HOLOMORPHIC EFFECTIVE ACTION OF N=2 SYM THEORY
FROM HARMONIC SUPERSPACE WITH CENTRAL CHARGES

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Abstract
We compute the one-loop holomorphic effective action of the massless Cartan sector of $N = 2$ SYM theory in the Coulomb branch, taking into account the contributions both from the charged hypermultiplets and off-diagonal components of the gauge superfield. We use the manifestly supersymmetric harmonic superfields diagram techniques adapted to $N = 2$ supersymmetry with the central charges induced by Cartan generators. The (anti)holomorphic part proves to be proportional to the central charges and it has the generic form of Seiberg’s action obtained by integrating $U(1)_R$ anomaly. It vanishes for $N = 4$ SYM theory, i.e. the coupled system of $N = 2$ gauge superfield and hypermultiplet in the adjoint representation.
1 Introduction

Quantum calculations in $N = 2$ and $N = 4$ super Yang-Mills (SYM) theories within the $N = 2$ harmonic superspace (HSS) approach [1, 2] were a subject of several recent studies [3]-[12]. The main advantage of the HSS approach is that it offers a unique opportunity to keep manifest off-shell $N = 2$ SUSY at all steps of computation (as distinct, e.g., from the $N = 1$ superfield techniques).

It also allows to clearly see that the genuine $N = 2$ SUSY in the Coulomb branch of $N = 2$ gauge theory (including matter hypermultiplets) [13] is the central charge-extended $N = 2$ SUSY, with the central charges induced by non-zero vacuum values of the diagonal (Cartan) components of the gauge superfield [5, 6]. These vacuum values simultaneously trigger a spontaneous breaking of gauge symmetry down to its Cartan torus, producing BPS masses for all superfields with non-trivial charges with respect to the Cartan generators, including off-diagonal components of the gauge superfield. This modification of $N = 2$ SUSY implies that in the course of quantum computations in $N = 2$ gauge theories in the Coulomb branch one should use the HSS techniques (propagators, etc) pertinent just to the central charge-extended case. They are a proper generalization of the quantum HSS methods worked out in [1, 2] for ordinary $N = 2$ SUSY.

In [5] these techniques were applied to compute the leading contributions to the low-energy Wilsonian effective action of hypermultiplets minimally coupled to $U(1)$ gauge supermultiplet. The hypermultiplets acquire BPS masses on account of non-zero vacuum values of the gauge superfield which induce central charges in $N = 2$ superalgebra. The HSS quantum computation with the central charge-massive hypermultiplet propagator showed that the leading term is uniquely defined as an integral over the analytic harmonic superspace with the quartic self-coupling of the hypermultiplet superfields as the Lagrangian density. This gives rise to an interesting consequence for the physical bosonic fields of the hypermultiplet: their metric ceases to be flat and is quantum-mechanically deformed into a non-trivial hyper-Kähler metric. It is the Taub-NUT metric in the case of one hypermultiplet, or its higher-dimensional generalizations in the case of few hypermultiplets. Simultaneously, a scalar fields potential arises. The same approach was used in [6] to demonstrate the central-charge origin of the Seiberg-type [14] perturbative holomorphic contributions to the effective action of gauge superfield in the same system of $U(1)$ gauge supermultiplet interacting with charged hypermultiplets. Once again, these contributions were found to be proportional to the central charges of $N = 2$ superalgebra, viz. vacuum values of $U(1)$ gauge superfields\(^1\). Thus, the same mechanism of non-zero central charges proves to be responsible for the two types of analytic contributions to the low-energy effective action in the Coulomb branch: the complex-analytic (holomorphic) ones in the gauge fields sector and the harmonic-analytic ones in the hypermultiplet sector. It is the quantum HSS approach that makes manifest the common origin of these two different sorts of the analytic corrections.

In all these studies, only contributions of charged hypermultiplets were taken into account, i.e. the gauge theory was assumed to be abelian. On the other hand, it is natural to start from the full non-abelian $N = 2$ gauge theory and to take account of the contributions of the charged components of the gauge superfield itself, along with those of hypermultiplets. The appearance of the holomorphic effective action for the massless Cartan gauge superfields in the pure $N = 2$ SYM theory (as well as non-holomorphic corrections to it) in the HSS approach was discussed in [12] using the background fields method [4]. It would be of interest to compute these contributions directly, by the same methods as in [5, 6], i.e. by treating the central charge-massive off-diagonal components of the gauge superfield on equal footing with the massive hypermultiplets.

This is what we do in the present paper. In Sect. 2, 3 we briefly discuss the quantum techniques of the central charge-extended HSS in application to $N = 2$ SYM theory. We give the relevant Lagrangians, propagators and vertices and demonstrate that at one loop only three-linear vertex

\(^1\)This calculation was firstly made within the standard quantum HSS approach, by treating non-zero vacuum value of the gauge superfield as a perturbation [9].
from the infinite sequence of $N = 2$ SYM vertices should actually be taken into account while computing the holomorphic part of the effective action. Then, in Sect. 4, choosing for simplicity $SU(2)$ as the gauge group, we calculate the one-loop holomorphic correction to the action of the neutral component of the gauge superfield. It comes from the box-type diagrams with the neutral component on the external legs and the massive charged components (together with the charged components of the Faddeev-Popov ghosts) running inside. The answer coincides with the Seiberg’s effective action obtained in [14] by integrating $U(1)_R$ anomaly. We compare it with the contribution of $q$-hypermultiplet in the adjoint representation and find both to cancel each other, in agreement with the absence of holomorphic terms in the effective action of $N = 4$ SYM to which this $N = 2$ system amounts (actually, this is a modified $N = 4$ SYM theory, with a central-charges extended $N = 4$ SUSY). The calculations can be easily generalized to an arbitrary semi-simple gauge group. The corresponding holomorphic contribution is given by a sum over positive roots in the spirit of refs. [15]. In two Appendices some technical points are treated. In particular, we check that our results are independent of the choice of $\alpha$-gagues in the massive gauge superfield propagators.

2 Classical and microscopic N=2 SYM actions in the presence of central charges

The classical HSS action of $N = 2$ SYM theory minimally coupled to hypermultiplets reads

$$S = S^{N=2}_{SYM} + S^{N=2}_{HP}.$$  (2.1)

Here [16]

$$S^{N=2}_{SYM} = \frac{1}{2g^2} \text{Tr} \int d^{12}z \sum_{n=2}^\infty \frac{(-i)^n}{n} \int du_1 \ldots du_n \frac{V^{++}(z, u_1) \ldots V^{++}(z, u_n)}{(u_1^+ u_2^+)(u_2^+ u_3^+) \ldots (u_n^+ u_1^+)}$$  (2.2)

and $V^{++}(z, u) \equiv V^{++}_A(z, u)T_A$ is the $N = 2$ analytic gauge potential $^2$. The integration in (2.2) goes over the full $N = 2$, $D = 4$ HSS in the central basis, the harmonic distributions and integrals are defined in [2]. The second term in (2.1) is the hypermultiplet action

$$S^{N=2}_{HP} = \frac{1}{2} \int d\zeta(-4) q^B \nabla^{++} q^{B^a} , \nabla^{++} q^{B^a} = D^{++} q^{B^a} + igV^{++}(t_D)^{BC} q^{C^a}.$$  (2.3)

Here the integration goes over the harmonic analytic superspace $\{\zeta\} \equiv \{z^m_A, \theta^+_a, \bar{\theta}^+_a, u_{\pm}^a\}$ are the analytic hypermultiplet superfields in a representation $R$ of the group $G$, $(t_D)^{BC}C'$ are the corresponding generators. We shall mainly deal with $q^B$ in the adjoint representation, i.e. $q^{+Ba}$. In this case the action (2.1) is that of $N = 4$ SYM theory in the $N = 2$ superfield formulation, with a hidden second $N = 2$ SUSY [2].

