THE KOSZUL–TATE COHOMOLOGY IN COVARIANT HAMILTONIAN FORMALISM

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We show that, in the framework of covariant Hamiltonian field theory, a degenerate almost regular quadratic Lagrangian $L$ admits a complete set of non-degenerate Hamiltonian forms such that solutions of the corresponding Hamilton equations, which live in the Lagrangian constraint space, exhaust solutions of the Euler–Lagrange equations for $L$. We obtain the characteristic splittings of the configuration and momentum phase bundles. Due to the corresponding projection operators, the Koszul–Tate resolution of the Lagrangian constraints for a generic almost regular quadratic Lagrangian is constructed in an explicit form.

1 Introduction

The covariant Hamiltonian approach to field theory has been vigorously developed from the seventies (see [1-3] for a survey). This is the Hamiltonian counterpart of Lagrangian field theory on fibre bundles $Y \to X$, which looks promising for quantization [4]. If $\dim X = 1$, covariant Hamiltonian formalism provides the adequate description of Hamiltonian time-dependent mechanics [5,6]. Here, we aim to apply some elements of the classical BRST technique for Hamiltonian systems to covariant Hamiltonian theory.

Let us recall that, given a fibre bundle $Y \to X$ coordinated by $(x^\lambda, y^i)$, a first order Lagrangian $L$ of fields is defined as a horizontal density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega, \quad \omega = dx^1 \wedge \cdots dx^n, \quad n = \dim X; \quad (1)$$

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on the jet bundle $J^1Y \to X$ seen as a finite-dimensional configuration space of fields and coordinated by $(x^\lambda, y^i, y^i_\lambda)$. The corresponding Euler–Lagrange equations take the coordinate form

\[(\partial_i - d_\lambda \partial^i_\lambda)\mathcal{L} = 0, \quad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_\lambda \partial^i_\mu.\] (2)

Every Lagrangian $L$ (1) yields the Legendre map

\[\tilde{L} : J^1Y \to \Pi, \quad p^\lambda_i \circ \tilde{L} = \pi^\lambda_i = \partial^\lambda_\lambda \mathcal{L},\]

of $J^1Y$ to the Legendre bundle

\[\Pi = \tilde{n} T^*X \otimes V^*Y \otimes TX,\]

equipped with the holonomic coordinates $(x^\lambda, y^i, p^\lambda_i)$ and seen as a momentum phase space of fields [7,8]. Hamiltonian dynamics on $\Pi$ is phrased in terms of Hamiltonian forms

\[H = p^\lambda_i dy^i \wedge \omega^\lambda - \mathcal{H}(x^\lambda, y^i, p^\lambda_i)\omega, \quad \omega^\lambda = \partial^\lambda_\lambda \omega,\]

and the corresponding covariant Hamilton equations

\[y^i_\lambda = \partial^\lambda_\lambda \mathcal{H}, \quad p^\lambda_i = -\partial_i \mathcal{H}.\] (3)

In the case of hyperregular Lagrangians, Lagrangian formalism and covariant Hamiltonian formalism are equivalent. For any hyperregular Lagrangian $L$, there exists a unique Hamiltonian form $H$ such that the Euler–Lagrange equations (2) for $L$ are equivalent to the Hamilton equations (3) for $H$. The case of degenerate Lagrangians is more intricate. Let us restrict our consideration to almost regular Lagrangians $L$, i.e., (i) the Lagrangian constraint space $N_L = \tilde{L}(J^1Y)$ is a closed imbedded subbundle of the Legendre bundle $\Pi \to Y$; (ii) the Legendre map $\tilde{L} : J^1Y \to N_L$ is a fibred manifold, and (iii) the inverse image $\tilde{L}^{-1}(q)$ of any point $q \in N_L$ is a connected submanifold of $J^1Y$. At least locally, one can assign to $L$ a complete set of Hamiltonian forms $H$ such that there is one-to-one correspondence between solutions of the Hamilton equations for $H$ from this set, which live in the Lagrangian constraint space $N_L$, and solutions of the Euler–Lagrange equations for $L$ (see [2, 3, 9] for a detailed exposition). It is important that these Hamiltonian forms $H$ are not necessarily degenerate.

