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ABSTRACT

In this thesis we will discuss various aspects of noncommutative geometry and compactified Little-String theories.

First we will give an introduction to the use of noncommutative geometry in string theory. Thereafter we will present a proof of the connection between D-brane dynamics and noncommutative geometry. This proof was made in collaboration with Edna Cheung. Then we will explain the concept of instantons in noncommutative gauge theories which will be relevant for the last chapters.

The last chapters shift the focus to Little-String- and (2, 0)-theories. We study compactifications of these theories on tori with twists. First we study the case of two coinciding branes in detail. This is based on work with Edna Cheung and Ori Ganor. Finally we study the case of an arbitrary number of coinciding branes. The main result here is that the moduli spaces of vacua for the twisted compactifications are equal to moduli spaces of instantons on a noncommutative torus. A special case of this is that a large class of gauge theories with $\mathcal{N} = 2$ supersymmetry in $D = 4$ or $\mathcal{N} = 4$ in $D = 3$ has moduli spaces which are moduli spaces of instantons on noncommutative tori. This work was done in collaboration with Edna Cheung, Ori Ganor and Andrei Mikhailov.
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1. Introduction to Noncommutative geometry in string theory

Noncommutative geometry was invented by mathematicians as a generalization of the notion of a topological space. Alain Connes’ book [1] is an exposition to these ideas. Here we will focus on its implications in string and gauge theory. Noncommutative geometry entered string theory in the fall of 1997 in the paper by Connes, Douglas and Schwarz [2]. Here matrix theory for M-theory on $T^d$ was considered. One can put a background value for the 3-form potential of M-theory with one index along the lightlike circle and 2 indices along $T^d$. We see that we need $d \geq 2$ to do this. In that paper it was argued that the matrix model is a gauge theory on a noncommutative torus. Following Sen and Seiberg [3,4] the matrix model for M-theory in a certain background is given by the dynamics of D0-branes in the same background in a certain limit. The limit is such that the resulting theory becomes a gauge theory when $d \leq 3$ or another kind of field theory without gravity for higher $d$. The 3-form potential with one lightlike index becomes an NS-NS B-field.

We see that the question of the role of noncommutative geometry in string theory can be phrased solely in terms of D-branes in a background $B^{NS}$-field. The connection to Matrix theory then follows trivially.

We thus consider the following question. Consider Type IIA compactified on $T^2$ of radii $R_1, R_2$ with a constant $B^{NS}$-field along $T^2$. Higher dimensional tori can be dealt with in a similar way. Let the string mass and coupling be $m_s$ and $g$. Define $\theta^{NS} = \frac{1}{2\pi} \int_{T^2} B^{NS}$. $\theta^{NS}$ is a pure number. The gauge invariance of $B^{NS}$ makes $\theta^{NS}$ periodic with period 1. Let us put $N_0$ D0-branes and $N_2$ D2-branes in this background. We will take the limit
This limit is the usual matrix theory limit, which gives a gauge theory decoupled from gravity. We want to give a description of this system.

A method for dealing with D0-branes on a torus was developed by Taylor [5] and Ganor, Ramgoolam and Taylor [6]. This was taken further by Connes, Douglas and Schwarz [2]. The method is to consider D0-branes described the quantum mechanics Lagrangian

\[ L = \frac{(2\pi)^2}{4gm^3_s} Tr(\sum_i 2(D_0 X^i)^2 + \sum_{i,j} [X^i, X^j]^2 + 2i\bar{\Psi}\Gamma^0 D_0 \Psi + \bar{\Psi}\Gamma^i [X^i, \Psi]) \]

(2)

Here \( X^i, i = 1, \ldots, 9 \) are Hermitian matrices, \( \Psi_\alpha \) are Majorana-Weyl spinors of SO(1,9) and

\[ D_0 X^i = \frac{\partial X^i}{\partial t} + i[A_0, X^i] \]

(3)

and similarly for \( D_0 \Psi \). This action has a supersymmetry

\[ \delta X^i = \frac{i}{2} \varepsilon \Gamma^i \Psi \]

\[ \delta \Psi = -\frac{1}{4} (2D_0 X^i \Gamma^{0i} + [X^i, X^j] \Gamma^{ij}) \varepsilon \]

(4)

with \( \Gamma^{ij} = [\Gamma^i, \Gamma^j] \) There is also a nonlinearly realised supersymmetry

\[ \delta X^i = 0 \]

\[ \delta \Psi_\alpha = \zeta_\alpha \]

(5)
Remark that the last symmetry only acts in the $U(1)$ part of $U(N)$. To deal with D0-branes on a $T^2$ we should make $X^1, X^2$ periodic. Since field configurations that differ by a $U(N)$ gauge transformation should be identified the periodicity of $X^1, X^2$ means that there exist unitary operators $U_1, U_2$ such that

\begin{align}
U_1X^1U_1^{-1} &= X^1 + R_1m_s^2 \\
U_2X^2U_2^{-1} &= X^2 + R_2m_s^2 \\
U_iX^jU_i^{-1} &= X^j \quad i \neq j \\
U_i\Psi_\alpha U_i^{-1} &= \Psi_\alpha
\end{align}

(6)

We will take solutions of these equations to describe a certain D0-brane configuration on $T^2$. Remark that $N_0, N_2$ and $\theta^{NS}$ appear nowhere. We then expect that there are several classes of solutions to these equations, one class for each choice of $N_0, N_2$ and $\theta^{NS}$.

By taking the trace of the above equations it is readily seen that these equations do not admit finite dimensional matrix solutions. The natural thing is to look for solutions where $U_i, X^i$ and $\Psi_\alpha$ are operators in an infinite dimensional Hilbert space. This infinite dimensionality just reflects that there are infinitely many D0-branes on the covering space of the $T^2$.

Let us now discuss the solutions of these equations. It is easily seen that $U_1U_2U_1^{-1}U_2^{-1}$ commutes with all $X^i$ and $\Psi_\alpha$. Since we are interested in an irreducible representation of $X^i$ and $\Psi_\alpha$ Schur’s lemma tells us that $U_1U_2U_1^{-1}U_2^{-1}$ must be a constant. Since $U_1, U_2$ are unitary the constant must be on the unit circle in the complex plane. We thus have

\begin{align}
U_1U_2 = e^{2\pi i \theta}U_2U_1
\end{align}

(7)

We will later see that the interpretation of $\theta$ is that $\theta = \theta^{NS} = \frac{1}{2\pi} \int_{T^2} B^{NS}$.  

3
$U_1$ and $U_2$ generate an algebra. If $\theta = 0$ this is the algebra of functions on a $T^2$: For instance we could take $U_1 = e^{ix_1}, U_2 = e^{ix_2}$. For nonzero $\theta$ this algebra is known as the noncommutative torus, $T^2_\theta$. The point of noncommutative geometry is that often one does not need the actual space but only the algebra of functions on the space. There is a one-to-one correspondence between topological spaces and commutative $C^*$-algebras. Whereas it is hard to generalize the space to a “noncommutative” space it is straightforward to drop the requirement that the $C^*$-algebra be commutative. The algebra $T^2_\theta$ is a natural generalisation of $T^2$, and is hence forth called the noncommutative torus. Remark that $\theta$ is periodic with period 1.

Solving eq.(6) involves finding a Hilbert space and the set of operators $X,\Psi$ obeying eq.(6). Another way of saying this is to say that the Hilbert space is a module over $T^2_\theta$, and $X^1,X^2$ are connections in this module and $X^i i > 2$ and $\Psi_\alpha$ are endomorphisms of the module. We will not explain the mathematics here. An exposition is given in [2,7] among others. For $\theta = 0$ the solution is given by bundles over $T^2$ [6]. The modules are the sections of these bundles. In the non-commutative case there is no space anymore and we can not define bundles in the usual way. The modules are the right generalisation.

Now we will explicitly describe these modules following [8,9]. Let $\sigma_1,\sigma_2$ be coordinates periodic with period $\frac{2\pi}{m_s^2 R_1}, \frac{2\pi}{m_s^2 R_2}$ and obeying

$$[\sigma_1, \sigma_2] = 2\pi i \theta \frac{1}{m_s^4 R_1 R_2}$$

(8)

Let $\partial_1, \partial_2$ be derivatives with respect to $\sigma_1, \sigma_2$. They obey

$$[\partial_i, \sigma_j] = \delta_{ij}$$

$$[\partial_i, \partial_j] = 0$$

(9)

Define

$$U_1 = e^{i\sigma_1 m_s^2 R_1} e^{2\pi i \frac{1}{m_s^2 R_2} \partial_2}$$

$$U_2 = e^{i\sigma_2 m_s^2 R_2} e^{-2\pi i \frac{1}{m_s^2 R_1} \partial_1}$$

(10)
Then

\[ U_1 U_2 = e^{i2\pi \theta} U_2 U_1 \]  \hspace{1cm} (11)

The Hilbert space consists of \( n \)-component vector functions \( \Phi(\sigma_1, \sigma_2) \) obeying the boundary conditions

\[
\Phi(\sigma_1 + \frac{2\pi}{m^2 s R_1}, \sigma_2) = \Omega_1(\sigma_1, \sigma_2) \Phi(\sigma_1, \sigma_2) \\
\Phi(\sigma_1, \sigma_2 + \frac{2\pi}{m^2 s R_2}) = \Omega_2(\sigma_1, \sigma_2) \Phi(\sigma_1, \sigma_2) \]  \hspace{1cm} (12)

Here

\[
\Omega_1(\sigma_1, \sigma_2) = e^{i\frac{m \sigma_2 m^2 s R_2 U}{m}} \\
\Omega_2(\sigma_1, \sigma_2) = V \]  \hspace{1cm} (13)

where \( m \) is an integer. Here \( U, V \) are \( n \times n \) unitary matrices satisfying

\[ UV = e^{-2\pi i \frac{m}{m} VU} \]  \hspace{1cm} (14)

Such matrices are readily found, for example as clock and shift matrices. \( m \) and \( n \) are integers characterizing the module. \( \Omega_1, \Omega_2 \) satisfy the cocycle condition

\[
\Omega_1(\sigma_1, \sigma_2 + \frac{2\pi}{m^2 s R_2}) \Omega_2(\sigma_1, \sigma_2) = \Omega_2(\sigma_1 + \frac{2\pi}{m^2 s R_1}, \sigma_2) \Omega_1(\sigma_1, \sigma_2) \]  \hspace{1cm} (15)

which is necessary for eq. (12) to make sense. Sections of the adjoint
bundle, or endomorphisms of the module, satisfy

\[ \psi(\sigma_1 + \frac{2\pi}{m_2^2 R_1}, \sigma_2) = \Omega_1(\sigma_1, \sigma_2) \psi(\sigma_1, \sigma_2) \Omega_1^{-1}(\sigma_1, \sigma_2) \]
\[ \psi(\sigma_1, \sigma_2 + \frac{2\pi}{m_2^2 R_2}) = \Omega_2(\sigma_1, \sigma_2) \psi(\sigma_1, \sigma_2) \Omega_2^{-1}(\sigma_1, \sigma_2) \]  

(16)

Connections, or covariant derivatives, satisfy the same equation

\[ D_i(\sigma_1 + \frac{2\pi}{m_2^2 R_1}, \sigma_2) = \Omega_1(\sigma_1, \sigma_2) D_i(\sigma_1, \sigma_2) \Omega_1^{-1}(\sigma_1, \sigma_2) \]
\[ D_i(\sigma_1, \sigma_2 + \frac{2\pi}{m_2^2 R_2}) = \Omega_2(\sigma_1, \sigma_2) D_i(\sigma_1, \sigma_2) \Omega_2^{-1}(\sigma_1, \sigma_2) \]  

(17)

One connection is

\[ D_1 = \partial_1 \]
\[ D_2 = \partial_2 - if\sigma_1 \]  

(18)

where f is a constant. By plugging \( D_1, D_2 \) into eq.(17) we see that

\[ 2\pi f = \frac{m}{n - m\theta} m_2^4 R_1 R_2 \]  

(19)

The most general connection is this one plus an arbitrary adjoint section:

\[ \nabla_i = D_i + A_i \]  

(20)

Now let us solve our original equations, eq.(6). \( X^i, i \neq 1, 2 \) and \( \Psi_\alpha \) commute with \( U_1 \) and \( U_2 \). \( U_1 \) and \( U_2 \) are constructed such that any function of \( \sigma_1 \) and \( \sigma_2 \) commute with it. We thus conclude that \( X^i, i \neq 1, 2, \Psi_\alpha \) can be any adjoint section, i.e. it solves eq.(16) and is
a function of $\sigma_1,\sigma_2$, but must not contain derivatives. In mathematical terms they are endomorphisms of the module over $T_\theta^2$. $X^1, X^2$ are solved by

$$
X^1 = i \nabla_1 \\
X^2 = i \nabla_2
$$

(21)

where $\nabla_1, \nabla_2$ are arbitrary connections of the module. The fields $X^i, \Psi_\alpha$ should also be Hermitian as always for D0-branes. Now we have the fields of the theory. The action is given by the time integral of eq.(2). The only thing we still need to specify is the definition of the trace. The trace is always taken of an endomorphism. Let $\Psi$ be an endomorphism, i.e. it obeys eq.(16). First we can take the trace of $\Psi$ as a $n \times n$ matrix. The trace of eq.(16) shows that $tr \Psi(\sigma_1, \sigma_2)$ is a periodic function of $\sigma_1, \sigma_2$. It makes sense to integrate $tr \Psi(\sigma_1, \sigma_2)$ over $\sigma_1, \sigma_2$. We thus define

$$
Tr \Psi(\sigma_1, \sigma_2) = C \int d\sigma_1 d\sigma_2 tr \Psi(\sigma_1, \sigma_2)
$$

(22)

We only need to fix the constant, $C$. A natural normalisation, which is consistent with the commutative case $\theta = 0$, is the following. Let $F_{12}$ be the field strength of a connection $\nabla$

$$
F_{12} = [\nabla_1, \nabla_2]
$$

(23)

Then $Tr(F_{12})$, which is a topological invariant, should satisfy

$$
\frac{(2\pi)^2}{m_s^4 R_1 R_2} \frac{1}{2\pi i} Tr F_{12} = -m
$$

(24)

where $m$ is the integer from above. We remember that $\frac{(2\pi)^2}{m_s^4 R_1 R_2}$ is the
volume of the torus spanned by $\sigma_1,\sigma_2$. Let us use $D_i$ from above

$$F_{12} = [D_1, D_2] = -if$$  \hfill (25)

We have

$$\frac{m_s^4 R_1 R_2}{(2\pi)^2} (-m) = \frac{1}{2\pi i} Tr F_{12}$$
$$= \frac{1}{2\pi i} C \int d\sigma_1 d\sigma_2 tr(-if)$$  \hfill (26)
$$= -\frac{1}{(2\pi)^2} C \frac{2\pi}{m_s^2 R_1} \frac{2\pi}{m_s^2 R_2} \frac{n}{n - m\theta} m_s^4 R_1 R_2$$

Here we used the value for $f$ given in eq.(19). This fixes

$$C = \frac{n - m\theta}{n} m_s^4 R_1 R_2 \frac{1}{(2\pi)^2}$$  \hfill (27)

In this discussion we need $n - m\theta > 0$. The trace is then defined as

$$Tr \Psi(\sigma_1, \sigma_2) = \frac{m_s^4 R_1 R_2}{(2\pi)^2} \frac{n - m\theta}{n} \int d\sigma_1 d\sigma_2 tr \Psi(\sigma_1, \sigma_2)$$  \hfill (28)

Here $\Psi(\sigma_1, \sigma_2)$ is any endomorphism. Notice that when $\theta = 0$ we get the usual definition of the trace. Notice also that $Tr 1 = n - m\theta$.

Let us shortly recapitulate. We have solved the equations for the compactification of the D0-brane quantum mechanics. The most general solution depended on integers $m, n$ and an angle $\theta$. We got an action for the fields. The only ambiguity was the definition of the trace. What we will do now is to identify the parameters in terms of the physical parameters $N_0, N_2$ and $\theta^{NS}$. We will also calculate the mass of some BPS-states and compare with the correct result, which
is known from type IIA string theory. The agreement of these BPS masses will prove the correctness of the definition of the trace and give some evidence for the relevance of modules over noncommutative tori to D-brane dynamics. In chapter 2 we will provide a proof that this noncommutative gauge theory is indeed the correct answer.

Let us first match the parameters $n, m, \theta$ with the number of D0-branes, $N_0$, the number of D2-branes, $N_2$ and $\theta^{NS} = \frac{1}{2\pi} \int_{T^2} B^{NS}$. For $\theta^{NS} = 0$ we understand the situation very well. The system is described by a gauge theory obtained after T-duality on both directions of $T^2$. After T-duality there are $N_0$ D2-branes and $N_2$ D0-branes giving a $U(N_0)$ gauge theory with first Chern class, $c_1 = N_2$. Comparing with the module above we get

$$n = N_0$$
$$m = N_2$$

and $\theta = 0$, since there is no noncommutativity. For general $\theta^{NS}$ the relation (29) must still hold since $n, m, N_0, N_2$ are integers and cannot change continuously. We know from string theory that in the presence of a $B^{NS}$-field the “effective” number of D0-branes is $N_0 - \theta^{NS} N_2$. Since the combination $n - \theta m$ appears all over in the above construction we are led to $\theta^{NS} = \theta$. In other words the noncommutativity parameter is set exactly by the $B^{NS}$-flux.

Let us now calculate the energy of some BPS states. First we take the lowest energy state which just corresponds to the pure D0,D2-brane system with no excitations. This state preserves 16 supercharges. In the noncommutative gauge theory this is given by setting

$$\Psi_{\alpha} = 0$$
$$X^i = 0 \quad i = 3, \ldots, 9$$
$$X^1 = i D_1$$
$$X^2 = i D_2$$

The energy, which can be calculated classically since it is a BPS state,
Let us compare this with the correct result known from string theory. $N_2$ D2-branes wrapped on a $T^2$ with a $B^{NS}$ has an effective D0-brane number, $-\theta^{NS} N_2$. If there are $N_0$ D0-branes in addition, the effective D0-brane number is $N_0 - \theta^{NS} N_2 = n - m\theta$. The energy of D0- and D2-branes add in quadrature. One way of understanding this is to T-dualize along a single direction and get a diagonal wound D-string. This T-dual picture also explains the shift $N_0 \rightarrow N_0 - \theta^{NS} N_2$, since $\theta^{NS}$ makes the T-dual torus non-rectangular. The energy is thus

\[
E = \sqrt{\left(\frac{(n - m\theta) m_s}{g}\right)^2 + \left(\frac{m R_1 R_2 m_s^2}{g}\right)^2}
\]

\[
= (n - m\theta) \frac{m_s}{g} + \frac{1}{2} \frac{m^2}{n - m\theta} (R_1 R_2)^2 m_s^5 \frac{1}{g} + \ldots
\]

where we have expanded the square root. In the limit...
\[ m_s \rightarrow \infty \]
\[ g \rightarrow 0 \]
\[ R_1, R_2 \rightarrow 0 \]
\[ m_s^2 R_1, m_s^2 R_2 \quad \text{fixed} \]
\[ gm_s^3 \quad \text{fixed} \]

which we consider, the first term goes to infinity, the second term stays finite and all higher terms vanish. The first term is well known from matrix theory. It is equivalent to the term \( \frac{N}{R} \) which is always subtracted. The second term is the interesting one which should be reproduced by the SYM action. This story is also well known from expanding the Born-Infeld action, where the first term is the background energy of the brane, the second term is the SYM action and higher terms are suppressed at low energy.

By comparing the second term in eq. (32) with the energy obtained from the noncommutative SYM, eq. (31), we find agreement. We also obtained a further understanding for the ubiquity of the expression \( n - \theta m \).

Let us now consider the D0,D2-brane system with momentum and fundamental string winding along the cycles of the \( T^2 \). For generic momenta and winding this state preserves 8 supercharges, but for special values 16 supercharges are preserved. Let us just look at a particular example of a 16 supercharge state. The others can be done similarly but with a bit more work. We consider a state with \( N_0 \) D0-branes, \( N_2 \) D2-branes, \( k \) units of momentum along the first axis and \( w \) units of fundamental string winding along the second axis. By performing a T-duality along the first axis this becomes a wrapped D-string and a wrapped fundamental string. They combine into a maximally supersymmetric state when they are parallel. The condition for that is

\[ wN_0 = kN_2 \]  

(34)

If they are not parallel they will form a string web. We will only consider the parallel case. The energy of the state is readily calculated
since the different contributions add in quadrature. The fact that they add in quadrature is particularly clear from the T-dual picture.

\[ E^2 = ((N_0 - \theta N_2) \frac{m_s}{g})^2 + \left( \frac{N_2 m_s^3 R_1 R_2}{g} \right)^2 + \left( \frac{k - \theta w}{R_1} \right)^2 + (w m_s^2 R_2)^2 \]  

(35)

Taking into account the limit, eq.(33), the energy becomes

\[ E = (N_0 - \theta N_2) \frac{m_s}{g} + \frac{1}{2} \frac{m_s^5 (R_1 R_2)^2}{g} \frac{N_2^2}{N_0 - \theta N_2} + \frac{1}{2} \frac{g}{m_s R_1^2} \frac{(k - \theta w)^2}{(N_0 - \theta N_2)} + \text{vanishing terms} \]  

(36)

Again it is the finite terms which should be compared with the gauge theory.

Let us now turn to the noncommutative gauge theory and reproduce this state. Let us consider the fields of the form

\[ X^1 = iD_1 + A(t)1 \]
\[ X^2 = iD_2 \]  

(37)

where \( D_1, D_2 \) are the connections given in eq.(18). \( A(t) \) is a function of time. Remark that \( A(t) \) multiplies the unit element in \( \text{End}_{T^2_\theta}(E) \).

We are going to take \( A(t) \) to be linear in time. The corresponding field strength is then constant and we see from the supersymmetry variations, eq.(4),eq.(5), that the corresponding state preserves 16 supercharges. This especially means that the energy receives no corrections. Let us write down the lagrangian for these fields, obtained
by plugging into eq.(2).

\[ L = \frac{(2\pi)^2}{4gm_s^2} Tr(2\dot{A}^2 + 2(i\dot{f})^2) = \frac{1}{2} \frac{(2\pi)^2}{gm_s^3} (n - m\theta) \dot{A}^2 - \frac{1}{2} m_s^5 (R_1 R_2)^2 \frac{1}{g} \frac{m^2}{n - m\theta} \]  

(38)

This action is very simple and we can easily find the energy levels. The only thing we need to know is the period of \( A \). \( A \) is periodic, because there are gauge transformations that shift \( A \). A similar story holds, of course, in the commutative case, \( \theta = 0 \), where it is more well known. Let \( D = gcd(n, m) \) then the period of \( A \) is

\[ A \rightarrow A + m_s^2 R_1 \frac{D}{n - \theta m} \]  

(39)

This is not hard to show, but we will not present the calculation here. The energies are now readily found

\[ E = \frac{1}{2} m_s^5 (R_1 R_2)^2 \frac{1}{g} \frac{m^2}{n - m\theta} + \frac{1}{2} m_s R_1^2 \frac{n - \theta m}{D^2} l^2 \]  

(40)

where \( l \) is the integer quantum number for the conjugate momentum to \( A \). Using eq.(34) which implies

\[ l = \frac{k - \theta w}{n - \theta m} D \]  

(41)

we now see that there is perfect agreement with the string theory result, eq.(36).

This finishes our treatment of comparing BPS masses. We will not discuss the less supersymmetric states. This is done in detail in the literature, [10,11,12].
In this chapter we have argued that the D-brane system with $B^{NS}$ flux is described by gauge theory on a noncommutative torus. Firstly we obtained it as a solution to the quotient conditions eq.(6), secondly BPS masses agree exactly. In the next chapter we will provide a proof of this connection from a worldsheet point of view. Before we do this we need to explain another equivalent formulation of gauge theory on a noncommutative torus.

