Non-Abelian Berry connections for quantum computation

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In the holonomic approach to quantum computation information is encoded in a degenerate eigenspace of a parametric family of Hamiltonians and manipulated by the associated holonomic gates. These are realized in terms of the non-abelian Berry connection and are obtained by driving the control parameters along adiabatic loops. We show how it is possible, for a specific model, to explicitly determine the loops generating any desired logical gate, thus producing a universal set of unitary transformations. In a multi-partite system unitary transformations can be implemented efficiently by sequences of local holonomic gates. Moreover a conceptual scheme for obtaining the required Hamiltonian family, based on frequently repeated pulses, is discussed, together with a possible process whereby the initial state can be prepared and the final one can be measured.

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The field of quantum information and computation (QC) [1] brings together ideas and techniques from very different areas ranging from fundamental quantum physics to solid-state engineering and computer science. QC synergistically benefits from all these contributions and conversely quite often offers fresh viewpoints on old subjects. Recently it has been suggested [2] that even tools related to gauge theories [3] might play a fruitful role in the arena of QC. Indeed in ref. [2] the possibility of realizing quantum information processing by using non-abelian Berry holonomies [4] induced by moving along suitable loops in a control space $M$ has been analysed. The computational capability stems from the features of the global geometry of the bundle of eigenspaces associated with a family $\mathcal{F}$ of Hamiltonians parametrized by points of $M$. The geometry is described by a non-trivial gauge potential $A$ or connection, with values in the algebra $u(n)$ of anti-hermitian matrices ($n$ is the dimension of the computational space). Since the unitary transformations realizing the computations are nothing but the holonomies associated with the connection $A$, this conceptual framework for QC is referred to as Holonomic Quantum Computation (HQC). In a sense HQC can be considered as the (continuous) differential-geometric counterpart of the (discrete) topological QC with anyons described in refs. [5,6].

In this paper we shall provide further analysis of this proposal. After concisely reviewing the conceptual basis of HQC, we shall show how, in a specific relevant model, one can explicitly determine the sequence of loops necessary for generating any given quantum gate. Then we shall introduce HQC models with a natural multi-partite structure and discuss how this bears on the question of complexity. Finally we shall discuss how in principle one can implement HQC by repeated pulse control of a system with degenerate spectrum.

Let us begin by recalling the basic ideas of HQC [2]. Quantum information is encoded in a $n$-fold degenerate eigenspace $C$ of a Hamiltonian $H_0$, with eigenvalue $\varepsilon_0$. Operator, $H_0$, belongs to a family $\mathcal{F} = \{H_\lambda\}_{\lambda \in \mathcal{M}}$, $H_0 = H_{\lambda_0}$, in which no energy level crossings occur as $\lambda$ ranges over $\mathcal{M}$. In the following we shall satisfy this latter condition by assuming, for simplicity, that the Hamiltonians $H_\lambda$ are isospectral ($H_\lambda = U(\lambda)H_0U(\lambda)^\dagger$). The $\lambda$'s represent the “control” parameters that one has to drive in order to manipulate the coding states $|\psi\rangle \in C$. In general the points of $\mathcal{M}$, from the physical point of view, can be thought of as describing external fields, such as electric or magnetic fields, or couplings between subsystems. Let $C$ be a loop in the control manifold $\mathcal{M}$, with base point $\lambda_0$, : $[0, 1] \mapsto \mathcal{M}$, $C(0) = C(1) = \lambda_0$. We assume that $C$ is traversed along slowly with respect to the longest dynamical time scale involved: in this case the evolution is adiabatic i.e., no transitions among different energy levels are induced. If $|\psi\rangle_{in} \in C$ is the initial state, at the end of this control process one gets $|\psi\rangle_{out} = e^{i\Gamma_\lambda} \Gamma_\lambda(C)|\psi\rangle_{in}$. The first factor here is just an overall dynamical phase and in the following it will be omitted; let us just mention that such a decoupling of the fast dynamical evolution opens new possibilities about coherent and error avoiding encoding [6]. The second contribution, the holonomy $\Gamma_\lambda(C) \in U(n)$, has a purely geometric origin and its appearance accounts for the non-triviality (curvature) of the bundle of eigenspaces over $\mathcal{M}$. By introducing the Wilczek-Zee connection [7]

