arise from the four diagrams of Fig. 2.
The denominators take the form

$$\begin{aligned}
 D_1 &= -2 (k \cdot p_-) s', \\
 D_2 &= -2 (k \cdot p_+) s', \\
 D_3 &= 2 (k \cdot q_-) s, \\
 D_4 &= 2 (k \cdot q_+) s,
 \end{aligned}
 \tag{21}$$

with $s = (p_+ + p_-)^2$ and $s' = (q_+ + q_-)^2$.

The numerators consist of an electron and a muon part contracted with a photon polarization four-vector, ϵ_α .

$$\begin{aligned}
 N_1 &= E^{\mu\alpha}(1) \epsilon_\alpha M_\mu(1), \\
 N_2 &= E^{\mu\alpha}(2) \epsilon_\alpha M_\mu(2), \\
 N_3 &= E_\mu(3) \epsilon_\alpha M^{\mu\alpha}(3), \\
 N_4 &= E_\mu(4) \epsilon_\alpha M^{\mu\alpha}(4),
 \end{aligned}
 \tag{22}$$

with

$$\begin{aligned}
 E^{\mu\alpha}(1) &= \bar{v}(p_+) \gamma^\mu (\not{p}_- - \not{k} + m) \gamma^\alpha u(p_-), \\
 E^{\mu\alpha}(2) &= \bar{v}(p_+) \gamma^\alpha (-\not{p}_+ + \not{k} + m) \gamma^\mu u(p_-),
 \end{aligned}
 \tag{23}$$

$$E_{\mu}(3) = E_{\mu}(4) = \bar{v}(p_+) \delta_{\mu} u(p_-),$$

$$M_{\mu}(1) = M_{\mu}(2) = \bar{u}(q_-) \delta_{\mu} v(q_+),$$

(23 cont.)

$$M^{\mu\alpha}(3) = \bar{u}(q_-) \delta^{\alpha} (\not{p}_- + \not{k} + \not{\mu}) \delta^{\mu} v(q_+),$$

$$M^{\mu\alpha}(4) = \bar{u}(q_-) \delta^{\mu} (-\not{p}_+ - \not{k} + \not{\mu}) \delta^{\alpha} v(q_+).$$

The quantity $\sum |T|^2$ is now written in the form

$$\sum |T|^2 = \sum_{i,j=1}^4 \sum_{\text{perms}} \frac{N_i N_j^*}{D_i D_j} = \sum_{i,j=1}^4 \frac{F_{ij}(p_+, p_-, q_+, q_-, k)}{D_i D_j}. \quad (24)$$

The expressions for F_{ij} are given in Appendix A.

As explained in Section 2, the cross-terms between diagrams 2a-b and 2c-d cancel for the charge symmetric $\mu^+ \mu^-$ configuration, i.e., in the summation over i and j in Eq. (24), the terms with the combination $i=1,2$ and $j=3,4$ are omitted (of course, one also drops those with $i=3,4$, $j=1,2$).

From Eq. (19), multi-differential cross-sections are derived, and the question of the choice of integration variables now arises. Since we are primarily interested in the simulation of $e^+e^- \rightarrow \mu^{\pm} \mu^{\mp}$ by the emission of an extra photon, it is preferable to choose anyhow $d\Omega_{\mu}$, the solid angle of the μ^+ , as variable.

For the other three variables, one could choose, e.g., q_{+0} , q_{-0} , and φ , the energies of the μ^+ and μ^- , and the angle between the $(\vec{p}_+, \vec{q}_+)$ and $(\vec{q}_+, \vec{q}_-)$ plane. Since we want to integrate over these three variables to find the simulated $e^+e^- \rightarrow \mu^{\pm} \mu^{\mp}$ events, it turns out that it is more practical to choose variables in which the places of rapid variation in the multi-differential cross-section are easily located. The cross-section peaks sharply whenever the photon is emitted parallel to the electron or positron direction (and, to a lesser extent, to

the μ^+ or μ^- direction). So, it is advantageous to use the angular variables of the photon. The polar and azimuthal photon angles, θ_γ and φ_γ , are taken with respect to a frame where $\vec{q}_+$ defines the z axis, and $\vec{q}_+ \wedge \vec{p}_+$ the y axis.

As third variable, $k = |\vec{k}|$ is used, which is convenient in order to exhibit the infrared divergence associated with $k \rightarrow 0$.

