Uncertainty is complementary to Complementarity

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For any ideal two-path interferometer it is shown that the wave-particle duality of quantum mechanics implies Heisenberg’s uncertainty relation and vice versa. It is conjectured that complementarity and uncertainty are two aspects of the same general principle.

Bohr’s principle of complementarity, applied to a two-path interferometer, describes the nature of a quantum system as being ‘dualistic’ in its particle and wave aspects [1]. Although Bohr originally intended the principle of complementarity to be of greater generality than the wave-particle duality of quantum mechanics both concepts are now often treated as equivalent [1,2]. In the famous debates between Einstein and Bohr in the late 1920’s [1] complementarity was contested but finally the argument has settled in its favor [1,2]. The foundation of the concept of complementarity, however, is still argued upon [3–15]. Whereas Bohr invariably used different versions of Heisenberg’s uncertainty relation in refuting Einstein’s attempts to disprove the concept of complementarity [1], a theoretical proposal from 1991 [3] and an experiment performed last year [8] have been interpreted as showing that complementarity can arise without uncertainty [7–9]. The question therefore remains as to whether complementarity always arises from Heisenberg’s uncertainty principle [11–15] or whether a ‘more fundamental mechanism’ – entanglement without uncertainty – can be at work [6,7,9]?

To answer this question, optimal two-path interferometers are analyzed; this is all that is needed since we are only interested in fundamental physical limits, thus neglecting unbalanced, lossy, and otherwise imperfect setups. Moreover, two paths, described by two quantum mechanical modes, capture all the essential physics, including all double slit setups and all systems currently under discussion [3–15].

In a two-mode interferometer two ‘paths’ does not necessarily refer to spatially separated paths, it can, for instance, mean two different spin states in Ramsey interferometry [3,8]. Heisenberg’s position-momentum uncertainty should therefore not always be appropriate for the description of the loss of interference fringes when determining the path of a quantum particle [3,8,11]. This has recently been debated [3,5,11–15] but seems to be resolved now [8]: complementarity can be enforced without invoking position-momentum uncertainty.

We will, however, see that the assumption that path and wave measurements (\(\hat{P}\) and \(\hat{W}\)) are complementary always leads to the general Heisenberg–Robertson uncertainty relation [1,16,17]

\[
\Delta \hat{P} \cdot \Delta \hat{W} \geq \frac{1}{2} |\langle [\hat{P}, \hat{W}] \rangle|,
\]

and that the ensuing uncertainties are sufficient to make inaccessible either the path or the wave aspect of the quantum system: complementarity implies uncertainty and this uncertainty enforces complementarity.

In an optimal interferometer the two paths are identified with two quantum mechanical modes and can, without loss of generality, be assigned the two basis states of a formal spin-1/2 system \(|\psi_+\rangle = (1,0)\) and \(|\psi_-\rangle = (0,1)\). To scan the interference pattern the relative phase \(\Phi\) between these two basis states is changed by a phase shifter \(\Phi = \exp(-i \sigma_z / 2)\) without loss of generality [6]. When measuring the particle aspect, i.e. experimentally discriminating the paths, see Fig.1., the paths are assigned two different values \(p_\pm\). Using the above representation for \(|\psi_\pm\rangle\) we hence find that the path operator \(\hat{P}\) has the general form \(\hat{P} = p_+ |\psi_\rangle \langle \psi_+| + p_- |\psi_-\rangle \langle \psi_-|\) which, after suitable rescaling and without loss of generality, leads to our choice \(p_\pm = \pm 1\) and the customary form [6]

\[
\hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z,
\]

where the \(\sigma\)’s are the Pauli-matrices for spin-1/2. Analogous considerations hold for the operator \(\hat{W}\) describing the measurement of the wave features of the quantum particle. This is easiest seen from the fact that the determination of the interference pattern also amounts to a path measurement, \(\sigma_z^{\text{after}}\), after the final beamsplitter \(B\), see Fig.1. We conclude that, without loss of generality, \(\hat{W}\) has two eigenvalues \(w_\pm = \pm 1\) with associated eigenvectors \(|\omega_\pm\rangle\).

What is the general form of \(\hat{W}\) in terms of the basis vectors \(|\psi_\pm\rangle\) of \(\hat{P}\)? To find this connection between path and wave measurement all we need to use is the fact that they are complementary to each other:

"We say that two variables are ‘complementary’ if precise knowledge of one of them implies that all possible outcomes of measuring the other one are equally probable." [3]

With our convention \(p_\pm = \pm 1\) the expectation value of a path-measurement \(\hat{P}\), after a preceding measurement has
projected the particle into a wave-eigenstate $|\omega_{\pm}\rangle$, must therefore be zero

$$\langle \omega_{+} | \hat{P} | \omega_{+} \rangle = 0 = \langle \omega_{-} | \hat{P} | \omega_{-} \rangle.$$  \hspace{1cm} (3)

The complementary statement is

$$\langle \psi_{+} | \hat{W} | \psi_{+} \rangle = 0 = \langle \psi_{-} | \hat{W} | \psi_{-} \rangle.$$  \hspace{1cm} (4)

Both statements are in accord with other quantifications of complementarity suggested in recent years [18,19,6]. Using the orthonormality condition $\langle \omega_{+} | \omega_{-} \rangle = 0$ we can solve equation (3) and find

$$|\omega_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( \exp[-i\phi_0/2], \exp[i\phi_0/2] \right).$$  \hspace{1cm} (5)

