Measurement–induced interference in an inhomogeneous gravitational field.

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Abstract

A very interesting quantum mechanical effect is the emergence of gravity–induced interference, which has already been detected. This effect also shows us that gravity is at the quantum level not a purely geometric effect, the mass of the employed particles appears explicitly in the interference expression. In this work we will generalize some previous results. It will be shown that the introduction of a second order approximation in the propagator of a particle, immersed in the Earth’s gravitational field, and whose coordinates are being continuously monitored, allows us to include, in the corresponding complex oscillator, a frequency which now depends on the geometry of the source of the gravitational field, a fact that is absent in the case of a homogeneous field. Using this propagator we will analyze the interference pattern of two particle beams whose coordinates are being continuously monitored. We will compare our results against the case of a homogeneous field, and also against the measurement outputs of the Colella, Overhauser, and Werner experiment, and find that the difference in the dependence upon the geometry of the source of the gravitational field could render detectable differences in their respective measurement outputs.

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1 Introduction.

Currently a very interesting and also controversial point comprises the role that gravity plays at the quantum level. We already know that at this level (here quantum level means the effect that a classical gravitational field has on a quantum particle, i.e., no quantum theory of gravity is contemplated here) gravity is not a purely geometric effect. Indeed, the experimentally detected gravity–induced interference pattern emerging from two particle beams depends explicitly on the mass of the employed particles [1]. This is not the case in classical mechanics, in this context mass does not appear in the motion equation of a particle trajectory, i.e., gravity in classical mechanics can be geometrized.

An additional fundamental problem in current physics comprises the so called quantum measurement problem. In connection with this conceptual difficulty, there are already many attempts which try to solve it, one of them is the so called restricted path integral formalism (RPIF) [2]. This formalism explains a continuous quantum measurement with the introduction of a restriction on the integration domain of the corresponding path integral. This last condition can also be reformulated in terms of a weight functional that has to be considered in the path integral.

This formalism has been employed in several situations, i.e., the analysis of the response of a gravitational wave antenna of Weber type [2], the measuring process of a gravitational wave in a laser–interferometer [3], or even to explain the emergence of the classical concept of time [4]. In the context of this model it is also important to add that there are already some theoretical predictions that could render a feasible framework which could allow us to confront RPIF against experimental outputs [5], but it seems also that more work is needed in this direction.

At this precise point we may join RPIF and the emergence of a gravity–induced interference. It has already been proved that if the position of two particle beams, immersed in a homogeneous gravitational field, is measured continuously, then a new testing framework for the theoretical predictions of RPIF is obtained [6].

In this work we will analyze the case of two particle beams, immersed in an inhomogeneous gravitational field, and consider the continuous monitoring of their positions. The idea here is to obtain a more realistic testing framework for RPIF, in which an experiment similar to Colella, Overhauser, and Werner (COW) case [1] could be performed. This more general situation could also shed some light on the controversy around the validity, at quantum level, of the equivalence principle [7], and also on the conceptual relation between quantum measurement theory and the weak equivalence principle [8].

2 Measurements and Inhomogeneous Gravity.

As has already been mentioned, in the attempts to solve the quantum measurement problem we may find RPIF [2]. This formalism explains a continuous quantum measurement with the introduction of a restriction on the integration domain of the corresponding path integral. This last condition can also be reformulated in terms of a weight functional that has to be considered in the path integral.

Let us explain this point a little bit better, and suppose that we have a particle which shows one–dimensional movement. The amplitude \( A(q'', q') \) for this particle to move from the point \( q' \) to the point \( q'' \) is called propagator. It is given by Feynman [9]

\[
A(q'', q') = \int d[q] e^{i\frac{i}{\hbar} S[q]}, \tag{1}
\]

here we must integrate over all the possible trajectories \( q(t) \), and \( S[q] \) is the action of the system, which is defined as

\[
S[q] = \int_{t'}^{t''} dt L(q, \dot{q}). \tag{2}
\]

Let us now suppose that we perform a continuous measurement of the position of this particle, such that we obtain as result of this measurement process a certain output \( a(t) \). In other words, the measurement process gives the value \( a(t) \) for the coordinate \( q(t) \) at each time \( t \), and this output has associated a certain error \( \Delta a \), which is determined by the experimental resolution of the measuring device. The amplitude \( A_\omega(q'', q') \) can be now thought of as a probability amplitude for the continuous measurement process to give the result \( a(t) \). Taking the square modulus of this amplitude allows us to find the probability density for different measurement outputs.