From Eq. (19), we obtain

$$\frac{\partial^5 \sigma^B}{\partial \Omega_\mu \partial \Omega_\gamma \partial k} = \frac{\alpha^3}{2\pi^2} \frac{m^2 \mu^2}{[\lambda(\lambda - 4m^2)]^{\frac{1}{2}}} \frac{|\vec{q}_+| k}{2p_{+0} - k + \frac{q_{+0}}{|\vec{q}_+|} k \cos \theta_\gamma} \sum_{i,j=1}^4 \frac{F_{ij}}{D_i D_j}. \quad (25)$$

The expression for the cross-section takes a very simple form when one assumes that in the quantities $N_\perp$ the photon momentum can be neglected, and that the emission of a photon does not alter the momenta of the muons. This approximation is obviously good when the photons are sufficiently soft (soft photon approximation). In this case, we find

$$\frac{\partial^5 \sigma^S}{\partial \Omega_\mu \partial \Omega_\gamma \partial k} = \frac{\alpha^3}{2\pi^2} \frac{m^2 \mu^2}{[\lambda(\lambda - 4m^2)]^{\frac{1}{2}}} \frac{|\vec{q}_+| k^2}{2p_{+0} k_0} \frac{\chi}{\lambda^2} \quad (26)$$

$$\left[\frac{m^2}{(p_- k)^2} + \frac{m^2}{(p_+ k)^2} - \frac{2(p_- p_+)}{(p_- k)(p_+ k)} + \frac{\mu^2}{(q_- k)^2} + \frac{\mu^2}{(q_+ k)^2} - \frac{2(q_- q_+)}{(q_- k)(q_+ k)} \right],$$

where

$$\begin{aligned} \chi &= \frac{-1}{m^2 \mu^2} \left[2(p_+ q_+)^2 + 2(p_+ q_-)^2 + \lambda(m^2 + \mu^2) \right] \\ &= \frac{-\lambda^2}{2m^2 \mu^2} \left[\frac{1}{2} + \frac{2(m^2 + \mu^2)}{\lambda} - 2 \cos^2 \theta \left(-\frac{1}{4} + \frac{m^2 + \mu^2}{\lambda} - \frac{4m^2 \mu^2}{\lambda^2} \right) \right]. \end{aligned} \quad (27)$$

In order to deal with the infrared divergences, the photon has been given a small mass, λ , such that one now has to distinguish between k and k_0 . The quantity X also occurs in $d\sigma^0/d\Omega_\mu$ and the bremsstrahlung cross-section of Eq. (26) becomes proportional to $d\sigma^0/d\Omega_\mu$.

In the following, it will be useful to integrate the cross-section for isotropic photon emission, from zero momentum up to a specific maximal photon momentum k_m . The result is

$$\begin{aligned} \frac{d\sigma^s}{d\Omega_\mu} &= \int_0^{k_m} dk \int_{-1}^1 d(\cos\theta_r) \int_0^{2\pi} d\varphi_r \frac{\partial^5 \sigma^B}{\partial\Omega_\mu \partial\Omega_r \partial k} \\ &= \frac{d\sigma^0}{d\Omega_\mu} \left[\delta_s(m^2, k_m) + \delta_s(\mu^2, k_m) \right] \end{aligned} \quad (28)$$

with

$$\begin{aligned} \delta_s(m^2, k_m) &= -\frac{\alpha}{\pi} \left\{ \left[2 + \frac{1+a^2}{a} \log b \right] \log \frac{2k_m}{\lambda} \right. \\ &\quad \left. + \frac{1}{a} \log b + \frac{1+a^2}{a} \left[\text{Li}_2\left(\frac{2a}{1+a}\right) + \frac{1}{4} (\log b)^2 \right] \right\}. \end{aligned} \quad (29)$$

It can be seen from Eq. (29) and Eq. (12) that the infrared divergences cancel in the sum of δ_{VC} and δ_S .

Because of its simplicity, the soft photon approximation is often used for situations, where the photons are not soft anymore. In Section 6, it will be shown how inaccurate this procedure can become.

5. A DISCUSSION OF THE PHASE SPACE

In this Section, some kinematics for reaction (10) will be discussed. The results can easily be applied to other reactions, like

$$e^+ e^- \rightarrow e^+ e^- \gamma \quad (30)$$

or

$$e^+ e^- \rightarrow \gamma \gamma \gamma \quad (31)$$

by replacing μ by m or zero.

The formulae below can be used to translate certain experimental constraints in limits on the integration variables k , θ_γ , and φ_γ [see Eq. (28)]. Some often occurring conditions, like $\theta_\gamma = \text{constant}$, or $\delta = \text{constant}$, where δ is the angle between $\vec{q}_+$ and $\vec{q}_-$, represent curves in the $\mu^+ \mu^- \gamma$ Dalitz plot, i.e., they are relations between the variables q_{+0} and q_{-0} . A special case is the boundary of the Dalitz plot itself.

