THEORETICAL VERIFICATION OF COULOMB ORDER OF IONS IN A STORAGE RING

R.W. Hasse
Theoretical Verification of Coulomb Order of Ions in a Storage Ring

Rainer W. Hasse

*GSI Darmstadt, D-64291 Darmstadt, Germany*

(July 5, 1999)

We verify theoretically that the anomalous longitudinal temperature reduction of strongly electron cooled heavy ions in the ESR at very low density is explained by the fact that there is no intrabeam scattering and that the particles by their Coulomb repulsion cannot pass each other any more. At the achievable momentum spreads Coulomb order is reached at particle distances of the order of centimeters. It is also shown that under the given experimental conditions in the proton NAP-M experiment of 1980 intrabeam heating counteracts Coulomb order.

In 1996, Steck et al. [1] reported on measurements with very low density and extremely electron cooled heavy ions in the Experimental Storage Ring (ESR) of GSI. By Schottky noise measurements they found a sharp drop of the longitudinal momentum spread $\delta p/p$ by an order of magnitude from $5 \times 10^{-6}$ down to $5 \times 10^{-7}$ for particle numbers from $10^3$ down to 3 in the ring of about 100 m circumference. Thus, arranged in linear chains the average distances between the ions would be between 10 cm and 33 m. Due to machine limitations $\delta p/p$ could not fall below this lower value. A typical example is shown in Fig. 1.

![Graph](image)

**FIG. 1.** Experimental momentum spread vs. number of stored ions in the ESR for electron cooled $^{92+}$ ions at 240 MeV/u (From ref. [1])

The beam radius for the heavy beams could be determined to about 30 $\mu$m and by emittance measurements, the transverse temperature was limited to about 1.5 eV. This anomaly resembles a strong suppression of intrabeam scattering below a certain threshold. Since heating of the beam is caused by intrabeam scattering, also heating is strongly inhibited,
thus reaching the very low $\delta p/p \approx 5 \times 10^{-7}$.

It has been speculated that the final beam structures might be the storage ring analogues of Coulomb crystals as they were calculated in ref. [2] and as they were found in ion traps [3]. In this Letter we confirm with the methods applied in ref [4] that indeed the beams resemble strings with particles which move slowly in the beam direction but, however, cannot pass at each other any more. This type of order of a liquid caused by the nearest neighbours only we call Coulomb order in contrast to a Coulomb crystal which is generated by long range Coulomb interaction over many neighbours.

In order to explain this effect we perform classical Monte-Carlo trajectory calculations of two charged particles heading at each other with constant focusing with the betatron frequency of the ESR and calculate the probability of these two particles being reflected at each other. It is sufficient to consider the interaction of two particles only since their mutual Coulomb repulsion acts only considerably at near distance of the order of tens of micrometers. To have a constant beam radius for all masses the experimental transverse temperature must obey the approximate relation $T_{\text{trans}} = 7.5 \times 10^4$ meV. This energy is distributed among the two transverse degrees of freedom according to a Boltzmann distribution in harmonic potentials with equal betatron frequencies $\omega_q = 2\pi \beta c/L$, where $\beta c$ is the beam velocity, $Q=2.3$ is the average tune, and $L$ is the circumference of the ring. The longitudinal kinetic energy is obtained from $M(c/\beta \delta p/p)^2/(8 \ln 2)$, where $M$ is the mass, see ref. [1].