Clearly, the integration in the Feynman path–integral should be restricted to those trajectories that match with the experimental output. RPIF says that this condition can be introduced by means of a weight functional \( \omega_\omega[q] \) [2]. This means that expression (1) becomes now under a continuous measurement process

\[
A_\omega = \int d[q] \omega_\omega[q] e^{iS[q]} . \tag{3}
\]

The more probable the trajectory \( [q] \) is, according to the output \( a \), the bigger that \( \omega_\omega[q] \) becomes [2]. This means that the value of \( \omega_\omega[q] \) is approximately one for all trajectories \( [q] \) that agree with the measurement output \( a \), and it is almost 0 for those that do not match with the result of the experiment. Clearly, the weight functional
contains all the information about the interaction between measuring device and measured system.

Consider the case of a particle with mass \( m \) located in the Earth’s gravitational field. Then its Lagrangian has the form

\[
L = \frac{\vec{P}^2}{2m} + \frac{GMm}{r}.
\]

(4)

Let us now write \( r = R + l \), where \( R \) denotes the Earth’s radius and \( l \) the distance above the Earth’s surface. Under the condition \( R >> l \) we may approximate the Lagrangian, up to second order in \( l \), as follows

\[
L = \frac{\vec{P}^2}{2m} + \frac{GMm}{R} \left( 1 - \frac{l}{R} + \frac{l^2}{R^2} \right).
\]

(5)

Now consider a quantum particle, whose Lagrangian is given by (5), then its propagator, if the particle goes from point \( P \) to point \( Q \), is

\[
U(Q, \tau''; P, \tau') = \left( \frac{m}{2\pi\hbar T} \right) \exp\left\{ \frac{im}{\hbar T} \left[ (x_Q - x_P)^2 + (y_Q - y_P)^2 \right] \right\} \\
\times \int d[l(t)] \exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[ \frac{m}{2} \dot{l}^2 + \frac{GMm}{R} - \frac{GMm}{R^2} l + \frac{GMm}{R^3} l^2 \right] dt \right\},
\]

(6)

here \( \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \) denotes the projection on the Earth’s surface of the distance between \( Q \) and \( P \), and \( T = \tau'' - \tau' \).

Our path integral may be rewritten as follows

\[
\int d[l(t)] \exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[ \frac{m}{2} \dot{l}^2 + \frac{GMm}{R} - \frac{GMm}{R^2} l + \frac{GMm}{R^3} l^2 \right] dt \right\} = \\
\exp\left\{ i \frac{GMm}{\hbar R} T \right\} \int d[l(t)] \exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[ \frac{m}{2} \dot{l}^2 + F(t) l - \frac{m}{2} w^2 l^2 \right] dt \right\},
\]

(7)
being \( F(t) = -\frac{Gm}{R^2} \) and \( w^2 = -\frac{2GM}{R^2} \). Therefore the problem reduces to the calculation of the path integral of a driven harmonic oscillator, which has imaginary frequency \( w = i\sqrt{\frac{2GM}{R^2}} = i\Omega \) (\( \Omega \in \mathbb{R} \)), and we already know the result of this evaluation [10].

In our case we have

\[
\int d[l(t)] \exp \left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[ \frac{m}{2} \dot{l}^2 + F(t)l - \frac{m}{2} w^2 l^2 \right] dt \right\} = \\
\sqrt{\frac{m\Omega}{2\pi i\hbar \sinh(\Omega T)}} \exp \left\{ \frac{im\Omega}{2\hbar \sinh(\Omega T)} \left( (l_Q^2 + l_P^2) \cosh(\Omega T) \right) \right. \\
-2l_Q l_P + R[1 - \cosh(\Omega T)] \left[ l_Q + l_P - R \right] - \sqrt{\frac{GM R}{8} T \sinh(\Omega T)} \right\}. \tag{8}
\]

This allows us to rewrite the propagator

\[
U(Q, \tau''; P, \tau') = \left( \frac{m}{2\pi i\hbar T} \right)^\frac{1}{2} \sqrt{\frac{\Omega T}{\sinh(\Omega T)}} \exp \left\{ \frac{im}{2\hbar T} \left[ (x_Q - x_P)^2 + (y_Q - y_P)^2 \right] \right\} \\
\times \exp \left\{ \frac{iGMm}{hR} T \right\} \exp \left\{ \frac{im\Omega}{2\hbar \sinh(\Omega T)} \left( (l_Q^2 + l_P^2) \cosh(\Omega T) \right) \right. \\
-2l_Q l_P + R[1 - \cosh(\Omega T)] \left[ l_Q + l_P - R \right] - \sqrt{\frac{GM R}{8} T \sinh(\Omega T)} \right\}. \tag{9}
\]

Let us now introduce a measuring process, namely we will monitor continuously the \( l \)-coordinate of the particle. Then expression (6) becomes now

\[
U_{[\alpha(t)]}(Q, \tau''; P, \tau') = \left( \frac{m}{2\pi i\hbar T} \right)^\frac{1}{2} \sqrt{\frac{\Omega T}{\sinh(\Omega T)}} \exp \left\{ \frac{im}{2\hbar T} \left[ (x_Q - x_P)^2 + (y_Q - y_P)^2 \right] \right\} \\
\times \exp \left\{ \frac{iGMm}{hR} T \right\} \int d[l(t)] w_{[\alpha(t)]}[l(t)] \exp \left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[ \frac{m}{2} \dot{l}^2 + F(t)l - \frac{m}{2} w^2 l^2 \right] dt \right\}. \tag{10}
\]

The modulus square of (10) gives the probability of obtaining as measurement output (for the \( l \)-coordinate) function \( \alpha(t) \). The weight functional \( w_{[\alpha(t)]}[l(t)] \) contains
the information concerning the measurement, and is determined by the experimental construction [2].