In the present work, the case of almost regular quadratic Lagrangians, appropriate for application to many physical models, is studied in detail. We obtain a complete set of non-degenerate Hamiltonian forms with the above mentioned properties for a generic almost
regular quadratic Lagrangian. We show that, in this case, the Legendre bundle $\Pi$ admits
the characteristic splitting $\Pi = \text{Ker}\sigma \oplus N_L$ (12a). Using the corresponding projection
operators, we construct the Koszul–Tate resolution for the Lagrangian constraints $N_L$ of
a generic almost regular quadratic Lagrangian $L$ in an explicit form.

2 Quadratic degenerate systems

Given a fibre bundle $Y \to X$, let us consider a quadratic Lagrangian $L$ which has the
coordinate expression
$$L = \frac{1}{2}a_{ij}^{\lambda \mu} y_\lambda^i y_\mu^j + b_i^i y_\lambda^i + c,$$
(4)
where $a$, $b$ and $c$ are local functions on $Y$. This property is coordinate-independent due
to the affine transformation law of coordinates $y_\lambda^i$. The associated Legendre map
$$p_i^\lambda \circ \hat{L} = a_{ij}^{\lambda \mu} y_\mu^j + b_i^i$$
(5)
is an affine morphism over $Y$. It defines the corresponding linear morphism
$$\mathcal{T} : T^*X \otimes VY \to Y, \quad p_i^\lambda \circ \mathcal{T} = a_{ij}^{\lambda \mu} y_\mu^j,$$
(6)
where $T^*X \otimes VY$ is the underlying vector bundle of the affine jet bundle $J^1Y \to Y$ and
$y_\mu^j$ are bundle coordinates on it.

Let the Lagrangian $L$ (4) be almost regular, i.e., the matrix function $a_{ij}^{\lambda \mu}$ is of constant
rank. Then the Lagrangian constraint space $N_L$ (5) is an affine subbundle of the Legendre
bundle $\Pi \to Y$, modelled over the vector subbundle $\overline{N}_L$ (6) of $\Pi \to Y$. Hence, $N_L \to Y$
has a global section. For the sake of simplicity, let us assume that it is the canonical zero
section $\hat{0}(Y)$ of $\Pi \to Y$. Then $\overline{N}_L = N_L$. Accordingly, the kernel of the Legendre map
(5) is an affine subbundle of the affine jet bundle $J^1Y \to Y$, modelled over the kernel of
the linear morphism $\mathcal{T}$ (6). Then there exists a connection
$$\Gamma : Y \to \text{Ker}\hat{L} \subset J^1Y, \quad a_{ij}^{\lambda \mu} \Gamma_\mu^j + b_i^\lambda = 0,$$
(7)
on $Y \to X$. Connections (7) constitute an affine space modelled over the linear space of
soldering forms $\phi$ on $Y \to X$ satisfying the conditions
$$a_{ij}^{\lambda \mu} \phi_\mu^j = 0, \quad \phi_\lambda^i b_i^\lambda = 0.$$
The key point of our consideration is a linear bundle map

$$\sigma : \Pi \to T^*X \otimes VY, \quad \bar{y}_\lambda \circ \sigma = \sigma_{ij}^\mu p_j^\mu,$$

such that $$\bar{L} \circ \sigma|_{N_L} = \text{Id } N_L.$$ It is a solution of the algebraic equations

$$a_{ij}^\mu \sigma_{jk}^\alpha a_{\alpha \nu}^{b\mu} = a_{ib}^\nu.$$

(9)