$$A^\lambda_{\alpha\alpha} := \langle \psi^\alpha(\lambda) \bigg| \frac{\partial}{\partial \lambda^\mu} \bigg| \psi^\alpha(\lambda) \rangle,$$

one finds $\Gamma_\lambda(C) = \mathbf{P} \exp \int_0^1 A [4]$, where $\mathbf{P}$ denotes path ordering. The set Hol($A$) := $\{\Gamma_\lambda(C)\}_C \subset U(n)$ is known as the holonomy group [8]. In the case in which it coincides with the whole unitary group $U(n)$ the connection
A is called irreducible. In [2] it has been argued that for a large enough control manifold, the irreducible case is the generic one, therefore one can in principle implement any computation over the code $C$ just resorting to this very special class of quantum evolutions.

Quantum Gates. A workable HQC model which represents a natural non-abelian generalization of the original Berry phase, for which explicit construction of the holonomic gates is possible, is now discussed. The model is worked out with some details in that it is extendable to the more general case when $M$ is a coset space. The features of the construction presented are twofold. On the one hand it fully exploits the loop composition structure at the basis of the holonomy group, showing a procedure whereby loops can be decomposed into 2-dimensional components, simple to deal with. On the other this topological construction overcomes the difficulties connected with the path ordering prescription.

Let us consider the Hamiltonian $H_0 = \varepsilon_0(n + 1)$ acting on the state-space $H \cong \mathbb{C}^{n+1} = \text{span}([\alpha])_{\alpha=1}^{n+1}$. We shall take as the family $F$ the whole orbit $O(H_0) := \{H_0U | U \in U(n+1)\}$ of $H_0$ under the (adjoint) action of the unitary group $U(n+1)$. This orbit is isomorphic to the $n$-dimensional complex projective space:

$$O(H_0) \cong \frac{U(n+1)}{U(n) \times U(1)} \cong SU(n+1) \cong \mathbb{CP}^n.$$  

The points of $\mathbb{CP}^n$ can be parametrized by the unitary matrices $U(z) = U_1(z_1)U_2(z_2)\ldots U_n(z_n)$ where $U_\alpha(z_\alpha) = \exp[G_\alpha(z_\alpha)]$ with $G_\alpha(z_\alpha) = z_\alpha[\alpha | (n + 1) - z_\alpha| n + 1]$. $z_\alpha$ is a parameter of the rotated Hamiltonians $[\alpha | (n + 1) - z_\alpha| n + 1]$. The eigenstates of the rotated Hamiltonians are $[\alpha | (\theta, \phi)] := U(\theta, \phi)[\alpha] = \exp(-i\phi_\alpha) \sin \theta_\alpha \sum_\beta \sin \theta_\beta \cos \theta_\gamma \pi^2$ and $[\theta, \phi_\alpha]$ is given by (2). From (2) we see that we can always choose the parameters which define the position of the plane $(\theta_\beta, \lambda_\beta)$, where loop $C$ lies, in such a way that the matrix $A_\beta \phi$ is identically zero. If one takes $\theta_\beta = 0$, $\forall \beta \neq \beta$, matrices $A_\beta \phi$ and $A_\beta \phi$ commute, so that we can calculate the integral and exponentiate avoiding the path ordering problem.