At this point, in order to obtain theoretical predictions, we must choose a particular expression for \(w[\alpha(t)][l(t)]\). We know that the results coming from a Heaveside weight functional [11] and those coming from a gaussian one [12] coincide up to the order of magnitude. These last remarks allow us to consider a gaussian weight functional as an approximation of the correct expression. But a sounder justification of this choice comes from the fact that there are measuring processes in which the weight functional has precisely a gaussian form [13]. In consequence we could think about a measuring device whose weight functional is very close to a gaussian behavior.

Therefore we may now choose as our weight functional the following expression

\[
\omega[\alpha(t)][l(t)] = \exp\left\{-\frac{2}{T\Delta \alpha^2} \int_{\tau'}^{\tau''} [l(t) - \alpha(t)]^2 dt\right\},
\]

(11)

here \(\Delta \alpha\) represents the error in our measurement.

Hence, with the introduction of a continuous quantum measurement the new propagator is

\[
U[\alpha(t)](Q, \tau''; P, \tau') = \left( \frac{m}{2\pi \hbar T} \right) \exp\left\{ \frac{im}{2\hbar T} \left[(x_Q - x_P)^2 + (y_Q - y_P)^2\right] \right\}
\]

\[
\times \exp\left\{ i \frac{GMm}{\hbar R} T \right\} \exp\left\{ -\frac{2}{T\Delta \alpha^2} \int_{\tau'}^{\tau''} [l(t) - \alpha(t)]^2 dt\right\}
\]

\[
\times \int d[l(t)] \exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[\frac{m}{2} \dot{l}^2 + F(t)l - \frac{m}{2} \omega^2 l^2\right] dt\right\}.
\]

(12)

Once again we find the case of a driven harmonic oscillator, but now the involved forces and frequencies have a nonvanishing imaginary part. Besides, the driving force term is not constant, this nontrivial time dependence arises from the presence of the term \(\frac{4\hbar \alpha(t)}{T\Delta \alpha^2}\) (see below).

(12) may be rewritten as follows

\[
U[\alpha(t)](Q, \tau''; P, \tau') = \left( \frac{m}{2\pi \hbar T} \right) \exp\left\{ \frac{im}{2\hbar T} \left[(x_Q - x_P)^2 + (y_Q - y_P)^2\right] \right\}
\]

\[
\times \exp\left\{ i \frac{GMm}{\hbar R} T \right\} \exp\left\{ -\frac{2}{T\Delta \alpha^2} \int_{\tau'}^{\tau''} \alpha^2(t) dt\right\}
\]

\[
\times \int d[l(t)] \exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau''} \left[\frac{m}{2} \dot{l}^2 + \tilde{F}(t)l - \frac{m}{2} \omega^2 l^2\right] dt\right\}.
\]

(13)
In this last expression we have introduced the following definitions \( \hat{F}(t) = \frac{-GMm}{R^2} \) and \( \hat{\omega}^2 = \frac{-2GM}{R^3} - \frac{4\hbar}{mT\Delta\alpha^2} \).

After a lengthy calculation we obtain the propagator for a particle whose \( l \)-coordinate has been continuously monitored.

\[
U_{[\alpha(t)]}(Q, \tau''; P, \tau') = \left( \frac{m}{2\pi i \hbar T} \right)^\frac{3}{2} \sqrt{\frac{\hat{\Omega}T}{\sinh(\hat{\Omega}T)}} \exp\left\{ \frac{im\hat{\Omega}}{2\hbar \sinh(\hat{\Omega}T)} \int_{\tau'}^{\tau''} \alpha^2(t) dt \right\} \exp\left\{ \frac{2iGMm\bar{h}\Delta\alpha^2}{\hbar T} \right\} \times \exp\left\{ \frac{2l_Ql_P}{R} \right\} \left( \frac{R^3}{2GM(1 + \gamma)} \right)^\frac{1}{2} \int_{\tau'}^{\tau''} \frac{R^3}{2GM(1 + \gamma)} \left( \int_{\tau'}^{\tau''} F^{(1)}(\tau'', \tau') \, d\tau \right) + \frac{4\hbar}{T\Delta\alpha^2m} \left( \frac{R^3}{2GM(1 + \gamma)} \right)^\frac{3}{2} \left( \int_{\tau'}^{\tau''} F^{(2)}(\tau'', \tau') \, d\tau \right) - \int_{\tau''}^{\tau'} \frac{R^3}{2GM(1 + \gamma)} \left( \int_{\tau'}^{\tau''} d\tau' \right) \right\}. \quad (14)
\]