The matrix $$a$$ in the Lagrangian $$L$$ (4) can be seen as a global section of constant rank of the tensor bundle

$$\wedge^n T^*X \otimes [\vee (T^*X \otimes V^*Y)] \to Y.$$

By virtue of the well-known theorem on the splitting of an exact sequence of vector bundles, there exists the bundle splitting

$$T^*X \otimes VY = \text{Ker } a \oplus E',$$

(10)

together with a (non-holonomic) atlas of this bundle such that transition functions of $$\text{Ker } a$$ and $$E'$$ are independent. Since $$a$$ is a non-degenerate section of $$\wedge^n T^*X \otimes (\vee E^*) \to Y,$$ there exists an atlas of $$E'$$ such that $$a$$ is brought into a diagonal matrix with non-vanishing components $$a^{AA}.$$ Due to the splitting (10), we have the corresponding bundle splitting

$$TX \otimes V^*Y = (\text{Ker } a)^* \oplus E^*.$$

Then the desired map $$\sigma$$ is represented by a direct sum $$\sigma_1 \oplus \sigma_0$$ of an arbitrary section $$\sigma_1$$ of the fibre bundle

$$\wedge^n TX \otimes (\vee \text{Ker } a) \to Y$$

and the section $$\sigma_0$$ of the fibre bundle

$$\wedge^n TX \otimes (\vee E') \to Y$$

which has non-vanishing components $$\sigma_{AA} = (a^{AA})^{-1}$$ with respect to the above mentioned atlas of $$E'.$$ Moreover, $$\sigma$$ satisfies the additional relations

$$\sigma_0 = \sigma_0 \circ L \circ \sigma_0, \quad a \circ \sigma_1 = 0, \quad \sigma_1 \circ a = 0.$$
With the map (9), we have the splittings
\[ J^1Y = S(J^1Y) \oplus F(J^1Y) = \text{Ker} \hat{L} \oplus \text{Im}(\sigma \circ \hat{L}), \tag{11a} \]

\[ y_i^\lambda = S_i^\lambda + F_i^\lambda = [y_i^\lambda - \sigma_{\lambda a}^{ik}(a_{kj}^\mu y_j^\mu + b_k^a)] + [\sigma_{\lambda a}^{ik}(a_{kj}^\mu y_j^\mu + b_k^a)], \tag{11b} \]

\[ \Pi = R(\Pi) \oplus \mathcal{P}(\Pi) = \text{Ker} \sigma_0 \oplus N_L, \tag{12a} \]

\[ p_\lambda^i = R_\lambda^i + \mathcal{P}_\lambda^i = [p_\lambda^i - a_{ij}^\lambda \sigma_{\mu a}^{jk} p_\mu^a] + [a_{ij}^\lambda \sigma_{\mu a}^{jk} p_\mu^a]. \tag{12b} \]

It is readily observed that, with respect to the coordinates \( S_i^\lambda, F_i^\lambda \) (11b) and \( R_\lambda^i, \mathcal{P}_\lambda^i \) (12b), the Lagrangian (4) reads
\[ L = \frac{1}{2} a_{ij}^\lambda \mathcal{F}_\lambda^i \mathcal{F}_{\mu}^j + c', \tag{13} \]
while the Lagrangian constraint space is given by the reducible constraints
\[ \mathcal{R}_\mu^\lambda = p_{\lambda}^{i} - a_{ij}^{\lambda} \sigma_{\mu a}^{jk} p_{\mu}^{a} = 0. \tag{14} \]

Note that, in gauge theory on principal bundles, we have the canonical splitting (11a) where \( 2F \) is the strength tensor [3, 8]. The Yang–Mills Lagrangian of this gauge theory is exactly of the form (13) where \( c' = 0 \). The Lagrangian of Proca fields is also of the form (13) where \( c' \) is the mass term. This is an example of a degenerate Lagrangian system without gauge symmetries.

Given the linear map \( \sigma \) (9) and a connection \( \Gamma \) (7), let us consider the Hamiltonian form
\[ H = p_\lambda^i dy^i \wedge \omega_\lambda - [\Gamma_\lambda^i p_\mu^i + \frac{1}{2} \sigma_{0\lambda\mu} p_\mu^i p_\mu^j + \sigma_{1\lambda\mu} p_\mu^i p_\mu^j - c'] \omega = \tag{15} \]

\[ (\mathcal{R}_\lambda^i + \mathcal{P}_\lambda^i) dy^i \wedge \omega_\lambda - [(\mathcal{R}_\lambda^i + \mathcal{P}_\lambda^i) \Gamma_\lambda^i + \frac{1}{2} \sigma_{0\lambda\mu} \mathcal{P}_\mu^i + \sigma_{1\lambda\mu} \mathcal{P}_\mu^i - c'] \omega. \]