We shall concentrate on the first term in (2.1) which is the pure $N = 2$ SYM action. Let $C$ be the Cartan subgroup of the gauge group $G$. We shall use the decomposition

$$V^{++} = V^{++}_{(c)} + V^{++}_{G/C},$$

where $V^{++}_{(c)}$ takes values in the Lee algebra of $C$ and $V^{++}_{G/C}$ belongs to the coset $G/C$. In the standard Cartan-Weyl basis $V^{++}_{(c)}$ and $V^{++}_{G/C}$, collect, respectively, the diagonal and off-diagonal components of $V^{++}$. In what follows we assume that all Cartan components develop non-zero vacuum expectation values $v^{++} \equiv \langle V^{++}_{(c)} \rangle >:

$$V^{++} = v^{++} + \bar{V}^{++} = -(\theta^+ \bar{\theta}^+)W_0 - (\bar{\theta}^+ \theta^+)W_0 + \bar{V}^{++} , \quad < \bar{V}^{++} > = 0.$$  (2.4)

$^2$The generators $T_A$ of the gauge group $G$ are chosen Hermitean, $[T_A, T_B] = i f_{ABC}T_C$, and belonging to the adjoint representation. We normalize them so that $\text{Tr} (T_A T_B) = \delta_{AB}$. 

2
Here $W_0, W_0$ are some constants (moduli) with values in the Lee algebra of $C$. This corresponds to the Coulomb branch of the theory, with the original gauge symmetry being spontaneously broken down to that with respect to the Cartan subgroup. The leading terms of the Cartan subalgebra valued superfield strengths $W_{(c)} \equiv W_{(c)} - W_0, \bar{W}_{(\bar{c})}$ are expressed through the potential $V_{(c)}^{++} \equiv \tilde{V}_{(c)}^{++}$ as

$$W_{(c)} = -\int du(D^-)^2 V_{(c)}^{++}, \quad \bar{W}_{(\bar{c})} = -\int du(D^-)^2 \bar{V}_{(\bar{c})}^{++}. \quad (2.5)$$

Full such strengths include, of course, nonlinear terms $\sim \tilde{V}_{(c)}^{++} \equiv \tilde{V}_{(c)}^{++}$. In our computation of the Cartan subgroup sector of the effective action of $N = 2$ SYM we shall be interested only in such leading terms, assuming that the full strengths are restored by non-abelian gauge invariance.

The non-vanishing background superfield $v^{++}$ triggers the spontaneous breakdown of $G$ to $C$ and simultaneously generates constant $U(1)$ central charges in $N = 2$ superalgebra [5, 17]. They explicitly break $U(1)_R$ automorphism symmetry of the original $N = 2$ SUSY. The generators of the central charges-modified $N = 2$ superalgebra are

$$Q^k_\alpha = Q^k_\alpha \pm i \theta^k_\alpha W_0, \quad \bar{Q}_{\bar{k} \dot{\alpha}} = \bar{Q}_{\bar{k} \dot{\alpha}} \pm i \dot{\theta}_{\bar{k} \dot{\alpha}} W_0, \quad (2.6)$$

and the algebra of the flat covariant derivatives is modified as

$$\{D^+_\alpha, D^-_\beta\} = -2i \epsilon_{\alpha \beta} W_0, \quad \{\bar{D}^+_\dot{\alpha}, \bar{D}^-_{\dot{\beta}}\} = 2i \epsilon_{\dot{\alpha} \dot{\beta}} \bar{W}_0$$

$$\{\bar{D}^+_\dot{\alpha}, D^-_\beta\} = \{D^+_\alpha, \bar{D}^-_{\dot{\alpha}}\} = 2i D_\alpha \beta$$

$$[D_{++}, D^-_\alpha] = \bar{D}^-_\alpha, \quad \{D_{++}, \bar{D}^-_{\dot{\alpha}}\} = \bar{D}^-_{\dot{\alpha}}$$

$$\{D^+_\alpha, \bar{D}^+_\dot{\alpha}\} = \{D^-_\alpha, \bar{D}^-_{\dot{\alpha}}\} = 0. \quad (2.7)$$

This algebra can be interpreted as the algebra of the full covariant $N = 2$ SYM derivatives in a covariantly constant gauge background $v^{++}$ [4, 5]. Then, following [1], we can introduce the “bridge” $v$ corresponding to the particular analytic gauge potential $v^{++}$

$$v = -(\bar{\theta}^+ \theta^-) W_0 - (\bar{\theta}^- \theta^+) \bar{W}_0, \quad v^{++} = D^{++} v. \quad (2.8)$$

It allows one to choose different frames for the harmonic superfields of the central-charge extended $N = 2$ supersymmetry, with the manifest or “covariant” harmonic analyticities, making a finite Cartan subgroup transformation with $v$ as the group parameter. In one of these frames, the “$\lambda$-frame”, the derivatives $D^\lambda_\alpha, \bar{D}^\lambda_{\dot{\alpha}}$ contain no central charge terms, that is $(D^\lambda_\alpha)_{\lambda} = D^\lambda_\alpha, \quad (\bar{D}^\lambda_{\dot{\alpha}})_{\dot{\lambda}} = \bar{D}^\lambda_{\dot{\alpha}}$.

In this frame, like in the case of ordinary $N = 2$ SUSY, one can define the manifestly analytic harmonic superfields which depend only on half of the original Grassmann coordinates. At the same time, the harmonic derivatives $(D^{\pm \pm})_{\lambda}$ necessarily contain the central-charge terms. On the contrary, in the “$\tau$-frame” the above spinor derivative include the central-charge terms, while the harmonic derivatives do not. In this frame the harmonic analyticity is covariant. The relation between the two frames is given by

$$(\bar{V}_{(c)})^{++}_{\tau} \equiv e^{-iv} (\bar{V}_{(c)})^{++} \lambda \epsilon^{\tau \lambda}, \quad (D)_{\tau} = e^{-iv} (D)_{\lambda} \epsilon^{\tau \lambda}, \quad (2.9)$$

where $D$ stands for the spinor or harmonic derivative. For what follows, it will be useful to give the $\tau$ frame derivatives in the central basis of $N = 2$ HSS (for brevity, we omit the subscript $\tau$)

$$D^+_\alpha = D^+_\alpha - i \theta^+_\alpha W_0, \quad \bar{D}^+_\dot{\alpha} = \bar{D}^+_\dot{\alpha} - i \bar{\theta}^+_\dot{\alpha} W_0, \quad D^{\pm \pm} = u^{\pm \pm} \frac{\partial}{\partial u^{\pm \pm}}. \quad (2.10)$$

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3 We use the following notation: $(D)^2 = \frac{1}{4} D^\alpha D_\alpha = \frac{1}{4} (DD), (D)^2 = \frac{1}{4} D_\alpha D^\alpha = \frac{1}{4} (DD), (D)^4 = (D)^2 (D)^2$. 