In this last expression we have \( \hat{\Omega} = \sqrt{\frac{2GM}{R^3}(1 + \gamma)} \), \( \gamma = \frac{2\hbar R^3}{GMmT\Delta\alpha^2} \), \( F^{(2)}(\tau'', \tau') = \int_{\tau''}^{\tau} \alpha(\tau) \sinh(\hat{\Omega}[(\tau'' - \tau)]) \, d\tau \), \( F^{(1)}(\tau'', \tau') = \int_{\tau''}^{\tau} \alpha(\tau) \sinh(\hat{\Omega}[\tau - \tau']) \, d\tau \), \( F^{(3)}(\tau'', \tau') = \int_{\tau''}^{\tau} d\tau' \int_{\tau''}^{\tau} d\alpha(\tau) \alpha(s) \sinh(\hat{\Omega}[(\tau'' - \tau')]) \sinh(\hat{\Omega}[s - \tau']) \), and finally we have also introduced \( F^{(4)}(\tau'', \tau') = \int_{\tau''}^{\tau} \alpha(\tau) \sinh(\hat{\Omega}[\tau'' - \tau]) \cosh(\hat{\Omega}[\tau - \tau']) \, d\tau \).

Let us now suppose that we have two particles, which start at point \( P \) and are detected at point \( Q \). The \( l \)-coordinate of both particles is going to be continuously monitored.
monitored, and in this measuring process we use measuring devices with different experimental resolutions, in other words, we have not only two different trajectories, $\alpha(t)$ and $\beta(t)$, as measurement outputs, but we also have $\Delta \alpha \neq \Delta \beta$.

Under these conditions the emerging interference pattern is obtained using two expressions like that given in (14).

The resulting interference pattern can be written as follows

$$I = I_1 + I_2 + I_3 + I_4 + I_5. \quad (15)$$

Here we have that

$$I_1 = \frac{m}{2\hbar} (l_Q^2 + l_P^2) \left[ \frac{\tilde{\Omega} \sin(2\tilde{\Omega}T) - \tilde{\Omega} \sinh(2\tilde{\Omega}T)}{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)} - \frac{\tilde{\Gamma} \sin(2\tilde{\Gamma}T) - \tilde{\Gamma} \sinh(2\tilde{\Gamma}T)}{\cosh(2\tilde{\Gamma}T) - \cos(2\tilde{\Gamma}T)} \right], \quad (16)$$

$$I_2 = -2l_Ql_P \frac{m}{\hbar} \left[ -\tilde{\Omega} \sinh(\tilde{\Omega}T) \cos(\tilde{\Omega}T) + \tilde{\Omega} \cosh(\tilde{\Omega}T) \sin(\tilde{\Omega}T) \right. \frac{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)}{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)} - \left. \tilde{\Gamma} \sinh(\tilde{\Gamma}T) \cos(\tilde{\Gamma}T) + \tilde{\Gamma} \cosh(\tilde{\Gamma}T) \sin(\tilde{\Gamma}T) \right] \frac{\cosh(2\tilde{\Gamma}T) - \cos(2\tilde{\Gamma}T)}{\cosh(2\tilde{\Gamma}T) - \cos(2\tilde{\Gamma}T)}$$

$$+ \frac{mT}{2\hbar} \sqrt{\frac{GMR}{8}} \left[ [1 + \tilde{\gamma}^2]^{-3/4} [-\tilde{\Omega} \cos\left(\frac{3}{2} \arctan(\tilde{\gamma})\right) + \tilde{\Omega} \sin\left(\frac{3}{2} \arctan(\tilde{\gamma})\right)] - [1 + \tilde{\eta}^2]^{-3/4} [-\tilde{\Gamma} \cos\left(\frac{3}{2} \arctan(\tilde{\eta})\right) + \tilde{\Gamma} \sin\left(\frac{3}{2} \arctan(\tilde{\eta})\right)] \right], \quad (17)$$

$$I_3 = \frac{m}{\hbar} \left[ -\tilde{\Omega} \sinh(\tilde{\Omega}T) \cos(\tilde{\Omega}T) + \tilde{\Omega} \cosh(\tilde{\Omega}T) \sin(\tilde{\Omega}T) \right. \frac{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)}{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)} \times \left[ [1 - \cosh(\tilde{\Omega}T) \cos(\tilde{\Omega}T)] \frac{R}{1 + \tilde{\gamma}^2} [l_Q + l_P - \frac{R(1 - \tilde{\gamma}^2)}{2(1 + \tilde{\gamma}^2)}] \right. \right.$$