One can show that the Hamiltonian forms (15) parametrised by connections \( \Gamma \) (7) constitute a complete set for the Lagrangian (4) [3]. It is readily observed that, if \( \sigma_1 \) is non-degenerate, so are the Hamiltonian forms (15). Thus, for different \( \sigma_1 \), we have different complete sets of Hamiltonian forms (15). Hamiltonian forms \( H \) (15) of such a complete set differ from each other in the term \( \phi_\lambda^\lambda \mathcal{R}_\lambda^i \), where \( \phi \) are the soldering forms (8). It follows from the splitting (12a) that this term vanishes on the Lagrangian constraint space.
3 Geometry of antighosts

We aim to obtain the Koszul–Tate resolution for the constraints (14). Since these constraints are reducible, we need an infinite number of antighost fields in general [10, 11] (we follow the terminology of [11]). They are graded by the antighost number \( r \) and the Grassmann parity \( r \mod 2 \). Therefore, we should generalize the notion of a graded manifold [12] to commutative graded algebras generated both by odd and even elements.

Let \( E = E_0 \oplus E_1 \to Z \) be the Whitney sum of vector bundles \( E_0 \to Z \) and \( E_1 \to Z \) over a manifold \( Z \). One can think of \( E \) as being a bundle of vector superspaces with a typical fibre \( V = V_0 \oplus V_1 \) where transition functions of \( E_0 \) and \( E_1 \) are independent. Let us consider the exterior bundle

\[
\wedge E^* = \bigoplus_{k=0}^{\infty} (\bigwedge^k Z E^*),
\]

which is the tensor bundle \( \otimes E^* \) modulo elements

\[
e_0 e'_0 - e'_0 e_0, \quad e_1 e'_1 + e'_1 e_1, \quad e_0 e_1 - e_1 e_0 \quad e_0, e_1, e'_0, e'_1 \in E^*_0, \quad e_1, e'_1 \in E^*_1, \quad z \in Z.
\]

Global sections of \( \wedge E^* \) constitute a commutative graded algebra \( \mathcal{A}(Z) \) modelled on the locally free \( C^\infty(Z) \)-module \( E_0^* (Z) \oplus E_1^* (Z) \) of global sections of \( E^* \). This is the product of the commutative algebra \( \mathcal{A}_0(Z) \) of global sections of the symmetric bundle \( \vee E_0^* \to Z \) and the graded algebra \( \mathcal{A}_1(Z) \) of global sections of the exterior bundle \( \wedge E_1^* \to Z \). The pair \( (Z, \mathcal{A}_1(Z)) \) is a graded manifold [12]. For the sake of brevity, we agree to call \( (Z, \mathcal{A}(Z)) \) a graded manifold, though its generating set contain an even subset \( \mathcal{A}_0 \). Accordingly, elements of \( \mathcal{A}(Z) \) are called graded functions. Let us introduce the differential calculus in these functions.

We start from the \( \mathcal{A}(Z) \)-module \( \text{Der}\mathcal{A}(Z) \) of graded derivations of \( \mathcal{A}(Z) \). Recall that by a graded derivation of the commutative graded algebra \( \mathcal{A}(Z) \) is meant an endomorphism of \( \mathcal{A}(Z) \) such that

\[
u(f f') = u(f) f' + (-1)^{|u||f|} f u(f')
\]

for the homogeneous elements \( u \in \text{Der}\mathcal{A}(Z) \) and \( f, f' \in \mathcal{A}(Z) \). We use the notation \(|.|\) for the Grassmann parity.