We saw that the fields of the gauge theory were connections and endomorphisms over the noncommutative torus. This is beautiful from a mathematical point of view, but from a practical point of view it would be preferable to work with ordinary functions or sections of a bundle. Indeed there is a way of reformulating the theory to a theory of ordinary functions or sections. Let us here explain the case of $N_2 = 0$. In this case the fields could be considered as functions, $\Phi(\sigma_1, \sigma_2)$, of noncommuting variables $\sigma_1, \sigma_2$.

\[ [\sigma_1, \sigma_2] = 2\pi i \theta \frac{1}{m_s^4 R_1 R_2} \] (42)

$\sigma_1$ and $\sigma_2$ are periodic with period

\[
\sigma_1 \rightarrow \sigma_1 + \frac{2\pi}{m_s^2 R_1}, \\
\sigma_2 \rightarrow \sigma_2 + \frac{2\pi}{m_s^2 R_2} \] (43)

Any field can be written as

\[
\Phi(\sigma_1, \sigma_2) = \sum_{n_1, n_2} C_{n_1, n_2} e^{i n_1 \sigma_1 m_s^2 R_1} e^{i n_2 \sigma_2 m_s^2 R_2} \] (44)

Let us now consider ordinary commuting variables, $s_1, s_2$, of the same period as $\sigma_1, \sigma_2$. We can define a map from $\Phi(\sigma_1, \sigma_2)$ to a function
\[ F(s_1, s_2) \] as follows

\[ F(s_1, s_2) = \sum_{n_1, n_2} C_{n_1, n_2} e^{i n_1 s_1 m_1^2 R_1} e^{i n_2 s_2 m_2^2 R_2} e^{-i \pi n_1 n_2} \]  

\[ (45) \]

The product of fields \( \Phi(\sigma_1, \sigma_2) \) map to the following *-product

\[
(F_1 * F_2)(s_1, s_2) = e^{i \pi \theta \frac{1}{m_1^2 n_1 n_2} \left( \frac{\partial}{\partial \sigma_1} \frac{\partial}{\partial b_2} - \frac{\partial}{\partial \sigma_2} \frac{\partial}{\partial b_1} \right)} F_1(a_1, a_2) F_2(b_1, b_2)|_{a_i = b_i = s_i} \]

\[ (46) \]

The gauge theory action can thus be formulated in terms of ordinary functions but with the *-product replacing any normal product. This presentation is more suitable for standard quantum field theory treatment. We remark that the action is nonlocal. The effect of the exponential is to introduce nonlocal interactions along the first axis for processes which have a momentum transfer in the 2. direction or vice versa.
2. Worldsheet derivation of Noncommutative Geometry from D0-branes in a Background B-field

In this chapter we continue the study of D0-branes in type IIA on $T^2$ with a background B-field turned on. We calculate explicitly how the background B-field modifies the D0-brane action. The effect of the B-field is to replace ordinary multiplication with a noncommutative $\ast$-product. This theory is exactly the non-local 2+1 dim SYM theory on a dual $T^2$ proposed by Connes, Douglas and Schwarz, which was discussed in chapter 1. We calculate the radii and the gauge coupling for the SYM on the dual $T^2$ for all choices of longitudinal momentum and membrane wrapping number on the $T^2$.

2.1 Introduction

Last fall Connes, Douglas and Schwarz [2] made a very interesting proposal relating the matrix theory of M-theory on $T^2$ with a background three form potential, $C_{-12}$, to a gauge theory on a noncommutative torus. Shortly after Douglas and Hull justified this claim by relating these theories to a theory on a D-string [13]. They also mentioned that this could be seen in the original 0-brane picture.

The purpose of the present paper is to precisely incorporate the background B-field in the dynamics of 0-branes. In this way we will obtain the matrix model for M-theory on $T^2$ with $C_{-12} \neq 0$. It confirms the claims made by Connes, Douglas and Schwarz. The theory is a SYM theory on a dual torus with a modified interaction depending on the B-field. The theory contains higher derivative terms of arbitrarily high power and is thus non-local. We calculate the radii of the dual torus and the gauge coupling constant. We get a noncommutative gauge theory for any choice of longitudinal momentum and number of membranes wrapped around the $T^2$. The radii and gauge coupling depend on these numbers.

Aspects of the connection between compactifications of M-theory and noncommutative geometry has, among others, been worked out in [14,15,16].
2.2 Zero-branes on $T^2$ with background $B$-field

Let us consider M-theory on $T^2$ with radii, $R_1$, $R_2$. The torus is taken to be rectangular for simplicity. Making the torus oblique does not introduce anything interesting. We are interested in the matrix description of this theory with a background $C$-field, $C_{-12} \neq 0$. Here -$\cdot$ denotes the lightlike circle and 1, 2 the directions along the torus. Let the Plank mass be $M$ and the radius of the lightlike circle be $R$. Following [17,18], we take this to mean a limit of spatial compactifications. We also perform a rescaling to keep the interesting energies finite. The upshot is that we consider M theory on $T^2 \times S^1$ with Plank mass $\tilde{M}$ and radii $\tilde{R}_1$, $\tilde{R}_2$, $\tilde{R}$ in the limit $R \to 0$ with

$$\tilde{M}^2 \tilde{R} = M^2 R$$
$$\tilde{M} \tilde{R}_i = MR_i \quad i = 1, 2.$$  \hspace{1cm} (47)

This turns into type IIA on $T^2$ with string mass $m_s$, coupling $\lambda$ and radii $r_1$, $r_2$ given by

$$m_s^2 = \tilde{M}^3 \tilde{R}$$
$$\lambda = (\tilde{M} \tilde{R})^{\frac{3}{2}}$$
$$r_i = \tilde{R}_i \quad i = 1, 2.$$  \hspace{1cm} (48)

Furthermore there is a flux of the $B_{ij}$ field through the torus. We call the flux $B$:

$$B = RR_1 R_2 C_{-12}.$$  \hspace{1cm} (49)

We are interested in the sector of theory labelled by two integers, namely the number of D0-branes, $N_0$, and the number of D2-branes, $N_2$, wrapped on $T^2$. In this section we will solely be interested in the case $N_2 = 0$. In the next section we will treat the general case. Let us for the moment set $N_0 = 1$. This makes us avoid some essentially irrevant indices. The generalization to general $N_0$ is straightforward.
The method for dealing with this situation has been developed in [5,6]. We work with the covering space of $T^2$, namely $\mathbb{R}^2$ and place 0-branes in a lattice (Figure 1).

Let us label the 0-branes $(a, b)$ where $a, b$ are integers. The open strings obey Dirichlet boundary condition on the 0-branes. This is the point where we need $N_2 = 0$. If there had been D2-branes, the 0-branes would have been dispersed as fluxes inside the 2-branes and the open strings would have Dirichlet boundary conditions on the 2-branes and the above picture does not apply.

The fields in the theory come from quantizing the open strings and calculating their interactions. In the limit we are taking, $m_s \rightarrow \infty$, only the lowest modes survive and when $B = 0$ the theory becomes a SYM quantum mechanics [19,20,21,22,23]. The gauge group is "$U(\infty)$" since there are infinitely many 0-branes. To be more precise let us define a Hilbert space on which the fields will be operators. There is a basis vector for each 0-brane, i.e. the basis is $|a, b>$ where $a, b \in \mathbb{Z}$. Let $\phi$ be any field in theory, then the matrix element $\phi_{a_1 b_1, a_2 b_2}$ has the interpretation as a field which annihilates a state of an open string starting at $a_1 b_1$ and ending at $a_2 b_2$, see figure 2.

The fields of the theory are

---

**Fig. 2:** 0-branes on the covering space, $\mathbb{R}^2$
Fig. 3: String starting at 0-brane \((a_1, b_1)\) and ending at 0-brane \((a_2, b_2)\)

1. bosons: \(X^i\) \(i = 1, \ldots, 8\)
2. fermions: \(\Psi_\alpha\) \(\alpha = 1, \ldots, 16\)
3. The gauge field: \(A_0\).

The fields are constrained to obey the following conditions:

\[
U_i^{-1} X^a U_i = X^a + 2\pi r_0 \delta^a i = 1, 2; \\
U_i^{-1} \Psi^\alpha U_i = \Psi^\alpha, \\
U_i^{-1} A_0 U_i = A_0;
\]

where \(U_i\) are translation operators on the states in the Hilbert space:

\[
U_1|a, b> = |a + 1, b>
\]
\[
U_2|a, b> = |a, b + 1>.
\]

The gauge field \(A^0\) can be gauged away, and we will work in the gauge \(A^0 = 0\). When \(B = 0\) the action is

\[
L = \frac{m_s}{2\lambda} \text{Tr}[\dot{X}^a \dot{X}^a + \frac{m_s^4}{(2\pi)^2} \sum_{a < b} [X^a, X^b]^2 + \frac{m_s^2}{2\pi} \Psi^T i\dot{\Psi} - \frac{m_s^4}{(2\pi)^2} \Psi^T \Gamma_\alpha [X^a, \Psi]].
\]
What about $B \neq 0$? We will now show how to incorporate $B$-dependence in the action. The two-form $B_{ij}$ couples to the world-sheet through the interaction $\int_{W.S.} B_{ij}$, i.e. the $B$-field is pulled back to the worldsheet and integrated. Let us look at the example shown in Figure 3 below.

![Figure 3: The interaction between these three strings give rise to a cubic vertex.](image)

This interaction is represented, in the case $B = 0$, by a term:

$$\kappa \phi^{(3)}_{ik} \phi^{(2)}_{kj} \phi^{(1)}_{ji}$$

(53)

where $\kappa$ is a constant. This term could, for instance, annihilate string 1 and 2 and create 3 with opposite orientation. The worldsheet would look as shown in figure 4.

To calculate $\int_{W.S.} B_{ij}$ we just need the projection into the plane of the torus, since this is the only direction in which $B_{ij} \neq 0$. This projection is exactly given by the area between the three strings in Figure 3. The important point is that $B_{ij}$ is closed so $\int B_{ij}$ only depends on the homotopy type of the worldsheet imbedding and is insensitive to the finer details of how the interaction takes place. For
Fig. 5: The worldsheet for a cubic vertex

the example in Figure 3, $\int_{W.S.} B_{ij} = \frac{1}{2} B$, where we remark that $B$ was defined to be the flux through the torus. This means that the interaction eq.(53) now is replaced with

$$e^{i\frac{1}{2}B_{ij}}\phi_{ik}^{(3)}\phi_{kj}^{(2)}\phi_{ji}^{(1)}.$$  \hspace{1cm} (54)

It is now a straightforward exercise to figure out what happens to a general interaction between the fields:

$$\phi_{a_1b_1,a_kb_k}^{(k)}\cdots\phi_{a_3b_3,a_2b_2}^{(2)}\phi_{a_2b_2,a_1b_1}^{(1)}.$$  \hspace{1cm} (55)

We have to find the integral of the $B$-field through the polygon shown in Figure 5.
We should remember to count with sign. Orienting a polygon oppositely would change the sign of $\int_{W.S.} B_{ij}$. It is easily deduced that the result is

$$\int_{W.S.} B_{ij} = \frac{1}{2} B \sum_{i=2}^{k-1} \begin{vmatrix} a_{i+1} - a_i & a_i - a_1 \\ b_{i+1} - b_i & b_i - b_1 \end{vmatrix}$$

(56)

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. This means that the interaction eq.(55) now becomes
The reason for distributing the exponentials in this way will become clear in a moment. One could put an exponential between \( \phi(k) \) and \( \phi(k-1) \), but it would be identically 1, so we omitted it. We can introduce a notation which will make this look simpler. First note that the interactions always appear with a sum over indices.

\[
\sum_{a_1 b_1, a_k b_k} \phi^{(k)}_{a_1 b_1, a_k b_k} \phi^{(k-1)}_{a_k b_k, a_{k-1} b_{k-1}} \phi^{(3)}_{a_1 b_1, a_2 b_2} \phi^{(1)}_{a_2 b_2, a_1 b_1} \tag{58}
\]

If we think of the fields as matrices this is just

\[
Tr(\phi(k) \cdot \phi(k-1) \cdots \phi(2) \cdot \phi(1)) \tag{59}
\]

Now we define a product, called \(*\), by

\[
(\phi(2) \ast \phi(1))_{a_3 b_3, a_1 b_1} = \sum_{a_2 b_2} e^{i \frac{1}{2} B b_3 - b_2} \phi^{(2)}_{a_3 b_3, a_2 b_2} \phi^{(1)}_{a_2 b_2, a_1 b_1} \tag{60}
\]

Now the interaction with a \( B \)-field, eq.(57), can be written

\[
Tr(\phi(k) \ast \phi(k-1) \cdots \ast \phi(2) \ast \phi(1)) \tag{61}
\]

This is really nice. It shows that to generalize the action eq(52) to include a \( B \)-field we just need to replace ordinary matrix product...
with $\ast$-product. For $B = 0$ the $\ast$-product coincides with the ordinary product. This point of view, that the fields take value in another algebra, was of course the main point of [2].

In the case $B = 0$ the fields with the constraints eq.(50) and action can be rewritten to a $SYM$ theory on a dual $T^2$ [5,6]. Let us briefly repeat that construction for the case, $B \neq 0$. Let us first express the basis of the Hilbert space in another form. We want to think of the Hilbert space as $L^2$ functions on a dual torus with radii, \( \frac{1}{m^2 r_1}, \frac{1}{m^2 r_2} \). Let the basis vector $|ab> \) correspond to $e^{i\alpha m^2 r_1}e^{ibm^2 r_2}$, then the operators $U_1, U_2$ become multiplication operators

$$
U_1 = e^{ixm^2 r_1}, \quad U_2 = e^{ym^2 r_2}.
$$

It is now easy to solve the constraints for the fields eq.(50)

$$
X^1 = -i2\pi \frac{1}{m^2} \frac{\partial}{\partial x} + \sum_{a,b} X_{ab,00}^1 e^{iam^2 r_1 x} e^{ibm^2 r_2 y}
$$

$$
X^2 = -i2\pi \frac{1}{m^2} \frac{\partial}{\partial y} + \sum_{a,b} X_{ab,00}^2 e^{iam^2 r_1 x} e^{ibm^2 r_2 y}
$$

$$
X^j = \sum_{a,b} X_{ab,00}^j e^{iam^2 r_1 x} e^{ibm^2 r_2 y}
$$

$$
\Psi = \sum_{a,b} \Psi_{ab,00} e^{iam^2 r_1 x} e^{ibm^2 r_2 y}.
$$

We see that $X^1, X^2$ become covariant derivatives and all other fields are multiplication operators. These are exactly the fields of 2+1 dim $SYM$ on a torus of radii $\frac{1}{m^2 r_1}, \frac{1}{m^2 r_2}$.

All this is independent of $B$. We saw that the only $B$-dependence was to change products of fields to the $\ast$-product. Let us see how the $\ast$-product looks in this basis. We only need to consider the types of
operators which appear in the action. We see from eq.(63) that these are the differential operators, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and multiplication operators. $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are diagonal and it is easily seen from eq.(60) that for diagonal operators the $*$-product is equal to the usual product. Let us now look at two multiplication operators $\phi^{(1)}(x, y)$ and $\phi^{(2)}(x, y)$. We have

$$\phi^{(i)}(x, y) = \sum_{a, b} \phi^{i}_{ab, 00} e^{iam_{s}r_{1}x} e^{ibm_{s}r_{2}y}$$  \hspace{1cm} (64)$$

with $\phi^{i}_{a_{2}b_{2}, a_{1}b_{1}} = \phi^{i}_{(a_{2}-a_{1})(b_{2}-b_{1}), 00}$. Plugging into eq.(60) it is seen that

$$\left(\phi^{(2)} * \phi^{(1)}\right)(x, y) = e^{-\frac{1}{2} \frac{B}{m_{s}^{2}r_{1}r_{2}} \left(\frac{\partial}{\partial x_{2}} \frac{\partial}{\partial y_{1}} - \frac{\partial}{\partial y_{2}} \frac{\partial}{\partial x_{1}}\right)} \phi^{(2)}(x_{2}, y_{2}) \phi^{(1)}(x_{1}, y_{1}) \big|_{x_{2}=x_{1}=x, y_{2}=y_{1}=y}.$$  \hspace{1cm} (65)$$

Let us recapitulate what we have obtained so far. We consider one 0-brane on a $T^{2}$, $N_{0} = 1$, and no membranes, $N_{2} = 0$. The flux of the $B_{ij}$-field through the torus is $B$. We consider the limit coming from matrix theory. If $B = 0$ the resulting theory is a 2+1 dim. SYM on a dual $T^{2}$. In terms of the M-theory variables the radii of the $T^{2}$ are

$$r'_{1} = \frac{1}{m_{s}^{2}r_{1}} = \frac{1}{M^{3}RR_{1}}$$  \hspace{1cm} (66)$$

$$r'_{2} = \frac{1}{m_{s}^{2}r_{2}} = \frac{1}{M^{3}RR_{2}}$$

and the gauge coupling is

$$\frac{1}{g^{2}} = \frac{m_{s}r_{1}r_{2}}{\lambda} = \frac{R_{1}R_{2}}{R^{'}}.$$  \hspace{1cm} (67)$$

The gauge bundle on $T^{2}$ is trivial. This is a consequence of the fact that all the fields in eq.(63) are functions on $T^{2}$ instead of sections.
of a non-trivial bundle. Equivalently \( c_1 = \frac{1}{2\pi} \int trF = 0 \). For any \( B \) the only difference is that every time two fields are being multiplied in the action one should instead use the *-product.

When \( B = 0 \) the *-product, of course, coincides with the usual product. Looking at eq.(60) we see that the product has a periodicity in \( B \) of \( 4\pi \). For \( B = 2\pi \) it is different from \( B = 0 \). At first sight this is problematic, since the theory is known to have a periodicity in \( B \) of \( 2\pi \). The puzzle is resolved by noting that there is a field redefinition which takes the theory at \( B = 2\pi \) into \( B = 0 \). The field redefinition is

\[
\phi_{a_2b_2,a_1b_1} \rightarrow (-1)^{(a_2-a_1)(b_2-b_1)} \phi_{a_2b_2,a_1b_1}.
\]

Thus the gauge theories actually have the correct \( 2\pi \) periodicity in \( B \).

So far we have only discussed the case with \( N_0 = 1 \) and \( N_2 = 0 \), i.e. one 0-brane and no 2-branes. The case with any \( N_0 \) goes in exactly the same way. It is now a \( U(N_0) \) theory instead of a \( U(1) \) theory. Nothing else is changed. Especially the same form of the *-product should be used, except that now the fields are \( N_0 \times N_0 \) matrices.

2.3 Non-trivial gauge bundles

In the previous section we only considered cases with no membranes, \( N_2 = 0 \). What about \( N_2 \neq 0 \)? In the case \( B = 0 \) we know the answer. Here the final theory is obtained by T-duality on both circles. After T-duality we get the decoupled theory of \( N_0 \) D2-branes with \( N_2 \) D0-branes dispersed in the 2-branes. 0-branes in 2-branes is just magnetic flux. In other words now it is a \( U(N_0) \) theory with a non-trivial bundle on \( T^2 \). The first Chern class is \( c_1 = \frac{1}{2\pi} \int trF = N_2 \).

In the previous section we saw that for \( B \neq 0 \) and \( N_2 = 0 \) the theory became a \( U(N_0) \) theory with \( c_1 = 0 \) and deformed by the *-product. All these theories have radii and coupling given by eq.(66). The obvious guess is now that the case with \( B \neq 0 \) and \( N_2 \neq 0 \) was described by a \( U(N_0) \) theory with \( c_1 = N_2 \) and an action deformed
by the ∗-product. However this cannot be true, at least not in this naive sense. The reason is that if the bundle is non-trivial we really need to define the fields in coordinate patches. The ∗-product does not transform correctly under change of patch. To make it do so we would have to replace \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \) by covariant derivatives. Even if this is possible there are other reasons to doubt that this is correct. Firstly \( B \rightarrow B + 2\pi \) is not a symmetry in the presence of 2-branes. It is only a symmetry if one changes the number of 0-branes: \( N_0 \rightarrow N_0 - N_2 \). For \( N_2 = 1 \) one could always change to \( N_0 = 0 \). If the above guess was correct this would lead to strange connections between “\( U(0) \)” and \( U(N) \) theories.

Another reason to doubt this naive guess is the following. When one uses Sen’s and Seiberg’s prescription [17,18] to derive a matrix model the energy has the form

\[
E = \sqrt{\left( \frac{N}{R} \right)^2 + P_\perp^2 + m^2} = \frac{N}{R} + \frac{1}{2} \frac{R}{N} (P_\perp^2 + m^2) + \ldots. \tag{69}
\]

Here it is written for uncompactified M-theory, but a similar expression is valid for all compactifications. The point is that in the limit \( R \rightarrow 0 \), the first term goes to infinity, the second term stays finite (after rescaling of all energies) and the terms indicated by dots vanish. The first term goes to infinity but is fixed and independent of any dynamics. Therefore we can ignore it and just keep the second term. A matrix theory Hamiltonian always gives the second term. When we change \( N \) the theory changes drastically. For instance the gauge group changes. In other words when the infinite term is changed we expect the finite term to change drastically. Let us now look at our situation. With \( N_0 \) 0-branes, \( N_2 \) 2-branes and a \( B_{ij} \)-field flux \( B \). Here the energy takes the form

\[
E = \frac{N_0 + BN_2}{R} + \text{finite}. \tag{70}
\]

For \( B \neq 0 \) the infinite term changes when \( N_2 \) is changed. So following the remarks above we expect the theory to change drastically.
Specifically it is probably not enough to change the bundle, but also radii and gauge coupling changes.

Whether or not the case of $N_2 \neq 0$ is solved by just changing the first Chern class, there is another way of solving it. This is the subject of next section.

2.4 INCORPORATING 2-BRANES

In this section we will obtain the matrix model for the general case, with $N_0$ 0-branes, $N_2$ 2-branes and a flux $B$. We will do that by performing a T-duality to transform to the case $N_2 = 0$.

For a review of T-duality, see [24]. The T-duality group for IIA on $T^2$ contains an $SL(2, \mathbb{Z})$ subgroup which acts as follows. It leaves the complex structure of $T^2$ invariant. Define a complex number in the upper halfplane, $\rho = B + iV$. Here $V$ is the volume of the torus measured in string units and $B$ is the flux of $B_{ij}$ through the torus.

In our case $V = m_8^2 r_1 r_2$. An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ acts as follows

$$\rho' = \frac{a\rho + b}{c\rho + d}, \quad \begin{pmatrix} N_0' \\ -N_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} N_0 \\ -N_2 \end{pmatrix}. \quad (71)$$

Let us use this in our case. Let $Q$ be the greatest common divisor of $N_0$ and $N_2$. Write

$$N_2 = Q\tilde{N}_2, \quad N_0 = Q\tilde{N}_0. \quad (72)$$

Since $\tilde{N}_0, \tilde{N}_2$ are relatively prime we can choose $a, b$ such that $a\tilde{N}_0 - b\tilde{N}_2 = 1$. Let us now perform a T-duality transformation with the
matrix
\[
\begin{pmatrix}
  a & b \\
  \tilde{N}_2 & \tilde{N}_0
\end{pmatrix}.
\tag{73}
\]

An easy calculation gives the new radii, \( B_{ij} \) flux, 0-brane and 2-brane numbers

\[
\begin{align*}
  r_1' &= \frac{r_1}{\tilde{N}_0 + \tilde{N}_2 B} \\
  r_2' &= \frac{r_2}{\tilde{N}_0 + \tilde{N}_2 B} \\
  B' &= \frac{aB + b}{\tilde{N}_0 + \tilde{N}_2 B} \\
  N_0' &= Q \\
  N_2' &= 0.
\end{align*}
\tag{74}
\]

The string mass is invariant
\[
m_s' = m_s
\tag{75}
\]
and the new coupling is
\[
\lambda' = \frac{\lambda}{\tilde{N}_0 + \tilde{N}_2 B}.
\tag{76}
\]

In these formulas we have taken the matrix theory limit in the quantities which have a non-zero limit. We remark that the denominator \( \tilde{N}_0 + \tilde{N}_2 B \) is positive since
\[
P_- = \frac{\tilde{N}_0 + \tilde{N}_2 B}{R}
\tag{77}
\]
and \( P_- \) is positive as always in matrix theory. We now see that the parameters of the theory go to zero and infinity in exactly the same
way as in last section. This means that we are in exactly the same situation as in last section.

In other words the matrix theory is a 2+1 dim. SYM on a $T^2$ with gauge group $U(Q)$ where $Q$ is the greatest common divisor of $N_0$ and $N_2$. The action is deformed with the $*$-product with a value of $B$ equal to

$$B' = \frac{aB + b}{\tilde{N}_0 + \tilde{N}_2B}.$$  (78)

The $T^2$ has radii

$$r_1'' = \frac{1}{m'_{s^2}r_1'} = \frac{\tilde{N}_0 + \tilde{N}_2B}{M^3RR_1}$$
$$r_2'' = \frac{1}{m'_{s^2}r_2'} = \frac{\tilde{N}_0 + \tilde{N}_2B}{M^3RR_2}$$  (79)

and the gauge coupling is

$$\frac{1}{g^2} = \frac{m'_{s^2}r_1'r_2'}{\lambda'} = \frac{R_1R_2}{R(\tilde{N}_0 + \tilde{N}_2B)}.$$  (80)

The SL(2,Z) duality employed has a very easy geometric interpretation if one performs a T-duality on a single circle as in [13]. Here $N_0, N_2$ parametrize which homology cycle the D-strings wrap. The factor $\tilde{N}_0 + \tilde{N}_2B$ is just the length of the D-string. The T-duality transformation is just a geometric change of $\tau$-parameter of the torus.
2.5 Conclusion

It was explained in [5,6] how to describe 0-branes on $T^2$ by working on the covering space $\mathbb{R}^2$ and modding out by translations. We did this in the presence of a background B-field. This enabled us to get a matrix theory of M-theory on $T^2$ with a background $C_{-12}$. The result agrees with [2,13] and is a gauge theory on a noncommutative torus.

There are some interesting aspects of this. In the case $B = 0$ this procedure leads to a 2+1 dim SYM which is exactly the same as the theory on the D2-brane in the T-dual picture. In other words the procedure of compactifying the 0-branes agrees with T-duality. For $B \neq 0$ this is not the case. T-duality does not give a theory on a finite torus when $B \neq 0$. This is the whole reason for all this interest in $B \neq 0$. This means that working with 0-branes on the covering space is not the same as T-duality. We believe, of course, that T-duality still is true. The point is just that the T-dual description is not simpler. The T-dual description is the theory on D2-branes wrapped on a dual $T^2$ which is again shrinking. To extract a well defined action out of that one has to expand the full Born-Infeld action as advocated in [25]. It would indeed be interesting to use the noncommutative theory to put constraints on the full Born-Infeld action. All the higher derivative terms should come out of this.