$$+ \frac{R\tilde{\eta}}{1 + \tilde{\gamma}^2} \left[ \frac{R}{1 + \tilde{\gamma}^2} - (l_Q + l_P) \sinh(\tilde{\Omega}T) \sin(\tilde{\Omega}T) \right]$$

$$+ \left. \frac{m}{\hbar} \left[ \tilde{\Omega} \cosh(\tilde{\Omega}T) \sin(\tilde{\Omega}T) + \tilde{\Omega} \sinh(\tilde{\Omega}T) \cos(\tilde{\Omega}T) \right. \frac{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)}{\cosh(2\tilde{\Omega}T) - \cos(2\tilde{\Omega}T)} \right.$$

$$\right. \left. \times \left[ 1 - \cosh(\tilde{\Omega}T) \cos(\tilde{\Omega}T) \right] \frac{R}{1 + \tilde{\eta}^2} [l_Q + l_P - \frac{R(1 - \tilde{\eta}^2)}{2(1 + \tilde{\eta}^2)}] \right. \right.$$
\[
\times \left[1 - \cosh(\tilde{\Omega} T) \cos(\tilde{\Omega} T)\right] \frac{R \tilde{\gamma}}{1 + \tilde{\gamma}^2} \left[\frac{R}{1 + \tilde{\gamma}^2} (l_Q + l_P)\right]
- \frac{R}{1 + \tilde{\gamma}^2} [l_Q + l_P - \frac{R(1 - \tilde{\gamma}^2)}{2(1 + \tilde{\gamma}^2)}] \sinh(\tilde{\Omega} T) \sin(\tilde{\Omega} T)\right]
- \frac{m}{\hbar} \left[-\tilde{\Gamma} \cosh(\tilde{\Gamma} T) \sin(\tilde{\Gamma} T) + \tilde{\Gamma} \cosh(\tilde{\Gamma} T) \sin(\tilde{\Gamma} T)\right] \cosh(2\tilde{\Gamma} T) - \cos(2\tilde{\Gamma} T)
\times \left[1 - \cosh(\tilde{\Gamma} T) \cos(\tilde{\Gamma} T)\right] \frac{R \tilde{\gamma}}{1 + \tilde{\gamma}^2} [l_Q + l_P - \frac{R(1 - \tilde{\gamma}^2)}{2(1 + \tilde{\gamma}^2)}]
+ \frac{R \tilde{\eta}}{1 + \tilde{\eta}^2} [l_Q + l_P - \frac{R}{1 + \tilde{\eta}^2}] \sinh(\tilde{\Gamma} T) \sin(\tilde{\Gamma} T)\right]
+ \frac{m}{\hbar} \left[\tilde{\Gamma} \cosh(\tilde{\Gamma} T) \sin(\tilde{\Gamma} T) + \tilde{\Gamma} \cosh(\tilde{\Gamma} T) \cos(\tilde{\Gamma} T)\right] \cosh(2\tilde{\Gamma} T) - \cos(2\tilde{\Gamma} T)
\times \left[1 - \cosh(\tilde{\Gamma} T) \cos(\tilde{\Gamma} T)\right] \frac{R \tilde{\gamma}}{1 + \tilde{\gamma}^2} [l_Q + l_P - \frac{R}{1 + \tilde{\gamma}^2}]
- \frac{R}{1 + \tilde{\gamma}^2} [l_Q + l_P - \frac{R(1 - \tilde{\gamma}^2)}{2(1 + \tilde{\gamma}^2)}] \sinh(\tilde{\Gamma} T) \sin(\tilde{\Gamma} T)\right].
\]

Here \(\tilde{\Omega} = \sqrt{\frac{2GM}{R^3}} [1 + \tilde{\gamma}^2]^{1/4} \cos\left(\frac{1}{2} \arctan(\tilde{\gamma})\right)\), \(\tilde{\Omega} = \sqrt{\frac{2GM}{R^3}} [1 + \tilde{\gamma}^2]^{1/4} \sin\left(\frac{1}{2} \arctan(\tilde{\gamma})\right)\), with \(\tilde{\gamma} = \frac{2hR^4}{GMmT\Delta^2}\). Also \(\tilde{\Gamma} = \sqrt{\frac{2GM}{R^3}} [1 + \tilde{\eta}^2]^{1/4} \cos\left(-\frac{1}{2} \arctan(\tilde{\eta})\right)\), and \(\tilde{\Gamma} = \sqrt{\frac{2GM}{R^3}} [1 + \tilde{\eta}^2]^{1/4} \sin\left(-\frac{1}{2} \arctan(\tilde{\eta})\right)\), with \(\tilde{\eta} = \frac{2hR^4}{GMmT\Delta^2}\).