![Graph](image)

**FIG. 2.** Calculated reflection probabilities vs. distance between particles for given temperatures

In order to systematize the calculations, three dimensionless parameters are introduced: The relative transverse, $\Theta_{\text{trans}}$, and longitudinal, $\Theta_{\text{long}}$, kinetic energies measured in units of the mutual Coulomb energy of two particles at a distance $d$, $e_C = q^2 \gamma / d$, where $q$ is the charge and $\gamma$ is the relativistic parameter. These relative temperatures, thus, are the reciprocal gamma parameters in Wigner crystal theory; i.e. a one-component plasma is in the gaseous state for $\Gamma \ll 1$, in the liquid state for $1 < \Gamma < 100$, and in the crystalline state for $\Gamma \geq 170$. Note, however, that here $\Gamma$ does not play a decisive role since distances involved are much larger than the Wigner-Seitz radius. Furthermore, the linear string density $\lambda = a_{\text{WS}} / d$
is the axial number of particles within a Wigner-Seitz radius $a_{WS} = \left(\frac{3q^2}{2M \omega_0^2}\right)^{1/3}$. Note that at zero temperature $\lambda = 0.709$ is the limiting value for a Coulomb string turning into a zigzag and $\lambda \approx 4$ would give a helix with a string at the center [2].

For typical experimental values of the kinetic energies a result is shown in Fig. 2. Within a factor of two in the distance, e.g. from 10 cm to 20 cm, the reflection probability rises sharply from 10% to 90%. On the other hand, Fig. 3 shows a contour plot of the reflection probability for fixed distance. Similarly, for given distance the reflection probability varies very slowly with $\Theta_{trans}$, i.e. it goes from 10% to 90% about within a factor 100 in $\Theta_{trans}$ but more rapidly, with a factor of 5 only, in $\Theta_{long}$. As a rule of the thumb $\Theta_{long}/\Theta_{trans}$ stays constant for given distance and fixed reflection probability. In the analysis of the experiments, hence, the results are little sensitive to the assumed transverse temperature of $7.5 \times 10^{-6}$ A meV.

![Contour plot of the calculated reflection probabilities vs. relative transverse and longitudinal temperature at fixed density $\lambda = 0.00015$](image)

**FIG. 3.** Contour plot of the calculated reflection probabilities vs. relative transverse and longitudinal temperature at fixed density $\lambda = 0.00015$

With the help of these tools the ESR experiments were analysed with the results shown in Fig. 4. In most cases the calculated reflection probability rises sharply in the vicinity of the last upper (open) data point thus indicating that for larger particle distances the ions cannot pass each other any more. In the last frame the reflection probability is also calculated for the first ultracold data point (left line). This is shifted to smaller distances by almost two orders of magnitude. It indicates that here the ions move so slowly in the beam direction that reflection would happen even for much smaller interparticle distances which, however, cannot be reached experimentally. However, this distance is still two orders of magnitude larger than the Wigner-Seitz radius (the typical Coulomb crystal string distance) which means that real Coulomb crystals instead of Coulomb order only cannot be produced in the ESR with the present electron cooling methods. The same argumentation applies to the two lines in the nickel frame where even multiple points were measured for the same distance. An exception from this systematics is the case of argon which suggests that the last upper data point should be somewhat smaller than $\delta p/p = 4 \times 10^{-6}$. In the titanium data there is no drop in the momentum spread. The point used for evaluation evidently belongs already
to the ultracold branch which can also be seen from the low longitudinal temperature of table I.

![Graphs showing momentum spreads and reflection probabilities vs. distance for various ions.]

FIG. 4. Experimental momentum spreads (points, left scale) and calculated reflection probabilities (lines, right scale) vs. distance for the various ESR experiments [1]. The reflection probabilities are calculated at the last upper or first lower (open) data point.

A summary of the data and of the results is shown in table I. The linear density $\lambda$ is about 5000 times smaller than the critical string density which shows again that the order reached is far from the one of Coulomb crystals. The average transverse $\rho_{rms}$ displacement in the last column was calculated with [4] $\rho_{rms} = d\sqrt{4\lambda^3 \Theta_{trans}/3}$. As assumed, it settles around 30 $\mu$m for all elements as noted by the authors of ref. [1]. In the next to the last column is shown the ratio of collisional time to betatron period of a particle, $\tau_{coll}/\tau_{\beta} = \sqrt{T_{trans}/T_{long}} \times d/\rho_{rms}$. As a result, thousands of betatron oscillations are performed during one binary collision, thus indicating that intrabeam scattering is negligible.