Originally it was thought that compactifications of M-theory could be gotten by compactifying the 0-brane quantum mechanics. That was indeed the case for toroidal compactifications up to $T^3$. For other compactifications certain degrees of freedom were missing. It was later realized that the correct way of obtaining the matrix model was to use string dualities in order to realise the theory as a theory living on a brane decoupled from gravity. In the case of $C_{-12} \neq 0$ we are in some sense back to the first philosophy. We can obtain the final theory starting with the 0-brane theory but we do not know how to realise it as a sensible limit of a theory on a brane.

It is an interesting question whether these new theories make sense as renormalizable quantum theories. In the case of $B = 0$ we know that the procedure of putting 0-branes on the covering space gives a renormalizable theory up to $T^3$ and not for higher tori. So certainly arguing that this procedure should give a well defined theory
is wrong. However, one might hope that the question of renormalisability is related to the “number of degrees of freedom”. In that sense the theory on $T^d$ with $B \neq 0$ behaves like the theory on $T^d$ with $B = 0$ and we might expect that the noncommutative theories are well defined up to $T^3$. Realizing these theories as theories on branes would resolve this issue, but as discussed above this might require knowing the full Born-Infeld action.

It will be very interesting to see what the methods of noncommutative geometry can teach us about string theory and the other way round.
3. Instantons in noncommutative gauge theories

In this chapter we will discuss certain aspects of the D0-D4 brane system because that will be relevant to us in chapter 5.

Consider type IIA on $T^4$ with radii, $R_i, i = 1, \ldots, 4$, with $N_4$ D4-branes wrapped on $T^4$. There is no $B^{NS}$-field at the moment. Let the string mass be $m_s$ and the coupling be $\lambda$. If $m_s R_i \gg 1$ it is well known that this system is described by 4+1 dimensional maximally supersymmetric Yang-Mills theory on a $T^4$. A $U(N_4)$ bundle on $T^4$ is characterised by the chern numbers, which are integers. In physical terms the first chern class corresponds to the number of D2-branes wrapped on the 6 two-cycles on $T^4$ and the second chern class to the number of D0-branes. The moduli space of lowest energy configurations are the instanton configurations where

$$F^+ + \omega^+ = 0 \quad (81)$$

Here $F^+$ means the selfdual part of $F$ and $\omega^+$ is a selfdual constant in the $U(1)$ part of $U(N)$. Especially the $SU(N)$ part is antiselfdual. Of course one could exchange selfdual with antiselfdual above. This depends on whether there is a positive or negative number of D0-branes.

Let the number of D0-branes be $N_0$. Suppose that $m_s R_i \ll 1$, then we can obtain a description of the same system by a T-duality on all 4 circles. This will give us a $U(N_0)$ gauge theory with chern numbers determined by the D2-branes and $N_4$. The lowest energy configurations are the instanton configurations,

$$F^+ + \omega^+ = 0 \quad (82)$$

now with the $U(N_0)$ gauge field. We will now make the following claim. Whereas the gauge theory is only a good description when the radii are big the moduli space of lowest energy configurations is always equivalent to an instanton moduli space. To our knowledge
there is no rigorous proof of this but it is very plausible for the following reasons. The moduli space is a hyperkähler manifold because of the amount of supersymmetry. An instanton moduli space is also hyperkähler. They are equal at large radii as explained above. Furthermore for very small radii we can perform T-duality to another instanton moduli space as discussed above. However these two instanton moduli spaces are known to be equal by the so called Nahm transformation [26]. These facts taken together strongly suggests that the space of lowest energy configurations is given exactly by the instanton moduli space for all radii, not just large one. This holds irrespective of what the coupling is.

Let us now include the $B^{NS}$-field. We consider type IIA on $T^4$ of radii $R_i$ as before. There is a flux of the NS-NS B-field on $T^4$. Define

$$\theta_{ij} = \frac{1}{2\pi} \int_{T^4} B^{NS}$$

(83)

$\theta_{ij}$ is antisymmetric, so there are 6 independent numbers. There are D0,D2,D4-branes wrapped on cycles of $T^4$. In the previous chapters only a two-torus was discussed. However the discussion can be repeated in any dimension. This time the low energy physics is described by a gauge theory on a noncommutative $T^4$. The noncommutative $T^4$ is defined as the algebra generated by $U_1,U_2,U_3$ and $U_4$ satisfying

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i$$

(84)

Bundles, or finitely generated projective modules, over the noncommutative $T^4$ are classified by 8 integers, exactly corresponding to the numbers of D0,D2 and D4 branes wrapped on the original $T^4$. For a certain regime of radii, $m_s$ and $\lambda$ the low energy physics of this system is described by a 4+1 dimensional SYM on a noncommutative $T^4$. This is not a renormalizable theory, so new degrees of freedom is needed at high energy. However the lowest energy configurations, which are BPS, can be obtained from the gauge theory. Let us recall

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the supersymmetry variation from eq.(4), eq.(5) in the case of static bosonic configurations

$$\delta \Psi = -\frac{1}{4} [X^i, X^j] \Gamma^{ij} \epsilon + \zeta$$  \hspace{1cm} (85)$$

We are interested in configurations with \(X^i \neq 0\) only for \(i = 1, 2, 3, 4\). These \(X^i\) are connections in the module over the noncommutative torus as explained in chapter 1. \([X^i, X^j]\) is the field strength \(F^{ij}\). Let us look for solutions to

$$\delta \Psi = 0$$  \hspace{1cm} (86)$$

We can try to take \(\epsilon\) chiral

$$\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \epsilon = -\epsilon$$  \hspace{1cm} (87)$$

then

$$\delta \Psi = -\frac{1}{4} F_{ij}^{+} \Gamma_{ij} \epsilon + \zeta$$  \hspace{1cm} (88)$$

where \(F_{ij}^{+}\) is the selfdual part of \(F\). We see that if

$$F^{+} = -\omega^{+}$$  \hspace{1cm} (89)$$

where \(\omega^{+}\) is a constant then \(\zeta\) can always be adjusted to cancel the first term. In other words if

$$F^{+} = -\omega^{+}$$  \hspace{1cm} (90)$$

then all \(\epsilon\) obeying eq.(87) are preserved. So 8 supercharges are preserved. This works exactly as in the commutative case, which is a special case of this. The connections satisfying the instanton equation, eq.(89), are the ones that preserve half the supersymmetry.
Locally a connection has the form
\[ \nabla_i = \partial_i + A_i \]  (91)
where \( A_i \) is a matrix valued one-form. The field strength is
\[ F_{ij} = \partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i \]  (92)
the difference between the commutative and noncommutative case is that the matrix entries are functions on respectively a commutative and a noncommutative space. Alternatively one can make the \( A_i \) ordinary functions but then the product should be replaced with the *-product defined in eq.(46)
\[ F_{ij} = \partial_i A_j - \partial_j A_i + A_i * A_j - A_j * A_i \]  (93)
We thus see that the instanton equations are being deformed by the noncommutativity.

Let us now discuss the possible singularities in the instanton moduli space. Consider first the D0-D4 system in type IIA on \( \mathbb{R}^{1,9} \) without any D2-branes and zero \( B_{NS} \)-field. A D0-brane alone preserves the supersymmetry
\[ \Gamma_0 \epsilon_L = \epsilon_R \]  (94)
where \( \epsilon_L, \epsilon_R \) is left and right chirality spinors of \( SO(1,9) \). A D4-brane oriented along directions 1, 2, 3, 4 preserve
\[ \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 e_L = \epsilon_R \]  (95)
Each brane preserves 16 supercharges. If they are both present, separated from each other, they preserve 8 supercharges, namely the \( \epsilon_L, \epsilon_R \) that obey both equations, eq.(94), eq.(95). This means there will be no force between a D0-brane and a D4-brane. One can move the D0-brane into the D4-brane where it can dissolve in a bound state. This looks like an instanton on the D4-brane worldvolume. The instanton moduli space has singularities coming from small instantons. The singularities in the instanton moduli space exactly reflects that there is a branch where the branes are separated.
Let us now turn on a constant $B^{NS}$-field along directions $1, 2, 3, 4$. In other words $B^{NS}$ is a closed 2-form on $\mathbb{R}^4$. The D0-brane still preserves

$$\Gamma_0 \epsilon_L = \epsilon_R$$  \hspace{1cm} (96)

The D4-brane preserves

$$\frac{1}{\sqrt{\det(1 + B_{ij})}} e^{-B_{ij} \frac{\delta}{\delta \Gamma_i} \frac{\delta}{\delta \Gamma_j}} \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \epsilon_L = \epsilon_R$$  \hspace{1cm} (97)

where $\frac{\delta}{\delta \Gamma_i}$ removes a factor of $\Gamma_i$, similarly to the way we differentiate Grassmann variables. This formula can be derived as follows. First there is a Lorentz frame where $B$ has the form

$$B = \begin{pmatrix}
0 & B_{12} & 0 & 0 \\
-B_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & B_{34} \\
0 & 0 & -B_{34} & 0
\end{pmatrix}$$  \hspace{1cm} (98)

Here it is easy to show, compactifying on a torus and using T-duality for instance, that the preserved supersymmetry is

$$\Gamma_0 \frac{(\Gamma_1 \Gamma_2 + B_{12}) (\Gamma_3 \Gamma_4 + B_{34})}{\sqrt{1 + B_{12}^2}} \sqrt{1 + B_{34}^2} \epsilon_L = \epsilon_R$$  \hspace{1cm} (99)

The formula above is just the Lorentz-invariant version of this. Let us check for which B-fields the D0-brane and D4-brane preserves supersymmetry when they are separated. We will do it in the special
Lorentz frame, even though it does not make a difference. We need to solve eq. (96) and eq. (97). Combining them we get

\[
\frac{(\Gamma_1 \Gamma_2 + B_{12})}{\sqrt{1 + B_{12}^2}} \frac{(\Gamma_3 \Gamma_4 + B_{34})}{\sqrt{1 + B_{34}^2}} \epsilon_L = \epsilon_L
\]  

(100)

In other words the matrix on the lefthand side needs to have the eigenvalue +1. \(\Gamma_1 \Gamma_2\) and \(\Gamma_3 \Gamma_4\) have eigenvalues \(\pm i\) and can be diagonalized simultaneously. It is easy to see that we need either

\[
\Gamma_1 \Gamma_2 = i, \quad \Gamma_3 \Gamma_4 = -i
\]  

(101)

or

\[
\Gamma_1 \Gamma_2 = -i, \quad \Gamma_3 \Gamma_4 = i
\]  

(102)

and

\[
B_{12} = B_{34}
\]  

(103)

The Lorentz invariant version of the last equation is that \(B_{NS}\) is selfdual. We have thus derived that if and only if \(B_{NS}\) is selfdual can a separated D0-brane and D4-brane preserve supersymmetry. They preserve 8 supercharges in this case.

What about a bound state of D0-branes and D4-branes. When \(B_{NS} = 0\) it exists as a BPS state. It has a description as instantons, as we discussed above. In other words it is a single particle state in a short multiplet. When we vary \(B_{NS}\) the dimension of the representation can not jump. We thus conclude that the bound state always exists as a BPS state. This argument has been used often in string dualities, in going from weak to strong coupling. Here we do not vary the coupling but the \(B_{NS}\)-field.

Suppose a D0-brane is bound to \(N_4\) D4-branes in the presence of a \(B_{NS}\)-field. It can be described as an instanton in the noncommutative \(U(N_4)\) theory. If \(B_{NS}\) is not selfdual the D0-brane can not leave the D4-branes meaning that there is no small instanton. We
thus see that the singularities in the instanton moduli space has been resolved. Here we argued from string theory. It would be interesting to confirm this picture directly from the instanton equations. A discussion of various aspects of noncommutative instantons can be found in [27,28,29].

This discussion was for the case of $\mathbb{R}^4$. It stays valid on $T^4$, since the preserved supersymmetry is identical on $\mathbb{R}^4$ and $T^4$. $T^4$ can be obtained from $\mathbb{R}^4$ by a periodic identification which does not change any of the above. In chapter 5 we will return to noncommutative instantons.
4. Twisted \((2, 0)\) and Little-String Theories

Now we will leave the realm of noncommutative geometry for a while. In this chapter we will study the compactification of the \((2, 0)\) theory and the little-string theory on \(S^1, T^2\) and \(T^3\). The \((2, 0)\)-theory describes the low-energy modes coming from type-IIB on an \(A_{k-1}\) singularity \([30]\) or, equivalently, \(k\) 5-branes of M-theory \([31]\). The little-string theory is the theory of \(k\) type-II NS5-branes decoupled from gravity \([32]\). In order to get an interesting low-energy question we will twist the boundary conditions along \(T^d\) by elements of the \(Spin(5)\) (or \(Spin(4)\) for the little-string theory) R-symmetry. In this way we obtain new kinds of theories with 8 supersymmetries. The aim of this paper is to find the low-energy description of these theories. We will present an explicit construction in the case \(k = 2\). The construction for \(k = 2\) involves the moduli space of the heterotic 5-brane wrapped on tori.

In certain limits we recover the known moduli spaces of Super-Yang-Mills theories with a massive adjoint hypermultiplet. In the compactified little-string theories, examination of the moduli space shows that for certain values of the external parameters there is a phase transition to a phase where little-strings condense.

The chapter is organized as follows. In section (4.1) we explain our notation, present the problem and discuss the parameters on which the compactifications depend. In section (4.2) we present the general solution for \(k = 2\). In section (4.3) we study in more detail various limiting cases of the solution. In particular, we study the limits where Super-Yang-Mills is obtained. In subsection (4.3.2) we observe the phase transition. In section (4.4) we explain the relation between the twist and the mass of the adjoint hypermultiplets in the effective low-energy description of Super-Yang-Mills. In section (4.5) we discuss in more detail what it means to twist the little-string theories. We study what happens to the twists after T-duality and suggest that the R-symmetry twists are a special case of a more general twist. We end with a discussion and open problems.
4.1 The problem

The problem that we are going to study is to find the Seiberg-Witten curves of certain $\mathcal{N} = 2$ theories in 3+1D and to find the hyperkähler moduli space of certain $\mathcal{N} = 4$ theories in 2+1D. The $\mathcal{N} = 2$ theories will be obtained by compactifying the (2, 0) theory or, slightly generalizing, the little-string theory, on $T^2$ with twisted R-symmetry boundary conditions along the sides of the torus. The $\mathcal{N} = 4$ theories in 2+1D are similarly obtained by compactification on $T^3$. In this section we will describe the setting and the notation.

4.1.1 Definitions

Let us denote by $T(k)$ the (2, 0) low-energy theory of $k$ 5-branes of M-theory [30,31]. We denote by $S_A(k)$ ($S_B(k)$) the theory of $k$ type-IIA (type-IIB) NS5-branes in the limit when the string coupling goes to zero keeping the string tension fixed [32]. Compactified on a circle, these two theories are T-dual. $T(k)$ is often called “the (2, 0) theory” and $S(k)$ is referred to as “the little-string theory”.

4.1.2 The (2, 0) theory and the little-string theories

When $T(k)$ is compactified on $T^2$ we get a 3+1D theory which at low-energy becomes $k$ free vector multiplets (at generic points in the moduli space). The vector-multiplet moduli space is $(S^1 \times \mathbb{R}^5)^k/S_k$ where the size of $S^1$ is $A^{-1/2}$ and $A$ is the area of $T^2$. When we compactify $T(k)$ on $T^3$ the low-energy is (generically on the moduli space) given by a $\sigma$-model on the hyperkähler manifold $(T^3 \times \mathbb{R}^5)^k/S_k$. The $T^3$ in the moduli space has the same shape as the physical $T^3$ but its volume is $V^{-1/2}$, where $V$ is the volume of the physical $T^3$. (See [33] for review.) $S_A(k)$ has a low-energy description given by 5+1D SYM and has a scale $M_s$. The scale is related to the SYM coupling constant $M_s^{-1}$. The parameters of the compactification are now the metric on $T^3$ and also the NSNS 2-form on $T^3$. The 2-form couples as a $\theta$-angle in the effective 5+1D
low-energy SYM, i.e. as $\int B \wedge \text{tr} \{F \wedge F\}$. Together they parameterize

$$SO(3, 3, \mathbb{Z}) \backslash SO(3, 3, \mathbb{R})/(SO(3) \times SO(3)) = SL(4, \mathbb{Z}) \backslash SL(4, \mathbb{R})/SO(4).$$

(104)

The moduli space is given by $(T^4 \times \mathbb{R}^4)^k/S_k$ where $T^4$ has the shape given by the point in $SL(4, \mathbb{Z}) \backslash SL(4, \mathbb{R})/SO(4)$ and has a fixed volume $M_s^2$.

We have to mention that the arguments of [34] (see also [35]) show that the theories $S(k)$ are far more complicated than the $T(k)$ theories, in the sense that they have a continuous spectrum starting at energy around $M_s$ and this spectrum describes graviton states propagating in a weakly coupled throat. Below the scale $M_s$ there is a discrete spectrum (up to the effect of the $4k$ non-compact scalars). Since there is a mass gap, one can still ask low-energy questions, as we are doing.

4.1.3 R-symmetry Wilson lines

The compactifications discussed above have 16 supersymmetries and therefore the moduli spaces obtained in 2+1D are flat and only their global structure is interesting. To get interesting metrics on the moduli space we need to break the supersymmetry down by $1/2$. This can be done as follows. Suppose we identify a global symmetry of the $(2, 0)$ theory. When we compactify on $S^1$ of radius $R$ and coordinate $0 \leq x \leq 2\pi R$, we can glue the points $x = 0$ and $x = 2\pi R$ by adding a twist of the global symmetry. When we compactify on $T^3$ we can twist along all 3 directions so long as the twists commute. The global symmetry of $T(k)$ is the Spin(5) R-symmetry. Such a twist has been recently used in [36] to break the supersymmetry of the $(2, 0)$ theory in compactifications.

When we compactify the little-string theory $S_A(k)$ ($S_B(k)$) it is not immediately obvious that we can use such a twist because the space-time interpretation is not unique. However, since we can embed the twist as a geometrical twist in type-IIA, the question is well defined. We will elaborate more on that point in section (6).
Let us now take the $(2,0)$ theory $T(k)$ on $\mathbf{T}^3$ with three commuting twists $g_1, g_2, g_3 \in \text{Spin}(5)$ along $\mathbf{T}^3$. The 16 super-charges of $T(k)$ transform as a space-time spinor which also has indices in the $4$ of $\text{Spin}(5)$. The condition that 8 supersymmetries will be preserved is the condition that $g_1, g_2, g_3$ preserve a two-dimensional subspace of the representation $4$ of $\text{Spin}(5)$. This becomes the following condition. Take $SU(2)_B \times SU(2)_U = \text{Spin}(4) \subset \text{Spin}(5)$ and let $g_1, g_2, g_3$ be $3$ commuting elements in the first $SU(2)_B$ factor. This is the generic twist which preserves $N = 4$ in $2+1$D. Similarly, for the little-string theory $S(k)$ the R-symmetry is $\text{Spin}(4)$ and we need,

$$g_1, g_2, g_3 \in SU(2)_B \subset SU(2)_B \times SU(2)_U = \text{Spin}(4).$$

Since the $g_i$'s are commuting they can be taken inside a $U(1)$ subgroup of $SU(2)_B$. Then $g_i = e^{i\alpha_i} \in U(1) \subset SU(2)_B$. The subscripts $B$ and $U$ are short for “broken” and “unbroken” respectively. We can now ask what is the low-energy description of $T(k)$, $S_A(k)$ and $S_B(k)$ compactified, in turn, on $\mathbf{S}^1$, $\mathbf{T}^2$ and $\mathbf{T}^3$ with twists $\alpha_i$. The most general question is about $S(k)$ on $\mathbf{T}^3$ since all others can be obtained by taking appropriate limits. The low-energy description in $2+1$D is a $\sigma$-model on a $4(k-1)$-dimensional hyperkähler manifold. We will always ignore the decoupled “center of mass”. Furthermore, as will be elaborated in section (4), in appropriate limits we obtain $3+1$D or $2+1$D $SU(k)$ SYM with a massive adjoint hypermultiplet.

What is the external parameter space? The parameter space for the metric and $B$ fields on $\mathbf{T}^3$ is given by (104). The parameter space for conjugacy classes of three commuting $SU(2)$ R-symmetry twists along $\mathbf{T}^3$ is given by $\widetilde{\mathbf{T}}^3/\mathbb{Z}_2$ where $\widetilde{\mathbf{T}}^3$ is the torus dual to $\mathbf{T}^3$ and $\mathbb{Z}_2$ is the Weyl group of $SU(2)$. However, with R-symmetry twists, we can no longer divide by the full T-duality group $SO(3,3,\mathbb{Z})$ (see the discussion in section (6)). This means that the parameter space is a fibration of $(\widetilde{\mathbf{T}}^3/\mathbb{Z}_2)$ over

$$SL(3,\mathbb{Z})\backslash SO(3,3,\mathbb{R})/(SO(3) \times SO(3)).$$
4.1.4 Why is the problem not trivially solved by M-theory?

Let us explain why we cannot just read off the SW-curves and moduli spaces from M-theory. To be specific, let us take the 6-dimensional non-compact space defined as an $\mathbb{R}^4$-fibration over $T^2$ with $Spin(4)$ twists along the cycles of the $T^2$. This is the geometric realization of the R-symmetry twist, that we mentioned above (see section (6) for a more detailed discussion). M-theory compactified on this space preserves 16 supersymmetries if the two twists $\alpha_1, \alpha_2$ are taken inside $SU(2)_B \subset SU(2)_B \times SU(2)_U = Spin(4)$. Let us wrap $k$ 5-branes on $T^2$. Given the success of the method described in [37] one may at first sight wonder whether the classical moduli space of the $k$ 5-branes immediately gives the right answer. The answer is negative. There is, in fact, a big difference between the situation in [37] and ours. The construction of [37] was used to solve certain QCD questions. As explained there, QCD is not the low-energy description of 5-branes in M-theory. It is not even an approximate one. QCD is only a good approximation in the region of moduli space where the 5-branes are close together and the 11th dimension is very small. When this parameter was increased the dynamics of the system is completely changed except for the vacuum states (i.e. the moduli of the vector-multiplets). This relied on the fact that the parameter that deforms the system from close NS5-branes and D4-branes in type-IIA to M5-branes decoupled from the vector-multiplet moduli space (similarly to the decoupling in [38] and [39,40]). The classical result was correct for the M5-brane limit because all the relevant sizes were much larger than $M_{Pl}$ (the Planck scale).

In our case, not all the relevant sizes of the M5-brane configuration are large. Let $A$ be the area of $T^2$ and let $\Phi$ be the modulus of the tensor multiplet in 5+1D. $\Phi$ is related to the separation $y$ between the 5-branes as $\Phi \sim M_p^3 y$. The interesting region in moduli space is $\Phi A \sim 1$. This region is $M_p^3 Ay \sim 1$ and at least one of $y$ or $A$ cannot be made large.
4.2 Solution

In this section we will consider the theory $S_A(2)$ compactified on $T^3$. We recall that $S_A(2)$ is the theory living on 2 coincident NS 5-branes in type IIA in the limit of vanishing string coupling with string scale, $M_s$, kept fixed. The compactified theory has a moduli space of vacua which is a hyperkähler manifold. The purpose of this section is to find this hyperkähler manifold as a function of the parameters of the compactification. These parameters are described above. There is the IIA string scale, $M_s$ (which is already a parameter in 6 dimensions). There is the metric, $G_{ij}^A$ and NS-NS 2-form, $B_{ij}^A$, on the $T^3$. Here $A$ denotes the underlying type IIA theory. Finally, there are the holonomies of the $Spin(4)$ R-symmetry around the 3 circles. The holonomies are taken inside an $SU(2)_B$ subgroup of $Spin(4)$ to preserve half of the supersymmetries. The 3 holonomies must commute and can thus be taken inside $U(1) \subset SU(2)_B$. We denoted the holonomies $e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}$, where $\alpha_i$ is periodic with period $2\pi$. Furthermore the Weyl group of $SU(2)_B$ relates $\alpha_i$ to $-\alpha_i$. These are the parameters of the theory.