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The algebra (2.7) does not depend on the choice of frame and/or basis.

The realization of $N = 2$ SUSY preserves its old form only on the Cartan part of the shifted superfield $V^{++}$, since the latter is evidently invariant under rigid $C$ transformations. It is modified on all other superfields having non-trivial charges with respect to $C$ and, hence, non-zero central charges. All these superfields acquire BPS masses by the standard Scherk-Schwarz mechanism. This concerns as well the off-diagonal component $V^{++}_{G/C} \equiv \hat{V}^{++}$ of $\hat{V}^{++}$. One of the two real $G$-algebra-valued physical scalar fields in it, along with the physical fermions, become massive by the same mechanism. The remaining physical dimension scalar field is the $G/C$ Goldstone boson and it produces a mass for the $G/C$ gauge fields via the Higgs effect (this scalar field fully disappears in the “unitary” gauge). As a result, only the physical fields contained in $\hat{V}^{(c)}_{(c)}$ remain massless (for the gauge group $G$ of rank $r$ these are $r$ complex scalar fields, $r$ gauge fields and $2r$ gaugini Weyl spinors).

Let us return to the $N = 2$ SYM action (2.2).

The central-charge induced splitting (2.4) of $\hat{V}^{++}$ into the massless and massive pieces $\hat{V}^{++}_{(c)} \equiv V^{++}_{(c)}$ and $\hat{V}^{++}$, due to the shift $v^{++}$, essentially redefines the kinetic part of the action. In particular, a mass term for the charged components $\hat{V}^{++}$ is generated (this phenomenon is analogous to the central-charge modification of the free $q^+$ action, see ref. [5, 6]). The precise form of the split action can be found using the fact that the redefinition (2.4) and the relations (2.9) are a particular case of the splitting of $\hat{V}^{++}$ into the quantum and background parts within the $N = 2$ background fields method [4], with $v^{++}$ and $\hat{V}^{++}$ playing the roles of the background and quantum parts. Then, following [4], we rewrite (2.2) as

$$S_{SYM}(\hat{V}^{++} + v^{++}) = S(v^{++}) + \frac{(-i)}{2} \text{Tr} \int d^{12} z d u (\hat{V}^{++})_\tau e^{iv} D^- e^{-iv}$$

$$+ \frac{1}{2g^2} \text{Tr} \int d^{12} z \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \int d u_1 \ldots d u_n \frac{(\hat{V}^{++})_{\tau}}{(u_1^{++} u_2^{++}) \ldots (u_n^{++} u_1^{++})}. \tag{2.11}$$

First two terms in (2.11) are of no relevance for the further consideration and can be omitted (actually, the second term is zero as a consequence of the manifest analyticity of $\hat{V}^{++}$ and the fact that $v^{++}, v$ are defined on the commutative Cartan subalgebra). This resulting form of $N = 2$ SYM action is the convenient starting point for constructing full quantum version of the central-charge modified theory.

The Faddeev-Popov-t’Hooft procedure changes insignificantly for the massive case as compared to the case of pure massless $V^{++}$ worked out in [2]. The main difference is related to the systematic use of the algebra of the centrally-extended covariant derivatives (2.7). Following the same steps as in [2], the gauge-fixed quadratic part of the split $N = 2$ SYM action, together with the Faddeev-Popov ghosts part, can be written as (for the standard $\alpha$-gauges)

$$S_{SYM}^{(2)} + S_{FP} = S_{(c)}^{(2)}(\hat{V}^{++}_{(c)}) + \frac{1}{4 \alpha} \text{Tr} \int d \zeta^{(-4)} (\hat{V}^{++}_{(c)})_\tau \hat{\square} (\hat{V}^{++}_{(c)})_\tau$$

$$+ \frac{1}{4} \left( 1 + \frac{1}{\alpha} \right) \text{Tr} \int d \zeta^{(-4)} d u_2 (\hat{V}^{++}_{(c)})_\tau (D_{\tau}^{++})_\tau \frac{1}{(u_1^{++} u_2^{++})^2} (\hat{V}^{++}_{(c)})_\tau$$

$$+ i \text{Tr} \int d \zeta^{(-4)} F_\tau D^{++} [D^{++} P_\tau + [(\hat{V}^{++})_\tau, P_\tau]] . \tag{2.12}$$

Here $\hat{\square} = \square + W_0 \bar{W}_0$ and $S_{(c)}^{(2)}(\hat{V}^{++}_{(c)})$ is the standard quadratic action for the massless $C$ algebra valued superfields. The latter has the same form as that of $\hat{V}^{++}$, but with taking the purely flat expressions for the Box operator and spinor derivatives (with no terms $\sim W_0, \bar{W}_0$ producing mass for $\hat{V}^{++}$). Notice the appearance of the index $\tau$ in the FP action: the ghost superfields $F$ and $P$
are in the adjoint representation like $\tilde{V}^{++}$ and therefore their off-diagonal components acquire non-trivial central charges and become massive like the charged $q^+$ hypermultiplets. We also note that, though $(\tilde{V}^{++})_\tau, F_\tau, P_\tau$ are covariantly analytic, the integrands in (2.12) are manifestly analytic: this is because they are invariant under the global $G$ symmetry and, hence, are the central charge singlets.

The full microscopic quantum action for the central-charges extended system of $N = 2$ SYM and hypermultiplet superfields can be obtained as a sum of (2.12), the self-interaction $N = 2$ SYM part and the hypermultiplet action

$$S_{\text{micr}} = S^{(2)}_{\text{SYM}} + S^{\text{int}}_{\text{SYM}} + S_{FP} + S^N_{HP},$$

(2.13)

where

$$S^{\text{int}}_{\text{SYM}} = \frac{1}{2g^2} \text{Tr} \int d^2 z \sum_{n=3}^{\infty} \frac{(-i)^n}{n} \int du_1 \ldots du_n \frac{1}{(u_1^{-1} u_2^{-1})(u_2^{-1} u_3^{-1}) \ldots (u_n^{-1} u_1^{-1})}$$

(2.14)

and [5]

$$S^N_{HP} = \frac{1}{2} \int d\varphi (-4)(q^+ B^\alpha)_\tau \tilde{V}^{++}(q^+ B^\alpha)_\tau,$$

$$\tilde{V}^{++}(q^+ B^\alpha)_\tau = D^{++}(q^+ B^\alpha)_\tau + ig(\tilde{V}^{++D})_\tau (t_D)^BC^\alpha (q^+ C^\alpha)_\tau.$$

(2.15)

It is easy to show that action (2.13) is invariant under the appropriate BRST transformations which replace the gauge ones in the quantum case. These are a straightforward extension of those for the massless case [2].

### 3 Feynman diagram techniques in HSS with central charges

Now it is easy to read off the Feynman rules. It is more convenient to construct them using covariantly analytic superfields and then to pass to the manifestly analytic ones. The covariant analytic Feynman rules do not contain any new types of the vertices.