The two italicized lines are predictions for a proton experiment in the ESR storage ring with the energy of the NAP-M experiment [5] and for a future krypton experiment in the synchrotron SIS at GSI at injection energy with the recently installed electron cooler, respectively. The proton prediction is very close to the existing carbon data. On the other hand, due to the stronger focusing forces in the SIS (the horizontal and vertical tunes are
4.3 and 3.3), respectively) the threshold of 50% reflection will be shifted to larger linear densities closer to the critical string density i.e. to smaller interparticle distances. Coulomb order can be reached with even larger momentum spreads of the order of $10^{-5}$. Here the calculation of the reflection probability was carried out with anisotropic focusing, however with little change in the results as compared to isotropic focusing.

**TABLE I.** Experimental data, momentum spread $\delta p/p$, distance $d$, Wigner-Seitz radius $a_{WS}$, linear density $\lambda$, longitudinal and transverse temperatures $T_{long}, T_{trans}$, reflection probability, rms radius $\rho_{rms}$, and ratio of collision time to betatron period. The italicized lines are predictions.

<table>
<thead>
<tr>
<th>Ring</th>
<th>Ion</th>
<th>E (MeV/u)</th>
<th>$\delta p/p$ (10^-6)</th>
<th>d [cm]</th>
<th>$a_{WS}$ [µm]</th>
<th>$\lambda$</th>
<th>$T_{long}$ (meV)</th>
<th>$T_{trans}$ (meV) [%]</th>
<th>$\rho_{rms}$ [µm]</th>
<th>$\tau_{coll}$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESR</td>
<td>$^{12}$C$^{6+}$</td>
<td>240</td>
<td>2</td>
<td>0.17</td>
<td>7.7</td>
<td>0.0046</td>
<td>1.5</td>
<td>90</td>
<td>68</td>
<td>30</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{20}$Ne$^{10+}$</td>
<td>240</td>
<td>2</td>
<td>0.25</td>
<td>9.1</td>
<td>0.0036</td>
<td>0.40</td>
<td>2000</td>
<td>80</td>
<td>30</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{40}$Ar$^{18+}$</td>
<td>360</td>
<td>4</td>
<td>4</td>
<td>8.9</td>
<td>0.0020</td>
<td>19</td>
<td>30</td>
<td>5</td>
<td>21</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{48}$Ti$^{22+}$</td>
<td>240</td>
<td>2.5</td>
<td>0</td>
<td>11.5</td>
<td>0.0026</td>
<td>9</td>
<td>370</td>
<td>100</td>
<td>30</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{58}$Ni$^{28+}$</td>
<td>205</td>
<td>4</td>
<td>8</td>
<td>13.6</td>
<td>0.0016</td>
<td>26</td>
<td>440</td>
<td>24</td>
<td>33</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{86}$Kr$^{36+}$</td>
<td>240</td>
<td>4</td>
<td>6</td>
<td>13.3</td>
<td>0.0022</td>
<td>39</td>
<td>640</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{132}$Xe$^{54+}$</td>
<td>240</td>
<td>6</td>
<td>10</td>
<td>15.0</td>
<td>0.0015</td>
<td>126</td>
<td>1000</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>FSR</td>
<td>$^{197}$Au$^{79+}$</td>
<td>360</td>
<td>6</td>
<td>2</td>
<td>14.0</td>
<td>0.0070</td>
<td>290</td>
<td>1500</td>
<td>82</td>
<td>21</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{238}$U$^{52+}$</td>
<td>360</td>
<td>5</td>
<td>10</td>
<td>14.6</td>
<td>0.0015</td>
<td>240</td>
<td>1800</td>
<td>99</td>
<td>21</td>
</tr>
<tr>
<td>ESR</td>
<td>$^{p}$</td>
<td>65</td>
<td>1</td>
<td>0.2</td>
<td>9.1</td>
<td>0.0045</td>
<td>0.01</td>
<td>7.5</td>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>SIS</td>
<td>$^{86}$Kr$^{36+}$</td>
<td>11.4</td>
<td>15</td>
<td>0.5-10</td>
<td>50</td>
<td>0.004-0.01</td>
<td>40</td>
<td>90-200</td>
<td>50</td>
<td>60-90</td>
</tr>
<tr>
<td>NAP-M</td>
<td>$^{p}$</td>
<td>65</td>
<td>1</td>
<td>2µm</td>
<td>8.0</td>
<td>4.2</td>
<td>0.01</td>
<td>25</td>
<td>—</td>
<td>150</td>
</tr>
</tbody>
</table>

