Functional Integral in terms of the Field Strength:
An Approach to Chiral Symmetry Breaking

Naoki Tanimura
Department of Physics, Kyushu University
Fukuoka 812-8581, JAPAN
E-mail: naoki@higgs.phys.kyushu-u.ac.jp

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Abstract
The chiral symmetry breaking in the 4-dimensional QED with the chirally invariant four-fermion interaction is discussed by using a novel path integral expression in terms of the field-strength tensor. In the local potential approximation, we find that the chiral symmetry is spontaneously broken for any nonzero gauge and four-fermion couplings on the tree level of an auxiliary field \( \sigma \). The present approach allows us to easily include higher orders of the gauge coupling so that the effective potential up to the sixth order is obtained.

1 Introduction
The standard model of elementary particles has had good agreement with experiments, in which masses of quarks and leptons are generated by the Yukawa interactions with the Higgs bosons. There, however, have been no experimental observations of the Higgs boson, and moreover, there is a theoretical problem of the “naturalness.” Therefore models without any elementary scalars have been considered[1, 2] as effective field theories to understand the origin of fermion masses: Those are generated dynamically through the interaction

\[
g^2 2 \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right], \tag{1.1}
\]

which is suppressed by inverse powers of a high energy scale \( \Lambda \), at energies much below it, that is, \( g^2\Lambda^2 \sim O(1) \). It is called the Nambu–Jona-Lasinio (NJL) model[3] and has been generalized to the one with the gauge interaction[4]:

\[
\mathcal{L} = -14F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\gamma_\mu (\partial^\mu - ieA^\mu)\psi + g^2 2 \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]. \tag{1.2}
\]
Although, in this model, nonperturbative approaches such as the Schwinger–Dyson equation with the ladder approximation[5, 6] or renormalization group[7] give a non-trivial phase structure for chiral symmetry in a plane of gauge and four-fermi couplings, they suffer from gauge dependence. In the perturbative framework the inversion method[8] does give a gauge-independent result[9], higher order calculations are tough tasks.

If we employ the perturbative effective action approach to the problem of chiral symmetry breaking(\(\chi_{SB}\)), there is some room for improvement or simplification of the calculation from two viewpoints: (i) In view of the nonperturbative renormalization group approach[7], the critical behavior is successfully described in the local potential approximation: taking the lowest order of derivative expansion in the effective action. We, therefore, expect that it could also be good in the perturbative calculation. (ii) After integrating with respect to matter (fermionic) fields the result must be gauge invariant, that is, be written as a functional of the field-strength tensor \(F_{\mu\nu}\). Therefore it is preferable to rewrite the functional integral of the gauge potential \(A_\mu\) into that of \(F_{\mu\nu}\). This could be a great help to the gauge-invariant calculation.

Indeed in the lattice gauge theory, a change of variables from gauge potentials to field strengths has been given[10], whose recipe, however, is specialized to the lattice. In the continuum theory, the field-strength formulation has already been proposed[11]. It is, however, based on the special choice of gauge—the coordinate gauge \(x_\mu A_\mu = 0\), which cannot fix the gauge at the origin and moreover yields complicated fermionic currents with line integrals in the lagrangian.

In this article we give a perturbative and gauge-independent method for calculating the dynamical fermion mass under the local potential approximation. Inclusion of higher orders of the gauge coupling constant is simpler in this method. We use the functional integration not as a method simply reproducing perturbative diagrams systematically, but as “integration” with nice features: Changing variables and introducing auxiliary variables can easily be made. We first construct a Euclidean path integral expression in terms of the gauge field strength for the gauge sector with a conserved source \(J_\mu\): We start from the canonical formalism for quantization and find a suitable change of variables, which is the contents of Sec. 2. Second, in Sec. 3 the fermionic partition function, minimally coupled to the gauge potential, in the local potential approximation is obtained by introducing auxiliary fields and by utilizing the Fock-Schwinger proper-time method[12, 13]. At last combining the above two partition functions and integrating with respect to the gauge field strength to obtain an effective potential of the dynamical fermion mass. The final Sec. 4 is devoted to a discussion.