The moduli space of vacua has real dimension 4, since we are dealing with 2 5-branes and we throw away the center of mass motion. We want to find the metric on this as a function of $M_s, G_{ij}^A, B_{ij}^A$ and $\alpha_i$. Our strategy will be to start at the special point $\alpha_i = 0$ and then later understand how to do the general case. At $\alpha_i = 0$ the theory actually has $N = 8$ supersymmetry in 3 dimensions (like $N = 4$ in 4 dimensions). Here the moduli space is just the classical one. At the origin of the moduli space the low energy theory is an $SU(2), N = 8$ theory. There are also heavy Kaluza-Klein modes with masses that go like multiples of $\frac{1}{R_i}$, where $R_i$ are the radii of the circles. In $N = 4$ language the multiplet is a vector-multiplet and an adjoint hypermultiplet. On the the moduli space of vacua $SU(2)$ is broken to $U(1)$. Dualizing the photon gives an extra scalar, so the vector-multiplet has 4 scalars. In the $N = 8$ theory the moduli space of vacua is 8 dimensional. Four of the directions come from scalars in the hypermultiplet. These are lifted as soon as $\alpha_i \neq 0$, because $\alpha_i$ supply a mass to the hypermultiplet. We are really only interested in
the 4 directions coming from scalars in the vector-multiplet. These 4 scalars are all compact. From the 5-brane point of view these scalars come about as follows. One of them is the relative position of the 5-branes on the 11th circle. The other 3 are the 2-form living on the 5-brane with indices along the $T^3$. These 4 scalars are obviously compact. The Weyl group of the SU(2) gauge group changes the sign of all these. We thus see that the moduli space of vacua is $T^4/\mathbb{Z}_2$. When we deform to $\alpha_i \neq 0$, the moduli space remains compact. The only compact 4 dimensional hyperkähler manifolds are K3 and $T^4$. $T^4/\mathbb{Z}_2$ is topologically a K3 manifold with a singular metric. We thus conclude that for all parameters $G^A_{ij}, B^A_{ij}, \alpha_i$ the moduli space is topologically K3. We just need to find the hyperkähler metric as a function of these parameters.

Let us first recall the moduli space of hyperkähler metrics on K3. It is

$$O(3, 19, \mathbb{Z}) \backslash O(3, 19, \mathbb{R}) / ((O(3) \times O(19)) \times \mathbb{R}^+).$$

$\mathbb{R}^+$ parameterizes the volume. This moduli space nicely coincides with the moduli space for Heterotic string theory on $T^3$. This is a well-known consequence of the duality of M-theory on K3 with heterotic on $T^3$. On the heterotic side the $\mathbb{R}^+$ denotes the dilaton. The space $O(3, 19, \mathbb{R}) / O(3) \times O(19)$ can be parameterized by the metric and NS-NS 2-form on the $T^3$, $G^H_{ij}, B^H_{ij}$ and the Wilson lines around the 3 circles $V_1, V_2, V_3$. We will work with the $E_8 \times E_8$ Heterotic theory. The reason for that will become clear in a moment. There is a very nice way of obtaining the K3 on the M-theory side as a moduli space of vacua for a 3-dimensional $\mathcal{N} = 4$ theory. This is the membrane of M-theory imbedded in $R^{1,6} \times K3$ with the world-volume along $R^{1,6}$ and at a point in K3. On the dual Heterotic side it corresponds to the 5-brane wrapped on $T^3$ [40]. This is thus the moduli space of the $(1, 0)$ little-string theory obtained from an NS5-brane in the heterotic string by taking the coupling constant to zero [32].

Our aim can now be formulated as finding $G^H_{ij}, B^H_{ij}, V_1, V_2, V_3$ for given $G^A_{ij}, B^A_{ij}, \alpha_i$. According to the arguments of [42], the exter-
nal parameters can be combined into scalar components of auxiliary vector-multiplets which are non-dynamical. Supersymmetry then requires that the periods of the three 2-forms which determine the hyperkähler metric on the moduli space are linear in these combinations of external parameters [43]. To find the map subject to this restriction, we first examine $\alpha_i = 0$. We saw earlier that this was the $\mathcal{N} = 8$ theory and the moduli space is $\mathbb{T}^4/\mathbb{Z}_2$. Therefore, we can find the data of the $\mathbb{T}^4$ by classical analysis, starting from the $(1,0)$ tensor-multiplet living on the IIA 5-brane. (We have ignored the VEVs along the $(1,0)$ hypermultiplet direction.) The $(1,0)$ tensor-multiplet is also the low-energy description of the $E_8 \times E_8$ Heterotic 5-brane and the scalar is compact since it corresponds to motion in the $11^{th}$ direction, which is an interval. Let us compactify this theory on $\mathbb{T}^3$ with data $G^H_{ij}, B^H_{ij}, V_1, V_2, V_3$. To obtain the same moduli space of vacua as in the $S_A(2)$ case we need to set $G^H = G^A$, $B^H = B^A$. What about $V_1, V_2, V_3$? The $S_A(2)$ theory had a $\mathbb{T}^4/\mathbb{Z}_2$ as moduli space. $\mathbb{T}^4/\mathbb{Z}_2$ has 16 $A_1$ singularities. This means that M-theory on this K3 has $SU(2)^{16}$ gauge symmetry. To achieve this we need very special Wilson lines. We can take $V_1$ to break $E_8 \times E_8$ to $SO(16) \times SO(16)$ and $V_2$ to break each $SO(16)$ to $SO(8) \times SO(8)$ and $V_3$ to break each $SO(8)$ to $SO(4) \times SO(4)$. The unbroken symmetry group is thus $SO(4)^8 = SU(2)^{16}$ as desired. These Wilson lines are unique up to $E_8 \times E_8$ conjugation. We can write down $V_1, V_2, V_3$ explicitly. The two $E_8$’s are treated symmetrically, so we restrict to one of them. Consider $\Gamma^8 \subset \mathbb{R}^8$ where $\Gamma^8$ is the weight lattice of $E_8$. Recall that $\Gamma_8$ can be characterized as all sets $(a_1, \ldots, a_8)$ such that either all $a_i$ are half-integers or all $a_i$ are integers. Furthermore $\sum a_i$ is even. A Wilson line around a circle can be specified by an element $V \in \mathbb{R}^{16}$ such that a “state” given by a weight vector $a$ is transformed as $e^{ia \cdot V}$ on traversing the circle. In this notation
\[ V_1 = (0, 0, 0, 0, 0, 0, 0, 0, 1) \]
\[ V_2 = (0, 0, 0, 0, 1/2, 1/2, 1/2, 1) \]
\[ V_3 = (0, 0, 1/2, 0, 0, 1/2, 1/2) \]
\[ (105) \]

Now we have the map in the case \( \alpha_i = 0 \). We will make a proposal for the general case presently. The 16 singularities in \( \mathbb{T}^4/\mathbb{Z}_2 \) are due to an adjoint hypermultiplet becoming massless. When \( \alpha_i \neq 0 \) the hypermultiplet is massive and we expect the singularities to disappear. Near the original singularities the theory now looks like pure \( SU(2) \) SYM. This does not have any singularities \([40,43]\). We thus see that the Wilson lines must change when we turn on \( \alpha_i \).

We now make the following proposal. For nonzero \( \alpha_i \) we still have 
\[ G^H_{ij} = G^A_{ij}, \quad B^H_{ij} = B^A_{ij}. \]
The Wilson lines \( W_1, W_2, W_3 \) are taken to be,
\[ W_i = V_i + \frac{\alpha_i}{\pi} (\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0) \],
in the notation from above. This is the same as embedding \( e^{\frac{1}{2}i\alpha_i} \) in the diagonal \( SU(2) \subset SU(2)^{16} \subset E_8 \times E_8 \). The coefficients of \( \alpha_i \) are chosen such that the period is \( \alpha_i \rightarrow \alpha_i + 2\pi \).

We do not have a proof of this proposal, but this certainly satisfies the requirements of linearity in external parameters, because this is also the moduli space of the compactified \((1, 0)\) little-string theory. In the coming sections we will show that our proposal is consistent with string theory and field theory expectation.

There is another very similar theory. This is the theory on 2 coincident type IIB NS 5-branes in the limit of vanishing string coupling and fixed string mass. We call this theory \( S_B(2) \). As soon as we compactify it on a circle it is T-dual to the theory studied above. When we compactify it on a \( \mathbb{T}^3 \) with R-symmetry twists we get a 3-dimensional theory with a K3 as the moduli space of vacua. Arguing exactly as in the IIA case we propose that this K3 is given in the same way as in the IIA case, except that we replace Heterotic \( E_8 \times E_8 \)
with Heterotic SO(32). This is because the low energy description of the theory living on a IIB 5-brane is a gauge theory. The Heterotic SO(32) 5-brane is also described, at low energy, by a gauge theory. When we do the comparison at the point without an R-symmetry twist, the $\mathcal{N} = 8$ point, the moduli spaces will automatically agree. This is analogous to the $\mathcal{N} = 8$ point in the IIA case where we compared two tensor-multiplets. The T-duality between the IIA and IIB 5-brane theories on $\mathbf{T}^3$ fits very nicely with the T-duality between Heterotic SO(32) and Heterotic $E_8 \times E_8$ on $\mathbf{T}^3$ at the point $\alpha_i = 0$. For $\alpha_i \neq 0$, the R-symmetry twists do not remain R-symmetry twists after T-duality.

4.3 LIMITS

Now that we have identified the moduli space of vacua for $S_A(2)$ compactified on $\mathbf{T}^3$ with arbitrary R-symmetry twists, we can decompactify one or two of the circles to obtain the moduli space of vacua for $S_A(2)$ compactified to 4 and 5 dimensions. Another limit is to take $M_s \to \infty$ in the $S_A(2)$ theory. This takes us to the $(2, 0)$ theory, which we call $T(2)$. In this section we will consider these limits.

4.3.1 Decompactification limits

Let us first recall the correspondence between M-theory on K3 and Heterotic $E_8 \times E_8$ on $\mathbf{T}^3$. M-theory on K3 has a Planck mass, $M_{Pl}$, and a moduli space

\[ O(3, 19, \mathbb{Z}) \backslash O(3, 19, \mathbb{R}) / ((O(3) \times O(19)) \times \mathbb{R}^+ \]

$\mathbb{R}^+$ denotes the volume of K3, $\text{Vol} (K3)$. In Heterotic $E_8 \times E_8$ on $\mathbf{T}^3$ there is a string mass, $M_s$, and a moduli space, which is the same as for M-theory on K3. There is a 10-dimensional string coupling, $\lambda$. The $\mathbf{T}^3$ has a volume, $\text{Vol} (\mathbf{T}^3)$, which is part of $O(3, 19, \mathbb{R}) / (O(3) \times$
Under the duality an M5-brane wrapped on K3 is mapped to the Heterotic string. Equating the tensions gives,

$$M_{Pl}^6 Vol(K3) = M_s^2.$$  \hspace{1cm} (106)

Equating the 7-dimensional gravitational couplings gives,

$$M_{Pl}^9 Vol(K3) = \frac{M_s^8 Vol(T^3)}{\lambda^2}.$$  \hspace{1cm} (107)

We thus see, that the $\mathbb{R}^+$ on the Heterotic side is $\frac{Vol(T^3)}{\lambda^2}$, which of course is T-duality invariant. Eq.(106) agrees with the fact that the volume of the moduli space of vacua of the Heterotic 5-brane is $M_s^2$ and $M_{Pl}^6 Vol(K3)$ is the volume of the moduli space of the M2-brane probe. We remember that scalar fields have dimension $\frac{1}{2}$ in 3 dimensions. A concrete way of tracing the duality between these two theories is to use T-duality from Heterotic $E_8 \times E_8$ on $T^3$ to Heterotic SO(32) on $T^3$, and then S-duality to type-I on $T^3$, then T-duality to type-IA on $T^3$ which can be viewed as M-theory on K3.

Let us now consider the decompactification to 4 dimensions. This can be done by taking $\alpha_3 = 0$ and $R_3 \to \infty$. In this limit the K3 becomes elliptically fibered with the fiber shrinking. The area of the fiber $A$ is

$$M_{Pl}^3 A = \frac{1}{R_3}.$$  

This can be seen by noting that a membrane wrapped on the fiber corresponds to momentum around the circle $R_3$ in the Heterotic theory. This limit of M-theory on an elliptically fibered K3 is exactly what gives F-theory on this K3. The M2-brane probe becomes the D3-brane probe in F-theory on K3 [44,39]. Since the volume of K3 stays fixed and the fiber shrinks this means that the base grows. One might thus think that the moduli space seen by the D3-brane probe is infinite. However we should remember that a scalar field in
4 dimensions has dimension one, so we need a factor of the type-IIB string mass in the area of the moduli space. Inserting this makes the area is up to a constant, $M_s^2$. This agrees with the expectation from $S_A(2)$ compactified on $T^2$. We can thus summarize our result for the 4-dimensional case. Take the theory $S_A(2)$ with mass scale $M_s$. Compactify it on a $T^2$ with R-symmetry twists given by $\alpha_1, \alpha_2$. The $T^2$ is specified by $G_{ij}^A, B_{ij}^A$. The moduli space of vacua for this $\mathcal{N} = 2$ theory in $D = 4$ is the same as the moduli space of vacua for the $E_8 \times E_8$ Heterotic 5-brane wrapped on $T^2$ with string mass, $M_s$, and a point in $O(18, 2)/O(18) \times O(2)$ given as follows. The metric and 2-form on $T^2$ is $G_{ij}^A, B_{ij}^A$. The Wilson lines on $T^2$ depend on $\alpha_1, \alpha_2$. In the case $\alpha_i = 0$ they are the essentially unique Wilson lines that break $E_8 \times E_8$ to $SO(8)^4$. For non-zero $\alpha_i$ the Wilson lines are constructed as in the last section by embedding in a diagonal $SU(2)^{16} \subset SO(8)^4$.

This wrapped 5-brane in the Heterotic theory is dual to the 3-brane probe in F-theory on the corresponding elliptic-fibered K3. This K3 is the Seiberg-Witten curve for the moduli space. As in the 3 dimensional case, we are not saying that the compactified $S_A(2)$ theory is equal to the little-string theory on the Heterotic 5-brane, but just that the low-energy description is the same. It is obvious that they are not equal since the $S_A(2)$ theory has enhanced supersymmetry when $\alpha_i = 0$.

Decompactifying to 5 dimensions is now easy. The correspondence becomes the following. Consider the theory $S_A(2)$ compactified on $S^1$ of radius $R$, string scale $M_s$ and R-symmetry twist $\alpha$. This is a 5 dimensional theory with $\mathcal{N} = 1$ supersymmetry. The coulomb branch is 1-dimensional. Topologically it is $S^1/\mathbb{Z}_2$. This moduli space is the same as the moduli space of the Heterotic $E_8 \times E_8$ 5-brane compactified on a circle with an $E_8 \times E_8$ Wilson line. The Wilson line for one $E_8$ is,

$$W = (0, 0, 0, 0, 0, 0, 0, 1) + \frac{\alpha}{2\pi}(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0),$$

and the same for the other $E_8$. 51
Completely analogous statements can be made for the type-IIB 5-brane theory, $S_B(2)$. Here the moduli space is given by the 5-brane in the Heterotic SO(32) theory. Let us describe this in detail for the case of 5 dimensions. Consider $S_B(2)$ on a circle of radius $R$, with R-symmetry twist $\alpha$ and string scale $M_s$. The moduli space of vacua of this is the same as the 5-brane of SO(32) Heterotic string theory on a circle with radius $R$, string scale $M_s$ and SO(32) Wilson line

$$W = \left( \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0 \right) + \frac{\alpha}{\pi} \left( \frac{1}{2}, 0, \frac{1}{2}, 0, \ldots, \frac{1}{2}, 0 \right)$$

The string coupling $\lambda$ goes to zero to give a decoupled theory on the 5-brane.

There is a dual type-IA picture of the Heterotic theory. The 5-brane becomes a D4-brane living on an interval with 8-branes. The parameters of the type-IA theory are

$$M'_{s} = \frac{M_{s}}{\sqrt{\lambda}}$$

$$R' = \frac{\lambda}{M_{s}^{2} R}$$

$$\lambda' = \frac{1}{\sqrt{\lambda} M_{s} R}$$

(108)

All quantities of interest to the D4-brane theory have a limit as $\lambda \to 0$. The positions of the D8-branes are given by the Wilson line.

At each end there are 4 D8-branes. There are two more stacks of 4 D8-branes at distance $\frac{\alpha}{2} R'$ from each end. When $\alpha = 2\pi$ the 8-branes reach the other end. This has to be the case since $\alpha$ is periodic with period $2\pi$. We also remark that something interesting happens when $\alpha = \pi$. Here 8 D8-branes are on top of each other. We will return to a discussion of this point later. There behavior in $D = 3, 4$ is similar.
Fig. 7: The dual type-IA picture.

$S_A(2)$ compactified to 5 dimensions was described above by mapping the R-symmetry twists to $E_8 \times E_8$ Wilson lines. For later purposes it will be more convenient to use the type-IA dual description as the theory living on a D4-brane. The chain of dualities going from Heterotic $E_8 \times E_8$ on $S^1$ to type-IA is to first invoke T-duality from Heterotic $E_8 \times E_8$ to Heterotic $SO(32)$, and then proceed as above to reach type-IA. The parameters of the type-IA theory in terms of the parameters of the $S_A(2)$ theory become

$$M'_s = \frac{M_s}{\sqrt{\lambda}} (2\left(\frac{\alpha^2}{\pi^2} + (M_s R)^2\right))^{\frac{1}{4}},$$

$$R' = \frac{\lambda}{M_s^2 R} (2\left(\frac{\alpha^2}{\pi^2} + (M_s R)^2\right))^{\frac{1}{2}},$$

$$\lambda' = \frac{(2\left(\frac{\alpha^2}{\pi^2} + (M_s R)^2\right))^{\frac{5}{4}}}{\sqrt{\lambda M_s R}}.$$

The configuration of D8-branes is as in the $S_B(2)$ case. At each
end there are 4 D8-branes. At distance $\frac{\alpha'}{2} R'$ from the ends are 4 D8-branes. Here

$$\alpha' = \frac{\alpha}{2(\frac{\alpha^2}{\pi^2} + (M_8 R)^2)}.$$  \hspace{1cm} (110)

We see an interesting effect here. When $\alpha = 0$, $\alpha'$ is also zero. For small $\alpha, \alpha'$ is an increasing function. At $\alpha = \pi M_8 R$, $\alpha'$ reaches its maximum and start to decrease.

The moduli spaces of all the $S_A(2)$ theories with all possible $\alpha$-twists occupy some subspace

$$\mathcal{M}' \subset SO(1, 17, \mathbb{Z}) \backslash SO(1, 17, \mathbb{R}) / SO(17).$$

We wish to know what is the locus $\mathcal{M}''$ which the $S_A(2)$ theories with the T-dual $\eta$-twists span. In the above chain of dualities we started with the heterotic $E_8 \times E_8$ 5-brane wrapped on $S^1$. This represented $S_A(2)$ on a circle with a $\alpha$-twist. By definition, $S_A(2)$ with a $\eta$-twist is T-dual to $S_B(2)$ with some $\alpha$-twist and therefore corresponds to a point in the moduli space of the $SO(32)$ 5-brane. We have seen that the points on the $SO(32)$ which correspond to our $S_B(2)$ theories map under heterotic T-duality to the points on the $E_8 \times E_8$ moduli space which correspond to the $S_A(2)$ theories. Thus $\mathcal{M}'$ and $\mathcal{M}''$ are the same locus. Nevertheless, $S_A(2)$ with a $\alpha$-twist is not equivalent to $S_A(2)$ with a $\eta$-twist.

4.3.2 A peculiar phase transition

As we have explained above, when we compactify $S_B(2)$ with a twist $\alpha$ on $S^1$ of radius $R$ we get a 4+1D theory whose low-energy is the same as that of a D4-brane probe in a configuration of D8-branes on an interval. In this configuration there are 2 stacks of four coincident D8-branes. Whenever the D4-brane crosses the stack, a particle of $U(1)$ charge 2 (coming from the adjoint of $SU(2) \supset U(1)$) becomes massless. When the two stacks of D8-branes coincide we get two massless hypermultiplets. Since the low-energy description
of a $U(1)$ with two massless particles is weakly coupled, we can trust field-theory and the conclusion is that there exists a Higgs phase where the massless hyper-multiplets get a VEV and break the $U(1)$.

In the $S_B(2)$ case, this phase transition occurs for $\alpha = \pi$ and for all values of $R$. In contrast, for $S_A(2)$ this only happens for small enough $R$. We can see this from eq.(110). The phase transition occurs when $\alpha' = \pi$ since this is where two stacks of D8-branes coincide in the type-I\(A\) picture. This has a real solution $\alpha$, only if $M_s R \leq \frac{1}{4}$. Such a bound is certainly to be expected for $S_A(2)$. The reason is that this phase transition happens when two NS 5-branes are on opposite points of the $11^{th}$ circle. The 2 hypermultiplets that become massless originate from membranes stretched between the two 5-branes and wrapped on the compactified circle (the circle which takes us from a 6-dimensional theory to a 5-dimensional theory). The tension of the membrane gives a mass to these states. However there is also a contribution to the mass from the zero-point energy of the fields on the membrane. This contribution depends on $\alpha$. For certain values of the parameters the zero-point energy can cancel the mass from the tension. This is how the hypermultiplet can become massless. Obviously the mass from the tension can not be canceled if the $11^{th}$ circle is too big, or equivalently the compactified circle is too large. This is the reason for the above inequality.

4.3.3 The $(2,0)$ limit

Let us briefly consider another limit, namely the limit where $S_A(2)$ on $T^3$ becomes $T(2)$ on $T^3$. This happens when the $11^{th}$ circle opens up. We see from eq.(106) and eq.(107) that $\text{Vol}(K3) \to \infty$, i.e. the moduli space becomes non-compact. This is as expected. Basically we just get half of the K3. The other half goes to infinity. In the $E_8 \times E_8$ Heterotic 5-brane picture it means that the distance between the ends of the world go to infinity and we only look at one end.
4.3.4 Field theory limits

In this section we will compare the moduli spaces of vacua found in the other sections with field theory results. At each point of the moduli spaces for the $T(k)$ and $S(k)$ theories, we can find a field theory description for the light modes. We are fortunate that such field theories in $D = 3, 4, 5$ are known. The metric on the moduli space around the chosen point will be determined by the light matter. We are going to compare our exact metric with this field theory expectation.

Let us start with $S_B(2)$ compactified on $S^1$. The effective field theory is that of the $D4$-brane probe in type-I A. From $S_B(2)$ we have $SU(2)$ gauge theory with (1,1) supersymmetry. (The field content of the (1,1) vector multiplet are a (1,0) vector multiplet and a (0,1) adjoint hypermultiplet.) Upon compactification on $S^1$ of radius $R$ with a R-symmetry twist $\alpha$, the moduli space is parameterized by the sixth-component of the gauge field, $A_6$, in $U(1) \subset SU(2)$. The full R-symmetry of $S_B(2)$ theory is $SO(4) = SU(2)_U \otimes SU(2)_B$ which is broken down to $SU(2)_U$ by $e^{i\alpha} \in U(1) \subset SU(2)_B$. We get in 5D an $SU(2)$ vector-multiplet and massive adjoint vector-multiplets (with masses $\frac{n}{R}$ with $n \in \mathbb{Z} \neq 0$) transforming non-trivially under $SU(2)_U$ R-symmetry. The boundary conditions on the two complex scalars in the adjoint hypermultiplet are shifted by $\alpha$:

$$\phi_1(2\pi R) = e^{i\alpha} \phi_1(0), \quad \phi_2(2\pi R) = e^{-i\alpha} \phi_2(0).$$

This shifts the periodicity of the fields around the circle. The reduction also gives a tower of adjoint hypermultiplets in 5D with masses,

$$m^2 = \left( n + \frac{\alpha}{2\pi} \right)^2 \frac{1}{R^2}, \quad n \in \mathbb{Z}.$$

For small $\alpha > 0$, we get one light adjoint hypermultiplet of mass $\frac{\alpha}{2\pi R}$. Now let us look at the moduli space around $A_6 = 0$. From field
theory it looks like $SU(2)$ theory with an adjoint hypermultiplet of mass $\frac{\alpha}{2\pi R}$. The gauge coupling is then given by [45],

$$\frac{1}{g^2} = b + cA_6,$$

where $b$ and $c$ are constants. The slope, $c$, changes when charged matter becomes massless. The change in the slope is proportional to the cube of the charge of the multiplet becoming massless. In $U(1) \subset SU(2)$ an adjoint field has components of charge $-2, 0, +2$ under the $U(1)$ in units where the 2 of $SU(2)$ has charge $\pm 1$. This means that the change in the slope, $c$, is 8 times bigger for an adjoint hypermultiplet than for a fundamental. Let us calculate at what value of $A_6$ the charge 2 component of the adjoint hypermultiplet becomes massless. The holonomy around the circle is

$$\phi \rightarrow e^{-4\pi i R A_6} \phi.$$

To cancel $e^{i\alpha}$ we thus need

$$A_6 = \frac{\alpha}{4\pi R}.$$

Let us now compare to the solution from the previous section. Here $\alpha$ parameterizes the position of 4 D8-branes. For $\alpha = 2\pi$ they reach the other end of the interval. In terms of $A_6$ the other end of the interval is at $\frac{1}{2R}$, so the position of the 4 D8-branes is,

$$A_6 = \frac{\alpha}{2\pi} \times \frac{1}{2R} = \frac{\alpha}{4\pi R},$$

in exact agreement. However the number of D8-branes is 4 and not 8 as naively expected from the discussion above. It seems like the change in slope is half of what should be expected from field theory.
There is no discrepancy for a subtle reason. We compare, on one hand, the $U(1)$ low energy effective action for a D4-brane moving in an orientifold setting, with, on the other hand, a $U(1)$ from a $SU(2)$ gauge theory. The $U(1)$ on the D4-brane probe corresponds not to the $U(1) \subset SU(2)$ but to one of the $U(1)$ factors in $U(1) \times U(1) \subset U(2)$. The action for the diagonal $U(1) \subset U(2)$ is twice the action for a single $U(1)$ factor. The normalization would contain an extra $\sqrt{2}$ factor. Taking this factor of 2 into account the change in the slope becomes 8 instead of 4.