Finally we analyse the cooling experiment with protons in the then existing Novosibirsk NAP-M storage ring [5] which burned down afterwards. Here the authors suggested long ago that order has been reached. Their argumentation was based on the fact that if the proton current fell below 10 µA the noise power dropped to unmeasurable levels and thereafter stayed constant.

![Diagram](attachment:image.png)

**FIG. 5.** Momentum spread vs. longitudinal distance between protons of the NAP-M experiment (after ref. [5])

5
With the given data of circumference 47.25 m, average tune 1.29, energy 65 MeV, the number of particles in the ring for 10 μA current was \( N = 2.5 \times 10^7 \) from the relation \( I_p = eNf_{\text{rev}} \), where \( f_{\text{rev}} \) is the revolution frequency. An average transverse kinetic energy of 25 meV was derived from the measured beam radius of 100 μm and an average longitudinal kinetic energy of \( 10^{-4} \) eV was obtained from Schottky noise measurements. From this one gets the momentum spread of Fig. 5 with a critical \( \delta p/p = 10^{-6} \).

![Graph showing time vs. kinetic energy with labels](image)

**FIG. 6.** Half transverse (upper curve) and increase of longitudinal kinetic energy (lower curves) due to intrabeam scattering up to thermal equilibrium \( (T_{\text{long}} = \frac{1}{2} T_{\text{trans}}) \) with the input data of the NAP-M experiment. Shown is the average over 20 Monte-Carlo simulations. The long lower line is without cooling and the short one with cooling with e-folding time 400μsec.

According to the last row of table I the linear density is \( \lambda \approx 4 \) indicating that the system is no longer in the linear regime and the average axial distance is 2 μm, much smaller than the Wigner-Seitz radius of 8 μm. The average spacial particle distance is about 40 μm. According to table I, collision time and betatron period are about the same. Two particle calculations without taking into account other neighbours, hence, do not suffice to simulate this system. Therefore we performed full molecular dynamics calculations with periodic boundary conditions as in ref. [4] with 1000 particles under constant focusing and computed the Coulomb interaction with Ewald summation [2].

Fig. 6 shows the average over 20 simulations with random initial coordinates of the particles. With (short lower line) or without (long lower line) cooling, by intrabeam scattering after 200 betatron oscillations the longitudinal kinetic energy already reaches one third of the value of the transverse kinetic energy. This yields an initial longitudinal heating rate of more than 50 eV/sec and after about 1000 betatron oscillations, i.e. about 400 μsec, without cooling thermal equilibrium \( (\Theta_{\text{long}} = \frac{1}{2} \Theta_{\text{trans}}) \) has been reached. Cooling with an e-folding time of 400μsec just has the effect that the longitudinal temperature reaches only half of the value of thermal equilibrium. This has to be compared with typical electron cooling times of a few milliseconds [6]. The authors’ theory of collective interaction of the protons together with beam magnetization [7] may explain a possible suppression of intrabeam scattering in this case. Our predictions, on the other hand, cf. table I, would yield Coulomb order for an interparticle distance of 0.2 cm, i.e. proton currents below 10 pA.
In summary, our calculations of the reflection probabilities have shown that with the ESR experiments for the first time Coulomb order has been established in a heavy ion storage ring. This order is of liquid type where the particles still move slowly against each other but cannot pass any more.

The author likes to thank M. Steck for valuable discussions.