In this framework, it is possible to identify first four families of loops in such a way as to produce the basis of four matrices (the Pauli matrices and the identity) of all possible two-by-two submatrices belonging in the algebra of $U(2)$. The first choice is $(\theta_\beta, \phi_\beta)$, where the non-zero component of the connection is $A_{\beta \beta} = -i \sin^2 \theta_\beta$. The second choice is the loop on submanifold $(\theta_\beta, \phi_\beta)$ for $\beta > \beta$, with $\theta_\beta = \pi/2$, giving a different connection with two non-zero elements, $A_{\beta \beta} = i \sin^2 \theta_\beta$ and $A_{\beta \beta} = -i$. Of course the latter element will give zero when integrated along a loop. For $\beta > \beta$ both matrices are identically zero, and give rise to trivial holonomy. With these two connections and for appropriate loops one can obtain all possible $U(n)$ diagonal transformations. For loop $C_1 \in (\theta_\beta, \phi_\beta)$, $\Gamma_A(C_1) = \exp[-i|\beta|/|\beta|\Sigma_1]$, $\Sigma_1$ denoting the area enclosed by $C_1$, on the $S^2$ sphere with coordinates $(2\theta_\beta, \phi_\beta)$. For $C_2 \in (\theta_\beta, \phi_\beta)$, $\Gamma_A(C_2) = \exp[i|\beta|/|\beta|\Sigma_2]$. Recalling the constraint $|\beta| > \beta$, we see that one can produce $n - 1$ distinct holonomies from $C_2$. To obtain the non-diagonal transformations one has to consider a loop on the $(\theta_\beta, \theta_\beta)$ plane, with $\theta_\beta = 0$ for all $i \neq \beta, \beta$. Then the only non-vanishing elements of the connection are $A_{\beta \beta} = \exp(-i \sin^2 \theta_\beta) \sin \theta_\beta$, $A_{\alpha \beta} = 0$. By choosing further the $(\theta_\beta, \theta_\beta)$ plane at $\phi_\beta = \phi_\beta = 0$ the holonomy becomes, for $C_3 \in (\theta_\beta, \theta_\beta)$, $\Gamma_A(C_3) = \exp[-i|\beta|/|\beta|\Sigma_3]$, where $\Sigma_3$ is the area on the plane with coordinates $(\pi/2, \theta_\beta, \phi_\beta)$. Note that any loop $C$ on the $(\theta_\lambda, \lambda_\lambda)$ plane with the same enclosed area (when mapped on the appropriate sphere) $\Sigma_C$ will give the same holonomy independent of its position and shape. These four holonomies restricted each time to a
specific $2 \times 2$ submatrix, generate all $U(2)$ transformations. Finally it is easy to check that in this way one can indeed obtain $U = \exp[i \mu_j T_j]$, where $T_j$ ($j = 1, \ldots , n^2$) is a $u(n)$ generator and $\mu_j$ an arbitrary real number. Therefore any element of $U(n)$ can be obtained by controlling the $2n$ parameters labelling the points of $\text{CP}^n$.

It is instructive to consider the form that the Hamiltonian family $\mathcal{F}$, takes when restricted to the particular 2-submanifolds. For the loop $C_1$ (similarly for $C_2$) one finds $H_1 = -\epsilon_0 \sqrt{2} \bar{B}(2 \theta_2, \phi_2) \cdot \bar{\sigma}$ for $\bar{B}(\theta_2, \phi_2) = (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2)^T$, where the only non-zero elements are on the $\beta$-th and $(n + 1)$-th row and column. $H_1$ generates an abelian $\text{CP}^2$ phase between the states $|\beta\rangle$ and $(n + 1)$. On the other hand for the path $C_3$ (similarly for $C_4$) we have $H_3 = \epsilon_1 \bar{B}(\theta_3, \phi_3) \bar{B}(\theta_2, \phi_2)^T$, where the non-zero elements connect the states $|\beta\rangle$, $|\beta\rangle$ and $(n + 1)$. In this Hamiltonian there is direct coupling between three states, giving rise to a non-abelian interaction.