2 Functional Integral in terms of the Field Strength

In this section we construct a Euclidean path integral expression in 4-dimensional abelian gauge theory coupled to a conserved current \(J_\mu\), based on the canonical formalism[14],

\[
L = -14F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu.
\]  

(2.1)
In order to show the gauge independence manifestly we fix the gauge in terms of an arbitrary real function \( \phi_\mu(x) \) satisfying\(^{[14, 15]} \)
\[
\partial_\mu \phi^\mu(x) = \delta^4(x); \tag{2.2}
\]
the gauge fixing condition is\(^{[14]} \)
\[
\int d^4 y \, \phi_\mu(x - y) A^\mu(y) = 0, \tag{2.3}
\]
which is satisfied by
\[
A^\mu(x; \phi) = A^\mu(x) - \partial_\mu \int d^4 y \, \phi_\nu(x - y) A^\nu(y), \tag{2.4}
\]
for an arbitrary \( A^\mu(x) \). Here we take a \( \phi_\mu(x) \) whose support is 3-dimensional and spacelike:
\[
\phi^\mu(x) = (0, f^i(x) \delta(x_0)), \quad \tilde{\phi}^\mu(p) = (0, \tilde{f}^i(p)) \tag{2.5}
\]
with
\[
\nabla^i f^i(x) = \delta^3(x), \quad p^i \tilde{f}^j(p) = -i. \tag{2.6}
\]
If it were not for this restriction, \( A^\mu(x; \phi) \) in Eq. (2.4) is nonlocal in time, so that we cannot follow the canonical procedure. (Here and hereafter repeated indices \( i, j, k \), etc., imply the summation over 1 to 3.) The relation (2.4) turns into
\[
A^0(x; \phi) = A^0(x) + \tilde{f}^i(-i \nabla) \dot{A}^i(x), \tag{2.7}
\]
\[
A^i(x; \phi) = A^i(x) - \nabla^i \tilde{f}^j(-i \nabla) A^j(x), \tag{2.8}
\]
where
\[
\tilde{f}^j(-i \nabla) A^j(x) \equiv \int d^3 p (2\pi)^3 e^{ip \cdot x} \tilde{f}^j(p) \tilde{A}^j(p) = \int d^3 y \, f^j(x - y) A^j(y), \tag{2.9}
\]
and the time argument has been omitted. Hereafter we use an abbreviation like Eq. (2.9) for simplicity; all the functions of \(-i \nabla\) including \(1/|\nabla|\) and \(1/\nabla^2\) should be understood in the momentum space representation.

The gauge-fixed variable \( A^i(x; \phi) \) must have two degrees of freedom out of three \( A^i(x) \)'s, which can be singled out by considering the norm of the functional space
\[
\int d^3 x \, \delta A^i(x; \phi) \delta A^i(x; \phi) = \int d^3 x \, \delta A^i(x) M^{jk}(-i \nabla) \delta A^k(x), \tag{2.10}
\]
where
\[
M^{jk}(p) \equiv \delta^{jk} + i \tilde{f}^{j*}(p)p^k - ip^j \tilde{f}^k(p) + p^2 \tilde{f}^{j*}(p) \tilde{f}^k(p). \tag{2.11}
\]
This matrix can be diagonalized as

$$ n_{(a)}^i(p)M^{ij}(p)n_{(β)}^j(p) = \begin{pmatrix} 1 & p^2\tilde{f}(p)|^2 \\ 0 \end{pmatrix}_{αβ}, $$

(2.12)

where $n_{(a)}^k$’s are given by

$$
\begin{align*}
n_{(1)}^k(p) & = \delta^{km}n_{(2)}^m(p), \\
n_{(2)}^k(p) & = \left[ ip^k + p^2\tilde{f}^k(p) \right]/\sqrt{p^2(p^2|\tilde{f}(p)|^2 - 1)}, \\
n_{(3)}^k(p) & = ip^k/|p|.
\end{align*}
$$

(2.13)

and form an orthonormal base obeying

$$
\sum_{α=1}^3 n_{(α)}^i(p)n_{(α)}^j(p) = δ^{ij}, \quad n_{(α)}^{ks} = n_{(α)}^k(-p).
$$

(2.14)

In view of Eqs. (2.10) and (2.12), genuine physical variables are given as

$$
A^{(α)}(x) ≡ n_{(α)}^k(-i\nabla)A^k(x) = n_{(α)}^k(-i\nabla)A^k(x; φ), \quad (α = 1, 2).
$$