Let us now consider the case of $S_A(2)$ on $T^3$ with twists $\alpha_1, \alpha_2, \alpha_3$. For simplicity the torus is taken to be rectangular with radii $R_1, R_2, R_3$ with $B_{ij} = 0$. We will also take $\alpha_i$ to be small. We want to find the light fields. Finding the light fields in this case is not as easy as in the previous case, because $S_A(2)$ does not have a Lagrangian description. However we can figure out the result by first compactifying on a small $R_1$, with $\alpha_1 = 0$. Then the low energy description is a 5-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory. In $\mathcal{N} = 1$ language it comprises a vector-multiplet and a hypermultiplet. Now we can compactify this on a second circle of radius $R_2 \gg R_1$. At scale $R_2$ the $SU(2)$ gauge theory is weakly coupled and we can perform a classical analysis to include the twists $\alpha_2$. We get an $SU(2)$ gauge theory in $D = 4$ with a hypermultiplet of mass $\frac{\alpha_2}{2\pi R_2}$. In $D = 4$, $\mathcal{N} = 2$ a hypermultiplet mass is complex. Since there is no distinction between direction 1 and 2 we expect a contribution $\frac{\alpha_1}{2\pi R_1}$ from direction 1. They have to combine into a complex mass

$$m = \frac{\alpha_1}{2\pi R_1} + i\frac{\alpha_2}{2\pi R_2}.$$

On compactifying down to 3 dimensions on $R_3$ (we assume that $R_3 > R_2$) there will similarly be a contribution $\frac{\alpha_3}{2\pi R_3}$. In $D = 3$ a hypermultiplet mass consists of 3 real numbers that transform in the 3 of $SO(3)$ [43]. This $SO(3)$ is part of the R-symmetry group. We thus conclude that the 3 real numbers are $\frac{\alpha_i}{2\pi R_i}$. There is a region in moduli space where the theory looks like $\mathcal{N} = 4$, $SU(2)$ gauge
theory with an adjoint hypermultiplet with mass $m_i = \frac{\alpha_i}{2\pi R_i}$. As we have seen, this region is when $|\alpha_i| \ll \pi$ and when the mass scale set by the 2+1D SYM coupling constant (the smallest of $\frac{R_1}{R_2 R_3}$, $\frac{R_2}{R_1 R_3}$ and $\frac{R_3}{R_1 R_2}$) is much smaller than the smallest compactification scale (the smallest of $R_1^{-1}$, $R_2^{-1}$ and $R_3^{-1}$). In our setting, $R_1 \ll R_2 < R_3$, this condition is met. Note that if $|\alpha_i| \ll \pi$ but $R_1 \sim R_2 \sim R_3$ are of the same order of magnitude, the correct approximation is to start with the 2+1D CFT to which $N = 8$ 2+1D SYM flows \cite{46,47} and deform it by the relevant operator to which the mass deformation flows. When $m_i = 0$ we obtain a $\mathcal{N} = 8$ theory and the moduli space is $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$. This has two $A_1$ singularities. When $m \neq 0$ these are blown up. From our solution in the previous section the sizes of the blow up can be read off as a function of $\alpha_i$. This means that we have derived a formula for the size of the blow-up of the singularities in $D = 3$, $\mathcal{N} = 4$ $SU(2)$ gauge theory with a massive adjoint hypermultiplet.

We can do the same analysis in $D = 4$. For $\alpha_1 = \alpha_2 = 0$ there are 4 singularities. Close to any one of them the system should be describable as an $\mathcal{N} = 2$, $SU(2)$ gauge theory with an adjoint hypermultiplet. For small $\alpha_i$ the mass of the hypermultiplet is

$$m = \frac{\alpha_1}{2\pi R_1} + i \frac{\alpha_2}{2\pi R_2}. $$

We expect this to change the Seiberg-Witten curve. Our result also predicts how this goes. Our picture is that the Seiberg-Witten curve is the same as the D3-brane probe in F-theory on the K3 as described earlier. For $\alpha_i = 0$ this has a description as a type-IIB orientifold 8 plane with 4 D7-branes on top making a $D_4$ singularity \cite{44,39}. For non-zero $\alpha_i$ two of the 7-branes move away, giving a $U(2) \times SO(4) = U(2) \times SU(2) \times SU(2)$ singularity. In a field theory setting this corresponds to the $SU(2)$ Seiberg-Witten theory with 4 fundamental hypermultiplets, 2 of them massless and 2 of them massive with equal mass. Our analysis thus predicts that this situation should have the same curve as the massive adjoint hypermultiplet. In the second
Seiberg-Witten paper [48] this was indeed found to be the case. In comparing the curves with the low energy effective action there is again a factor of 2 in the coupling constant $\tau$ because of a difference in conventions between the adjoint and fundamental case. This is the same factor of 2 as explained in the 5-dimensional case above.

4.4 Reduction of the twisted $(2, 0)$ theory to 4+1D

In this section we will study $T(2)$ on $S^1$ with a twist $\alpha$. Neglecting the overall center-of-mass, the moduli space is 1-dimensional. The low-energy physics is a $U(1)$ vector-multiplet. Let $\phi$ be the scalar partner of the vector field. In this section we will study the BPS states in the theory. There are two different regions in moduli space to consider. Let $R$ be the radius of $S^1$. When $\phi R \ll 1$ we can use the effective 4+1D SYM Lagrangian. We will show that for small $\alpha$, the BPS states come from the $W^\pm$ bosons and the charged states of a massive adjoint hyper-multiplet. When $\phi R \gg 1$ we can identify the charged BPS states with strings wound around $S^1$.

The BPS masses in 4+1D are [45],

$$2\phi, \quad m_0 + 2\phi, \quad m_0 - 2\phi.$$ (112)

In the D4-brane and D8-brane picture, these come from strings connecting the D4-brane to its image, and to the two mirror D8-brane stacks. Here,

$$m_0 = \frac{\alpha}{2\pi R}.$$ 

This can be seen from eq.(109) and eq.(110). The states with mass $2\phi$ are vectors while those with masses $2\phi \pm m_0$ are hyper-multiplets.
4.4.1 Yang-Mills limit

When $\alpha = 0$, the low-energy description of $T(2)$ on $S^1$ is $SU(2)$ SYM with a coupling constant $g^2$ which is proportional to $R$. As long as our energy scale is below the compactification scale $R^{-1}$, the coupling constant is weak and the effective description is good. When $|\alpha| \ll 1$ it can be incorporated as a small perturbation in the effective Lagrangian. It corresponds to giving a bare mass of $m_0$ to the hyper-multiplet in the Lagrangian. After spontaneous breaking of $SU(2)$ down to $U(1)$, the masses in (112) are easily calculated in field theory. $2\phi$ is the mass of the $W^\pm$ bosons while $2\phi \pm m_0$ come from the hyper-multiplet. The adjoint hyper-multiplet also gives rise to a neutral multiplet with a mass $m_0$.

4.4.2 The large-tension limit

Let us assume that $\phi R \gg 1$. In this case, we can first reduce to the 5+1D low-energy of a single $\mathcal{N} = (2, 0)$ tensor multiplet and then reduce this tensor multiplet to 4+1D since the scale of the VEV $\phi$ is much higher than the compactification scale. In 4+1D, the neutral states come from the hyper-multiplet in 5+1D with twists along $S^1$ as in (111). The mass of these states is therefore (for small $\alpha$),

$$m = \frac{\alpha}{2\pi R}.$$

The charged states come from quantization of the strings wrapped on $S^1$. Up to a correction proportional to $\frac{\alpha^2}{R^2}$ (see [45,49,50]), the tension of the string in 5+1D is $\Phi = \phi/2R$. In the limit we are considering, $\Phi R^2 \gg 1$, it is enough to quantize only the low-energy excitations of the strings. This is just as well, since the low-energy excitations are the only things we know about these strings! This means that our results are correct up to $O(1/\Phi R^2)$. The low-energy description is given by a 1+1D $\mathcal{N} = (4, 4)$ theory. The VEV of the tensor multiplet of the 5+1D bulk breaks the $Spin(5)$ R-symmetry down to $Spin(4)$. The 1+1D low-energy description of a string contains 4 left-moving
bosons and 4 right-moving bosons, 4 left-moving fermions and 4 right-moving fermions. The bosons are not-charged under the $Spin(5)$ R-symmetry. The 8 fermions can be decomposed into representations, of

$$ (SU(2)_B \times SU(2)_U \times SU(2)'_1 \times SU(2)'_2)_{SO(1,1)} $$

Here $Spin(4) = SU(2)_B \times SU(2)_U$ is the unbroken R-symmetry of the 5+1D theory, $Spin(4) = SU(2)'_1 \times SU(2)'_2$ is the subgroup of $Spin(5,1)$ of rotations transverse to the string and $SO(1,1)$ is the world-sheet rotation group. The fermions are in the

$$ (2, 1, 2, 1)^{+\frac{1}{2}} + (1, 2, 1, 2)^{-\frac{1}{2}} $$

with an added reality condition. Under the embedding

$$ U(1) \subset SU(2)_B \subset SU(2)_U = U(1) \subset Spin(4) \subset Spin(5), $$

we find 2 left-moving fermions with charge +1 under $U(1)$, 2 left-moving fermions with charge $-1$ under $U(1)$, and 4 right-moving fermions with charge 0 under $U(1)$. The boundary conditions on the charged fermions are twisted. Quantization of this system gives low-lying vector-multiplets and hyper-multiplets with masses,

$$ \Phi_R, \quad \frac{\alpha}{2\pi R} \pm \Phi_R. $$

Recall that the derivation assumed that $\Phi R^2 \gg 1$ and $|\alpha| \ll \pi$. This agrees with eq.(112).
4.5 \( \text{R-symmetry twists in the little-string theories} \)

For the \((2,0)\) theories, which are believed to have a local description, a twist by a global symmetry along a circle makes perfect sense. For the little-string theories, the issue of locality is more complicated and the meaning of an R-symmetry twist has to be elaborated. In this section we will describe the construction in more detail. We will then see explicitly that T-duality of \(S(k)\) does not preserve the \(\alpha\)-twists. Instead it maps them to T-dual "\(\eta\)-twists". This raises the intriguing possibility to combine both kinds of twists simultaneously.

4.5.1 Geometrical realization

One way to define an R-symmetry twist is to realize it geometrically as follows. We can start with \(\mathbb{R}^{2,1} \times \mathbb{R}^3 \times \mathbb{R}^4\) and mod out by a discrete \(\mathbb{Z}^3\) symmetry which is generated by elements which act as a shift in \(\mathbb{R}^3\) and rotations in \(\mathbb{R}^4\). We obtain \(Z \times \mathbb{R}^{2,1}\) where \(Z\) is an \(\mathbb{R}^4\)-fibration over \(T^3\). Explicitly, we define the 7-dimensional space

\[
Z_{\alpha_1,\alpha_2,\alpha_3} = (\mathbb{R}^3 \times \mathbb{C}^2)/\mathbb{Z}^3,
\]

where \(\mathbb{Z}^3\) is the freely acting group generated by,

\[
\begin{align*}
s_1 &: (x_1, x_2, x_3, z_1, z_2) \rightarrow (x_1 + 2\pi R_1, x_2, x_3, e^{i\alpha_1}z_1, e^{-i\alpha_1}z_2), \\
s_2 &: (x_1, x_2, x_3, z_1, z_2) \rightarrow (x_1, x_2 + 2\pi R_2, x_3, e^{i\alpha_2}z_1, e^{-i\alpha_2}z_2), \\
s_3 &: (x_1, x_2, x_3, z_1, z_2) \rightarrow (x_1, x_2, x_3 + 2\pi R_3, e^{i\alpha_3}z_1, e^{-i\alpha_3}z_2),
\end{align*}
\]

(113)

Here \((x_1, x_2, x_3)\) are coordinates on \(\mathbb{R}^3\). We can similarly define

\[
Y_{\alpha_1,\alpha_2} = (\mathbb{R}^2 \times \mathbb{C}^2)/\mathbb{Z}^2, \quad X_\alpha = (\mathbb{R} \times \mathbb{C}^2)/\mathbb{Z}.
\]

(114)

The theory that we study in this paper, \(S_A(k)\) on \(T^3\) with a twist, can be obtained if we compactify type-IIA on \(Z_{\alpha_1,\alpha_2,\alpha_3}\), wrap \(k\) NS5-branes on \(T^3\) and take \(\lambda_s \rightarrow 0\) as in [32]. This shows that it makes sense to include an R-symmetry twist in \(S(k)\).
What is the meaning of these twists in terms of the theory $S(k)$ itself, without appealing to the underlying string-theory? Let us first refine our terminology. Let $p$ be a generic point in the parameter space

$$\mathcal{M}_A \equiv SO(3, 3, \mathbb{Z}) \backslash O(3, 3, \mathbb{R})/(O(3) \times O(3)).$$

We will denote the theory derived from $k$ type-IIA NS5-branes at the type-IIA moduli space point $p \in \mathcal{M}_A$ by $S_A(k; p)$. Similarly there is an identical moduli space $\mathcal{M}_B$ for type-IIB NS5-branes. We will denote the theory derived from $k$ type-IIB NS5-branes at the type-IIB moduli space point $p \in \mathcal{M}_B$ by $S_B(k; p)$. T-duality implies that there is a map,

$$T : \mathcal{M}_A \rightarrow \mathcal{M}_B,$$

with $T^2 = I$ such that $S_A(k, p) = S_B(k; T(p))$. This map can be defined as follows. Pick an element $v \in O(3, 3, \mathbb{Z})$ with $\det v = -1$ (all such elements are $SO(3, 3, \mathbb{Z})$ conjugate to each other). For $g \in O(3, 3, \mathbb{R})$ which is a representative of a point in $p \in \mathcal{M}_A$ take $v \circ g$ to be a representative of $T(p) \in \mathcal{M}_B$.

A generic point $p'$ in the cover,

$$SL(3, \mathbb{Z}) \backslash O(3, 3, \mathbb{R})/(O(3) \times O(3)),$$

of the parameter space (note that we divided by $SL(3, \mathbb{Z})$ instead of $SO(3, 3, \mathbb{Z})$) will be called a locality-frame. A generic point $p''$ in the cover,

$$O(3, 3, \mathbb{R})/(O(3) \times O(3))$$

will be called a coordinate-frame. There are the obvious maps,

$$p'' \rightarrow p' \rightarrow p.$$

Now suppose that we are in a specific point $p \in \mathcal{M}_A$, say, and we fix a locality-frame $p'$ for $p$ and a coordinate-frame $p''$ for $p'$. For a given
we can contemplate whether it makes sense to define R-symmetry twists along the cycles of $T^3$. If they commute with each other, an $SL(3, \mathbb{Z})$ transformation will permute the cycles and will act on the twists in an obvious way. However, a full $SO(3, 3, \mathbb{Z})$ transformation takes one locality-frame to another and an R-symmetry twist is not mapped back to an R-symmetry twist.

4.5.2 The T-dual of an R-symmetry twist

What does become of an R-symmetry twist after T-duality? The effect of the R-symmetry twist is to make a state which is R-charged have a fractional momentum, because its boundary conditions are not periodic. The momentum modulo $\mathbb{Z}$ is related to the R-charge and the twist in a linear way. Since T-duality replaces the momentum charge with another $U(1)$ charge – the winding number of little-strings, one would deduce that after T-duality, R-charged states should have fractional winding number.

To be more precise, let us take weakly coupled type-IIB on $X_{\alpha}$ from (114) and perform T-duality. Recall that,

$$X_{\alpha} = (\mathbb{R} \times \mathbb{C}^2)/\mathbb{Z},$$

with $\mathbb{Z}$ generated by,

$$s : (x, z_1, z_2) \rightarrow (x + 2\pi R, e^{i\alpha}z_1, e^{-i\alpha}z_2). \quad (115)$$

The world-sheet theory is the free type-IIB theory. Let

$$X = x + w\sigma + p\tau + \sum_{n \in \mathbb{Z}_{\neq 0}} \frac{\alpha-n}{n} e^{in(\tau-\sigma)} + \sum_{n \in \mathbb{Z}_{\neq 0}} \frac{\tilde{\alpha}-n}{n} e^{in(\tau+\sigma)},$$

$$Z_1 = \sum_{s \in \mathbb{Z} + \gamma_1} \frac{\zeta(s)}{s} e^{is(\tau-\sigma)} + \sum_{s \in \mathbb{Z} + \gamma_1} \frac{\tilde{\zeta}(s)}{s} e^{is(\tau+\sigma)}, \quad (116)$$

$$Z_2 = \sum_{s \in \mathbb{Z} + \gamma_2} \frac{\zeta(s)}{s} e^{is(\tau-\sigma)} + \sum_{s \in \mathbb{Z} + \gamma_2} \frac{\tilde{\zeta}(s)}{s} e^{is(\tau+\sigma)},$$

65
\( \gamma_{1,2} \) are real numbers which depends on the sector in a manner that we will write down below. When \( \gamma_i = 0 \), we need to add a piece \( z_i + p_i \tau \) to \( Z_i \). \( p_1, p_2 \) are complex while \( w, p \) are real. Also \( \alpha^{\dagger}_{-n} = \alpha_n \) and \( \tilde{\alpha}^{\dagger}_{-n} = \tilde{\alpha}_n \). Let \( L \) be the total number of \( \zeta^{(1)} \) creation operators minus the total number of \( \zeta^{(2)} \) creation operators in a state. If some \( \gamma_i = 0 \) we also need to add the rotation generator \( i(z_i p_i^\dagger - z_i^\dagger p_i) \).

\[
L \equiv \sum_{s \in \mathbb{Z} + \gamma_1} \frac{1}{s}(\zeta^{(1)}_{-s})^{\dagger}\zeta^{(1)}_{-s} - \sum_{s \in \mathbb{Z} + \gamma_2} \frac{1}{s}(\zeta^{(2)}_{-s})^{\dagger}\zeta^{(2)}_{-s} + (\zeta \leftrightarrow \tilde{\zeta}). \tag{117}
\]

Now we can determine which sectors are allowed. First we require invariance under \( s \) in (115). This is the world-sheet operator \( e^{2\pi i pR - i\alpha L} \), so we require,

\[
pR - \frac{\alpha}{2\pi}L \in \mathbb{Z}.
\]

The sector twisted by \( s^k \) has

\[
\frac{w}{R} = k, \quad \gamma = k\frac{\alpha}{2\pi}.
\]

What happens after T-duality? In a world-sheet formulation, T-duality replaces \( p \) with \( w \) and replaces \( R \) with \( R' = 1/R \). We now have the conditions

\[
\frac{w'}{R'} - \frac{\alpha}{2\pi}L \in \mathbb{Z}, \quad p'R' \in \mathbb{Z}, \quad \gamma = p'R'\frac{\alpha}{2\pi}.
\]

This suggests a more general twist, which can no longer be described as modding out by a discrete symmetry. This time we keep
the sectors with
\[ pR - \frac{\alpha}{2\pi} L \in \mathbb{Z}, \quad \frac{w}{R} - \frac{\eta}{2\pi} L \in \mathbb{Z}, \quad 2\pi\gamma = \frac{w}{R} + \eta pR. \quad (118) \]

We admit to not having checked that this is consistent with modular invariance. The following argument suggests that turning on both \( \alpha \) and \( \eta \) twists is consistent. For small \( \alpha \), turning on a \( \alpha \)-twists corresponds to making a small perturbation with a certain operator to the Hamiltonian of \( S(k) \). An infinitesimal \( \eta \)-twist also corresponds to a perturbation but with another operator. Now we can make a small perturbation with both a \( \alpha \)-twist as well as a \( \eta \)-twist. They preserve exactly the same supersymmetry. It could, however, happen that after we turn on both \( \alpha \)-twists and \( \eta \)-twists there is no longer any super-symmetric vacuum. We do not know of any way to settle this question.

4.6 Discussion

We have argued that the moduli space of vacua of \( S_A(2) \) (\( S_B(2) \)) compactified on \( T^3 \) with 3 R-symmetry twists, \( \alpha_1, \alpha_2, \alpha_3 \), is the same as the moduli space of vacua of the heterotic \( E_8 \times E_8 \) (\( SO(32) \)) (1, 0) NS5-brane theory compactified on the same \( T^3 \) with Wilson lines given by an embedding of the twists in the gauge group. We have also studied how T-duality of the little-string theory acts on the R-symmetry \( \alpha \)-twists. We have seen that they get mapped to other types of twists (\( \eta \)-twists). We have suggested that there exist theories with both kinds of twists simultaneously.

Let us suggest a few questions for further research:

1. Find an M-theoretic derivation of the moduli spaces, or perhaps using compactification on a Calabi-Yau manifold.

2. Study the BPS spectrum of the theories in 3+1D and 4+1D. We have identified the moduli spaces of the twisted (2, 0) theory with the moduli space of the compactified \( E_8 \) (1, 0) theory. However, these two theories are not identical. It would be interesting to see how this distinction is manifested in the multiplicities of BPS states [51,52,53,54,55].

3. Study the other phase where little-strings condense.
5. Instantons on a Non-commutative $T^4$ from Twisted (2,0) and Little-String Theories

In this chapter we will continue the study of compactified (2,0) and Little-string theories. The main result will be that their moduli spaces are equal to moduli spaces of instantons on noncommutative tori, which were discussed in chapter 3.

Starting with the work of [56], the moduli-spaces of vacua have been found for a large class of gauge theories with 8 super-charges in 3+1D and in 2+1D. These solutions were derived from string dualities in [38] and the works that followed. String theory also suggested the existence of new theories in six dimensions [30,32] (see also [57-58]). Compactification of these theories to 3+1D reduces, in certain limits of the external parameter spaces, to ordinary gauge theories. As we will see, all the previously solved gauge theories with $\mathcal{N} = 2$ supersymmetry and $SU(N_1) \times \cdots \times SU(N_r)$ gauge groups [37] can be recovered at special limits of the external parameters of the compactification of the Little-string theories. Let us recall that the Little-string theory is the world-volume theory on $k$ NS5-branes in type-IIA in the limit of vanishing string coupling keeping the string tension fixed [32]. We denote this theory $S_A(k)$. It has (2,0) supersymmetry. There is a similar theory coming from $k$ NS5-branes in type-IIB in the limit of vanishing string coupling keeping the string tension fixed. We denote this theory $S_B(k)$. It has (1,1) supersymmetry. $S_A(k)$ and $S_B(k)$ are often referred to as the little-string theories. They both have an inherent scale, $m_s$. In the limit $m_s \rightarrow \infty$, $S_A(k)$ becomes the theory on the world-volume of $k$ M5-branes – the so called (2,0) theory.

We will compactify these theories down to 3 dimensions. These theories have 16 super-charges, so if they are compactified on $T^3$ the resulting theories will have $\mathcal{N} = 8$ supersymmetry in three dimensions. The low energy behavior of $\mathcal{N} = 8$ theories is trivial. Instead we want to study theories with $\mathcal{N} = 4$ supersymmetry, i.e. 8 supercharges. So we have to compactify in a way that breaks half the supersymmetry. We will do that as in [59] by introducing holonomies of the R-symmetry around the three circles in $T^3$. To preserve half of
the supersymmetries the holonomies were chosen inside a $SU(2)$ subgroup of the $Spin(4)$ R-symmetry group. The low energy behaviour of a $\mathcal{N} = 4$ theory in $D = 3$ is a sigma-model with the moduli-space of vacua as the target-space. So the low energy behaviour is given by the moduli-space of vacua and the its metric.

Let us start by identifying all the parameters of the compactification. Consider $S_A(k)$ compactified on $T^3$. The scale of $S_A(k)$ is $m_s$, the string mass. The $T^3$ is specified by a metric. For simplicity we will take it to be rectangular. It is easy to incorporate the more general case. Furthermore there can be a flux of the 2-form $B^{NS}$ field of type IIA through 2-cycles in the $T^3$. For simplicity we set $B^{NS} = 0$. It is again not hard to incorporate the more general case. Now we come to the most interesting parameters – the twists. The R-symmetry group of $S_A(k)$ is $Spin(4)_R$, corresponding to transverse rotations. The twists are taken inside

$$U(1)_R \subset SU(2)_B \subset SU(2)_B \times SU(2)_U = Spin(4)_R \quad (119)$$

This preserves 8 of the 16 super-charges. There is a twist, $\alpha_i$, along each of the 3 circles. The $\alpha_i$’s are periodic

$$\alpha_i \rightarrow \alpha_i + 2\pi, \quad i = 1, 2, 3 \quad (120)$$

The twists can be described in the following way. States that are charged under $U(1)_R$ receive a phase shift in traversing a circle. In other words, momentum along the circle is shifted from $\frac{n}{R}$ to $\frac{n - \alpha}{2\pi R}$. By performing T-duality along all circles of the $T^3$ we get $S_B(k)$ on another $T^3$. Momentum has been exchanged with winding, so the T-dual of the twists has the following description. States that are charged under $U(1)_R$ have fractional winding numbers; $\frac{n - \alpha}{2\pi}$ instead of $n$. We call this kind of twist an “$\eta$-twist.” By combining these two types of twists we learn that the most general twist around a circle shifts both momentum and winding. In other words the $S_A(k)$
compactification on $T^3$ depends on 6 parameters

$$\alpha_i, \eta_i, \quad i = 1, 2, 3,$$

(121)

where $\alpha_i$ shifts momentum and $\eta_i$ shifts winding. The $\alpha_i$'s have a clear geometrical interpretation. In traversing the circle the transverse space is rotated. The $\eta_i$'s are harder to visualize. They are geometrical in the T-dual $S_B(k)$.