The Green function for the massless Cartan component $\tilde{V}_c^{++}$ has the standard form (see [2]). The equation defining the Green function for the off-diagonal massive $G/C$ components of the full $\tilde{V}^{++}$ directly follows from (2.12)

$$\frac{1}{2\alpha} \hat{G}^{(2,2)}(1 | 3) + \frac{1}{2} (1 + \frac{1}{\alpha}) (D^+_1)^4 \int du_2 \frac{1}{(u_1^{-1} u_2^{-1})}\hat{G}^{(2,2)}(z, u_2 | 3) = \delta^{(2,2)}(1 | 3),$$

(3.1)

where the covariantly analytic $\delta$-function is defined by [17, 5]

$$[\delta^{(2,2)} (1 | 2)]_{BC} = (D^+_1)^4 \delta^{12}(z_1 - z_2) \delta^{(2,2)}(u_1, u_2).$$

(3.2)

The Green function and $\delta$-function are matrices in the adjoint representation of the gauge group.

The solution of (3.1) in the Fermi-Feynman gauge ($\alpha = -1$) is

$$< (\tilde{V}^{++})_\tau | (\tilde{V}^{++})_\tau > = \frac{2}{\square_1}$$

(3.3)

The ghost propagator is also a direct generalization of the standard one. It can be immediately found from the ghost action in (2.12):

$$< (\hat{F}(1))_\tau | (\hat{P}(2))_\tau > = \frac{1}{\square_1} (D^+_1)^4 (D^+_2)^4 \delta^{12}(X_1 - X_2) \frac{(u_1^{-1} u_2^{-1})}{(u_1^{-1} u_2^{-1})}.$$  

(3.4)
We also give the expression for the $q^+$ propagator in the central-charge case \[5, 6\]

$$<\langle \hat{q}^+(1) \rangle_\tau | \langle \hat{q}^+(2) \rangle_\tau > = \frac{1}{(D_+^+)^4(D_-^+)^4} \delta^{12} (X_1 - X_2) \frac{1}{(u_1^+ u_2^+)^2}. \quad (3.5)$$

While constructing the effective action, it is of no actual need to know the explicit expressions for vertices in the Feynman rules. Nevertheless, for completeness and for further possible use we present some of them.

The vertex \( (F)_\tau (\tilde{V}^{++})_\tau (P)_\tau \) is

$$gf^{ABC} D^{++}_{(B)}.$$

(3.6)

There is an infinite number of the \( N = 2 \) SYM self-interaction vertices, as follows from (2.14). We shall present the first two ones (actually, it is the three-linear vertex which is of relevance for further calculations).

Here the zigzag lines denote the full \( \tilde{V}^{++} \) fields. Below we will deal only with the Cartan component of \( V^{++} \) on the external legs. So, to avoid a confusion, we shall denote the latter by the wavy lines

$$\tilde{V}^{++} \quad V^{++}_{(\bar{c})}.$$

In what follows we shall be interested in the leading contributions to that sector of the one-loop \( N = 2 \) SYM effective action which contains only the massless SYM superfields \( V^{++}_{(\bar{c})} \). As we shall show, such contributions can come only from the diagrams containing the three-linear vertices (3.7) (see Fig. 2). Clearly, when the external legs are massless \( V^{++}_{(\bar{c})} \), only the massive charged \( \hat{V}^{++} \) superfields with the propagators (3.1) can run inside. Thus the sector of \( N = 2 \) SYM effective action we are interested in is defined by the expression analogous to that defining the contribution from the charged \( q^+ \) hypermultiplets \[6\], namely, by

$$\Gamma[V^{++}_{(\bar{c})}] = i \text{Tr} \ln \{ \delta^{(4,2)}_A + V^{++}_{(\bar{c})} \tilde{G}^{(2,2)} \}.$$  (3.9)

Here \( \delta^{(4,2)}(1|2)_A \) is the appropriate analytic delta-function and \( \tilde{G}^{(2,2)}(1|2) \) is given by (3.2). Notice that \( (V^{++}_{(\bar{c})})_\tau = (V^{++}_{(\bar{c})})_{\lambda} = V^{++}_{(\bar{c})} \) in view of commutativity of \( V^{++}_{(\bar{c})} \) and the bridge \( v \) (they both
belong to the Cartan subalgebra).

Let us argue that the leading holomorphic part of the $N = 2$ SYM effective action in the considered sector can indeed come only from the diagrams on Fig. 2, with the three-linear vertices.

The generic argument is as follows.

For a fixed number of the external $V^{++}_{(c)}$ legs any other one-loop diagram apart from those on Fig. 2, i.e. a diagram including some number of higher-order $N = 2$ SYM vertices (of the type depicted on Fig. 1), contains not enough spinor derivatives to produce the (anti)holomorphic contributions in the local limit (after restoring the full Grassmann integration measures at the vertices).

This important property can be proved in the general case, but it is instructive to firstly illustrate it on a simple example. The simplest diagram of the above type includes two $V^{++}$ propagators, one three-linear and one quartic vertices, but its contribution is vanishing by the purely algebraic reason related to the properties of the gauge group structure constants. So we explain the above argument on the example of the supergraph depicted on Fig. 1. It could give a non-vanishing contribution to the quartic term of the effective action.

Using the identities

$$\delta^8(\theta_1 - \theta_2)(D)^n\delta^8(\theta_1 - \theta_2) = 0 \quad \text{if} \quad n < 8,$$

$$\delta^4(\theta_1 - \theta_2)(D^+_1 D^+_2)(D^+_c D^+_n)\delta(\theta_1 - \theta_2) = 16(u^+_1 u^+_n)(u^+_2 u^+_c)\delta(\theta_1 - \theta_2) \quad (3.10)$$

this contribution can be put in the form

$$\int \frac{d^4p_1 d^4p_2 d^8\theta_1 d\theta_2 d\theta_3 d\theta_4 d\theta_5 d\theta_6 d\theta_7}{(2\pi)^8(p_1^2 - m^2)(p_2^2 - m^2)} f(u_1, u_2, u_3, u_4, u_5, u_7) \times V^{++}_{(c)}(\theta_1, u_2)V^{++}_{(c)}(\theta_1, u_3)V^{++}_{(c)}(\theta_1, u_4)V^{++}_{(c)}(\theta_1, u_7), \quad (3.11)$$

with $f(u_1, u_2, u_3, u_4, u_5, u_7)$ being a function of the harmonic variables only. The holomorphic (anti-holomorphic) contribution is the integral over $d^4\theta$ ($d^4\bar{\theta}$) with a function of the strength $W_{(c)}$ ($\bar{W}_{(c)}$) as the integrand. Then, taking off the proper number of the derivatives from the Grassmann measure, we arrive at the expression

$$\int \frac{d^4p_1 d^4p_2 d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\theta_5 d\theta_6 d\theta_7}{(2\pi)^8(p_1^2 - m^2)(p_2^2 - m^2)} f(u_1, u_2, u_3, u_4, u_5, u_7) \times \int d^4\theta \left\{(D^+_1)^2(D^+_2)^2 V^{++}_{(c)}(\theta_1, u_2)V^{++}_{(c)}(\theta_1, u_3)V^{++}_{(c)}(\theta_1, u_4)V^{++}_{(c)}(\theta_1, u_7)\right\} \quad (3.12)$$

Recalling the definition of the abelian SF strengths (2.5), we see that one needs at least eight spinor derivatives to form these objects from the four gauge potentials in (3.12), while only four such derivatives are present in (3.12). Thus we conclude that no any local holomorphic (anti-holomorphic) contribution can be produced by the above supergraph.