(2.15)

With these variables the action is

$$
S = \int d^4x \left\{ -14 \left[ \partial_μA_ν(x; φ) - \partial_νA_μ(x; φ) \right]^2 + J_μ(\dot{x})A^μ(x; φ) \right\}
$$

$$
= \int d^4x \left\{ 12 \sum_{α=1}^2 A^{(α)}(x)\nabla^2A^{(α)}(x) + 12 \left[ \dot{A}^{(1)}(x) \right]^2 \\
- 12\dot{A}^{(2)}(x)\nabla^2\tilde{f}(-i\nabla)|^2\dot{A}^{(2)}(x) + 12 \left[ \nabla A^0(x; φ) \right]^2 \\
- A^0(x; φ)\sqrt{\nabla^2(\nabla^2\tilde{f}(-i\nabla))}^2 + 1)\dot{A}^{(2)}(x) \\
+ J_0(x)A^0(x; φ) - J(x)\cdot[A(x; φ)] \right\},
$$

(2.16)

where $[A(x; φ)]$ in the last line is given by

$$
A^i(x; φ) = n_{(1)}^{is}(-i\nabla)A^{(1)}(x)
$$

(2.17)

$$
+ \left[ n_{(2)}^{is}(-i\nabla) + n_{(3)}^{is}(-i\nabla)\sqrt{-\nabla^2\tilde{f}(-i\nabla)|^2 - 1} \right] A^{(2)}(x).
$$

The Hamiltonian is written as

$$
H(t) = \int d^3x \left\{ 12 \sum_{α=1}^2 \left[ \Pi^{(α)}(x) \right]^2 + \left[ \nabla A^{(α)}(x) \right]^2 \right\}
$$

$$
+ J_0(x)\sqrt{\nabla^2(\nabla^2\tilde{f}(-i\nabla)|^2 + 1)\nabla^2π^{(2)}(x) \\
+ 12J_0(x)|\tilde{f}(-i\nabla)|^2J_0(x) + J(x)\cdot[A(x; φ)] \right\},
$$

(2.18)
in terms of four dynamical variables, two $A^{(\alpha)}(x)$’s and their canonical conjugate momenta

$$\Pi^{(1)}(x) = \dot{A}^{(1)}(x),$$

$$\Pi^{(2)}(x) = \dot{A}^{(2)}(x) - \sqrt{\nabla^2 |\tilde{f}(-i\nabla)|^2 + 1} \nabla^2 J_0(t, x),$$

where $A^0(x; \phi)$ has been eliminated by

$$\sqrt{\nabla^2 |\tilde{f}(-i\nabla)|^2 + 1} \nabla^2 |\tilde{f}(-i\nabla)|^2 \Pi^{(2)}(x) + 1 |\tilde{f}(-i\nabla)|^2 A^0(x; \phi) + J_0(t, x) = 0.$$  \hspace{1cm} (2.21)

The field strengths are given as

$$E^i \equiv F^{i0} = -n^{i*}_{(1)}(-i\nabla)\Pi^{(1)} - n^{i*}_{(2)}(-i\nabla)\Pi^{(2)} + \tilde{f}^{i*}(-i\nabla)J_0,$$ \hspace{1cm} (2.22)

$$B^i \equiv -12\epsilon^{ijk}F_{jk} = |\nabla|n^i_{(1)}(-i\nabla)A^{(2)} - |\nabla|n^i_{(2)}(-i\nabla)A^{(1)}.$$ \hspace{1cm} (2.23)

Following the standard canonical procedure we obtain the path integral representation of the partition function $Z_T[J] = \text{Tr}(e^{-TH})$, by

$$Z_T[J] = \int\mathcal{D}n^{(\alpha)} \mathcal{D}A^{(\alpha)} \exp \left \{ \int d^3x_e \left \{ i \sum_{\alpha=1}^2 \Pi^{(\alpha)}(\tau, x) \dot{A}^{(\alpha)}(\tau, x) ight. \right.$$  

$$-12 \sum_{\alpha=1}^2 \left( [\Pi^{(\alpha)}(\tau, x)]^2 + [\nabla A^{(\alpha)}(\tau, x)]^2 \right)$$