These curves are obtained by using the four-vector n^μ , defined as

$$n^\mu = \varepsilon^{\mu\nu\rho\sigma} P_\nu q_{+\rho} q_{-\sigma} \quad (32)$$

or

$$n^\mu = - \varepsilon^{\mu\nu\rho\sigma} P_\nu q_{+\rho} k_\sigma, \quad (33)$$

where $P = p_+ + p_-$. In the c.m.s., only the spatial components are different from zero: they are proportional to the vector products $\vec{q}_+ \wedge \vec{q}_-$ or $\vec{q}_+ \wedge \vec{k}$, i.e., proportional to $\sin \delta$ or $\sin \theta_\gamma$. So, we find for n^2 , on the one hand

$$n^2 = - P_0^2 |\vec{q}_+|^2 |\vec{q}_-|^2 \sin^2 \delta, \quad (34)$$

$$n^2 = - P_0^2 |\vec{q}_+|^2 (P_0 - q_{+0} - q_{-0})^2 \sin^2 \theta_\gamma, \quad (35)$$

and, on the other hand [from Eq. (32)],

$$n^2 = - \begin{vmatrix} P^2 & P \cdot q_+ & P \cdot q_- \\ P \cdot q_+ & \mu^2 & q_+ \cdot q_- \\ P \cdot q_- & q_+ \cdot q_- & \mu^2 \end{vmatrix} = - P_0^2 \begin{vmatrix} 1 & q_{+0} & q_{-0} \\ q_{+0} & \mu^2 & q_+ \cdot q_- \\ q_{-0} & q_+ \cdot q_- & \mu^2 \end{vmatrix}. \quad (36)$$

Combining Eq. (36) with either Eq. (34) or Eq. (35), and eliminating $q_+ \cdot q_-$ with

$$q_+ \cdot q_- = -\mu^2 - \frac{1}{2} P_0^2 + P_0 (q_{+0} + q_{-0}) , \quad (37)$$

we obtain the curves for fixed $\sin \delta$ or $\sin \theta_\gamma$.

The fixed $\sin \delta$ relation reads

$$(\eta q_{+0} + \rho)^2 = |\vec{q}_+|^2 |\vec{q}_-|^2 \cos^2 \delta , \quad (38)$$

where

$$\begin{aligned} \eta &= q_{-0} - P_0 , \\ \rho &= \frac{1}{2} (\eta^2 - q_{-0}^2 + 2\mu^2) . \end{aligned} \quad (39)$$

More explicitly, this leads to the curves

$$\begin{aligned} q_{+0} &= \frac{-\eta \rho \pm \sqrt{D}}{\eta^2 - (q_{-0}^2 - \mu^2) \cos^2 \delta} \\ &\equiv B_\pm (q_{-0} , \cos^2 \delta) , \end{aligned} \quad (40)$$

with

$$D = |\vec{q}_-|^2 \cos^2 \delta (\rho^2 - \eta^2 \mu^2 + \mu^2 |\vec{q}_-|^2 \cos^2 \delta) .$$

The upper sign corresponds to curve 1 in Fig. 4, where $\cos \delta < 0$, and the lower sign to curves 2 ($\cos \delta < 0$) and 3 ($\cos \delta > 0$). The boundaries of the Dalitz plot are a special case:

$$q_{+0} = B_\pm (q_{-0} , 1) = \frac{1}{2} \left[-\eta \pm |\vec{q}_-| - \frac{\mu^2}{\eta \mp |\vec{q}_-|} \right] . \quad (41)$$

The upper sign corresponds to curve I, where $\delta = \pi$ and $\theta_\gamma = \pi$, the lower sign to curves II and III, where $\delta = \pi$, $\theta_\gamma = 0$ and $\delta = 0$, $\theta_\gamma = \pi$ respectively. In the points A and B, $|\vec{q}_-|$ and $|\vec{q}_+|$ vanish.

In a similar fashion, using Eqs. (35) and (36), fixed θ_γ curves are found. In Fig. 5, an example is drawn: along 1 and 3, $\cos \theta_\gamma < 0$, and, along 2, $\cos \theta_\gamma > 0$.

Experimentally, when measuring the $e^+e^- \rightarrow \mu^+\mu^-$ reaction, one uses some criterium to decide whether the observed μ pairs belong to this reaction. Ideally, one can imagine two rather distinct possibilities, although in practice they may be combined in some way.

In the first place, if the two muons are detected and their energies lie in the range $[p_{+0} - \epsilon, p_{+0} + \epsilon]$, then one counts them as real events.

In the second place, one may select the muons by the criterium that they are produced back to back. Then, no energy is measured, but it is established that their tracks make an angle $\delta < \zeta$, where the quantity ζ is the given maximum acollinearity. If accidentally one also sees the "undetected" photon together with the muons, one has to add this event to the μ pair cross-section.