We can actually generalize this system even more. Instead of $k$ NS5-branes we can consider $k$ NS5-branes on top of an $A_{q-1}$ singularity. In other words the transverse space to the NS5-branes is $\mathbb{R}^4/Z_q$, where $Z_q$ is a subgroup of $U(1)_R$. $U(1)_R$ is still a symmetry of this space, so we can twist as before. These theories have 8 super-charges in 6 dimensions. The $U(1)_R$ is a global symmetry which commutes with super-charges. The twists, therefore, do not break any more supersymmetry, so the compactified theory still has $\mathcal{N} = 4$ in 3 dimensions.

Theories of branes on top of an ADE singularity have been studied in [60,61]. These 6 dimensional theories are, loosely speaking, quiver gauge theories [62] coupled to tensor theories or vice versa, depending on whether it is in type-IIA or type-IIB.

The 3 dimensional theory, obtained after compactification with twists, has a low energy description as a sigma-model with a target-space, which is equal to the moduli-space of vacua. In this paper we will prove that the moduli-space of vacua is equal to the moduli-space of $k$ $U(q)$ instantons on a non-commutative $T^4$. The non-commutativity is set by the 6 parameters $\alpha_i$ and $\eta_i$. This generalizes the case of compactification without twists where the moduli-space of the theories turns out to be the moduli-space of ordinary instantons [61,63].

This result implies similar results for all the theories which are special cases of this. This includes firstly the $(2,0)$ theory which can be obtained from $S_A(k)$ by $m_s \to \infty$. Secondly, it includes all three-dimensional $U(k)$ gauge theories with adjoint matter. By incorporating the $A_q$ singularity it also includes all gauge theories with
group $U(k) \times \cdots \times U(k)$ and matter in $(k, \bar{k}, 1, \ldots, 1) + \text{permutations}$. By taking the gauge coupling to zero in some $U(k)$ we can get theories with the gauge group being $U(k) \times \cdots \times U(k)$ with fundamental and bi-fundamental matter in various combinations with generic masses.

Our results imply that all these 2+1D gauge theories have a moduli-space of vacua equal to the moduli-space of vacua of instantons on non-commutative $\mathbb{R}^3 \times S^1$. In the case of mass deformed $\mathcal{N} = 8$ this result was derived earlier in [64].

By decompactifying one circle similar results hold for the moduli space of 4 dimensional gauge theories on $\mathbb{R}^3 \times S^1$.

We also find that for certain discrete values of the twists there are Higgs branches emanating from some locus of the Coulomb branch. We will identify these and calculate their dimensions. We will also calculate the existence of these branches from pure field theoretic arguments and find agreement in the structure of the Higgs branches. These branches generalize a branch found in [59].

Moreover, combining our results with the formulas in [59] for the special case of $q = 1$ and $k = 2$, we get a prediction for the moduli-space of two $U(1)$ instantons on a non-commutative $T^4$. This is a $K3$ (projecting out the center of mass) and the exact point in moduli-space was given in [59] as a function of the twists.

The organization of the chapter is as follows. In section (5.1) we present the proof that the moduli space is equal to the moduli space of instantons on non-commutative $T^4$. In section (5.2) we have a short review of the relevant aspects of non-commutative gauge theories. In section (5.3) we use this information about non-commutative gauge theory to make the claim about the moduli-space of non-commutative instantons precise and discuss some features of it. In section (5.4) we describe the decompactification limit to 3+1D (compactification of the 5+1D theories on $T^2$ with twists). In section (5.5) we present a more detailed geometrical formulation of $\alpha$-twists and especially $\eta$-twists. We conclude with a summary of the results and possible further direction.
5.1 The solution

In this section, we derive the solution to the moduli space of the twisted theory. To construct the solution we will start with type-IIA on a space

$$\mathbb{R}^{2,1} \times T^3 \times_{\tilde{\alpha}} \mathbb{R}^4,$$

where $\times_{\tilde{\alpha}}$ means that locally the space looks like $\mathbb{R}^{2,1} \times T^3 \times \mathbb{R}^4$ but as we go around a cycle of the $T^3$ we have to twist the transverse space $\mathbb{R}^4$ by the appropriate element of $\text{Spin}(4)$ corresponding to the twist. Now we take $k$ NS5-branes and let them stretch along $\mathbb{R}^{2,1} \times T^3$ and the origin of $\mathbb{R}^4$. The question what is the low-energy effective action for this system in the limit that the type-IIA string coupling constant $\lambda \to 0$.

As will be clear later on, it is easier to solve the problem if we first replace the transverse $\mathbb{R}^4$ with another manifold $M_4$. In the limit that the curvature of $M_4$ is small at the position of the NS5-branes the switch from $\mathbb{R}^4$ to $M_4$ will not make a big difference. Moreover, we can argue that the quantum fluctuations in the transverse position of the NS5-brane are related to the fluctuations of the scalars of $S_A(k)$ as,

$$x \sim m_s^{-3} \lambda \Phi,$$

and for energy scales $m_s$, $\Phi$ is of the order of $m_s^2$. In the limit $\lambda \to 0$, the transverse fluctuations of the NS5-brane go to zero and if the point in $M_4$ is smooth, it would seem that the dynamics of the NS5-brane will be the same as on $\mathbb{R}^4$. This argument should be taken with caution since the actual solitonic solution of the NS5-brane has a cross-section of about $m_s$. In any case, we will not have to rely on this argument.

The manifold $M_4$ that we will use is the Taub-NUT space. The metric is,

$$ds^2 = \rho^2 U (dy - A_i dx^i)^2 + U^{-1} (d\tilde{x})^2, \quad i = 1 \ldots 3, \quad 0 \leq y \leq 2\pi.$$  \hfill (122)
where,
\[ U = \left( 1 + \frac{\rho}{2|\vec{x}|} \right)^{-1}, \]
and \( A_i \) is the gauge field of a monopole centered at the origin.

The Taub-NUT space has the following desirable properties (these properties were also used in [65]),

1. If we excise the origin, what remains is a circle fibration over \( \mathbb{R}^3 - \{0\} \). Eqn\text(122) is written such that \( \vec{x} \) is the coordinate on this base \( \mathbb{R}^3 - \{0\} \). For \( |\vec{x}| \) restricted to a constant, the fibration is exactly the Hopf fibration of \( S^3 \) over \( S^2 \).

2. The origin \( \vec{x} = 0 \) is a smooth point.

3. As \( |\vec{x}| \to \infty \) the radius of the fiber becomes \( \rho \).

4. The space has a \( U(1) \) isometry group that preserves the origin \( \vec{x} = 0 \). An element \( g(\theta) = e^{i\theta} \in U(1) \) acts by \( y \to y + \theta \). It also acts on the tangent space \( \mathbb{R}^4 \) at the origin by embedding \( e^{i\theta} \) inside
\[ U(1) \to SU(2)_L \to (SU(2)_L \otimes SU(2)_R)/\mathbb{Z}_2 = SO(4). \]

Now that we have replaced the transverse \( \mathbb{R}^4 \) with a Taub-NUT space we have \( k \) NS5-branes on the space,
\[ \mathbb{R}^{2,1} \times T^3 \times _\alpha T^N(\rho). \]

The \( \alpha \)-twists are incorporated as follows. As we go around a cycle of \( T^3 \) we have to act on the fiber \( T^N(\rho) \) with \( g(\alpha_i) \) where \( \alpha_i \) is the appropriate twist. In the limit \( \rho \to \infty \), \( T^N(\rho) \) becomes \( \mathbb{R}^4 \) and the isometry \( g(\alpha_i) \) becomes the element in \( SO(4) \) that we have used for the twist. The virtue of working with \( T^N(\rho) \) instead of \( \mathbb{R}^4 \) is that at \( \vec{x} = \infty \) the circle fiber becomes of finite size which will help in subsequent dualities.
To generalize the construction to the case of $k$ NS5-branes at an $A_{q-1}$ singularity, $\mathbb{R}^4/\mathbb{Z}_q$, we replace the transverse $\mathbb{R}^4/\mathbb{Z}_q$ with a $q$-centered Taub-NUT space, $\mathbb{T}N_q(\rho)$ with radius $\rho \to \infty$. This space has similar properties,

(1') If we excise the origin, what remains is a circle fibration over $\mathbb{R}^3 - \{0\}$. For $|\vec{x}|$ restricted to a constant, the fibration is a circle bundle over $S^2$ with first Chern-class $c_1 = q$.

(2') Near the origin $\vec{x} = 0$, $\mathbb{T}N_q$ looks like $\mathbb{R}^4/\mathbb{Z}_q$.

(3') As $|\vec{x}| \to \infty$ the radius of the fiber becomes $\rho$.

(4') The space has a $U(1)$ isometry group that preserves the origin $\vec{x} = 0$. An element $g(\theta) = e^{i\theta} \in U(1)$ acts at $\vec{x} = \infty$ by $y \to y + \theta$. It also acts on the tangent space $\mathbb{R}^4/\mathbb{Z}_q$ at the origin by embedding $e^{i\theta}$ inside

$$U(1) \to SU(2)_L \to (SU(2)_L \otimes SU(2)_R)/\mathbb{Z}_2 = SO(4).$$

Note that the discrete $\mathbb{Z}_q$ by which we mod out is a subgroup of the same $U(1) \subset SU(2)_L$ as well.

5.1.1 Chains of Dualities

We have seen that the twisted compactified little-string theories can be realized as follows. Start with type-IIA on $\mathbb{R}^{2,1} \times T^3 \times \mathbb{T}N_q$, where the radii of $T^3$ are $R_i$ (of the order of $m_s$) and the radius of the fiber of the Taub-NUT space is taken to be $\rho$. Put $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and study the limit,

$$\lambda \to 0, \quad m_s \rho \to \infty.$$ 

In principle, we could probably settle on a constant $m_s \rho$ as well, since the transverse fluctuations of the NS5-brane are small. However, the transverse size of the NS5-brane, as a solitonic object, is of the
order of $m_s^{-1}$. Therefore, to be on the safe side, we take $m_s \rho \to \infty$. The technique for solving theories with 8 supersymmetries is [38] to identify a parameter that decouples from the vector-multiplet and such that at one limit of this parameter the theory is described by gauge theory (or little-string theory, in our case) and in another limit a dual description becomes weakly coupled. In that second limit, the theory is no longer described by the gauge theory but the vacuum structure remains the same and is determined by the classical equations of motion. This method was also applied in [39,40,37].

In our case, to solve the problem we take the limit of strong coupling keeping the Taub-NUT radius large.

$$\lambda \to \infty, \quad m_s \rho \to \infty,$$  \hspace{1cm} (123)

We will also require that $\lambda (m_s \rho)^{-3} \to \infty$. We can think of $\rho$ as being fixed but very large and $\lambda \to \infty$ much faster. We will not show that this corresponds to a parameter that is in a hyper-multiplet (and hence decouples from the vector-multiplets) but this is the basic assumption. Recall that in 2+1D hyper-multiplets and vector-multiplets can be distinguished with the help of the $U(1)_R \otimes SU(2)_U$ symmetry which is the unbroken subgroup of (119). The scalar fields of a vector-multiplet are invariant under $SU(2)_U$ while the scalar fields of a hyper-multiplet are in the $2$ (see [43]). (The dilaton, which is a singlet, is a quadratic expression in these fields.) Similarly, the fermions of a hyper-multiplet are invariant under $SU(2)_U$ and the fermions of a vector-multiplet are in the $2$.

The next step is to use string-dualities to convert the region (123) to a weakly coupled theory.

At this point we have $k$ NS5-branes in type-IIA on $\mathbb{R}^{2,1} \times T^3 \times T \mathbb{N}_q$ with string coupling $\lambda$, string scale $m_s$, $T^3$-radii $R_i$, and twists $\alpha_i$. For simplicity, we assume that $T^3$ is of the form $S^1 \times S^1 \times S^1$ with no NS-NS 2-form fluxes. Since $\lambda \to \infty$ we view this as $k$ M5-branes in M-theory on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times T \mathbb{N}_q$. Let $M_p$ be the 11-dimensional
Planck scale. The radius of $S^1$ is, $R$. They are related according to,

$$R = \frac{\lambda}{m_s}, \quad M^3_p = \frac{m^3_s}{\lambda}.$$ 

The radius of $\text{T}N_q$ is, $\rho$.

**Step 1:** Since, in the limit (123),

$$M_p \rho = m_s \lambda^{-1/3} \rho \to 0,$$

we should view the fiber of the Taub-NUT as the $11^{th}$ small dimension and convert to type-IIA on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times \mathbb{R}^3$. We also have $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and $\text{T}N_q$ became $q$ D6-branes on $\mathbb{R}^{2,1} \times T^3 \times S^1$. The $\alpha$-twists became RR 1-form Wilson lines along the cycles of $T^3$. The string coupling constant is given by,

$$\lambda' = \lambda^{-1/2} (m_s \rho)^{3/2} \to 0.$$

The new string scale is,

$$M'_s = m^3_s \rho^{1/2} \lambda^{-1/2},$$

and the radii of $T^3$ satisfy,

$$M'_s R_i = m^3_s \rho^{1/2} \lambda^{-1/2} R_i \to 0.$$ 

This means that we must perform T-duality on $T^3$. 

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Step 2: After T-duality on $T^3$ we obtain type-IIB on $\mathbb{R}^{2,1} \times \tilde{T}^3 \times S^1 \times \mathbb{R}^3$ with radii $\hat{R}_i$ which satisfy,

$$M'_s \hat{R}_i = m_s^{-3/2} \rho^{-1/2} \lambda^{1/2} \hat{R}_i^{-1} \to \infty.$$ 

There are now $k$ NS5-branes on $\mathbb{R}^{2,1} \times \tilde{T}^3$ and $q$ D3-branes on $\mathbb{R}^{2,1} \times S^1$. At this point the $\alpha$-twists became RR 2-form fluxes,

$$\alpha_i \epsilon_{ijk} = \int_{C_{jk}} B^{RR}, \quad i, j, k = 1 \ldots 3,$$

where $C_{jk}$ is the 2-cycle made out of the $j^{th}$ and $k^{th}$ directions in $T^3$. The string coupling is now,

$$\lambda^{(2)} = \frac{\lambda'}{m_s^{9/2} \rho^{3/2} \lambda^{-3/2} R_1 R_2 R_3} = \lambda m_s^{-3} R_1^{-1} R_2^{-1} R_3^{-1} \to \infty.$$

This means that we must do S-duality.

Step 3: After S-duality we get type-IIB with $q$ D3-branes and $k$ D5-branes in the same geometry. The string coupling constant is now,

$$\lambda^{(3)} = \lambda^{-1} m_s^3 R_1 R_2 R_3 \to 0,$$

and the string scale is,

$$M_s^{(3)} = \lambda^{-1} R^{1/2} m_s^3 (R_1 R_2 R_3)^{1/2}.$$

The radii satisfy,

$$M_s^{(3)} \hat{R}_i = \rho^{-1/2} (R_1 R_2 R_3)^{1/2} \hat{R}_i^{-1} \to 0,$$
and the radius of $S^1$ satisfies,

$$M_s^{(3)}R = m_s^2 \rho^{1/2}(R_1R_2R_3)^{1/2} \to \infty.$$  

At this point,

$$\alpha_i \epsilon_{ijk} = \int_{C_{jk}} B^{NSNS}, \quad i, j, k = 1 \ldots 3.$$  

Since $M_s^{(3)}\hat{R}_i \to 0$, we must perform another T-duality on $T^3$. However, because of the NS-NS 2-form fluxes, just as in [2], another T-duality will not help. Instead, let us do a T-duality on $S^1$ which brings us to the final setup of gauge theory on a non-commutative $T^4$.

**Step 4:** After T-duality along $S^1$ we get type-IIA with $k$ D6-branes and $q$ D2-branes. The string coupling is now,

$$\lambda^{(4)} = \frac{\lambda^{(3)}}{M_s^{(3)}\hat{R}} = \lambda^{-1}m_s\rho^{-1/2}(R_1R_2R_3)^{1/2} \to 0,$$

and $\hat{M}_s = M_s^{(3)}$. The radii satisfy,

$$\hat{M}_s\hat{R}_i = \rho^{-1/2}(R_1R_2R_3)^{1/2} R_i^{-1} \to 0,$$

and the radius of the $S^1$ satisfies,

$$\hat{M}_s\hat{R} = m_s^{-2} \rho^{-1/2}(R_1R_2R_3)^{-1/2} \to 0.$$  

At this point, the $\alpha$-twists are still NS-NS 2-form fluxes. We thus end up with a system of $k$ D6-branes on $T^4 \times \mathbb{R}^{2,1}$ and $q$ D2-branes.
which are points on $T^4$. The radii of $T^4$ are given, in terms of the 3 radii $R_i$ of the original $T^3$, as follows,

$$
\hat{R}_i = \hat{M}_s^{-1} \rho^{-1/2}(R_1 R_2 R_3)^{1/2} R_i^{-1}, \quad i = 1, 2, 3,
\hat{R}_4 = \hat{M}_s^{-1} m_s^{-2} \rho^{-1/2}(R_1 R_2 R_3)^{-1/2}.
$$

Here $\hat{M}_s$ denotes the final type-IIA (with the D2-branes and D6-branes) string scale. The final string coupling constant is,

$$
\hat{\lambda} = \lambda^{-1} m_s \rho^{-1/2}(R_1 R_2 R_3)^{1/2}. \tag{124}
$$

Similarly, we can start with $S_A(k)$ with 3 $\eta$-twists. By definition, this is $S_B(k)$ on the dual $T^3$ with 3 $\alpha$-twists. We realize this in type-IIB on the background $\mathbb{R}^{2,1} \times T^3 \times T\mathbb{N}_q$ and $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$. As before, the fiber of the Taub-NUT space is denoted by $\rho$. We first perform S-duality to replace the NS5-branes with $k$ D5-branes. At this point the $\eta$-twists are off-diagonal components of the metric $g_{i9}$ with $i$ in the direction of $T^3$ and 9 in the direction of the Taub-NUT fiber. Then, we perform T-duality on the direction of $\rho$ to obtain type-IIA on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times \mathbb{R}^3$ with $q$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and $k$ D6-branes on $\mathbb{R}^{2,1} \times T^3 \times S^1$. The $\eta$-twists became NS-NS 2-form fluxes $B_{i4}$ where 4 is the direction of $S^1$. Then, we do T-duality on the three directions of $T^3$. We obtain $k$ D3-branes on $\mathbb{R}^{2,1} \times S^1$ and $q$ NS5-branes. The $\eta$-twists are now off-diagonal components $g_{i4}$. We then do another S-duality to get $k$ D3-branes and $q$ D5-branes and, finally, another T-duality on $T^3$. At this point we are back with $k$ D6-branes and $q$ D2-branes. The $\eta$-fluxes are now NS-NS 2-form fluxes $B_{i4}$.

The moduli space is thus the same as the moduli space of $q$ D2-branes inside $k$ D6-branes on $T^4$ with NS-NS 2-form fluxes. In the case of $\alpha$-twists, these fluxes have both indices in the direction of $T^3 \subset T^4$. In the case of $\eta$-twists, the fluxes had one index in the
direction of $\mathbf{T}^3$ and the other index in the 4th direction. In the generic case, we have both $\alpha$-twists and $\eta$-twists simultaneously. The result is that the NS-NS 2-form flux is nonzero for all 6 2-cycles of $\mathbf{T}^4$. The string scale, string coupling, and the parameters of the $\mathbf{T}^4$ are as calculated above. We could in principle follow the chain of dualities above with simultaneous $\alpha$-twists and $\eta$-twists but the intermediate steps would involve cumbersome non-linear expressions.

The moduli space of $q$ D2-branes inside $k$ D6-branes on $\mathbf{T}^4$ with NS-NS 2-form fluxes, and in the limit that the size of the $\mathbf{T}^4$ vanishes, was shown to be equivalent to the moduli space of $k$ instantons of $U(q)$ gauge theory on a non-commutative $\mathbf{T}^4$ [66-29]. It is likely that this result is true even for $\mathbf{T}^4$ of finite size, because the size decouples by arguments as above.

In the next sections we will review the non-commutative geometry and formulate a precise statement about the moduli space.

5.2 Review of Noncommutative Gauge Theory

In this section we will review the elements of non-commutative gauge theory which are relevant to our situation.

Non-commutative gauge theory first entered string theory in [2] where it was shown to provide a matrix model for M-theory on a torus with the $C^{(3)}$ field turned on along the light-like circle. Subsequently, a lot of interesting work on this topic was done [13-67]. What we need here is not the connection to matrix theory but just the study of D-branes with a $B_{NS}$ fields turned on.

Consider type-IIA on $\mathbb{R}^{1,9-d} \times \mathbf{T}^d$ with $q$ D0-branes. The radii of $\mathbf{T}^d$ are called $R_i$, $i = 1, \ldots, d$, the string mass $m_s$ and the coupling $\lambda$. Furthermore let there be a constant $B_{NS}$ field along $\mathbf{T}^d$. Let

$$b_{ij} = \int_{i,j} B_{NS}^{}, \quad i, j = 1, \ldots, d$$

(125)

be the flux of $B_{NS}^{}$ through the $\mathbf{T}^2$ spanned by directions $i, j$. The $b_{ij}$ are periodic with period $2\pi$ due to the gauge invariance of $B_{NS}^{}$. 

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In [68] this system was studied using the approach of [5]. The result is that the low energy physics is described by a $d+1$ dimensional $U(q)$ gauge theory on a dual torus, $\tilde{T}^d \times \mathbb{R}^{0,1}$ with radii

$$\tilde{R}_i = \frac{1}{m_s^2 \tilde{R}_i} \tag{126}$$

and gauge coupling

$$\frac{1}{g^2} = \frac{m_s^{2d-3} R_1 \ldots R_d}{\lambda} \tag{127}$$

The effect of $b_{ij}$ is to change the action. Every time two fields are being multiplied, the multiplication is with the $*$-product defined as,

$$(\phi^{(2)} \ast \phi^{(1)})(x) =
\begin{aligned}
e & \left. e^{-\frac{b_{ij}}{2m_s^2 R_i R_j} (\partial_i^{(2)} \partial_j^{(1)} - \partial_i^{(2)} \partial_j^{(1)})} \phi^{(2)}(x_2) \phi^{(1)}(x_1) \right|_{x^{(2)} = x^{(1)} = x}, \\
\partial_i^{(a)} & \equiv \frac{\partial}{\partial x_i^{(a)}}, \quad a = 1, 2.
\end{aligned} \tag{128}$$

The action is the usual gauge theory action just with this modification.

If there had been no $B^{NS}$-field the resulting $d + 1$ dimensional gauge theory could have been obtained by performing T-duality along $T^d$. The $q$ D0-branes would have turned into $q$ Dd-branes. The radii and gauge coupling of the $U(q)$ theory can be calculated in this way. The important point to remember is that the only change from having a $B^{NS}$-field is to change the product into eq.(128). The radii and gauge coupling are independent of $b_{ij}$. This result could not have been obtained by T-duality, since $B^{NS}$-fields change the formulas of T-duality and would have given other radii and gauge coupling.
There is another way of formulating this gauge theory. Instead of working with the \( \ast \)-product, eq.(128), one can say that the torus \( \tilde{T}^d \) is non-commutative. The algebra of functions on the torus is, \( A \), is generated by \( U_1, \ldots, U_d \) with relations

\[
U_i U_j = U_j U_i e^{ib_{ij}}
\]  

(129)

The generalization of finite dimensional vector fields is finitely generated projective modules over \( A \). Let \( E \) be such a module. One can define connections, \( \nabla \), and curvature \( F_{ij} \) of this module [2,29]. One can define the Chern character of the module \( E \)

\[
\text{ch}(E) = \sum_{k=0} \frac{\dot{\tau}(F^k)}{(2\pi i)^k k!}
\]  

(130)

\( \dot{\tau} \) is the trace on \( \text{End}_A(E) \). \( \text{ch}(E) \) can be regarded as an element in the cohomology, \( H^*(T^d, \mathbb{C}) \), of \( T^d \), the original torus. \( \text{ch}(E) \) is not integral but there exists an integral cohomology class \( \mu(E) \in H^*(T^d, \mathbb{C}) \) such that

\[
\text{ch}(E) = e^{\frac{1}{2\pi} \iota(b)} \mu(E)
\]  

(131)

Here \( \iota(b) \) denotes contraction with \( b \) considered as an element of \( H_*(T^d, \mathbb{C}) \) [29].

The mathematical fact that the module \( E \) is characterized by integers is in exact agreement with our expectation from D-brane physics. Besides the \( q \) D0-branes on \( T^d \) there could be any number of D2-branes, D4-branes , etc. wrapped on \( T^d \). These numbers are exactly given by \( \mu(E) \). \( \text{ch}(E) \) measures the fact that D2-branes with \( B^{NS} \)-fields turned on have an effective D0-brane charge and the
equivalent phenomena for other branes. Suppose for instance that only $\mu_0$ and $\mu_1$ are nonzero, then,

$$ch_0 = \mu_0 + \frac{b_{12}}{2\pi} \mu_1, \quad ch_1 = \mu_1.$$  \hspace{1cm} (132)

This equation reflects the fact that the number of D2-branes is unchanged by the presence of the $B^{NS}$-field but the number of D0-branes is shifted by the product of the number of D2-branes and the $B^{NS}$-field along the D2-branes.