$$+ iJ_4(\tau, x) \sqrt{\nabla^2 |\tilde{f}(-i\nabla)|^2 + 1} \nabla^2 \Pi^{(2)}(\tau, x)$$

$$+ 12J_4(\tau, x) |\tilde{f}(-i\nabla)|^2 J_4(\tau, x) - J(\tau, x) \cdot [A(\tau, x; \phi)] \right \},$$ \hspace{1cm} (2.24)

where $[A(\tau, x; \phi)]$ is given by Eq. (2.17) with $A(x; \phi) \rightarrow A(\tau, x; \phi),$

$$\int d^3x_e \equiv \int_0^T \int d^3x, \quad J_4 \equiv iJ_0,$$ \hspace{1cm} (2.25)

and the periodic boundary condition $A^{(\alpha)}(T, x; \phi) = A^{(\alpha)}(0, x; \phi)$ should be understood.

Our purpose is to rewrite the expression (2.24) of four variables $\Pi^\alpha$ and $A^\alpha$ into that of six variables $E^i$ and $B^i$. In view of Eq. (2.14) quantities which are proportional to $n^{i*}_{(3)}(-i\nabla)$ or $n^i_{(3)}(-i\nabla)$ are missing in their expressions (2.22) and (2.23). To this end, we introduce $\varepsilon$ and $\beta$ with the aid of the delta functions in the functional measure,

$$E^i = -n^{i*}_{(1)}(-i\nabla)\Pi^{(1)} - n^{i*}_{(2)}(-i\nabla)\Pi^{(2)} + n^{i*}_{(3)}(-i\nabla)\varepsilon - i\tilde{f}^{i*}(-i\nabla)J_4,$$ \hspace{1cm} (2.26)

$$B^i = |\nabla|n^i_{(1)}(-i\nabla)A^{(2)} - |\nabla|n^i_{(2)}(-i\nabla)A^{(1)} + n^i_{(3)}(-i\nabla)\beta.$$ \hspace{1cm} (2.27)
or equivalently
\[ \varepsilon = n_i(3)(\tau)E^i + i\tilde{f}_i(\tau)J_4 = 1|\nabla|(\nabla^iE^i - iJ_4), \quad (2.28) \]
\[ \beta = n_i(3)(\tau)B^i = 1|\nabla|\nabla^iB^i, \quad (2.29) \]

The norm of the functional space is given by
\[ \int d^3x \delta E^i \delta E^i = \int d^3x \left( \delta \Pi^{(1)} \delta \Pi^{(1)} + \delta \Pi^{(2)} \delta \Pi^{(2)} + \delta \varepsilon \delta \varepsilon \right), \quad (2.30) \]
\[ \int d^3x \delta B^i \delta B^i = \int d^3x \left( -\delta A^{(1)} \nabla^2 \delta A^{(1)} - \delta A^{(2)} \nabla^2 \delta A^{(2)} + \delta \beta \delta \beta \right), \quad (2.31) \]
so that
\[ D\Pi^{(a)}D\varepsilon D\beta \prod_x \delta(\varepsilon)\delta(\beta) = DE^iDB^i \prod_x \delta(\nabla^iE^i - iJ_4)\delta(\nabla^iB^i). \quad (2.32) \]

We then arrived at the desired result
\[ Z_T[J] = \int \mathcal{D}E^i \mathcal{D}B^i \prod_x \delta(\nabla^iE^i - iJ_4)\delta(\nabla^iB^i) \]
\[ \times \exp \left[ \int d^4x \left\{ iE^i(\tau, x)\epsilon^{ijk}\nabla^j\nabla^2\hat{B}^k(\tau, x) - 12 \left( [E^i(\tau, x)]^2 + [B^i(\tau, x)]^2 \right) \right. \right. \]
\[ \left. \left. \left. + J^i(\tau, x)\epsilon^{ijk}\nabla^j\nabla^2B^k(\tau, x) \right\} \right], \quad (2.33) \]

which is apparently free from the choice of \( f^i(x) \) as expected.