In principle, one could also consider a third case in which muons are produced with an acollinearity exceeding a certain minimum. One is then certain to deal only with inelastic events, and not with those simulating reaction (2). An analysis of this situation was recently carried out by Calva-Tellez¹²⁾.

Returning to the first case, one has to evaluate the influence of reaction (10) by integrating $\partial^5 \sigma^B / \partial \Omega_\mu \partial \Omega_\gamma \partial k$ over the shaded area of Fig. 5 and over the full φ_γ range. In the second case, one has to integrate over an area in the phase space between the curves $\cos \delta = -1$ and $\cos \delta = \cos(\pi - \zeta) \equiv z$. Since there usually exists a threshold energy for the muons, E_{th} , below which they cannot be detected, the available phase space is further restricted by the conditions $q_{+0} > E_{th}$ and $q_{-0} > E_{th}$. For every φ_γ , one then has to integrate the $\mu^+\mu^- \gamma$ cross-section over the shaded area in Fig.4.

It is the latter case which has been considered in e^+e^- colliding beam experiments so far, and which is also planned for at least one experimental set-up¹³⁾. In the following, we will carry out the integration explicitly for this case.

As can be seen from Fig. 4, the integration region can be divided into five areas. Going from point C to the origin, they are characterized by the following limits:

$$\begin{aligned}
 \text{i)} \quad & 0 \leq k \leq k_1, \quad -1 \leq \cos \theta_\gamma \leq 1, \\
 \text{ii)} \quad & k_1 \leq k \leq k_2, \quad -1 \leq \cos \theta_\gamma \leq f_1(k, z), \\
 \text{iii)} \quad & k_1 \leq k \leq k_2, \quad f_2(k, z) \leq \cos \theta_\gamma \leq 1, \\
 \text{iv)} \quad & k_2 \leq k \leq k_3, \quad g_1(k, E_{th}) \leq \cos \theta_\gamma \leq f_1(k, z), \\
 \text{v)} \quad & k_2 \leq k \leq k_3, \quad f_2(k, z) \leq \cos \theta_\gamma \leq g_2(k, E_{th}).
 \end{aligned} \tag{42}$$

Here,

$$\begin{aligned}
 k_1 &= P_0 \left\{ -1 - z + 2 \left[(1+z) \left(\frac{1}{2} - \frac{\mu^2}{P_0^2} (1-z) \right) \right]^{\frac{1}{2}} \right\} / (1-z), \\
 k_2 &= P_0 - E_{th} - B_+(E_{th}, 1), \\
 k_3 &= P_0 - E_{th} - B_+(E_{th}, z^2),
 \end{aligned} \tag{43}$$

and the functions $g_1(k, E_{th})$ and $g_2(k, E_{th})$ are given by the right-hand side of the equation

$$\cos \theta_\gamma = \frac{P_0 \left(\frac{1}{2} P_0 - k - q_{+0} \right) + k q_{+0}}{k |\vec{q}_+|} \tag{44}$$

by inserting for q_{+0} the expressions

$$q_{+0} = P_0 - E_{th} - k,$$

and

$$q_{+0} = E_{th}, \tag{45}$$

respectively.

The functions $f_1(k, z)$ and $f_2(k, z)$ are also obtained from Eq. (44) by inserting the q_{+0} values corresponding to intersections of a fixed k line with the curves 1 and 2 of Fig. 4. These two q_{+0} values are the roots of the equation

$$q_{+0}^2 - \ell q_{+0} + c = 0 \quad (46)$$

where

$$\begin{aligned} \ell &= P_0 - k, \\ c &= \gamma - \left[\gamma^2 - \frac{\tau^2 + z^2 \mu^2 (\ell^2 - \mu^2)}{1 - z^2} \right]^{\frac{1}{2}} \end{aligned} \quad (47)$$

with

$$\begin{aligned} \gamma &= \frac{\tau + \mu^2 z^2}{1 - z^2}, \\ \tau &= P_0 \left(\frac{1}{2} P_0 - k \right) - \mu^2. \end{aligned} \quad (48)$$

In Appendix B, it is briefly indicated how Eq. (46) is derived.