5.3 Noncommutative Instantons as the Moduli-space

Let us now go back to our system of $q$ D2-branes inside $k$ D6-branes given above. They have a common $\mathbb{R}^{1,2}$. This is the space-time in which the 3 dimensional theory is living. The 3 dimensional theory has a low energy description as a sigma model with the moduli space of vacua as target space. The moduli space of vacua is a Hyperkähler manifold. The moduli space of vacua comes from the dynamics on the $T^4$, which is the same as the dynamics of $q$ D0-branes in $k$ D4-branes on $T^4$. The radii of the $T^4$, $\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4$, and the string coupling $\hat{\lambda}$ and string scale $\hat{M}_s$ are given in terms of the parameters of the $S_A(k)$ compactification in (124) which we repeat here,

$$\hat{R}_i = m_s^{-3} \lambda \rho^{-1} R_i^{-1}, \quad i = 1, 2, 3,$$
$$\hat{R}_4 = m_s^{-5} \lambda \rho^{-1} (R_1 R_2 R_3)^{-1},$$
$$\hat{M}_s = \lambda^{-1} m_s^{3} \rho^{1/2} (R_1 R_2 R_3)^{1/2},$$
$$\hat{\lambda} = \lambda^{-1} m_s \rho^{-1/2} (R_1 R_2 R_3)^{1/2},$$  \hspace{1cm} (133)
Furthermore there is a $B^{NS}$-field turned on along $T^4$,

$$\int_{12} B^{NS} = \alpha_3, \quad \int_{31} B^{NS} = \alpha_2,$$

$$\int_{23} B^{NS} = \alpha_1, \quad \int_{i4} B^{NS} = \eta_i, \quad i = 1, 2, 3. \quad (134)$$

but the vacuum structure of the vector-multiplets should be independent of $\rho$ in this limit.

According to the above review of non-commutative geometry, the moduli space is equal to the moduli space of $k$ instantons in $U(q)$ gauge theory on a non-commutative torus, $\tilde{T}^4$, with non-commutativity parameters equal to $\alpha_i, \eta_i$. As explained above the radii and gauge coupling of this gauge theory are the same as if $\alpha_i = \eta_i = 0$. Hence they can be found by T-duality on $T^4$. By this T-duality one obtains $k$ D2-branes in $q$ D6-branes on $\tilde{T}^4$ of radii,

$$\tilde{R}_1 = \frac{\lambda}{m_s^3R_2R_3}, \quad \tilde{R}_2 = \frac{\lambda}{m_s^3R_1R_3}, \quad \tilde{R}_3 = \frac{\lambda}{m_s^3R_1R_2}, \quad \tilde{R}_4 = \frac{\lambda}{m_s}, \quad (135)$$

and string mass, $\tilde{m}_s$, and coupling, $\tilde{\lambda}$,

$$\tilde{m}_s = \tilde{M}_s = \lambda^{-1}m_s^3\rho^{1/2}(R_1R_2R_3)^{1/2},$$

$$\tilde{\lambda} = \lambda^{-1}m_s^3\rho^{3/2}(R_1R_2R_3)^{1/2}. \quad (136)$$

In the $U(q)$ theory, this gives a gauge coupling of,

$$\frac{1}{g^2} = \frac{\tilde{m}_s^3}{\tilde{\lambda}} = \lambda^{-2}m_s^6R_1R_2R_3. \quad (137)$$

Observe that $\rho$ has dropped out of the radii and the gauge coupling.
What about the limit $\lambda \to \infty$ and $m_s$ fixed. To see that the moduli space of vacua is well defined in this limit we should remember that scalar fields in three dimensions have dimension $1/2$, if we want a standard kinetic term. We can either view the moduli space of vacua from the $U(q)$ gauge theory point of view or from the $U(k)$ theory on the D2-branes. From the last point of view the moduli space is the Higgs branch. The action of the $U(k)$ theory has a term,

$$\frac{1}{2} \frac{1}{\lambda m_s} \int d^3x (\partial_\mu (\tilde{m}_s^2 X^i))^2$$

(138)

We define $\Phi^i = \tilde{\lambda}^{-1/2}\tilde{m}_s^{3/2} X^i$. This $\Phi$ has a standard kinetic term,

$$\frac{1}{2} \int d^3x (\partial_\mu \Phi^i)^2$$

(139)

The radii of the $\Phi^i$ are $R(\Phi^i) = \tilde{\lambda}^{-1/2}\tilde{m}_s^{3/2} \tilde{R}^i$.

$$R(\Phi^1) = \sqrt{\frac{R_1}{R_2 R_3}}, \quad R(\Phi^2) = \sqrt{\frac{R_2}{R_1 R_3}}, \quad R(\Phi^3) = \sqrt{\frac{R_3}{R_1 R_2}}$$

$$R(\Phi^4) = m_s^2 \sqrt{\frac{R_1 R_2 R_3}{}}$$

(140)

We see that the limit $\lambda \to \infty$ exists. This last discussion was really superfluous. Since $S_A(k)$ only depends on the combination $m_s^2$ and does not feel $\rho$, this had to be true. For finite $m_s \rho$, it could even be true for the full theory, not just the moduli space of vacua. The effect of the twists is just to deform the moduli space and so does not change the fact that the moduli space is independent of $\rho$ and has a limit when $\lambda \to \infty$, keeping $m_s$ fixed.

We can also see from (140) what happens in the limit of the $(2,0)$ theory. For this limit we take $m_s \to \infty$. We find that the $T^4$ degenerates to $T^3 \times \mathbb{R}$. 

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Let us now be more precise about the space of instantons on a non-commutative $\mathbb{T}^4$. For this sake we will temporarily neglect the uncompactified directions and think of our system as $q$ D0-branes and $k$ D4-branes on $\mathbb{T}^4$. According to the review of non-commutative geometry above, this is described by a gauge theory on the dual $\tilde{\mathbb{T}}^4$ with non-commutativity parameters, $b_{ij}$, equal to the twists. By gauge theory we really mean a projective module, $E$, which is characterized by

$$\mu(E) = H^*(\mathbb{T}^4, \mathbb{Z}).$$  \hfill (141)

$\mu(E)$ has components in dimensions 0, 2 and 4. $\mu_0 = q$ is the number of D0-branes on $\mathbb{T}^4$. $(\mu_1)_{ij}$ is the number of D2-branes in the $\mathbb{T}^2$ in direction $(i, j)$ with $i, j = 1, 2, 3, 4$. $\mu_2 = k$ is the number of D4-branes. So far we have not specified the number of D2-branes. Since we are interested in the low energy dynamics we should take the number of D2-branes to minimize the total energy in the D0,D2,D4 brane system. When $b_{ij} = 0$ this is done by setting $\mu_1 = 0$, i.e. no D2-branes. Let us turn on $b_{12}$, say. From the formula

$$ch(E) = e^{\frac{1}{2\pi} tr(b)} \mu(E)$$ \hfill (142)

we get

$$(ch_1)_{34} = (\mu_1)_{34} + \frac{b_{12}}{2\pi} \mu_2 = (\mu_1)_{34} + \frac{b_{12}}{2\pi} k.$$ \hfill (143)

To minimize the energy, $(ch_1)_{34}$ should be minimized. We see that when $b_{12} > \frac{1}{2k} 2\pi$ we can lower the energy by taking $(\mu_1)_{34} = -1$. This phenomena divides the space of $b_{ij}$ into “Brillouin” zones. Each zone is a six dimensional cube of length $\frac{2\pi}{k}$ in each direction. Inside a zone the low energy physics is described by the gauge theory corresponding to a module with the $\mu(E)$ which minimizes the energy. In crossing the boundary between 2 zones, $\mu(E)$ jumps.

We also see another interesting phenomena. Whenever $\frac{b_{12}}{2\pi} k$ is an integer we have $(\mu_1)_{34} = -\frac{b_{12}}{2\pi}$ and hence $(ch_1)_{34} = 0$. This means
that $\text{ch}(E)$ is nonzero only in dimensions 0 and 4 (We are keeping all other components of $b_{ij} = 0$. Only $b_{12} = n\frac{2\pi}{k}$). This is exactly like the pure D0,D4 system with no $B^{NS}$-field. This system has a phase where the D0-branes and D4-branes are separated. To reach this phase the system has to go through zero-size instantons. We thus conclude that whenever $b_{12} = n\frac{2\pi}{k}, n \in \mathbb{Z}$ there is another phase. Of course, there is nothing special about $b_{12}$. Similar statements could be made for the other 5 components of $b_{ij}$ and even for all of them simultaneously. The point is that for each center of the “Brillouin” zone there is another branch emanating from a locus on the Coulomb branch. It emanates from the points on the Coulomb branch where some instantons have shrunk to zero size. The other phase consists of the $k$ D4-branes with $-n$ D2-branes inside moving away from the $q$ D0-branes. Let us calculate the dimension of this branch. Suppose first $n = 1$, so there are $k$ D4-branes with $-1$ D2-brane inside (equivalently 1 anti D2-brane). This system has a bound state. It is not marginally bound. The system has an 8 dimensional moduli space. To see this we should really remember that it is really $k$ D6-branes with $-1$ D4-brane. 4 of the dimensions are $U(1)$ Wilson lines on the $T^4$. They are center of mass coordinates and are always present. We are not interested in these. The other 4 are 3 transverse positions and the dual photon in 3 dimensions. We conclude that the other phase is 4 dimensional. Furthermore it emanates from a point on the Coulomb branch, since all instantons have to shrink on top of each other. The only freedom is the point where they shrink, but that is a center of mass degree of freedom which we ignore.

Let us now take $n$ to be generic. Let $g = \gcd(n, k)$. The system of $n$ D2-branes inside $k$ D4-branes can split into $g$ separate systems. The dimension is thus $8g - 4$, subtracting the center of mass again. It emanates from the Coulomb branch on a locus of dimension $4g - 4$.

The special case of $q = 1, k = 2$ was studied in detail in [59]. Here it was found that there was another phase of dimension 4 for $\alpha = \pi$. We see that this agrees exactly with what was found here. However we get a much clearer picture of the other branch. In the next section we will understand these branches from a field theory point of view.
5.3.1 Phase Transitions from the Gauge Theory

With generic twists (non-commutativity parameters), the moduli-space that we obtain is smooth. However, for special values of the twists the moduli space has ADE-type singularities. We would now like to explain the origin of some of these singularities.

$S_B(k)$ is a gauge theory at low energies. Let us study it with an $\alpha$-twist along one circle and no twist along the other 2 circles. Since there is a circle without twist we can T-dualize on that direction to $S_A(k)$, so these remarks apply to $S_A(k)$ as well. We want to reproduce the existence of other branches of the moduli space. For a related discussion see [69].

The fields in 6 dimensions are a $U(k)$ vector-multiplet and an adjoint hypermultiplet. In 3 dimensions there is a tower of $U(k)$ vector-multiplets with masses $(n_1 R_1, n_2 R_2, n_3 R_3), n_i \in \mathbb{Z}$ and a tower of adjoint hypermultiplets with masses $(\frac{n_1 - \frac{2\alpha}{R_1}, n_2 R_2, n_3 R_3), n_i \in \mathbb{Z}$. We remember that a mass in $\mathcal{N} = 4$ theories in 3 dimensions is specified by 3 numbers. The moduli space is $4k$-dimensional including the center of mass degrees of freedom. On the Coulomb branch the $U(k)$ is broken to $U(1)^k$. Each adjoint hypermultiplet splits into $k^2$ hypermultiplets of the following charges. There are hypermultiplets with charge $(0, \ldots, 0)$, and there are $k$ hypermultiplets with charges $(1, -1, \ldots, 0)$ plus permutations. There is a total of $k(k-1)$ of these. Some of these hypermultiplets can become massless on the Coulomb branch. For that to happen we have to turn on a Wilson line, $A_1$, along the first circle and set the other $3k$ moduli zero. $A_1$ has the form

\[
A_1 = \begin{pmatrix}
  a_1 & 0 & \ldots & 0 & 0 \\
  0 & a_2 & \ldots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & a_{k-1} & 0 \\
  0 & 0 & \ldots & 0 & a_k
\end{pmatrix}
\] (144)

The tower of hypermultiplets is now as follows. There are $k$ of charge
(0, \ldots, 0) with mass \( \left( \frac{n_1 - \alpha}{R_1}, \frac{n_2}{R_2}, \frac{n_3}{R_3} \right) \) and for every \( i \neq j \) there is a hypermultiplet with charge \((0, \ldots, 1, \ldots, -1, \ldots, 0)\) plus permutations with the 1 on the \(i^{th}\) place and the -1 on the \(j^{th}\) place. It has a mass \( \left( \frac{n_1 - \alpha}{R_1} + \frac{a_i}{2\pi} - \frac{a_j}{2\pi}, \frac{n_2}{R_2}, \frac{n_3}{R_3} \right) \). The uncharged ones never become massless, as long as the twist is not a multiple of \(2\pi\). The charged ones become massless if
\[
n_1 - \frac{\alpha}{2\pi} + \frac{a_i}{2\pi} - \frac{a_j}{2\pi} = n_2 = n_3 = 0.
\]
(145)

Now it is easy to make some of them massless by choosing \(A_1\) appropriately. However to have a Higgs branch we need to have non trivial solutions to the D-flatness equations. For hypermultiplets charged under a \(U(1)^r\) group there should be at least \(r + 1\) of them to have a non trivial solution. We thus need to find a number of massless hypermultiplets which is bigger than the number of \(U(1)\)'s under which they are charged. No hypermultiplets are charged under the diagonal \(U(1)\). Let us first find a situation of \(k\) massless hypermultiplets which are charged under \(U(1)^{k-1}\). The hypermultiplet of charge \((1, -1, 0, \ldots, 0)\) is massless if,
\[
n_1 = 0, \quad a_1 - a_2 = \alpha.
\]
(146)
The one of charge \((0, 1, -1, 0, \ldots, 0)\) is massless if,
\[
n_1 = 0, \quad a_2 - a_3 = \alpha
\]
(147)
and so on, up to the multiplet of charge \((0, \ldots, 0, 1, -1)\) which is massless if,
\[
n_1 = 0, \quad a_{k-1} - a_k = \alpha
\]
(148)
This gives \(k - 1\) massless hypermultiplets. To have one more we need
\((-1, 0, \ldots, 0, 1)\) to be massless. This is the case if,

\[
\frac{\alpha}{2\pi} = \frac{a_k - a_1}{2\pi} + n_1, \tag{149}
\]

for some integer \(n_1\). Now,

\[
a_k - a_1 = (a_k - a_{k-1}) + \ldots + (a_2 - a_1) = -(k-1)\alpha \tag{150}
\]

so we need \(\frac{k\alpha}{2\pi}\) to be an integer. So for \(\alpha = \frac{2\pi}{k}\) we have another phase of dimension 4. The dimension is 4 because there are \(k\) massless hypermultiplets each having 4 scalar fields and the D-flatness conditions remove \(4(k-1)\) dimensions leaving 4 real dimensions. This phase agrees exactly with the exact result from the previous section. We thus see that a naive field theory treatment, keeping all Kaluza-Klein modes, reproduces the result. This phase emanates from the Coulomb branch whenever \(a_i = a_{i-1} = \alpha\) as we saw above. This fixes the \(a_i\) up to an overall shift. The overall shift is the \(U(1)\) part which we discard anyway. This shows that the other phase emanates from one particular point on the Coulomb branch. Note that the field theory treatment is justified when \(M_s R_i \gg 1\).

More generally, let us take \(\alpha = n \frac{2\pi}{k}\) and \(g = \gcd(n, k)\). Now we can play the same game as above but within \(g\) blocks of the \(U(k)\) matrix of size \(\frac{k}{g}\). We thus get \(g\) sets of \(\frac{k}{g}\) massless fields. Each set is charged under a \(U(1)\frac{k}{g} - 1\) subgroup. This gives a \(4g\) dimensional phase emanating from a locus on the Coulomb branch. This locus has dimension \(4g - 4\). The \(4g\) comes from the diagonal \(U(1)\) in each of the \(g\) blocks. The center of mass is subtracted again. This branch has a total dimension of \(4g + 4g - 4 = 8g - 4\). We again find agreement with the exact result described previously.

The branches described above are the only ones coming from the naive field theory description besides the cases \(\alpha = 2\pi n, n \in \mathbb{Z}\) which behave like \(\alpha = 0\).
5.4 The 3+1D limit

In this section we will explain how to obtain the 3+1D Seiberg-Witten curves of the various theories compactified on $T^2$ with a twist. This time we only have two independent $\alpha$-twists corresponding to the two cycles of $T^2$. The way to obtain the 3+1D SW curves is to start with the moduli space of the theory compactified on $T^2 \times S^1$ where $S^1$ is of radius $R$ and take the limit $R \to \infty$. Let the 2+1D hyper-Kähler moduli space be of dimension $4n$. In the limit $R \to \infty$, it can be written as a fibration of $T^{2n}$ over a base of dimension $2n$. In the decompactification limit the fiber $T^{2n}$ shrinks to zero. We interpret it as the Jacobian variety of a Riemann surface of genus $n$ which varies over the base. This will then be the Seiberg-Witten curve (see [43]). Starting with the Blum-Intriligator little-string theories of $k$ NS5-branes at an $A_{q-1}$ singularity compactified on $T^2$ with twists we can get, in appropriate limits, a 3+1D gauge theory with,

$$SU(k)_1 \times \cdots \times SU(k)_q,$$

and massive adjoint hyper-multiplets in consecutive $(k, \bar{k})$ representations. The Seiberg-Witten curves for these models have been derived in [37]. As we will show below, we can reproduce these curves by taking the appropriate decompactification limit of the moduli space of $k$ $U(q)$ instantons on the non-commutative $T^4$.

To start, we will recall how the reduction of the untwisted compactified Blum-Intriligator theories works.

5.4.1 From instantons to quiver gauge theories

When we set all the $\alpha$-twists to zero we obtain the statement that the Coulomb-branch moduli space of the theories of $k$ NS5-branes on an $A_{q-1}$ singularity, compactified on $T^3$ is the same as the moduli space of $k$ ordinary instantons with a $U(q)$ gauge group on $T^4$. This result has already been established in [61,63]. Suppose we compactified on $T^3 = T^2 \times S^1$ and take the radius of $S^1$, $R \to$
It can be checked (see (140)) that the auxiliary $T^4$ becomes a product $T^2_B \times T^2_F$. The complex structure of $T^2_F$ and $T^2_B$ are fixed as $R \to \infty$ while the area of $T^2_B$ is proportional to $R$ and the area of $T^2_F$ is proportional to $R^{-1}$. Now take a particular gauge configuration corresponding to an instanton of $U(q)$ with instanton number $k$. We can encode the information in the instanton as follows (see [70,71]). At a local point on the base, the gauge field reduces to two commuting $U(q)$ Wilson lines on the fiber. We can describe them uniquely as $q$ points on the dual $\tilde{T}^2$ of the fiber. These $q$ points vary over the base $B$. The instanton equations imply that they span a holomorphic curve $\Sigma_g$ of genus $g = qk + 1$. $\Sigma_g$ is called the “spectral curve”. To completely describe the instanton we also need to describe a line bundle over $\Sigma_g$ which corresponds to a point in the Jacobian of $\Sigma_g$ (recall that the Jacobian of a genus $g$ curve is $T^g$). The line bundle is called the “spectral-bundle”. Alternatively, we can represent the moduli space of $U(q)$ instantons at instanton number $k$ on $B \times F$ as the moduli space of $q$ D6-branes wrapped on $B \times F$ with $k$ D2-branes. The curves are obtained by T-duality along the two directions of $F$. We obtain a D4-brane wrapped on a curve $\Sigma_g$ of homology cycle $q[B] + k[F]$. The curve $\Sigma_g$ is the Seiberg-Witten curve of the point in the moduli space. It intersects a generic fiber $F$ in $q$ points and a zero section of the base $B$ at $k$ points. It is also easy to see that as the base $B$ decompactifies to $S^1 \times \mathbb{R}^1$ we reproduce exactly the curves from the brane construction of [37] for the quiver gauge theory.

5.4.2 The role of the non-commutativity

Now let us repeat the same procedure but with two non-commutativity parameters $\alpha_1$ and $\alpha_2$. We can take $\alpha_1$ to be along the first cycle of the base $B = T^2$ and the first cycle of the fiber $F = T^2$ and we take $\alpha_2$ to be along the second cycle of the base $B$ and the first cycle of the fiber $F$. The $\eta$-twists will similarly correspond to non-commutativity along the second cycle of $B$ and one of the two cycles of $F$.

To translate this to the curve $\Sigma_g$ we take the system of $q$ D6-branes and $k$ D2-branes and put in NSNS 2-form fluxes according
to the non-commutativity parameters. After T-duality along $F$ the NSNS fluxes become components of the metric $G_{I,J}$.

As a result, we obtain a tilted $T^4 \equiv \mathbb{R}^4 / \Lambda$, where $\Lambda$ is a lattice spanned by the following vectors:

\begin{align*}
\hat{e}_1 &= (1, 0, 0, 0), \\
\hat{e}_2 &= (\tau_1, \tau_2, 0, 0), \\
\hat{e}_3 &= (\alpha_1 + \eta_1 \tau_1, \eta_1 \tau_2, \chi, 0), \\
\hat{e}_4 &= (\alpha_2 + \eta_2 \tau_2, \eta_2 \tau_2, \chi \rho_1, \chi \rho_2). \\
\end{align*}

Here, $\tau \equiv \tau_1 + i \tau_2$ is the complex structure of $T^2_F$, $\rho \equiv \rho_1 + i \rho_2$ is the complex structure of $T^2_B$, and,

$$\chi = m_s (\tau_2 \rho_2)^{-1},$$

so that the overall volume of the unit cell will be $m_s^2$. We will denote the coordinates in $\mathbb{R}^4$ by $(x_1, x_2, x_3, x_4)$. The D2 and D6 branes became a single D4-brane in the homology class,

$$[\Sigma] = q [B'] + k [F'].$$

Here,

\begin{align*}
F' &\equiv \{ s \hat{e}_1 + t \hat{e}_2 \mid 0 \leq s, t \leq 2\pi \}, \\
B' &\equiv \{ s \hat{e}_3 + t \hat{e}_4 \mid 0 \leq s, t \leq 2\pi \}. \\
\end{align*}

are two faces of $T^4$. Similarly to [37] the D4-brane will find a minimal-area surface in this homology class. In the complex structure given by,

$$z = x_1 + i x_2, \quad w = x_3 + i x_4,$$

the cohomology class $\omega \in H^2(\mathbb{Z})$ which is Poincarè dual to $[\Sigma]$ will, generically, be a mixture of $(1, 1)$, $(0, 2)$ and $(2, 0)$ forms. However,
it is always possible to find a complex structure (with respect to the flat metric) for which \( \omega \) is entirely a \((1,1)\) form. In this complex structure the \( T^4 \) is “algebraic” (see p315 of [72]). Given the complex structure, it is possible to write down the curve \( \Sigma \) as the zero locus of a \( \theta \)-function on \( T^4 \). These \( \theta \)-functions are the sections of the line-bundle corresponding to \([\Sigma]\) and depend on \( kq \) parameters which are the moduli (see [72] for further details).

It is easy to see that the “elliptic-models” of [37] are recovered in the special limit in which we get a gauge theory with massive hypermultiplets. In this case \( \tau \to \infty \) and there are no \( \eta \)-twists. The fiber \( F' \) is replaced with a strip \( S^1 \times \mathbb{R}^1 \). The class \([\Sigma]\) is analytic (i.e. the class \( \omega \) is a \((1,1)\) 2-form) and the Seiberg-Witten curves of [37] are recovered.

### 5.5 Another Look at the \( \eta \)-twists

In this section, we write explicitly the solution for type-IIA (or type-IIB) theory, with both \( \alpha \)-twists and \( \eta \)-twists turned on. These solutions should be interpreted as string world-sheet \( \sigma \)-models with a \( B \)-field.

We will start with a Taub-NUT space without NS5-branes. It is straightforward to define the \( \alpha \)-twist. One starts with some given background, which is a principal \( U(1) \) bundle cross a torus \( T^d \). Locally, the \( \alpha \)-twist is just the change of coordinate in the \( S^1 \) fiber of the Taub-NUT space, of the form \( y \to y + \sum \alpha I \psi^I \). \( y \) is the coordinate on the circle (see (122)) and \( \psi^I \) is the coordinate on \( T^3 \) \((I = 1, 2, 3)\). Since it is just the change of variables, the string theory equations of motion are trivially satisfied. But globally, this is not a valid coordinate transformation, since \( \alpha I \psi^I \) is not a periodic function on \( T^3 \) modulo \( 2\pi \). Therefore, we get a different background – we call it the \( \alpha \)-twisted background. As for \( \eta \)-twists, they are related to \( \alpha \)-twists by T duality in \( T^3 \).

We will construct the background with both \( \alpha \) and \( \eta \) twists turned on in the following way. We first consider the background containing Taub-NUT space cross a three-torus, without any twists. We introduce \( \alpha \)-twists along the three-torus, with the parameters \( \eta_I \).
Then, we make a T-duality transformation, and get a background with \( \eta \)-twists. This new background is again a \( U(1) \) bundle cross a (dual) torus, and we now \( \alpha \)-twist it. In this way, we get a background with both \( \alpha \)-twists and \( \eta \)-twists.