To carry out the integration in the next section, it is convenient to introduce \( \overline{E}^i \) as
\[ \overline{E}^i = E^i - i\nabla^i\nabla^2J_4, \quad (2.34) \]
so that
\[ \nabla^i\overline{E}^i = \nabla^iE^i - iJ_4, \quad (2.35) \]
giving
\[ Z_T[J] = \int \mathcal{D}\overline{E}^i \mathcal{D}B^i \prod_x \delta(\nabla^i\overline{E}^i)\delta(\nabla^iB^i) \exp \left[ \int d^4x \left\{ i\overline{E}^i(\tau, x)\epsilon^{ijk}\nabla^j\nabla^2\hat{B}^k(\tau, x) \right. \right. \]
\[ \left. \left. - 12 \left( [\overline{E}^i(\tau, x)]^2 + [B^i(\tau, x)]^2 \right) - 12J_4(\tau, x)1\nabla^2J_4(\tau, x) \right. \right. \]
\[ \left. \left. \left. + J^i(\tau, x)\epsilon^{ijk}\nabla^j\nabla^2B^k(\tau, x) \right\} \right]. \quad (2.36) \]

Further introducing an auxiliary field \( \rho \):
\[ 1 = \int \mathcal{D}\rho \exp \left[ -12 \int d^4x_E (\rho + 1|\nabla|J_4)^2 \right], \quad (2.37) \]
so as to cancel the \((J_4)^2\) term in Eq. (2.36), decomposing \(E^i\) and \(B^i\) as

\[
E^i = \sum_{\alpha=1}^{3} n_{(\alpha)}^i(-i \nabla)\varepsilon_\alpha, \quad B^i = \sum_{\alpha=1}^{3} n_{(\alpha)}^i(-i \nabla)\beta_\alpha,
\]

(2.38)

and integrating with respect to \(\varepsilon_\alpha\)'s and \(\beta_3\), we obtain

\[
Z_T[J] = \int \mathcal{D}\beta_\alpha \mathcal{D}\rho \left[ \text{Det} |\nabla| \right]^{-2} \exp \left[ \int d^4x_E \left\{ -12 \sum_{\alpha=1}^{2} \left[ 1|\nabla|\dot{\beta}_\alpha(\tau, \mathbf{x}) \right]^2 + \beta_\alpha(\tau, \mathbf{x})^2 \right. \right.
\]

\[
-12 \rho(\tau, \mathbf{x})^2 - J_4(\tau, \mathbf{x})1|\nabla|\rho(\tau, \mathbf{x})
\]

\[
\left. - J_i(\tau, \mathbf{x})1|\nabla| \left[ n_{(2)}^i(-i \nabla)\beta_1(\tau, \mathbf{x}) - n_{(1)}^i(-i \nabla)\beta_2(\tau, \mathbf{x}) \right] \right\},
\]

(2.39)

This form with further changes of variables is used in the next section, where we regard the coefficients of the sources in Eq. (2.39) as gauge potentials:

\[
\overline{A}_4 \equiv 1|\nabla|\rho, \quad \overline{A}_i \equiv 1|\nabla| \left[ n_{(2)}^i(-i \nabla)\beta_1 - n_{(1)}^i(-i \nabla)\beta_2 \right].
\]

(2.40)

(2.41)

Therefore field strengths are given as

\[
F_{4i} \equiv \nabla_i\overline{A}_4 - \overline{A}_i
\]

\[
= n_{(1)}^i(-i \nabla)1|\nabla|\dot{\beta}_2 - n_{(2)}^i(-i \nabla)1|\nabla|\dot{\beta}_1 - n_{(3)}^i(-i \nabla)\rho,
\]

(2.42)

\[
F_{ij} \equiv \nabla_i\overline{A}_j - \nabla_j\overline{A}_i
\]

\[
= \epsilon_{ijk} \left[ n_{(1)}^k(-i \nabla)\beta_1 + n_{(2)}^k(-i \nabla)\beta_2 \right],
\]

(2.43)

so that

\[
\int d^4x_E F_{\mu\nu}F_{\mu\nu} = 2 \int d^4x_E \left\{ \sum_{\alpha=1}^{2} \left[ 1|\nabla|\dot{\beta}_\alpha \right]^2 + \beta_\alpha^2 + \rho^2 \right\}.
\]

(2.44)