In order to do the numerical integration of $\partial^5 \sigma^B / \partial \Omega_\mu \partial \Omega_\gamma \partial k$, one finally has to know all the scalar products occurring in the functions F_{ij} in terms of the variables k , $\cos \theta_\gamma$, and φ_γ . Once one knows q_{+0} in terms of these variables, it becomes a trivial matter. The required relation for that is

$$q_{+0} = \frac{\ell P_0 \left(\frac{1}{2} P_0 - k \right) - k \cos \theta_\gamma \left[P_0^2 \left(\frac{1}{2} P_0 - k \right)^2 + \mu^2 (k^2 \cos^2 \theta_\gamma - \ell^2) \right]^{\frac{1}{2}}}{\ell^2 - k^2 \cos^2 \theta_\gamma} \quad (49)$$

6. RESULTS

The differential cross-section for μ pair creation, S , up to order α^3 is given by the sum of expression (17), which contains the virtual corrections, and expression (25) for the bremsstrahlung, integrated over the phase space [Eqs. (42)]. Although both terms are separately infrared divergent, their sum converges. One can see this explicitly from the soft photon approximation, Eq. (29), which holds for small k . For numerical calculations of the total correction, however, it is advantageous to perform this infrared cancellation explicitly. We therefore write

$$\begin{aligned}
S = & 2 \frac{d\sigma^0}{d\Omega_\mu} \left[1 + \delta_{vc}(m^2) + \delta_{vc}(\mu^2) + \delta_{vp}(m^2) + \delta_{vp}(\mu^2) \right] \\
& + 2 \frac{d\sigma^{mag}(m^2)}{d\Omega_\mu} + 2 \frac{d\sigma^{mag}(\mu^2)}{d\Omega_\mu} \\
& + 2 \frac{d\sigma^0}{d\Omega_\mu} \left[\delta_s(m^2, k_1) + \delta_s(\mu^2, k_1) \right] \\
& + 2 \int_{AI} \frac{\partial^5 \sigma^B}{\partial \Omega_\mu \partial \Omega_\gamma \partial k} d\Omega_\gamma dk + 2 \int_I \frac{\partial^5 (\sigma^B - \sigma^S)}{\partial \Omega_\mu \partial \Omega_\gamma \partial k} d\Omega_\gamma dk,
\end{aligned} \tag{50}$$

where the anisotropic integral (AI) is over the regions ii)-v) in Eq. (42), and the isotropic one (I) over region i) only. The former is infrared convergent, since the integration region does not contain the soft photon region, and the latter also converges, as the integrand is regular for $k \rightarrow 0$.

The final result for S can again be written in terms of a δ_T , which measures the deviation from the lowest order cross-section:

$$S = 2 \frac{d\sigma^0}{d\Omega_\mu} (1 + \delta_A + \delta_N) = 2 \frac{d\sigma^0}{d\Omega_\mu} (1 + \delta_T). \tag{51}$$

A part of the total correction δ_T is known analytically, i.e., δ_A , which corresponds to the sum of the virtual corrections and the correction due to the emission of a soft photon with a maximal energy k_1 [Eq. (43)]. The other part, δ_N , has to be evaluated by numerical integration, and represents the effect of the hard anisotropic bremsstrahlung and the difference between hard and soft bremsstrahlung over an isotropic region.

The numerical integration was done over separate regions in phase space, chosen in such a way that small regions with a rapid variation of the integrand were attributed as many integration points as large regions with a small variation. Over every region, the integration was done using the multi-dimensional integration routine RIWIAD¹⁴⁾, which itself distributes the integration points as efficiently as possible over the integration domain. When δ_N becomes large, which is the case at small scattering angles, more integration points have to be used in order to get a reasonably accurate δ_T , in particular since large cancellations with δ_A occur in this case.

We have evaluated δ_A and δ_N for a set of different beam energies, p_{+0} , and scattering angles, θ . For one energy, we show the dependence of these quantities on the acollinearity angle, ζ . As long as the threshold energy, E_{th} , for the μ detection is small, its dependence is not strong. On the other hand, there is a sizable variation with ζ .

In Table I, we give $2d\sigma^0/d\Omega_\mu$ for the sake of completeness, and, in Tables II and III, we list our results.

From the Tables it is clear that for experiments at the percentage level, the quantity S has to be evaluated up to order α^3 . Also, the qualitative features of the results are easily understood. The angular dependence in δ_A comes only from the $d\sigma^{mag}/d\Omega_\mu$ terms in Eq. (17), and they are very small, especially for high beam energies. On the other hand, δ_N is strongly angular dependent, since the angle θ determines to what extent the region where $\partial^5\sigma^B/\partial\Omega_\mu\partial\Omega_\gamma\partial k$ is largest lies inside the phase space. This region is the one where the photon is emitted parallel to the electron or the positron, so δ_N strongly depends on the amount of overlap between the regions $\theta_\gamma \approx \theta$ and $\theta_\gamma \approx \pi - \theta$ on one hand, and the shaded area of Fig. 4 on the other. This overlap is larger for small angles θ and decreases with increasing θ . It should finally be noted that the main contribution to δ_N comes from the anisotropic integral (in many cases 70% or more).