In the generic case, every additional external line at some vertex of the given supergraph demands two extra spinor derivatives for this diagram be capable to produce a holomorphic term.
in the local limit. Such derivatives can be taken off only from the propagators. So, for obtaining a holomorphic contribution there should be a strict relation between the numbers of external lines and propagators. A simple analysis shows that these numbers should be equal, which selects the diagrams on Fig. 2 as the only source of holomorphic contribution from the pure $N = 2$ SYM to the Cartan sector of the effective one-loop $N = 2$ SYM action.

4 One-loop holomorphic contribution to the SYM effective action

For simplicity, we firstly choose $SU(2)$ as the gauge group, with $V^{++}_{(c)}$ being simply $V^{++}_3 T_3$. The techniques suitable for calculating $n$-particle holomorphic corrections to the effective action in the case of $N = 2$ SUSY with constant background central charges were worked out in [6]. Although the consideration in [6] was limited to the abelian $U(1)$ theory and to the supergraphs with $q^+$ hypermultiplets inside, it can be easily extended to our case. This is mainly due to the property that for extracting the holomorphic contributions it is sufficient to consider the diagrams depicted in Fig. 2. The only complication compared to the hypermultiplet case is the presence of harmonic non-localities in the pure $N = 2$ SYM vertices.

So, in the $N = 2$ SYM sector (with no hypermultiplets) the holomorphic contribution can come from the diagrams on Fig. 2 and those with the Faddeev-Popov ghosts inside (Fig. 3).

![Figure 2](image1.png)

![Figure 3](image2.png)

We begin by computing the first type of contribution. It can be represented in the form

$$\Gamma_{YM}^{(n)}[V^{++}] = \frac{(-i)^{n+1}}{n} \mathrm{Tr} \prod_{t=1}^{n} \int d^4 x_t d^8 \theta_t u_t d \omega_t d \nu_t \frac{V^{++}_{(c)}(t)}{(u_t^+ \omega_t^+)(\nu_t^+ \nu_t^+)} \times \frac{1}{\Box_t + m^2} [D^+(\theta_t, \nu_t)]^4 \delta^8(\theta_t - \theta_{t+1}) \delta^8(\theta_t - \theta_{t+1}) \delta^{(-2,2)}(\nu_t, \omega_{t+1}).$$

(4.1)

Hereafter, $V^{++}_{(c)}(t) \equiv V^{++}_{(c)}(x_t, \theta_t, u_t), \ x_{n+1} \equiv x_1, \ u_{n+1} \equiv u_1, \ \theta_{n+1} \equiv \theta_1, \ u_{1-1} \equiv u_n$.

Passing to the momentum space, we find that the terms resulting in the holomorphic and anti-holomorphic contribution in the low-energy limit are concentrated in the expression

$$\Gamma_{YM}^{(n)}[V^{++}] = \frac{i^{n+3}}{n(2\pi)^{2n}} \mathrm{Tr} \prod_{t=1}^{n} \int \frac{d^4 p_t d^4 \theta_t d u_t d \nu_t}{(p_t^2 - m^2)^2} [D^+(\theta_t, \nu_t)]^2 \delta^4(\theta_t - \theta_{t+1}) \frac{V^{++}_{(c)}(t)}{(u_t^+ \nu_t^+ - u_{t-1}^+ \nu_{t-1}^+)(\nu_t^+ \nu_t^+)} \int d^4 \tilde{\theta}_1 d^4 \tilde{\theta}_2 \delta^4(\tilde{\theta}_1 - \tilde{\theta}_2) [\mathrm{Ch}] \delta^4(\tilde{\theta}_2 - \tilde{\theta}_1) + \text{c.c.},$$

(4.2)

where by the symbol $[\mathrm{Ch}]$ we denoted the following chain of differential operators:

$$[\mathrm{Ch}] \equiv [\bar{D}^+(\nu_1)]^2 [\bar{D}^+(\nu_2)]^2 \ldots [\bar{D}^+(\nu_n)]^2.$$

(4.3)
Reducing this chain as explained in Appendix A, doing the $\bar{\theta}_2$ integral with making use of the relation
\[
\int du \frac{u_i^+ u_j^+}{(u^+ u^+)(u^+ v^+)} = \frac{u_i^+ u_j^-}{v^+ u^+} - \frac{v_i^+ v_j^-}{v^+ u^+},
\] (4.4)
and, finally, performing the integration over the harmonics $v_1, v_2, \ldots v_n$, we reduce (4.1) to
\[
\Gamma_{YM}^n[V^{++}] = \frac{i^{2n-1} W_0^{n-2} n}{4^n (2\pi)^{4n}} \text{Tr} \prod_{\ell=1}^{n} \int d^4 p d^4 \theta d \theta d^4 \bar{\theta} \frac{V_{(c)}^{++}(t)}{(p^2 - m^2)} [u_i^+ u_{\ell+1}^+] \times \left\{ [D^+(\theta_{\ell}, u_{\ell}) D^-(\theta_{\ell}, u_{\ell})] - [D^+(\theta_{\ell+1}, u_{\ell+1}) D^-(\theta_{\ell+1}, u_{\ell+1})] \right\} \delta^4(\theta_{\ell} - \theta_{\ell+1}) + c.c.(4.5)
\]
Thanks to the identity
\[
D^-(u_n)(u_n^+ u_p^+) = D^+(u_p) + D^+(u_n)(u_n^+ u_p^-),
\] (4.6)
the previous expression can be rewritten in the form
\[
\Gamma_{YM}^n[V^{++}] = \frac{i^{2n-1} n}{4^n (2\pi)^{4n}} \text{Tr} \prod_{\ell=1}^{n} \int d^4 \theta d^4 \theta d u_i \frac{V_{(c)}^{++}(t)}{(u_i^+ u_{\ell+1}^+)^2} [A_n + B_n] W_0^{n-2} + c.c. ,
\] (4.7)
where
\[
A_n = \frac{1}{4^n} \prod_{s=1}^{n} 2(D_s^+ D_{s+1}^+) \delta^4(\theta_s - \theta_{s+1})
\]
\[
B_n = \prod_{s=1}^{n} [(D_s^+)^2(u_s^- u_{s+1}^-) + (D_{s+1}^+)^2(u_{s+1}^- u_s^+)] \delta^4(\theta_s - \theta_{s+1}).
\]
Let us first consider the term proportional to $A_n$ in (4.7). Rearranging the derivatives in it and integrating over $\theta_3, \ldots, \theta_n$, we extract the following terms leading to the holomorphic and anti-holomorphic contributions
\[
\frac{2i^{2n-1} W_0^{n-2}}{(2\pi)^{4n}} \text{Tr} \prod_{\ell=1}^{n} \int d^4 \theta d^4 \theta d^4 \bar{\theta} \prod_{s=1}^{n} \frac{V_{(c)}^{++}(\theta_1, u_s)}{(u_i^+ u_{\ell+1}^+)^2} \times \delta^4(\theta_1 - \theta_2) (D_s^{++}(u_m)^2) (D_{s+1}^{--}(u_m)^2) \delta^4(\theta_1 - \theta_2) + c.c. .
\] (4.8)
The superfield $V_{(c)}^{++}$ is invariant under the central charge, hence $D^+ V_{(c)} = D^+ V_{(c)}$. Now, using the identity (4.6) and analyticity of $V_{(c)}^{++}$, we obtain
\[
(D_{m-1}^{++})^2 V_{(c)}^{++}(u_m) = (D_m^{--})^2 V_{(c)}^{++}(u_m)(u_{m-1}^+ u_m^+)^2.
\] (4.9)
Using eqs. (4.9) and (2.5), we find that (4.8) in the local limit is reduced to
\[
\frac{2 i}{n} \int \frac{d^4 p}{(2\pi)^{4n}(p^2 - W_0^n)} \int d^4 x d^4 \bar{\theta} W_0^{n-2} W_{(c)} + c.c.
\] (4.10)
(from here on, we denote by $W_0$, $W_{(c)}$, $\ldots$ the coefficients of $T_3$ in the corresponding $U(1)$ subalgebra-valued quantities).