The phase angle $\phi_0$ depends on the details of the interferometric setup. Note that the $|\omega_{\pm}\rangle$ are balanced, i.e., the particle is equally probably found in the upper or lower path of the interferometer. Only such states allow for maximum contrast of the interference pattern [19]. Consequently all states $|\phi\rangle$ we encounter in an optimal interferometer need to have this balanced form

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left( \exp[-i\phi/2], \exp[i\phi/2] \right).$$  \hspace{1cm} (6)

This also follows from the fact that a pure phase shift operation $\Phi$ applied to $|\omega_{\pm}\rangle$ can only generate states of the form (6). Using (5) we find

$$\hat{W} = \hat{W}_+ |w_+\rangle \langle w_+| + w_- |w_-\rangle \langle w_-|$$

$$= \cos \phi_0 \hat{\sigma}_z + \sin \phi_0 \hat{\sigma}_y.$$  \hspace{1cm} (8)

This form of $\hat{W}$ might look unfamiliar but it describes, what we would expect, a path measurement after the last state (6), this relation assumes the specific form

$$\Delta_\phi \hat{P} \cdot \Delta_\phi \hat{W} \geq |\sin(\phi - \phi_0)|.$$  \hspace{1cm} (11)

In separate calculations one can easily confirm that $\Delta_\phi \hat{P} = 1$ and $\Delta_\phi \hat{W} = |\sin(\phi - \phi_0)|$, a pictorial representation of these uncertainties is given in Fig.2. One might wonder whether Eq. (11) describes a valid Heisenberg uncertainty relationship since it gives a vanishing lower bound for states for which $\phi - \phi_0$ is an integer multiple of $\pi$. But this is no reason for worry, it reflects the well-known fact that this bound vanishes for eigenstates ($|\phi = |\omega_{\pm}\rangle$) of the considered observables [17]. For the important case of greatest interferometric sensitivity, when $|\delta(\hat{W})/\delta \phi| = |\delta \cos(\phi - \phi_0)/\delta \phi|$ is maximal, the uncertainty is maximal as well.

For any two-path interferometer we have derived the uncertainty relation that goes together with complementarity. Fig.2. shows that this uncertainty relation quantifies 'just what is needed': the variances have the magnitudes required to project the quantum state $|\phi\rangle$ into the eigenstates of the respective measurement. This projection uncertainty completely destroys the complementary information: when measuring the interference pattern all path knowledge is lost (3) and vice versa (4): uncertainty enforces complementarity.

Uncertainty relations (10) and (11) hold for any two-mode duality experiment. For the sake of generality we have not specified the coupling that entangles particle and detector, but, once the coupling is specified, relation (10) encourages us to search for other problem-specific uncertainty relations that allow us to get a deeper understanding of the respective mechanism [10,21] and even to discover new physics [11]. Our analysis also sheds a fresh light on recent discussions: in particular we find that the claim that complementarity could be enforced without the unavoidable 'measurement disturbances' characteristic of the principle of uncertainty [3,7,9] is not substantiated. Also the related idea that complementarity is a 'deeper principle' than uncertainty [6,7,9] is in disagreement with our present findings.

The analysis of complementarity and uncertainty presented here shows that for any two-path interferometer uncertainty and complementarity mutually imply each other; this leads me to the general conjecture that uncertainty and complementarity are two aspects of the same principle: Bohr's principle of complementarity is complementary to Heisenberg's uncertainty principle.

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[20] In principle the interference pattern measurement can be performed measuring \( \cos \phi_0 \hat{\sigma}_z + \sin \phi_0 \hat{\sigma}_x \) inside the interferometer rather than \( \hat{\sigma}_{\text{after}} \) after the second beam splitter, an analogous statement applies to \( \hat{P} \) [6].

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FIG. 1. A cartoon of a general two-mode interferometer: to create a balanced state (6) inside the interferometer the particle is, say, entering through the lower port of the first semi-transparent beam splitter \( B \). The subsequent phase-shifter \( \Phi \) allows us to scan the interference pattern which is measured by \( \hat{W} \), this is usually done after the second beam-splitter as a path measurement, \( \hat{\sigma}_{\text{after}} \), but could be done inside the interferometer [20]. Alternatively, the path of the particle can be determined, most conveniently by measuring \( \hat{P} = \hat{\sigma}_x \) inside the interferometer.

FIG. 2. A Bloch-sphere representation of the general balanced state \( |\phi\rangle \) of Eq. (6) in an optimal interferometer (we assume \( \phi_0 = 0 \) without loss of generality). Since \( |\phi\rangle \) is balanced it lies in the equatorial \( xy \)-plane. This is how the measurement uncertainties enforce complementarity:

If the path measurement \( \hat{P} \) is performed first the state \( |\phi\rangle \) is projected out of the \( xy \)-plane onto the \( z \)-axis (green arrows) and equally often ends up pointing ‘North’ or ‘South’. The ensuing variance of this process is unity.

If the interference measurement \( \hat{W} \) is performed first the state \( |\phi\rangle \) is projected onto the \( x \)-axis (red arrows), the variance then depends on the angle and equals \( |\sin \phi_0| \). In either case one measurement completely randomizes the outcome of the other, the variables are hence complementary.