One may wonder whether approximate calculations reproduce our results. A first approximation assumes that the soft photon expression describes the bremsstrahlung over the entire phase space region. After integration one then obtains a δ_N which can differ up to 40% from the exact one. This gives, in many cases, changes in δ_T which are too large to allow a meaningful comparison between theory and experiments at the percentage level.

It has also been suggested that it would suffice to consider radiation from the electrons only⁸⁾. In this case, we have found changes in δ_A and δ_N up to 50%. In our specific phase space, these changes almost cancel, but one could imagine situations in which only the isotropic phase space would contribute, and then the changes in δ_T would be considerable. So, for μ pair production, particularly at the high energies mentioned in the table, one has to consider the bremsstrahlung from both the electrons and the muons¹⁵⁾.

7. CONCLUSIONS

For future QED experiments with e^+e^- colliding beams at high energies and accuracies at the level of a few percent, the full radiative corrections to order α^3 have to be taken into account. In particular, when two-body events are selected by the criterium that they are back to back, quite hard photons can be emitted. The exact matrix element for hard photon emission has to be used. Measurement of the differential cross-section as a function of the scattering angle will show that the higher order correction changes sign.

A P P E N D I X A

In this Appendix, the explicit expressions for

$$F_{ij} (p_+, p_-, q_+, q_-, k) = \sum_{\text{spins}} N_i N_j^* \quad (\text{A.1})$$

are given.

In fact, several relations among the functions F_{ij} hold, such that it suffices to know only three of them to know them all, e.g., F_{11} , F_{12} and F_{13} .

From the reality of F_{ij} , it follows that

$$F_{ij} = F_{ji}^* . \quad (\text{A.2})$$

Since in the traces of Eq. (A.1) odd powers of m (or μ) are always combined with an odd number of γ matrices, and, therefore, vanish, we can replace m by $-m$ (or μ by $-\mu$) without changing anything. The following relations are then obtained:

$$\begin{aligned} F_{22} (p_+, p_-, q_+, q_-, k) &= F_{11} (p_-, p_+, q_+, q_-, k), \\ F_{14} (p_+, p_-, q_+, q_-, k) &= - F_{13} (p_+, p_-, q_-, q_+, k), \\ F_{23} (p_+, p_-, q_+, q_-, k) &= - F_{13} (p_-, p_+, q_+, q_-, k), \\ F_{24} (p_+, p_-, q_+, q_-, k) &= F_{13} (p_-, p_+, q_-, q_+, k). \end{aligned} \quad (\text{A.3})$$

Finally, from the similarity between the muon and the electron part, it follows that

$$\begin{aligned} F_{33} (p_+, p_-, q_+, q_-, k) &= F_{11} (q_+, q_-, p_+, p_-, -k), \\ F_{44} (p_+, p_-, q_+, q_-, k) &= F_{22} (q_+, q_-, p_+, p_-, -k), \\ F_{34} (p_+, p_-, q_+, q_-, k) &= F_{12} (q_+, q_-, p_+, p_-, -k). \end{aligned} \quad (\text{A.4})$$

The explicit expressions for F_{11} , F_{12} and F_{13} are

$$\begin{aligned} m^2 \mu^2 F_{11} = & -4m^2 \left[2(\lambda_+ q_+)(\lambda_- q_-) + 2(\lambda_+ q_-)(\lambda_- q_+) \right. \\ & \left. - \lambda'(\lambda_+ \lambda_-) - \lambda(q_+ q_-) + \lambda \lambda' \right] \\ & + [4m^2 + 4(k p_-)] \left[2(\lambda_+ q_+)(k q_-) + 2(\lambda_+ q_-)(k q_+) \right. \\ & \left. - 2(q_+ q_-)(k \lambda_+) + \lambda'(k \lambda_+) \right] \\ & - 4m^2 (k \lambda_-) [2(q_+ q_-) - 2\lambda'] , \end{aligned}$$

$$\begin{aligned} m^2 \mu^2 F_{12} = & 4(\lambda_+ \lambda_-) \left[2(\lambda_+ q_+)(\lambda_- q_-) + 2(\lambda_+ q_-)(\lambda_- q_+) \right. \\ & \left. - \lambda'(\lambda_+ \lambda_-) - \lambda(q_+ q_-) + \lambda \lambda' \right] \\ & - 2(\lambda_+ \lambda_-) \left[2(P q_+)(k q_-) + 2(P q_-)(k q_+) \right. \\ & \left. - 2(q_+ q_-)(k P) \right] \\ & - 2(k P) \left[2(\lambda_+ q_+)(\lambda_- q_-) + 2(\lambda_+ q_-)(\lambda_- q_+) \right. \\ & \left. - \lambda(q_+ q_-) + \lambda \lambda' - 2m^2 \mu^2 \right] \quad (A.5) \\ & - 8m^2 (k q_+)(k q_-) + 8(k \lambda_+)(\lambda_- q_+)(\lambda_- q_-) \\ & + 8(k \lambda_-)(\lambda_+ q_+)(\lambda_+ q_-) , \end{aligned}$$