9
The second term in (4.7) can be handled similarly. Using the identity \(u_j^+ u_2^i = u_j^- u_2^i - \delta_j^i\) and integrating by parts, we reduce it to the form in which it contains no explicit harmonics. In the local limit it yields

\[
\frac{i^{2n+3}}{2^{n-2} n} \int \frac{d^4 p}{(2\pi)^4 n(p^2 - W_0 W_0)^n} \int d^4 x d^4 \theta W_0^{n-2} \hat{W}_0^n + c.c.
\]

(4.11)

Putting together (4.10) and (4.11) results in

\[
\Gamma^n_{YM}[V^{++}] = 2 \frac{i}{n} \int \frac{d^4 p}{(2\pi)^4 n(p^2 - W_0 W_0)^n} \int d^4 x d^4 \theta W_0^{n-2} \hat{W}_0^n + c.c.
\]

(4.12)

The ghost contribution is given by the supergraphs on Fig. 3. The sum of holomorphic and anti-holomorphic contributions from \(\Gamma_{gh}^n[V^{++}]\) coincides, up to a coefficient, with the second term in eq. (4.7) (that is proportional to \(B_n\)). In the local limit it is

\[
\Gamma_{gh}^n[V^{++}] = \frac{i(-1)^n}{2^{n-2} n} \int \frac{d^4 p}{(2\pi)^4 n(p^2 - W_0 W_0)^n} \int d^4 x d^4 \theta W_0^{n-2} \hat{W}_0^n + c.c.
\]

(4.13)

Putting together (4.12) and (4.13), we get the full \(n\)-th order contribution from the \(N = 2\) SYM sector

\[
\Gamma^n_{YM}[V^{++}] + \Gamma^n_{gh}[V^{++}] = 2 \frac{i}{n} \int \frac{d^4 p}{(2\pi)^4 n(p^2 - W_0 W_0)^n} \int d^4 x d^4 \theta W_0^{n-2} \hat{W}_0^n + c.c.
\]

(4.14)

The total low-energy holomorphic effective action is obtained by summing over \(n\) and making the appropriate renormalization in the second-order term. The final answer is as follows

\[
\Gamma_{YM}[V^{++}] + \Gamma_{gh}[V^{++}] = \frac{1}{32\pi^2} \int d^4 x d^4 \theta W_0^2 \ln \frac{W_0^2}{M^2} + c.c.
\]

(4.15)

\(W_0 = W + \hat{W}\),

with \(M\) being normalization scale. It coincides (up to a difference in the conventions) with Seiberg’s action [14] obtained by integrating \(U(1)_R\) anomaly in the pure \(N = 2\) SYM theory. The same arguments as in ref. [6] show that the holomorphic contribution (4.15) is entirely due to non-zero central charges and is vanishing in the limit \(W_0 = \hat{W}_0 = 0\) (this does not apply to the divergent second-order correction which exists in the massless case as well [2]). The same holomorphic contribution was found in [4] using the background fields method. In our case we calculated it using the computation techniques manifestly covariant under \(N = 2\) SUSY with the central charges.

The calculation of the hypermultiplet contribution follows the same lines as in the abelian case [6]. Here we present only the final answer for the hypermultiplet in the adjoint representation. It amounts to the contribution of two charged massive hypermultiplets (two off-diagonal components of the adjoint representation of \(SU(2)\)) and differs only in sign from the total contribution of the gauge sector:

\[
\Gamma_{HP}[V^{++}] = -\frac{1}{32\pi^2} \int d^4 x d^4 \theta W_0^2 \ln \frac{W_0^2}{M^2} + c.c.
\]

(4.16)

The cancellation among the \(N = 2\) SYM and hypermultiplet contributions agrees with the absence of holomorphic corrections to the \(N = 4\) SYM effective action in the \(N = 2\) superfield notation and serves a good consistency check of our computation of the \(N = 2\) SYM contribution (4.15).
Note that in the present case we deal with a bit different $N = 4$ SYM action: both manifest and hidden $N = 2$ SUSY in it possess central charges proportional to $W_0, W_0$. They break the $U(4)_R$ symmetry of $N = 4$ SUSY down to $SO(4)$ which is the product of the $N = 2$ SUSY automorphism $SU(2)$ and the so-called Pauli-Gürsey $SU(2)$ realized on the doublet indices $a$ in (2.15).

Being armed with the above techniques, we can compute the holomorphic contribution to the $N = 2$ SYM effective action for the case of arbitrary semi-simple gauge group $G$ of rank $r$. It is convenient to choose the standard Cartan-Weyl basis for its algebra $\mathfrak{g}$

$$g = n_+ \oplus h \oplus n_-, \quad (4.17)$$

where $h$ is the Cartan subalgebra with the basis $\{H_i\}, \ i = 1 \ldots r$, and $n_+, \ n_-$ are the positive and negative roots subspaces spanned by the generators $E_\ell(\pm \lambda), \ E_{(-\lambda)} (\pm \lambda$ are the positive (negative) roots). For our purposes we need only the commutation relations involving the Cartan generators

$$[H_i, \ H_k] = 0, \ [H_i, \ E_{\ell(\pm \lambda)}] = \pm \lambda^{(i)} \ E_{\ell(\pm \lambda)}, \ [E_{\ell(\pm \lambda)}], \ E_{(-\lambda)}] = \lambda^{(i)} \ H_i, \quad (4.18)$$

$$\text{Tr}(H_i H_k) = \delta_{ik}, \quad \text{Tr}(E_{\ell(\pm \lambda)} E_{(-\beta)}) = \delta_{\lambda \beta}. \quad (4.19)$$