The fourth term is

\[
I_4 = \frac{8}{T\Delta^2} \left[\frac{\tilde{\Omega} \cosh(\tilde{\Omega} T) \sin(\tilde{\Omega} T) + \tilde{\Omega} \sinh(\tilde{\Omega} T) \cos(\tilde{\Omega} T)}{\cosh(2\tilde{\Omega} T) - \cos(2\tilde{\Omega} T)}\right]
\times \left[\frac{\tilde{\Omega}}{\tilde{\Omega}^2 + \tilde{\Omega}^2} [l_Q f^{(1)}(\tau'', \tau') + l_P f^{(1)}(\tau'', \tau')]\right]
+ \frac{\tilde{\Omega}}{\tilde{\Omega}^2 + \tilde{\Omega}^2} [l_Q f^{(2)}(\tau'', \tau') + l_P f^{(2)}(\tau'', \tau')]\right]
+ \frac{8}{T\Delta^2} \left[\frac{-\tilde{\Omega} \sinh(\tilde{\Omega} T) \cos(\tilde{\Omega} T) + \tilde{\Omega} \cosh(\tilde{\Omega} T) \sin(\tilde{\Omega} T)}{\cosh(2\tilde{\Omega} T) - \cos(2\tilde{\Omega} T)}\right]
\times \left[\frac{\tilde{\Omega}}{\tilde{\Omega}^2 + \tilde{\Omega}^2} [l_Q f^{(1)}(\tau'', \tau') + l_P f^{(1)}(\tau'', \tau')]\right]
- \frac{\tilde{\Omega}}{\tilde{\Omega}^2 + \tilde{\Omega}^2} [l_Q f^{(2)}(\tau'', \tau') + l_P f^{(2)}(\tau'', \tau')]\right].
\]
\[-\frac{8}{T\Delta\beta^2} \left[ \frac{\hat{\Gamma} \cosh(\hat{\Gamma}T) \sin(\hat{\Gamma}T) + \hat{\Gamma} \sinh(\hat{\Gamma}T) \cos(\hat{\Gamma}T)}{\cosh(2\hat{\Gamma}T) - \cos(2\hat{\Gamma}T)} \right]
\times \left[ \frac{\hat{\Gamma}}{\Gamma^2 + \hat{\Gamma}^2} \left[ l_Q f^{(3)}(\tau'', \tau') + l_P \tilde{f}^{(3)}(\tau'', \tau') \right] + \frac{\hat{\Gamma}}{\Gamma^2 + \hat{\Gamma}^2} \left[ l_Q f^{(4)}(\tau'', \tau') + l_P \tilde{f}^{(4)}(\tau'', \tau') \right] \right]. \tag{19}\]

Here we have defined \( f^{(1)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \alpha(\tau) \sinh(\hat{\Omega}[\tau - \tau']) \cos(\hat{\Omega}[\tau - \tau']) d\tau \). Also \( \tilde{f}^{(1)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \alpha(\tau) \sinh(\hat{\Omega}[\tau'' - \tau']) \cos(\hat{\Omega}[\tau'' - \tau']) d\tau \). Additionally we have introduced \( f^{(2)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \alpha(\tau) \cos(\hat{\Omega}[\tau'' - \tau']) \sin(\hat{\Omega}[\tau - \tau']) d\tau \). Similarly \( \tilde{f}^{(2)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \alpha(\tau) \cos(\hat{\Omega}[\tau'' - \tau']) \sin(\hat{\Omega}[\tau'' - \tau']) d\tau \).

Concerning the remaining definitions we have \( f^{(3)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \beta(\tau) \sinh(\hat{\Gamma}[\tau - \tau']) \cos(\hat{\Gamma}[\tau - \tau']) d\tau \). Also \( \tilde{f}^{(3)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \beta(\tau) \sinh(\hat{\Gamma}[\tau'' - \tau']) \cos(\hat{\Gamma}[\tau'' - \tau']) d\tau \). In addition we have employed \( f^{(4)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \beta(\tau) \cosh(\hat{\Gamma}[\tau - \tau']) \sin(\hat{\Gamma}[\tau - \tau']) d\tau \). Finally, \( \tilde{f}^{(4)}(\tau'', \tau') = \int_{\tau''}^{\tau''} \beta(\tau) \cosh(\hat{\Gamma}[\tau'' - \tau']) \sin(\hat{\Gamma}[\tau'' - \tau']) d\tau \).