We aim to show that graded derivations (16) are represented by sections of a vector bundle. Let \( \{e^a\} \) be the holonomic bases for \( E^* \to Z \) with respect to some bundle atlas.
\((z^A, v^i)\) of \(E \to Z\) with transition functions \(\{\rho^a_b\}\), i.e., \(c^a = \rho^a_b(z)c^b\). Then graded functions read

\[
 f = \sum_{k=0}^{\infty} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \cdots c^{a_k},
\]

where \(f_{a_1 \ldots a_k}\) are local functions on \(Z\), and we omit the symbol of an exterior product of elements \(c\). The coordinate transformation law of graded functions (17) is obvious. Due to the canonical splitting \(VE = E \times E\), the vertical tangent bundle \(VE \to E\) can be provided with the fibre bases \(\{\partial_a\}\) dual of \(\{c^a\}\). These are fibre bases for \(pr_2 VE = E\).

Then any derivation \(u\) of \(A(U)\) on a trivialization domain \(U\) of \(E\) reads

\[
 u = u^A \partial_A + u^a \partial_a,
\]

where \(u^A, u^a\) are local graded functions and \(u\) acts on \(f \in A(U)\) by the rule

\[
 u(f_{a_1 \ldots a_k} c^{a_1} \cdots c^{a_k}) = u^A(f_{a_1 \ldots a_k}) c^{a_1} \cdots c^{a_k} + u^a f_{a \ldots b} \partial_a (c^a \cdots c^b).
\]

This rule implies the corresponding coordinate transformation law

\[
 u' = u^A, \quad u'^a = \rho^a_j u^j + u^A \partial_A (\rho^a_j c^j)
\]

of derivations (18). Let us consider the vector bundle \(V_E \to Z\) which is locally isomorphic to the vector bundle

\[
 V_E |_U \approx \wedge^* Z \otimes (pr_2 VE \oplus TZ) |_U,
\]

and has the transition functions

\[
 z^A_{i_1 \ldots i_k} = \rho^{-1}_{-a_1} \cdots \rho^{-1}_{-a_k} z^A_{a_1 \ldots a_k},
\]

\[
 v^j_{i_1 \ldots j_k} = \rho^{-1}_{-b_1} \cdots \rho^{-1}_{-b_k} \rho^j_{b_1 \ldots b_k} c^j + \frac{k!}{(k-1)!} z^A_{b_1 \ldots b_{k-1}} \partial_A (\rho^j_{b_k})
\]

of the bundle coordinates \((z^A_{a_1 \ldots a_k}, v^j_{b_1 \ldots b_k})\), \(k = 0, \ldots\). These transition functions fulfill the cocycle relations. It is readily observed that, for any trivialization domain \(U\), the \(A\)-module \(\text{Der} A(U)\) with the transition functions (19) is isomorphic to the \(A\)-module of local sections of \(V_E |_U \to U\). One can show that, if \(U' \subset U\) are open sets, there is the restriction morphism \(\text{Der} A(U) \to \text{Der} A(U')\). It follows that, restricted to an open subset \(U\), every derivation \(u\) of \(A(Z)\) coincides with some local section \(u_U\) of \(V_E |_U \to U\), whose collection \(\{u_U, U \subset Z\}\) defines uniquely a global section of \(V_E \to Z\), called a graded vector field on \(Z\).
The $∧E^*$-dual $V_E^*$ of $V_E$ is a vector bundle over $Z$ whose sections constitute the $A(Z)$-module of exterior graded 1-forms $\phi = \phi_A dz^A + \phi_a dc^a$. Then the morphism $\phi : u \to A(Z)$ can be seen as the interior product

$$u \phi = u^A \phi_A + (-1)^{|\phi|} u^a \phi_a. \tag{20}$$

Graded $k$-forms $\phi$ are defined as sections of the graded exterior bundle $∧^k Z V_E^*$ such that

$$\phi \wedge \sigma = (-1)^{|\phi||\sigma|+|\phi||\sigma|} \sigma \wedge \phi,$$

where $|.|$ is the form degree. The interior product (20) is extended to higher graded forms by the rule

$$u \phi \wedge \sigma = (u \phi) \wedge \sigma + (-1)^{|\phi||\sigma|} \phi \wedge (u \sigma).$$

The graded exterior differential $d$ of graded functions is introduced by the condition $u df = u(f)$ for an arbitrary graded vector field $u$, and is extended uniquely to higher graded forms by the rules

$$d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma), \quad d \circ d = 0.$$