Next we notice that all the vertices we use in our calculations can be uniformly written as follows

$$\sim \text{Tr} \left([M, \ K] V^{++}(\ell)\right) = \text{Tr} \left([V^{++}(\ell), \ M] K\right) = \text{Tr} \left(M [K, \ V^{++}(\ell)]\right), \quad (4.20)$$

where

$$V^{++}(\ell) = \sum_i V^{++}_i H_i,$$

and

$$M = \sum_\lambda M_{(-\lambda)} E_{\ell(\pm \lambda)}, \quad K = \sum_\beta K_{(\pm \beta)} E_{(-\beta)}$$

stand for the positive (negative) roots parts either of the Faddeev-Popov ghost superfields, or the hypermultiplet in the adjoint representation, or the massive $N = 2$ SYM potentials $\hat{V}^{++}(1), \hat{V}^{++}(2)$. Defining

$$V^{++}_{(\lambda)} E_{\ell(\pm \lambda)} \equiv [V^{++}(\ell), \ E_{\ell(\pm \lambda)}] \quad (4.21)$$

and using (4.19), we represent these vertices as

$$\sim \sum_\lambda M_{(-\lambda)} V^{++}_{(\lambda)} K_{(\pm \lambda)}. \quad (4.22)$$

Then one should take into account that only the propagators $\langle M_{(-\lambda)}(1) | K_{(\pm \beta)}(2) \rangle$ appear as the internal lines in the one-loop diagrams we are interested in, and that these propagators are diagonal in the indices $\lambda, \beta$ (because they include only the unity matrix and the diagonal matrices of the Cartan generators). With this in mind, it is easy to find that the full holomorphic and anti-holomorphic correction from the SYM and ghost superfields is given by the sum over the positive roots, like in refs. [15]

$$\Gamma_{YM} + \Gamma_{gh} = \frac{1}{32\pi^2} \sum_{\lambda \in \mathfrak{h}^+} \int d^4 x dq^4 \theta \ W^2_{(\lambda)} \ln \frac{W^2_{(\lambda)}}{M^2} + \text{c.c.}, \quad W_{(\lambda)} E_{\ell(\pm \lambda)} \equiv [W(\ell), \ E_{\ell(\pm \lambda)}]. \quad (4.23)$$

As before, the hypermultiplet contribution differs only by sign from (4.23) (once again, we specialize to the adjoint representation). Thus, like in the $SU(2)$ case, we have for the $N = 4$ SYM holomorphic effective action

$$\Gamma_{N=4}^{hol}[V^{++}] = \Gamma_{HP}^{hol}[V^{++}] + \Gamma_{YM}^{hol}[V^{++}] + \Gamma_{gh}^{hol}[V^{++}] = 0. \quad (4.24)$$
Finally, let us make two comments.

The first one concerns the gauge invariance of our approach. Our computations are clearly covariant under the Cartan subgroup $C$ of the full gauge group $G$. What concerns non-abelian gauge $G/C$ transformations, their sole effect seems to be rotations of the considered sector of the full $N = 2$ SYM effective action into other sectors, with some numbers of off-diagonal components of the non-abelian strength as external legs. As for the internal lines, the above diagrams exhaust all possibilities in the sector we are considering, provided that one is interested only in the holomorphic contributions and keeps in mind the property discussed in the end of Sect. 3. In Appendix B we also argue that our results cannot depend on the choice of the $\alpha$ gauge in the massive $\tilde{V}^{++}$ propagator. Our reasoning is based on an explicit calculation. An implicit argument why the full $N = 2$ SYM contribution should be $\alpha$-independent is the absence of holomorphic corrections to the $N = 4$ SYM action. Indeed, this contribution should be cancelled by that from the $q$-hypermultiplet sector, but the latter contains no any $\alpha$-dependence.

Secondly, we wish to note that in the one-loop approximation all possible diagrams with massless gauge superfields outside necessarily contain only massive $\tilde{V}^{++}$ lines inside. This follows from the conservation of the Cartan $U(1)$ charges in the diagram and the property that among the elementary $N = 2$ SYM vertices there exist no those containing only Cartan $V^{++}_{(c)}$ components of $\tilde{V}^{++}$: any vertex should have at least two massive $\tilde{V}^{++}$. To be convinced of this, it is enough to observe that the nominator of a generic term in the $N = 2$ SYM self-interaction (2.14) can be rewritten through the successive commutators as

$$\sim \text{Tr}\{[(\tilde{V}^{++}_1)_{\tau} , [(\tilde{V}^{++}_2)_{\tau} , ...., [(V^{++}_{n-2})_{\tau} , (V^{++}_{n-1})_{\tau} ]...]](V^{++}_n)_{\tau}\}.$$

So, all possible one-loop diagrams with the $V^{++}_{(c)}$ external legs contain inside only the massive $\tilde{V}^{++}$ lines, and we can apply to them the general argument adduced in the end of Sect. 3.

5 Conclusions

In this paper we worked out the basic elements of the quantum HSS techniques for non-abelian $N = 2$ SYM theory in the Coulomb branch, with $N = 2$ SUSY modified by the Cartan central charges which give rise to the splitting of $N = 2$ SYM prepotential $V^{++}$ into the massless Cartan and massive charged $G/C$ projections. Using this approach, we calculated the holomorphic contribution of the massive $N = 2$ SYM superfields (and ghosts) into the effective action of the massless projection. Up to a numerical coefficient, it has the same form as the contribution of one $U(1) q^+$ hypermultiplet computed earlier in [6] with making use of similar techniques. Like the latter, it disappears in the limit of zero central charge, thus confirming the general property that all holomorphic perturbative one-loop contributions to the $U(1)$ gauge superfields sector of the quantum effective action of $N = 2$ SYM in the Coulomb branch are the entire effect of non-zero central charge in $N = 2$ superalgebra (similarly to the analytic corrections to the $q^+$ effective action [5]). We found the explicit cancellation of the $N = 2$ SYM contribution and that of the matter hypermultiplet in the adjoint representation of the gauge group, thus having checked the absence of the one-loop holomorphic corrections to $N = 4$ SYM action. The main novelty of our approach is the preservation of manifest $N = 2$ SUSY with the Cartan central charges at all steps of quantum calculations in non-abelian $N = 2$ SYM theories.

It would be interesting to apply the same straightforward quantum HSS techniques to computing non-holomorphic contributions to the Coulomb branch effective actions of $N = 2$ and $N = 4$ SYM theories, both in the gauge fields and hypermultiplet sectors.
Appendix A

Here we show how to simplify the chain operator \( [\mathbf{Ch}] = [\mathbf{D}^+(u_1)]^2[\mathbf{D}^+(u_2)]^2 \ldots [\mathbf{D}^+(u_n)]^2 \) appearing in eq. (4.2). We follow the method proposed in [6].

Let us write this chain of derivatives in the following form

\[
\prod_{t=1}^{n} [\mathbf{D}^+(u_t)]^2 = \frac{1}{4^n} \prod_{t=1}^{n} (D_{t-1}^+ D_{t}^+) = \left( \frac{-1}{4^n} \right)^{n-1} \left\{ \prod_{t=1}^{n} (u_{t}^+ u_{t+1}^-) (D_{t}^+ D_{t+1}^-) \right\} \{D_n^+ D_n^-\} . \tag{A.1}
\]

Here all the central basis spinor derivatives are evaluated at the same \( \theta \)-point.