The last contribution to the interference term reads

\[
I_5 = Re \left[ \frac{im\hat{\Omega}}{2h \sinh(\Omega T)} \left( \frac{4h}{T\Delta\alpha^2 m} \right)^2 \frac{R^3}{GM(1 + i\hat{\gamma})} F^{(3)}(\tau'', \tau') \right] + Re \left[ -\frac{2R\hat{\Omega}}{T\Delta\alpha^2 \sinh(\Omega T)} \sqrt{\frac{R^3}{2GM(1 + i\hat{\gamma})}} \left[ F^{(2)}(\tau'', \tau') - F^{(4)}(\tau'', \tau') \right] - \int_{\tau'}^{\tau''} F^{(1)}(\tau, \tau') \sinh(\hat{\Omega}(\tau'' - \tau)) d\tau + \frac{2R\hat{\Gamma}}{T\Delta\beta^2 \sinh(\Gamma T)} \right] \times \sqrt{\frac{R^3}{2GM(1 - i\hat{\gamma})}} \left[ \tilde{F}^{(2)}(\tau'', \tau') - \tilde{F}^{(4)}(\tau'', \tau') \right].
\]
\[-\int_{\tau'}^{\tau''} \tilde{F}^{(1)}(\tau, \tau') \sinh(\hat{\Gamma}(\tau'' - \tau))d\tau]. \quad (20)\]

Here \(\tilde{F}^{(2)}(\tau'', \tau') = \int_{\tau''}^{\tau'} \beta(\tau) \sinh(\hat{\Gamma}[\tau'' - \tau])d\tau, \tilde{F}^{(1)}(\tau'', \tau') = \int_{\tau''}^{\tau'} \beta(\tau) \sinh(\hat{\Gamma}[\tau - \tau'])d\tau, \tilde{F}^{(3)}(\tau'', \tau') = \int_{\tau''}^{\tau'} d\tau \int_{\tau'}^{\tau''} ds \beta(\tau) \beta(s) \sinh(\hat{\Gamma}[\tau'' - \tau]) \sinh(\hat{\Gamma}[s - \tau']),\) and finally we have also introduced \(\tilde{F}^{(4)}(\tau'', \tau') = \int_{\tau''}^{\tau'} \beta(\tau) \sinh(\hat{\Gamma}[\tau'' - \tau]) \cosh(\hat{\Gamma}[\tau - \tau'])d\tau,\) being \(\hat{\Gamma} = \hat{\Gamma} + i\hat{\Gamma}^\prime.\)

### 3 Conclusions.

Our results are a generalization (in the present work we have an inhomogeneous gravitational field) of expression (14) of reference [6] (which was deduced for the case of a homogeneous field).

Let us now analyze the effects that an inhomogeneous gravitational field has upon the emerging interference pattern. In order to do this, as a first step, we compare expressions (16), (17), and (18) with the corresponding ones in the case of a homogeneous field, expressions (15) and (16) in [6]. Clearly, if we consider \(I_1, I_2,\) and \(I_3\) as functions of the resolutions of the measuring devices, then we may see that the introduction of an inhomogeneous field renders an interference pattern that changes more rapidly (as function of these resolutions) than in the case of a homogeneous situation, this feature could be, in principle, detected.

We may understand this new characteristic noting that in the case of a homogeneous field the “frequency” of the complex harmonic oscillator is purely imaginary \((\omega = \sqrt{-\frac{GM}{mR^3}}).\) The case of an inhomogeneous gravitational field endows the corresponding “frequency” with a non–vanishing real part \((\hat{\omega} = \sqrt{-\frac{2GM}{R^3} - \frac{4\hbar}{mT\Delta\alpha^2}}).\) This means that the phase of each particle has a new dependence, and in consequence the phase shift, between the beams, that at the end appears, must contain this new information. In other words, the difference in the interference pattern between the present work and [6] stems from the presence of a non–vanishing real term in the “frequency” of the oscillator, expression (13).

The introduction of an inhomogeneous gravitational field does not imply the presence of a new term in the driving force for our oscillator. Indeed, if we take a look at the corresponding expression for the inhomogeneous situation, \(\hat{F}(t) = -\frac{GMm}{R^3} - \frac{4\hbar g(t)}{T\Delta\alpha^2},\) and compare it with the force in [6], then, remembering that \(g = \frac{GM}{R^2},\) we find that they are the same. This last remark also shows us that in the homogeneous case the information concerning the geometry of the source of the gravitational field
comes from the driving force term in the corresponding oscillator, and appears only in expressions (17), (18), and (19). The two first contributions to the interference term ((15) and (16) in [6]) do not contain any information about the geometry of the source.

But, in this new case case, we see that now all terms do contain this information, for instance,  \( \tilde{\Omega} \) and \( \tilde{\Omega} \) are now functions of \( \frac{2hR^3}{GMmT\Delta \alpha^2} \), i.e., the radius of the Earth appears in this expression (in the homogeneous case the respective parameter is a function of \( \frac{2\pi \hbar T}{m\Delta \alpha^2} \)). This is a trivial remark, nevertheless renders an important difference between the predictions of the homogeneous and inhomogeneous cases. Indeed, if we restrict ourselves at this point to expressions \( I_1 \), \( I_2 \), and \( I_3 \), then we may see that if we carry out the experiment not on the Earth’s surface, but at a certain height \( \tilde{R} \) above this surface, the inhomogeneous case predicts a change in the interference pattern (under this new condition we would have that \( \tilde{\Omega} \) and \( \tilde{\Omega} \) are now functions of \( \frac{2h(R+\tilde{R})^3}{GMmT\Delta \alpha^2} \)). This change in the interference pattern appears even if all the remaining involved parameters do not suffer any modification. In the homogeneous case, \( I_1 \) and \( I_2 \) do not suffer any modification under with change, because the “frequency” of the corresponding oscillator does not contain any information about the geometry of the source of the gravitational field.