Let us do it explicitly. Start with \( \mathbb{R}^{1,2} \times T\mathbb{N}(\rho) \times T^3 \). The metric is:

\[
    ds^2 = \rho^2 U_{[\rho]}(|\vec{r}|) A^2 + U_{[\rho]}(|\vec{r}|)^{-1} (d\vec{r})^2 \\
    + g_{IJ} d\psi^I d\psi^J - dx_0^2 + dx_1^2 + dx_2^2,
\]

where we have denoted

\[
    U_{[\rho]}(|\vec{r}|) \equiv \left( 1 + \frac{\rho}{2|\vec{r}|} \right)^{-1}
\]

and \( A \) is the connection one-form \( A = dy - \vec{A} \cdot d\vec{r} \). Also, we turn on the following \( B \) field:

\[
    B = b_{IJ} d\psi^I \wedge d\psi^J
\]

We wish to introduce \( \alpha \)-twists with the parameter \( \eta_I \). As was explained above, this means just the change of variables \( y \to y - \eta_I d\psi^I \). This amounts to replacing \( A^2 \) with \((A - \eta_I d\psi^I)^2\) in (153).

Now we make three T-dualities. We do this by the standard technique of treating \( V^I_{\alpha} \equiv \partial_{\alpha} \psi^I \) (where \( \alpha \) is a string world-sheet coordinate) as an independent variable and inserting a Lagrange multiplier, \( \tilde{\psi}_I \), for, \( \partial_{[\alpha} V^I_{\beta]} \). We get the following metric:

\[
    ds^2 = \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta) \rho^2 U_{[\rho]}(|\vec{r}|)} (A - b^{IJ} \eta_I d\tilde{\psi}_J)^2 + U_{[\rho]}(|\vec{r}|)^{-1} (d\vec{r})^2 \\
    + l_s^4 \left( g^{IJ} - \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + \rho^2 (\eta, \eta) U_{[\rho]}(|\vec{r}|)} \eta^I \eta^J \right) d\tilde{\psi}_I d\tilde{\psi}_J \\
    - dx_0^2 + dx_1^2 + dx_2^2,
\]

\[156\]
with the notation, \( \eta^I = g^{IJ} \eta_J \), \( (\eta, \eta) = \eta_I \eta^I \), and \( g^{IJ} + b^{IJ} \) is the matrix inverse to \( g_{IJ} + b_{IJ} \). Also, we have the following \( B \) field:

\[
B = -\frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + \rho^2 (\eta, \eta) U_{[\rho]}(|\vec{r}|)} \eta^I d\tilde{\psi}_I \wedge (A - b^{JK} \eta_J \tilde{\psi}_K) + b^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J
\]

(157)

Notice that

\[
\frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta) \rho^2 U_{[\rho]}(|\vec{r}|)} = \frac{\rho^2}{1 + (\eta, \eta) \rho^2 \left[ \frac{\rho}{1 + (\eta, \eta) \rho^2} \right]}(|\vec{r}|)
\]

(158)

If we start with a non-degenerate torus and a very small coupling constant, then T-duality gives us back a very small coupling constant.

Now we \( \alpha \)-twist this background. Again, \( \alpha \)-twisting is just a replacement,

\[
A \to A - \alpha^I d\tilde{\psi}_I,
\]

in all the formulas for the metric and the \( B \) field. It is convenient to absorb \( b^{IJ} \eta_I d\tilde{\psi}_J \) into \( \alpha^I d\tilde{\psi}_I \). Then, the background fields are:

\[
ds^2 = R^2(|\vec{r}|) (A - \alpha^I d\tilde{\psi}_I)^2 + U_{[\rho]}(|\vec{r}|) (d\vec{r})^2 + (dx^\mu)^2 + l_s^4 G^{IJ}(|\vec{r}|) d\tilde{\psi}_I d\tilde{\psi}_J,
\]

\[
B = (A - \alpha^I d\tilde{\psi}_I) \wedge B^J d\tilde{\psi}_J + B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J
\]

(159)

where

\[
R^2(|\vec{r}|) = \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta) \rho^2 U_{[\rho]}(|\vec{r}|)}
\]

\[
G^{IJ}(|\vec{r}|) = g^{IJ} - \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta) \rho^2 U_{[\rho]}(|\vec{r}|)} \eta^I \eta^J
\]

\[
B^I(|\vec{r}|) = \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta) \rho^2 U_{[\rho]}(|\vec{r}|)} g^{IJ} \eta_J
\]

\[
B^{IJ} = b^{IJ}
\]

(160)

Also, the dilaton is not constant. Let \( \lambda \) be the string coupling at
\(|\vec{r}| \to \infty\). Then, the string coupling at finite \(|\vec{r}|\) is:

\[
\lambda(|\vec{r}|) = \lambda \sqrt{\frac{1 + \eta^2 \rho^2}{1 + \eta^2 \rho^2 U[\rho](|\vec{r}|)}} \tag{161}
\]

The metric (159) is not, strictly speaking, Hyper-Kähler. Indeed, although it does have three complex structures, they are not covariantly constant with respect to the standard covariant derivative. But they must be covariantly constant, if we modify \(\Gamma_{\mu\nu}^\rho\) with the torsion, proportional to \(H = dB\).

We want to study the moduli space of the theory on the NS5-brane, sitting at \(\vec{r} = 0\) in this background. As we remarked in section (2), the NS5-brane has a size of \(l_s\) and, although it is very heavy, it could affect the metric. We will explore this later in this section. For now, we will assume that it is safe to forget about the NS5-brane. To study the moduli space, we perform the chain of dualities. It is most convenient to think of these dualities as acting on the asymptotic (\(|\vec{r}| \to \infty\)) values of the fields. Therefore, we would like to discuss how the background fields near the position of the NS5-brane (\(|\vec{r}| \to 0\)) are related to the asymptotic values of the fields at \(|\vec{r}| \to \infty\).

Let us look first at the geometry near the origin in \(\mathbb{R}^3\). From (156) and (158) we see that the geometry becomes flat when the following two conditions are satisfied:

\[
|\vec{r}| \ll \rho \quad \text{and} \quad |\vec{r}| \ll \frac{\rho}{1 + (\eta, \eta) \rho^2} \tag{162}
\]

In this limit, we have just \(\mathbb{R}^{1,6} \times T^3\) with the metric

\[
ds^2 = (dx^\mu)^2 + |d(e^{i\alpha^I \tilde{\psi}_J} z_1)|^2 + |d(e^{-i\alpha^I \tilde{\psi}_J} z_2)|^2 + g^{IJ} d\tilde{\psi}_I d\tilde{\psi}_J \tag{163}
\]
The $B$ field becomes:

$$B = -\eta^I d\tilde{\psi}_I \wedge \text{Im}(z_1^* dz_1 + z_2^* dz_2) + b^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J$$

(164)

We wish to study the moduli space for the NS five-brane sitting at $\vec{r} = 0$. Notice that the transversal fluctuations of this five-brane at energy scale $\simeq m_s^2$ have the characteristic size $\Delta X^\perp \simeq \lambda l_s$. If we take $\rho \simeq l_s$ and general $\eta$, then both of the inequalities (162) are satisfied for $|\vec{r}| \equiv \Delta X^\perp$. This suggests that the parameter $\rho \simeq l_s$ actually does not affect the moduli space. The reason why it might be not true is that the transversal size of the NS5-brane is, actually, of the order $l_s$. Therefore the curvature of the background should, presumably, affect the physics even in the limit $\lambda \to 0$. The answer we will get shows that the moduli space does not really depend on $\rho$.

Now let us look at the fields at infinity. They are given by the formulae (159) and (160) with $|\vec{r}| = \infty$. We will denote the limits of $R^2(|\vec{r}|), G^{IJ}(|\vec{r}|)$ and $B^{I}(|\vec{r}|)$ as $|\vec{r}| \to \infty$ by $R^2$, $G^{IJ}$ and $B^{I}$. It is convenient to have a dictionary relating the fields at $|\vec{r}| = \infty$ with the fields at $|\vec{r}| = 0$. Let us first summarize our notations. We have already introduced the matrices $g_{IJ}, b_{IJ}, g^{-1}_{IJ}$ and $G^{-1}_{IJ}$ satisfying:

$$(g^{IJ} + m_s^2 b^{IJ})(g_{JK} + i l_s b_{JK}) = \delta^I_K$$

We have also introduced $G^{IJ}$ and $B^{IJ}$ in (159). Now, we define $G_{IJ}, B_{IJ}, g^{-1}_{IJ}$ and $G^{-1}_{IJ}$ in the following way:

$$(G_{IJ} + B_{IJ})(G^{JK} + B^{JK}) = \delta^K_I, \hspace{1cm} g^{-1}_{IJ} g^{JK} = \delta^K_I, \hspace{1cm} G^{-1}_{IJ} G^{JK} = \delta^K_I$$

(165)

Then, we have the following dictionary, relating asymptotic back-
ground to the local background:

\[ \rho^2 = R^2 + (B, B), \quad R^{-2} = \rho^{-2} + (\eta, \eta), \]

\[ g^{IJ} = G^{IJ} + R^{-2}B^I B^J, \quad G^{-1}_{IJ} = g^{-1}_{IJ} + \rho^2 \eta_I \eta_J, \]

\[ \eta_I = \frac{R^{-2} G^{-1}_{IJ} B^J}{1 + R^{-2}(B, B)}, \quad B^I = \frac{\rho^2}{1 + \rho^2(\eta, \eta)} g^{IJ} \eta_J, \]

\[ B^{IJ} = b^{IJ}. \]  

The local value, \( \lambda_0 \), of the string coupling is related to the asymptotic value \( \lambda \) by the formula which follows from (161):

\[ \lambda_0^2 = (1 + (\eta, \eta)\rho^2)\lambda^2 \]  

(167)

5.5.1 The chain of dualities.

We start by replacing the Taub-NUT circle with the M-theory circle. We get a D6-brane wrapped on \( T^4 \), with the NS5-brane on top of it.

At this point it is useful that we remember how the fields of type-IIA theory are related to the fields of M-theory. M-theory on a \( U(1) \) bundle is type-IIA on the base of this bundle. Suppose that the action of \( U(1) \) is associated to the vector field \( v \). The M-theory three-form \( C_M \) splits as follows:

\[ C_M = \pi^* A^{(3)} + \mathcal{A} \wedge \pi^* B \]  

(168)

Also, we choose some local trivialization, and define the connection one-form \( A^{(1)} \) on the base, \( dA^{(1)} = \mathcal{F} \) (\( \mathcal{F} \) is the curvature two-form on the base, \( d\mathcal{A} = \pi^* \mathcal{F} \)). It should be identified with the RR one-form \( C^{(1)} \) of type-IIA. Also, \( B \) should be identified with the \( B \) field
of type-IIA (this follows from its coupling to the fundamental string). What is the relation between $A^{(3)}$ and the Ramond-Ramond three-form $C^{(3)}$ of type-IIA? Let us remember the general formula for the couplings of the Ramond-Ramond fields to the D-brane [66]:

$$S_{RR} = \int \mu p C \wedge \text{tr} e^{F-B}$$  \hspace{1cm} (169)

For example, for the D2 brane we get:

$$S_{RR} = \mu_2 \int C^{(3)} - C^{(1)} \wedge (B - F)$$  \hspace{1cm} (170)

Here $C^{(1)}$ should be identified with the connection one-form, $A = d\phi + C^{(1)}$. We have to keep in mind that various forms participating in this formula are, in general, subject to gauge transformations. For example, under the gauge transformation $C^{(1)} \rightarrow C^{(1)} - d\psi$ we should have $C^{(3)} \rightarrow C^{(3)} - d\psi \wedge B$ (this is needed for the coupling (170) to be correctly defined). This suggests that

$$C^{(3)} = A^{(3)} + C^{(1)} \wedge B$$  \hspace{1cm} (171)

(that is, $C_M = \pi^* C^{(3)} + d\phi \wedge \pi^* B$.) We may derive how Ramond-Ramond fields transform under T duality from their coupling to D branes. It follows that $Ce^{-B}$ transforms as a spinor of $O(d, d; \mathbb{Z})$. Notice that

$$Ce^{-B} = A^{(1)} + A^{(3)} + \text{forms of higher rank}.$$  \hspace{1cm} (172)

Let us return to our dualities. We assume that the M Theory circle in our original configuration has radius $S = \lambda l_s$, where $l_s$ is the string scale in the configuration we start with, and $\lambda$ is the original
coupling constant (which has to be very small, if we want to get Little String Theory on NS5 brane). The three-form of M Theory is read from (159):

\[ C_M = (A - \alpha^I d\tilde{\psi}_I) \wedge B^J d\tilde{\psi}_J \wedge d\theta + B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J \wedge d\theta \] (173)

If we now treat the Taub-NUT circle as the M-theory circle, we get (168) with

\[ A^{(3)} = B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J \wedge d\theta, \quad B = B^I d\tilde{\psi}_I \wedge d\theta. \]

(Notice that \( A - \alpha^I d\tilde{\psi}_I \) is just the connection 1-form after \( \alpha \)-twist.)

In the new type-IIA theory, obtained by compactifying M Theory on the Taub-NUT circle, we have the following asymptotic values of the background fields:

\[ ds^2 = S^2 d\theta^2 + l_s^4 G^{IJ} d\tilde{\psi}_I d\tilde{\psi}_J + dr^2 + (dx^\mu)^2, \]

\[ B = B^I d\tilde{\psi}_I \wedge d\theta, \]

\[ Ce^{-B} = \alpha^I d\tilde{\psi}_I + d\theta \wedge B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J. \] (174)

(We have used (172) to find \( Ce^{-B} \) in type-IIA.) The new string length is:

\[ l_1^2 = \frac{S}{R} l_s^2 = \lambda_0 \frac{l_s^3}{\rho} \] (175)

and the new string coupling constant is:

\[ \lambda_1 = \left( \frac{R}{l_s} \right)^{3/2} \frac{1}{\sqrt{\lambda}} \] (176)

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Making three $T$ duality transformations along $T^3$, we get:

$$ds^2 = \frac{l_s^2 \lambda^2}{\rho^2} \left[ \rho^2 d\theta^2 + G^{-1}_{IJ} d\psi^I d\psi^J + 2G^{-1}_{IJ} B^I d\psi^J d\theta \right] + dr^2 + (dx^\mu)^2$$

$$B^{RR} = \alpha f \epsilon_{IK} d\psi^I \wedge d\psi^K + d\theta \wedge \epsilon_{IJK} B^{IJ} d\psi^K$$

$$B^{NS} = 0$$

(177)

with the string coupling constant,

$$\lambda_2 = \frac{\lambda}{l_s^3 \sqrt{\det G}}.$$  

(178)

The NS5-brane remains an NS5-brane, wrapped on $T^3$, and D6-brane becomes D3-brane. It shares with NS5 the directions of $\mathbb{R}^{1,2}$.

Now we do S-duality, so that $B^{RR}$ becomes $B^{NS}$, and NS5 becomes D5. Also, we get the new string coupling and the new string length:

$$\lambda_3 = \frac{l_3^3 \sqrt{\det G}}{\lambda}, \quad l_3 = \lambda_0 \sqrt{\frac{(\det g^{-1})^{1/2}}{\rho}}$$

(179)

Then, doing T-duality along the circle parameterized by $\theta$. We have now D6 brane wrapped on the four-torus, and the D2 brane inside it, orthogonal to the torus. We end up with the following string coupling and string length,

$$\lambda_4 = \frac{l_s^2}{\lambda_0 \sqrt{\rho (\det g^{-1})^{1/2}}}, \quad l_4 = \lambda_0 \sqrt{\frac{(\det g^{-1})^{1/2}}{\rho}}$$

(180)
and the following metric and $B$ field,

$$ds_4^2 = \frac{l_s^2}{\rho^2} \lambda_0^2 \left[ l_s^{-4} (\det g^{-1}) (d\tilde{\theta} - \epsilon_{IJK} b^{IJ} d\psi^K)^2 + g_{IJ}^{-1} d\psi^I d\psi^J \right],$$

$$B = \alpha^I \epsilon_{IJK} d\psi^J \wedge d\psi^K + \eta_I d\theta \wedge d\psi^I. \quad (181)$$

Let us summarize. We have started with $k$ NS5-branes sitting at the center of the Taub-NUT space, string coupling $\lambda_0$ and string length $l_s$. The background fields are given by the equations (163) and (164), they correspond to both $\alpha$-twists and $\eta$-twists present. By the chain of dualities, we have mapped this configuration to $k$ D6 branes wrapped on $T^4$, and one D2 brane, the metric and the $B$ field given by (180) and (181). Notice that the volume of $T^4$ is $\frac{l_s^2}{\rho^2} l_4^4$. In the limit we are interested in ($\lambda_0 \to 0$) it remains finite in the string units (specified by $l_4$). The shape of the torus does not depend on $\rho$.

5.5.2 World-sheet T-duality in the limit $\rho \to \infty$

Let us now see what happens in the limit $\rho \to \infty$. The strategy will be to start with type-IIA string-theory on the purely geometrical background which realizes the $\alpha$-twist. We will then perform world-sheet T-duality on $S^1$ to obtain a nonlinear world-sheet $\sigma$-model. Finally, we will insert the NS5-branes back.

To describe the geometrical background we choose,

$$X_6, \ldots, X_9,$$

as the transverse coordinates (on which the R-symmetry $SO(4)$ acts). These replace the coordinates $y$ and $\vec{r}$ of $\text{TN}(\rho)$. We will denote,

$$Z_1 = X_6 + iX_7, \quad Z_2 = X_8 - iX_9.$$

The other coordinates will be denoted,

$$X_0 \ldots X_5,$$

where $X_5$ is periodic with period $2\pi$. They are the world-sheet fields corresponding to $x_0, x_1, x_2, \psi_1, \psi_2, \psi_3$ from the previous section. The
bosonic part of the world-sheet action is,

\[ L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu \nu} \partial_{\alpha} X_{\mu} \partial^{\alpha} X_{\nu} + R^2 \partial_{\alpha} X_{5} \partial^{\alpha} X_{5} + \sum_{i=1,2} \partial_{\alpha} Z_{i} \partial^{\alpha} Z_{i}. \]

Let us, for simplicity, twist only along \( X_5 (= \psi_3) \). The twist implies that \( Z_i \) are not single-valued but rather,

\[ W_i = Z_i e^{-i \frac{\alpha}{2\pi} X_5}, \quad i = 1, 2 \]

are single-valued. The world-sheet Lagrangian now reads,

\[ L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu \nu} \partial_{\alpha} X_{\mu} \partial^{\alpha} X_{\nu} + R^2 \partial_{\alpha} X_{5} \partial^{\alpha} X_{5} + \sum_{j=1,2} |\partial_{\alpha} W_j + i \frac{\alpha}{2\pi} W_j \partial_{\alpha} X_{5}|^2. \]  

Next we perform T-duality by the standard technique of treating \( V_{\alpha} \equiv \partial_{\alpha} X_5 \) as an independent field and inserting a Lagrange multiplier \( Y \) for \( \partial_{[\alpha} V_{\beta]} \).

The result is a world-sheet action corresponding to the metric and \( B \)-field,

\[ ds^2 = \sum_{\mu, \nu=0}^{4} \eta^{\mu \nu} dX_{\mu} dX_{\nu} + |dW_1|^2 + |dW_2|^2 + \frac{dY^2 + \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)^2}{R^2 + \frac{\alpha^2}{4\pi^2}(|W_1|^2 + |W_2|^2)}, \]

\[ B_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{dY \wedge \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)}{R^2 + \frac{\alpha^2}{4\pi^2}(|W_1|^2 + |W_2|^2)}. \]
5.5.3 Adding in the NS5-brane

Now we repeat the same exercise with the NS5-brane metric. In string units, the metric is,

\[ L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu\nu} \partial_\alpha X_\mu \partial^\alpha X_\nu + R^2 \partial_\alpha X_5 \partial^\alpha X_5 \]
\[ + \frac{1}{|Z_1|^2 + |Z_2|^2} \sum_{i=1,2} \partial_\alpha Z_i \partial^\alpha Z_i. \] (184)

The dilaton is given by,

\[ g_s^2 = \frac{1}{|Z_1|^2 + |Z_2|^2}, \]

and the solution is to be trusted when \( g_s \ll 1 \). (See discussion in [73].) After T-duality we obtain,

\[ ds^2 = \sum_{\mu, \nu=0}^{4} \eta^{\mu\nu} dX_\mu dX_\nu + \frac{|dW_1|^2 + |dW_2|^2}{|W_1|^2 + |W_2|^2} \]
\[ dY^2 + \frac{1}{|W|^2} \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)^2 \]
\[ + \frac{R^2 + \alpha^2/4\pi^2}{(R^2 + \alpha^2/4\pi^2)|W|^2}, \] (185)

\[ B_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{dY \wedge \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)}{(R^2 + \alpha^2/4\pi^2)|W|^2}. \]

This is to be trusted when,

\[ |W|^2 \equiv |W_1|^2 + |W_2|^2 \gg 1. \]

We see that as \( R \to 0 \), the \( Y \)-direction stays of finite size \( \frac{2\pi}{\alpha} \).
5.5.4 Large radius limit

An interesting question is what is the low-energy description of $S_B(k)$ compactified on $S^1$ of radius $R$ with a fixed $\eta$-twist in the limit $R \to \infty$. Naively, one can argue as follows. To perform an $\eta$-twist we have to go over the “fundamental” degrees of freedom of $S_B(k)$ (whatever they are!) and separate them according to their charge $Q$ under the $U(1)$ subgroup of the R-symmetry and according to their momenta $n$ and winding $w$ along $S^1$. We then add $\eta QR$ to the mass of this field. In the limit $R \to \infty$ and for generic $\eta$, this will push all the $Q$-charged fields to high energy and we will be left with only the $Q$-neutral sector. Thus, if we start with $\mathcal{N} = (1, 1) U(k)$ SYM in 5+1D, as the effective low-energy description, the conclusion would be that we are left with $\mathcal{N} = (1, 0) U(k)$ SYM. This conclusion cannot be correct since the gluinos of the $\mathcal{N} = (1, 0)$ vector-multiplet are chiral and the theory has a local gauge anomaly.

One possibility is that there is no 5+1D limit. For this to be true we must show that there are no BPS states corresponding to light KK states. On the type-IIA side we must show that there are no states made by strings wrapped on the T-dual $S^1$ which would become light. Perhaps, when the circle is small enough, they do not form bound states any more?

5.6 Conclusion

Let us summarize the results:

1. The moduli space of the little-string theories of $k$ NS5-branes compactified on $T^3$ with $Spin(4)$ R-symmetry $\alpha$-twists is equal to the moduli space of $k U(1)$ instantons on a non-commutative $T^4$. The shape of the $T^4$ is determined by the shape and size of the physical $T^3$ and by the NSNS 2-form fluxes along it. The non-commutativity parameters are determined from the values of the twists.

2. In principle, there are 6 non-commutativity parameters on $T^4$. They are determined from the 3 geometrical $\alpha$-twists and the 3 non-geometrical $\eta$-twists. The moduli space depends only on
the 3 self-dual combinations of the non-commutativity parameters and hence only on the sum of the $\eta$-twists and $\alpha$-twists.

3. Combining the result for $k = 2$ with the result of [59], we obtain a concrete prediction for the moduli space of $2 \, U(1)$ instantons on a non-commutative $T^4$. This 8-dimensional moduli space is a resolution of $(T^4 \times T^4)/\mathbb{Z}_2$ by blowing up the singular locus. It can also be described as a $T^4$ fibration over a $\mathbb{Z}_2^4$ quotient of a particular $K3$. The fiber corresponds to the “center-of-mass” of the NS5-branes and the structure group is $\mathbb{Z}_2^4$ acting as translations of the fiber. The particular point in the moduli space of hyper-Kähler metrics on the $K3$ was constructed in [59] as a function of the $\alpha$-twists, i.e. the non-commutativity parameters. This $K3$ turns out to have a $\mathbb{Z}_2^4$ isometry. The $K3$ can be described by blowing up $T^4/\mathbb{Z}_2$ and the $\mathbb{Z}_2^4$ acts by permuting the exceptional divisors of the blow-up. Note that this $\mathbb{Z}_2^4$ does not act freely.

4. Similarly, the moduli space of the little-string theories of [61] of $k$ NS5-branes at an $A_{q-1}$ singularity, compactified on $T^3$ with $\alpha$-twists (twists in the global $U(1)$), is equal to the moduli space of $k \, U(q)$ instantons on a non-commutative $T^4$.

5. We studied the phase transitions which occur at singular points of the moduli space.

6. If instead of the little-string theories we start with the $(2, 0)$ theory (or the SCFT theory of [60] in item (4) above), we obtain the moduli spaces of instantons on a non-commutative $T^3 \times \mathbb{R}$. The non-commutativity parameters are only along $T^3$, which is in accord with the fact that there are no $\eta$-twists for this problem.

Let us conclude with 3 open problems:

a. Generalize to other gauge groups, in particular to D-type and E-type little-string theories.

b. Generalize to NS5-branes at D-type or E-type singularities.
c. Study the $\eta$-twists, in particular how they are described at large compactification radii.
References