4 The Koszul–Tate resolution

Let us turn to the splitting (12a) and introduce the projection operators

$$\begin{align*}
P^\lambda_{ij} &= a_{ij}^{\mu} \sigma_{0 \mu}, \\
R_{ij}^\lambda &= (\delta^i_j \delta^\lambda - a_{ij}^{\mu} \sigma_{0 \mu}),
\end{align*}$$

such that

$$P_{ij}^\lambda R^\nu_k = 0, \quad P_{ij}^\lambda R^\nu_k = R^\lambda_i. \tag{21}$$

To construct the vector bundle $E$ of antighosts, let us consider the vertical tangent bundle $V Y \Pi$ of $\Pi \to Y$. Let us chose the bundle $E$ as the Whitney sum of the bundles $E_0 \oplus E_1$ over $\Pi$ which are the infinite Whitney sum over $\Pi$ of the copies of $V Y \Pi$. We have

$$E = V Y \Pi \oplus V Y \Pi \oplus \cdots.$$ 

This bundle is provided with the holonomic coordinates $(t, y^i, p_i^\lambda, \dot{p}_i^{(r)})$, $r = 0, 1, \ldots$, where $(t, y^i, p_i^\lambda, \dot{p}_i^{(2l)})$ are coordinates on $E_0$, while $(t, y^i, p_i^\lambda, \dot{p}_i^{(2l+1)})$ are those on $E_1$. By $r$ is meant the antighost number. The dual of $E \to V^*Q$ is

$$E^* = V^*_Y \Pi \oplus V^*_Y \Pi \oplus \cdots.$$
It is endowed with the associated fibre bases \( \{ c^\lambda_i^{(r)} \} \), \( r = 1, 2, \ldots \), such that \( c^\lambda_i^{(r)} \) have the same linear coordinate transformation law as the coordinates \( p_i^\lambda \). The corresponding graded vector fields and graded forms are introduced on \( \Pi \) as sections of the vector bundles \( V_E \) and \( V^*_E \), respectively.

The \( C^\infty(\Pi) \)-module \( \mathcal{A}(\Pi) \) of graded functions is graded by the antighost number as

\[
\mathcal{A}(\Pi) = \bigoplus_{r=0}^{\infty} \mathcal{N}^r, \quad \mathcal{N}^0 = C^\infty(\Pi).
\]

Its terms \( \mathcal{N}^r \) constitute a complex

\[ 0 \leftarrow C^\infty(\Pi) \leftarrow \mathcal{N}^1 \leftarrow \cdots \tag{22} \]

with respect to the Koszul–Tate differential

\[
\begin{align*}
\delta : C^\infty(V^*Q) &\to 0, \\
\delta(c^\lambda_i^{(2l)}) &= p_\mu^k c_k^{(2l-1)}, \quad l > 0, \\
\delta(c^\lambda_i^{(2l+1)}) &= R_\mu^k c_k^{(2l)}, \quad l > 0, \\
\delta(c^\lambda_i^{(1)}) &= R_\mu^k c_k.
\end{align*}
\]

The nilpotency property \( \delta \circ \delta = 0 \) of this differential is the corollary of the relations (21).

It is readily observed that the complex (22) with respect to the differential (23) has the homology groups

\[ H_{k>1} = 0, \quad H_0 = C^\infty(V^*Q)/I_{N_L} = C^\infty(N_L), \]

where \( I_{N_L} \) is an ideal of smooth functions on \( \Pi \) which vanishes on the Lagrangian constraint space \( H_L \). Thus, this is a desired Koszul–Tate resolution of the constraints (14) defined by the Lagrangian (4).

Note that, in different particular cases of the degenerate quadratic Lagrangian (4), the complex (22) may have a subcomplex, which is also the Koszul–Tate resolution. For instance, if the fibre metric \( a \) in \( VQ \to Q \) is diagonal with respect to a holonomic atlas of \( VQ \), the constraints (14) are irreducible and the complex (22) contains a subcomplex which consists only of the antighosts \( c^\lambda_i^{(1)} \).

References