Therefore, we may state that the difference in these two cases resides in the fact that in the extension that comprises the inhomogeneous field we are able to include in the “frequency” of the complex oscillator the effects of the geometry of the source of the gravitational field, something that is not possible in the first case.

Let us now compare our predictions with the measurement outputs of the COW experiment. If we analyze COW, taking into account also an inhomogeneous field, then we find that the interference pattern is given by \( I = \cos\left\{-\frac{2\mu^2Ll_{a}}{R^2} \left[1 - \frac{\mu}{R}\right]\right\} \), here \( \Lambda \) denotes the initial reduced wavelength of the packets, \( L \) the horizontal separation between starting point and detection point, and finally \( l_{a} \) is the vertical separation between these two points. The term \( \frac{\mu}{R} \) does not appear in COW, but it can be deduced if we include an inhomogeneous field [14]. The dependence of the cosine function upon the distance to the center of the Earth is given by \( \frac{GMm^2Ll_{a}\Lambda}{\hbar^2R^2} \left[1 - \frac{\mu}{R}\right] \). Clearly, expressions (16), (17), and (18) contain a very different dependence on \( R \). We have two very different mathematical dependences on the geometry of the source. In principle, this difference could be detected.

Concerning the feasibility of an experiment with these characteristics, for the time being, it seems that the present technology can not cope with the experimental difficulties that this proposal implies [15]. Nevertheless, in order to have a very rough estimation of the possible order of magnitude of the emerging effects that we could expect, let us at this point suppose that this kind of experiments could be carried out.
with the same accuracy that nowadays exist in the case of Paul traps [16], in which an individual ion is trapped employing a high-frequency electric quadrupole. If we take a look at expressions (16), (17), and (18) we will see that one of the points that could be important in this feasibility is the order of magnitude of $\bar{\Gamma}$, $\bar{\Omega}$, $\tilde{\Gamma}$, and also of $\tilde{\Omega}$. The bound for these parameters is going to be determined by $\sqrt{\frac{2GM}{R^3}}[1 + \gamma^2]^{1/4}$ and $\sqrt{\frac{2GM}{R^3}}[1 + \tilde{\eta}^2]^{1/4}$. Here the resolution of the measuring device is the most important factor ($\tilde{\eta} = \frac{2\hbar R^3}{GMmT\Delta \beta^2}$ and $\tilde{\gamma} = \frac{2\hbar R^3}{GMmT\Delta \alpha^2}$). If we assume the resolution of a typical Paul trap [17], $\Delta \alpha \sim 2 \mu m$, then we may see that $\sqrt{\gamma} \sim 10^5$ (here we consider, as in the COW experiment, thermal neutrons), in consequence the bound is $\sim 10^2 s^{-1}$.

Though this is a very rough estimation of $\tilde{\gamma}$ (also of $\tilde{\eta}$) it seems to claim that if the development of the current technology associated with Paul and Penning traps could, in a future, measure continuously the position of individual atoms, then this kind of proposals could render non-trivial measurement outputs.

It is also clear that even if we had the case $\alpha(t) = \beta(t)$, the condition $\Delta \alpha \neq \Delta \beta$ would render a non-vanishing interference pattern. This resolution-induced interference is a new feature in this context and renders an additional, and also less restricted, testing framework for the theoretical predictions of RPIF. Remembering the COW experiment [1] (where interference vanishes if the particles follow the same trajectory) we may conclude that the introduction of a measuring process implies the appearance, in the interference term, of the mass parameter, even if the condition $\alpha(t) = \beta(t)$ is fulfilled. This last result is, qualitatively, very similar to the behavior of the master equation of a freely falling particle, namely, the presence of a measurement coupling implies the appearance of the mass of the corresponding particles as a crucial factor in the dynamics of the system [8].

We shall not forget that our main result was deduced with the introduction of one approximation ($l < R$), and therefore can not have general validity. The general result, which will be published elsewhere, is important, not only because it could allow the introduction of less restricted experimental proposals, but also because it could render a more profound understanding of the conceptual relation between quantum measurement and the weak equivalence principle [8]. At the same time it could also shed some light on the controversy around the validity, at quantum level, of the equivalence principle [7].

Clearly gravity at quantum level continues to be a not purely geometric effect, mass appears in the interference pattern (always in the combination $\hbar/m$).

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