In order to represent a 2-qubit system we have to consider the control manifold $\text{CP}^4$. The holonomies in this case are $4 \times 4$ matrices, and we take as a representation basis of the unitary transformations the qubit basis $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. From the general scheme above it follows that by appropriate control of the parameters $(\theta, \phi)$ for obtaining various loops $C$, we can generate all possible $U(4)$ rotations i.e., any logical gate, in particular single-qubit rotations, and two-qubit gates such as the controlled operations XOR and CROT. (ular single-qubit rotations, and two-qubit gates such as $\text{CROT}$.

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The proof of the latter proposition proceeds as follows. First one observes that the subspaces $C^n_{ij}$ are degenerate eigenspaces of $V_i(0)$ corresponding to the eigenvalues $\pm e$. Hence the $C^n_{ij}$'s are $2^n$-dimensional eigenspaces of $H(0)$ with eigenvalues $\pm (n - 1)/2$ by assumption. For any pair $(i, j)$ of subsystems $|\lambda\rangle$ one can generate any unitary transformation over $C^n_{ij}$ by adiabatic loops $\tau_{ij}$ in $\mathcal{M}_{ij}$. Keeping all the remaining $\lambda$'s at 0, one has a trivial action over the other factors of $C^n_{ij}$, while $\Gamma_A(\tau_{ij})\otimes_k |\alpha_k\rangle \otimes |\pm\rangle \mapsto \sum_{\beta, \beta' \alpha_i} U_{\beta, \beta'} A_{\alpha_i}(\tau_{ij}) \otimes_k |\tilde{\alpha}_k\rangle \otimes |\pm\rangle$, where $\tilde{\alpha}_k = \alpha_k$ for $k \neq i$, $j$ and $\tilde{\alpha}_k = \beta_k$ when $k = i, j$. In particular one can obtain a universal set of gates e.g., XORs $\text{XOR}_{ij}$, and single-qubit operations, by using a fixed amount of resources. The claim then follows by well-known universality results for QC [12]. The above scheme involves the use of an ancilla and requires controllability of three-body interactions, extremely difficult to achieve in practice. In this respect a simplification, involving just two-body interactions, can be obtained by considering $N$ subsystems with $d$-levels [13].
**Implementation.** Now we discuss how one could in principle implement the holonomic loops even when the parametric Hamiltonian family \( F \) (or a part of it) is not available from the outset. We shall resort to ideas of quantum control theory in a way quite similar to the one adopted for symmetrization procedures \[14\] and decoherence control in open quantum systems \[15\]. Suppose that an experimenter has at disposal the following resources:
i) A quantum system characterized by the Hamiltonian \( H_0 \) admitting a \( n \)-fold degenerate eigenspace \( C \); ii) The way to turn on and off a set of interactions very quickly (with respect to the time-scales associated with \( H_0 \)) in such a way that a set of unitary “kicks” \( K \) := \{ \( U_\lambda \) \( \lambda \in \mathcal{M} \) \} can be realized. Let \( T = N \Delta t \) and \( t_0 = 0 \), \( t_{i+1} = t_i + \Delta t (i = 1, \ldots, N - 1) \) be a partition of the time interval \([0, T]\). Now let the system evolution to be as follows: at any time \( t_i \) the experimenter kicks the system with the pulse \( U_{i+1} \cdot U_i \) where \( U_i := U(\lambda_i) \) is a unitary chosen from the set \( K \) \((U_0 = U_N = 1)\). Between the kicks the system evolution is unperturbed, \( U(\Delta t) = e^{-i H_0 \Delta t} \). The global evolution is then given by

\[ U_{N, \Delta t}(T) = T \prod_{i=1}^{N-1} U_i U(\Delta t) U_i^\dagger, \]