As the next step, we expand the derivative \( D_{n-1}^+ \) in (A.1) over the \( n \)-th set of harmonics. The result reads

\[
\prod_{t=1}^{n} [\mathbf{D}^+(u_t)]^2 = \left( \frac{-1}{4^n} \right)^{n-1} \left\{ \prod_{t=1}^{n} (u_{t}^+ u_{t+1}^-) \prod_{t=1}^{n-2} (D_{t}^+ D_{t+1}^-) \right\} 
\times \left\{ (D_{n-2}^+ D_{n}^-)(D_{n}^+ D_{n}^-)(u_{n}^+ u_{n-1}^-) - (D_{n-2}^- D_{n}^+)(D_{n}^+ D_{n}^-)(u_{n}^- u_{n-1}^+) \right\} . \tag{A.2}
\]

Further, in the first term we expand \( D_{n-1}^+ \) again in terms of the \( n \)-th harmonics (we suppress spinor indices). Only the \( D_n^+ \) projection survives, then we anticommute it with \( D_n^- \) using the algebra (2.7). Since \( (D^+)^3 = 0 \), only the commutator remains. The situation is simpler with the second term.

We just anticommute \( D_n^+ \) with \( D_n^- \) and then expand \( D_{n-1}^- \) over the \( n \)-th harmonics. We arrive at

\[
\prod_{t=1}^{n} [\mathbf{D}^+(u_t)]^2 = \left( \frac{-1}{4^n} \right)^{n-1} 4i\bar{W}_0 \left\{ \prod_{t=1}^{n-1} (u_{t}^+ u_{t+1}^-) \prod_{t=1}^{n-3} (D_{t}^+ D_{t+1}^-) \right\} 
\times \left\{ (D_{n-2}^+ D_{n}^-)(D_{n}^+ D_{n}^-)(u_{n}^+ u_{n-1}^-) - (D_{n-2}^- D_{n}^+)(D_{n}^+ D_{n}^-)(u_{n}^- u_{n-1}^+) \right\} . \tag{A.3}
\]

Here the harmonic expression in the square brackets equals 1 due to the harmonics completeness relation, and we finally get

\[
\prod_{t=1}^{n} [\mathbf{D}^+(u_t)]^2 = \left( \frac{-1}{4^n} \right)^{n-1} 4t\bar{W}_0 \left\{ \prod_{t=1}^{n-1} (u_{t}^+ u_{t+1}^-) \prod_{t=1}^{n-3} (D_{t}^+ D_{t+1}^-) \right\} (D_{n-2}^+ D_{n}^-)(D_{n}^+ D_{n}^-) . \tag{A.4}
\]

Comparing (A.4) with (A.1), we observe that the former can be formally obtained from the latter by replacing the block \( D_{n-1}^- D_{n}^+ \) by \( 4i\bar{W}_0^2 \). This observation allows the process to go on.

The chain \( [\mathbf{D}^+(u_1)]^2[\mathbf{D}^+(u_2)]^2 \ldots [\mathbf{D}^+(u_n)]^2 \) is finally reduced to

\[
-\left( \frac{-t}{4} \right)^{n-2} \bar{W}_0^{n-2} \left\{ \prod_{t=1}^{n-1} (u_{t}^+ u_{t+1}^-) \right\} (D_{n-2}^+ D_{n}^-)(D_{n}^+ D_{n}^-) .
\]

Recalling that the chain operator in eq. (4.2) is sandwiched between two chiral harmonic \( \delta \)-function and using (3.10), the latter expression can be finally cast in the form

\[
-\left( \frac{-i}{2} \right)^{n-2} \bar{W}_0^{n-2} \left\{ \prod_{t=1}^{n-1} (u_{t}^+ u_{t+1}^-) \right\} . \tag{A.5}
\]
Appendix B

In this Appendix we argue that our computations do not depend on the choice of α-gauge for the massive \( \tilde{V}^{++} \) propagators. To this end, let us start with the full α-depended propagator which is a solution of the equation (3.1).

\[
< (\tilde{V}_1^{++})_\alpha | (\tilde{V}_2^{++})_\alpha >_{(\alpha)} = \frac{2}{\Lambda^4} \frac{1}{\alpha} \delta^{12}(X_1 - X_2) \delta^{(-2,2)}(u_1, u_2) - 2 \frac{1 + \alpha}{\Lambda^4} \Pi^{(2,2)}(1 | 2) . \quad (B.1)
\]

Here \( \Pi \) is the projection operator

\[
\Pi^{(2,2)}(1 | 2) = \frac{\delta^{12}(X_1 - X_2)}{(u_1^2 + u_2^2)^2} . \quad (B.2)
\]

Now we substitute this propagator into \( \Gamma^n_{YM} \) (4.1) and decompose the latter in series in \( \alpha' = \alpha + 1 \) at the point \( \alpha' = 0 \) (\( \alpha = -1 \))

\[
\Gamma^n_{YM}(\alpha') = \Gamma^n_{YM} + \Delta^n \delta \alpha' + \ldots . \quad (B.3)
\]

The idea is to show that, under the infinitesimal shift of \( \alpha' \), the holomorphic (anti-holomorphic) part of \( \Gamma^n_{YM} \) does not change, i.e. \( \Delta^n = 0 \) in the local limit. Then, assuming \( \Gamma^n_{YM}(\alpha') \) to be continuous in \( \alpha' \), we can state that its (anti)holomorphic local part does not depend on \( \alpha' \) at all.

We find

\[
\Delta^n \sim \frac{(-i)^{n+1}}{n} \prod_{t=1}^{n} \int d^4x_t d^8 \theta_t du_t d\omega_t d\nu_t \frac{V_{(\bar{c})}^{-+}(t)}{(u_t^2 + \omega_t^2)(\nu_t^2 + \nu_t^2)(u_t^2 + \omega_t^2)}
\]

\[
\times \prod_{t=1}^{n} \frac{1}{\Box_t + m^2} [D^+(\theta_t, \nu_t)]^4 \delta^8(x_t - x_{t+1}) \delta^8(\theta_t - \theta_{t+1}) \delta^{(-2,2)}(\nu_t, \omega_{t+1})
\]

\[
\times [D^+(\theta_n, \nu_n)]^4 [D^+(\theta_1, \omega_1)]^4 \delta^8(x_t - x_{t+1}) \delta^8(\theta_t - \theta_{t+1}) . \quad (B.4)
\]

Then, following the calculations (4.1)-(4.7), we get the expression which looks very similar to (4.7), except for the presence of an extra pair of spinor derivatives \( (D_l^+ D_k^+) \) in it:

\[
\Delta^n \sim \int d^4\theta d^8 \prod_{t=1}^{n} \int \frac{d^4p_t d^4\theta_t du_t}{(p_t^2 - m^2)} \times V_{(\bar{c})}^{-+}(t)
\]

\[
\times \frac{f(u_1, \ldots, u_n)W_0^{n-1}}{(p_n^2 - m^2)} \times (D_l^+ D_k^+) [A_n + B_n] + c.c. . \quad (B.5)
\]

But (4.7) already contains the critical number of spinor derivatives and the presence of two extra ones simply means that the holomorphic (anti-holomorphic) contribution from (B.5) vanishes

\[
\Delta^n_{hol} = 0 \quad (B.6)
\]

(due to the properties \( (D_l^+)^2 = 0 \) and \( D_l^+ (u) V_{(\bar{c})}^{-+} (u) = 0 ) \).

We have also explicitly checked the vanishing of the holomorphic contributions in the second order in \( \delta \alpha' \) for the diagrams with two and three external legs.
References