$$\begin{aligned} m^2 \mu^2 F_{13} = & - \left[4(\lambda_- q_-) - 2(k q_-) + 2(k \lambda_-) \right] \\ & \left[2(\lambda_+ q_+)(\lambda_- q_-) + 2(\lambda_+ q_-)(\lambda_- q_+) + m^2 \lambda' + \mu^2 \lambda \right] \\ & + 2(\lambda_- q_-) \left[2(\lambda_+ q_+)(k q_-) + 2(\lambda_+ q_-)(k q_+) + 2\mu^2 (k \lambda_+) \right. \\ & \left. - 2(\lambda_+ q_+)(k \lambda_-) - 2(\lambda_- q_+)(k \lambda_+) - 2m^2 (k q_+) \right] \\ & - 2(k \lambda_-) \left[2\mu^2 (\lambda_+ q_+) + \lambda'(\lambda_+ q_-) \right] \\ & + 2(k q_-) \left[2m^2 (\lambda_+ q_+) + \lambda(\lambda_- q_+) \right] \\ & + 8(\lambda_+ q_+)(k \lambda_-)(k q_-) + 4m^2 (k q_+)(k q_-) \\ & + 4\mu^2 (k \lambda_+)(k \lambda_-) , \end{aligned}$$

with $P = p_+ + p_-$.

With these expressions, one can now calculate $\sum |T|^2$ of Eq. (24). We have verified that, in the charge symmetric $\mu^+\mu^-$ case, they lead to an expression which coincides with that recently obtained by D'Ettore Piazzoli¹⁶⁾.

A P P E N D I X B

In order to derive Eq. (46), one has to solve for q_{+0} from Eq. (38) when $\cos \delta = z$ and

$$q_{-0} = P_0 - q_{+0} - k \quad (\text{B.1})$$

By introducing the variable $\xi = q_{+0} q_{-0}$, one finds the equation

$$\xi^2 (1 - z^2) - 2\xi (\tau + z^2 \mu^2) + \tau^2 + z^2 \mu^2 (b^2 - \mu^2) = 0, \quad (\text{B.2})$$

with the solutions

$$\xi = \gamma \pm \left[\gamma^2 - \frac{\tau^2 + z^2 \mu^2 (b^2 - \mu^2)}{1 - z^2} \right]^{\frac{1}{2}}. \quad (\text{B.3})$$

The quantities γ , τ , and b have been defined in Eqs. (47) and (48).

From the definition of ξ and Eq. (B.1), one arrives at

$$q_{+0}^2 - b q_{+0} + \gamma \pm \left[\gamma^2 - \frac{\tau^2 + z^2 \mu^2 (b^2 - \mu^2)}{1 - z^2} \right]^{\frac{1}{2}} = 0. \quad (\text{B.4})$$

By examining the limit $\mu \rightarrow 0$, one finds that the lower sign in Eq. (B.4) has to be taken.

R E F E R E N C E S

- 1) S.J. Brodsky, Proceedings of the 4th International Symposium on Electron and Photon Interactions at High Energies, Liverpool, p.3 (1969);
S.J. Brodsky, Proceedings of the 1971 International Symposium on Electron and Photon Interactions at High Energies, Cornell University, Ithaca, N.Y., p. 13, (1971).
- 2) B.E. Lautrup, A. Petermann and E. de Rafael, Phys.Reports 3, 193 (1972).
- 3) J. Perez-y-Jorba, Proceedings of the 4th International Symposium on Electron and Photon Interactions at High Energies, Liverpool, p.213 (1969).
- 4) C. Bernardini, Proceedings of the 1971 International Symposium on Electron and Photon Interactions at High Energies, Cornell University, Ithaca, N.Y., p. 38 (1971);
V. Sidorov, *ibid.*, p.66.
- 5) V. Alles Borelli, M. Bernardini, D. Bollini, P.L. Brunini, E. Fiorentino, T. Massam, L. Monari, F. Palmonari and A. Zichichi, Phys.Letters 36B, 149 (1971).
- 6) G. Putzolu, Nuovo Cimento 20, 542 (1961).
- 7) G. Källén, in Handbuch der Physik, edited by S. Flugge (Springer, Berlin, 1958), vol. 5, part 1, p. 304.
- 8) G. Bonneau and F. Martin, Nuclear Phys. B27, 381 (1971).
- 9) U. Mosco, Nuovo Cimento 33, 115 (1964).
- 10) G. Furlan, R. Gatto and G. Longhi, Phys.Letters 12, 262 (1964).
- 11) G. Rossi, CNEN Laboratori Nazionali di Frascati, Nota Interna 334.
- 12) M.E. Calva-Tellez, thèse de doctorat 3ème cycle, Université de Paris VI, June 1972.
- 13) See e.g.,
A. Litke, Thesis (Harvard University, 1970) unpublished, and
R. Wilson, Proceedings of the International School of Physics "Enrico Fermi", Varenna, p.453 (1969).
- 14) B. Lautrup, Proceedings of the 2nd Colloquium on Advanced Computing Methods in Theoretical Physics, Marseille, p.I-58, (1971).