where \( T \) denotes time-ordering. By considering the limit \( \Delta t \to 0 \), \( N \to \infty \), with \( N \Delta t = T \) one gets \( U_{N, \Delta t}(T) \to \exp -i \int_0^T dt H(t) \) where \( H(t) := U(\lambda(t)) H_0 U(\lambda(t))^\dagger \). In particular by making the function \( \lambda(t) \) vary adiabatically, one can obtain the desired holonomic evolution. This scheme is based on a strong separation between time-scales: each \( U_\lambda \) has to be enacted impulsively whereas the characteristic variation time of the control parameters \( \lambda \) has to be slow enough to satisfy the adiabaticity requirement. More precisely, if \( \tau_k \) denotes the kicking time, \( \omega \) the (highest) frequency associated to the dynamics generated by \( H_0 \) and \( \tau_\lambda \) the time-scale over which the function \( \lambda(t) \) varies, one must have \( \tau_k \leq \Delta t \ll \omega^{-1} \ll \tau_\lambda \). Notice that the pulses \( U_\lambda \) are not required to be a universal set of gates for QC; here they represent an extra resource needed for implementing HQC when \( F \) is missing.

Finally let us briefly consider the problem of codeword preparation and measurement. In order to encode the initial state into the degenerate subspace \( C \), or to make the measurement on the final state, it would be useful to lift the degeneracy. Indeed in this way one would be able to distinguish energetically the different coding states. This characteristic is quite often desirable from the experimental point of view in that one can resort to procedures involving energy transitions with state-dependent frequencies. The idea is to lift the degeneracy between the coding states, for example by switching on coherently an external (generic) perturbation. The basis states \( |\psi_\alpha\rangle \) of \( C \) are mapped onto a set of states \( |\psi_\alpha'\rangle \) that are no longer energy degenerate. Preparation/measurement are then performed and eventually degeneracy is coherently restored by switching off the perturbation.

**Summary.** In this paper we have provided further analysis of the proposal for Holonomic Quantum Computation of ref. \[2\]. We have explicitly designed control-loops whose holonomies generate universal gates for a CP\(^*\) control parameter manifold. The basic idea is to associate to the \( (n)\)-generators computable transformations obtained by loops on 2-dimensional subspaces of the control manifold. Explicit realization of two qubit gates has been given with the indication of which particular loops the experimenter has to perform. In terms of such elementary holonomic gates we analysed the complexity problem and we showed how to achieve efficient implementation of quantum computing by resorting to a HQC model involving only local interactions. Some implementative issues have been addressed, and we devised a scheme based on repeated pulses for realizing the parametric family of isospectral Hamiltonians required for HQC. Finally, we briefly indicated how to prepare the initial state and how to measure the final one by coherently switching on and off the energetic degeneracy of the computational subspace.

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4. For a review see, Geometric Phases in Physics, A. Shapiro and F. Wilczek, Eds. World Scientific, 1989
5. A. Kitaev, LANL e-print archive quant-ph/9707021;
9. We thank H. Barnum for pointing out this possibility.
10. The notation \( X \in \text{End}(\mathcal{H}_a) \) means that \( X \) has non-trivial action only on the \( j \)-th factor of \( \mathcal{H} \).
11. When the system geometry is given, one can restrict to “local” gates without loss of efficiency.
For example one could consider $N$ qu-trits, i.e., $\mathcal{H}_j \cong \mathbb{C}^3 = \text{span}\{|\alpha\rangle_j / \alpha = 0, 1, 2\}$ such that $H_{ij}(\mu_{ij})$ admits a four-dimensional degenerate eigenspace $C_{ij} := \text{span}\{|\alpha\rangle_i \otimes |\beta\rangle_j / \alpha, \beta = 0, 1\} \subset \mathcal{H}_i \otimes \mathcal{H}_j \cong \mathbb{C}^3$. Assuming for the $H_{ij}(\mu_{ij})$’s the conditions stated in the text, HQC can be efficiently implemented over $\mathcal{C} := \text{span}\{\otimes_{i=1}^N |\alpha_i\rangle_i / \alpha_i = 0, 1\} \cong (\mathbb{C}^2)^{\otimes N}$. Notice that in this case the dimension of the physical state-space is increased $3^N$ instead of $2^{N+1}$.