- 15) A similar conclusion was reached by V.N. Baier and V.A. Khoze,
Soviet Phys. JETP 21, 629 (1965).
- 16) B. D'Ettore Piazzoli, CNEN Laboratori Nazionali di Frascati,
LNF-71/61 (1971).

$\theta \backslash p_{+0}$	0.5	1.0	2.0	3.0	5.0
5°	20.19	5.14	1.29	0.57	0.21
20°	19.14	4.86	1.22	0.54	0.20
40°	16.27	4.10	1.03	0.46	0.16
60°	13.01	3.24	0.81	0.36	0.13
90°	10.59	2.61	0.65	0.29	0.10

TABLE I : The lowest order cross-section (in nb)
for different values of the beam energy,
 p_{+0} (in GeV) and the scattering angle,
 θ .

θ p_{+0}	0.5		1.0		2.0		3.0		5.0	
	δ_A	δ_T	δ_A	δ_T	δ_A	δ_T	δ_A	δ_T	δ_A	δ_T
5°	-5.2	3.6 ± 0.4	-6.0	5.5 ± 0.6	-7.0	6.6 ± 0.7	-7.5	7.1 ± 0.8	-8.2	8.0 ± 0.8
20°	-5.2	0.2 ± 0.3	-6.0	0.5 ± 0.3	-7.0	0.7 ± 0.4	-7.5	0.8 ± 0.4	-8.2	0.9 ± 0.5
40°	-5.2	-2.5 ± 0.1	-6.0	-2.7 ± 0.2	-7.0	-2.8 ± 0.2	-7.5	-2.9 ± 0.2	-8.2	-2.9 ± 0.3
60°	-5.2	-3.9 ± 0.1	-6.0	-4.3 ± 0.1	-7.0	-4.6 ± 0.1	-7.5	-4.8 ± 0.1	-8.2	-5.0 ± 0.2
90°	-5.2	-4.6 ± 0.1	-6.0	-5.1 ± 0.1	-7.0	-5.5 ± 0.1	-7.5	-5.7 ± 0.1	-8.2	-5.9 ± 0.1

TABLE II : The radiative corrections (in %) to the lowest order cross-section for different values of the beam energy, p_{+0} (in GeV), and the scattering angle, θ . The acollinearity angle $\xi = 10^\circ$, and the threshold energy $E_{th} = 0.2$ GeV.

$\theta \backslash \xi$	5°		15°	
	δ_A	δ_T	δ_A	δ_T
5°	-15.0	1.5 ± 0.9	-3.3	10.9 ± 0.7
20°	-15.0	-5.4 ± 0.5	-3.3	4.1 ± 0.4
40°	-15.0	-9.4 ± 0.3	-3.3	0.6 ± 0.2
60°	-15.0	-11.3 ± 0.2	-3.3	-1.2 ± 0.1
90°	-15.0	-12.3 ± 0.1	-3.3	-2.1 ± 0.1

TABLE III : The radiative corrections (in %) to the lowest order cross-section for different values of the acollinearity angle, ξ , and the scattering angle, θ . The beam energy $p_{+0} = 3$ GeV and the threshold energy $E_{th} = 0.2$ GeV.

FIGURE CAPTIONS

- Figure 1 :
Lowest order Feynman diagram for μ pair production.
- Figure 2 :
Feynman diagrams for the production of a μ pair accompanied by real photon emission.
- Figure 3 :
Feynman diagrams for the virtual radiative corrections to μ pair production.
- Figure 4 :
Dalitz plot for the μ pair. The curves 1 and 2 are the lines where the muons make an angle $\pi - \xi$. The shaded area is the experimental phase space.
- Figure 5 :
Dalitz plot for the μ pair, showing the curves 1, 2 and 3, where $\theta_\gamma \approx 0$ and $\theta_\gamma \approx \pi - \theta$.

FIG. 1

(a)

(b)

(c)

(d)

FIG.2

FIG.3

FIG.4

FIG.5