### 3 \(\chi^{SB}\) in QED\(_4\) with the chirally invariant four-fermion interaction

In this section, we consider fermionic system coupled minimally to the “gauge potentials” (2.40) and (2.41) with the chirally invariant four-fermion interaction. The partition function is

\[
Z[\overline{A}] = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp \left[ \int d^4x_E \left\{ -\overline{\psi}\gamma_\mu(\partial_\mu - i\overline{A}_\mu)\psi + g^2 \left( \left( \overline{\psi}\psi \right)^2 + \left( \overline{\psi}i\gamma_5\psi \right)^2 \right) \right\} \right],
\]

(3.1)

\(^1\text{Any choice for the complete orthonormal base} \{n_{(\alpha)}^i(-i \nabla)\} \text{ can be taken. Here, we only assume the same} n_{(3)}^i(-i \nabla) \text{ as Eq. (2.13).}\)
where the gauge coupling constant has been absorbed into $\overline{A}_\mu$. Our scenario is as follows: introduce auxiliary fields to cancel the four-fermion interaction, and then integrate with respect to the fermionic fields and finally the gauge field strengths with the aid of the representation in the previous section. The result is a tree potential of the auxiliary field, with which we examine the dynamical mass generation of fermions.

After introducing auxiliary fields $\sigma$ and $\pi$, as usual, and integrating with respect to $\psi$ and $\bar{\psi}$, we have

$$Z[A] = \int D\sigma D\pi \exp \left[ -\int d^4x \left( \sigma^2 + \pi^2 g^2 + \ln \text{Det} \left[ \gamma_\mu (\partial_\mu - iA_\mu) + \sigma + i\gamma_5 \pi \right] \right) \right].$$

(3.2)

Shifting as $\sigma \to m + \sigma'$ and $\pi \to \pi'$ and ignoring $\sigma'$ and $\pi'$, we obtain the tree level potential of $\sigma(m)$:

$$Z[A]_0 = \exp \left[ -\int d^4x m^2 g^2 + \ln \text{Det} \left[ \gamma_\mu (\partial_\mu - iA_\mu) + m \right] \right].$$

(3.3)

The exponent of Eq. (3.3) must be a functional of the field strength $F_{\mu\nu}$ rather than $A_\mu$ as far as a regularization preserves the gauge invariance.

We here employ the local potential approximation: We adopt the lowest order of derivative expansion, that is, discard any terms with differentials like $F_{\mu\nu} \Box \tilde{F}_{\mu\nu}$, too, to obtain a polynomial of $F_{\mu\nu}$. This approximation seems to be valid, since we are interested only in a low energy phenomena, the chiral symmetry breaking, where contributions from large $p_\mu$ should be much less important.

The functional form of the effective action under this approximation can be obtained nonperturbatively by the Fock–Schwinger’s proper time method[12, 13]:

$$Z[A]_0 = \exp \left[ -\int d^4x \left( m^2 g^2 + 12(2\pi)^{D2} \lim_{s \to 0} \int_0^\infty d\tau \tau^{s-D2-1} \text{e}^{-\tau m^2} G(\tau F) \right) \right],$$

(3.4)

where

$$G(F) = F_+ F_- \coth(F_+) \coth(F_-),$$

$$F_\pm = 12 \left( \sqrt{F_{\mu\nu} F_{\mu\nu} + F_{\mu\nu} \tilde{F}_{\mu\nu}} 2 \pm \sqrt{F_{\mu\nu} F_{\mu\nu} - F_{\mu\nu} \tilde{F}_{\mu\nu}} 2 \right).$$

(3.5)

(3.6)

To evaluate the $\tau$-integration in Eq. (3.4), we expand $G(F)$ as

$$G(F) = 1 + 13(F_+^2 + F_-^2) - 145 \left( (F_+^2 + F_-^2)^2 - 7F_+^2 F_-^2 \right) + 1945 \left( 2F_+^2 + F_-^2 \right)^2 - 13F_+^2 F_-^2 (F_+^2 + F_-^2) + O(F^8).$$

(3.7)

We need some regularization: In order to reproduce the NJL result in the limit $\epsilon \to 0$, we need an ultraviolet cutoff $\Lambda$ which is introduced by a modification of the range of the $\tau$-integration to $[1/\Lambda^2, \infty)$. However, this proper-time cutoff breaks the gauge invariance in the same way as the momentum-space cutoff, contrary to the dimensional regularization ($D = 4 - 2\epsilon$). To overcome this difficulty, we employ both regularizations at the same time. We use the cutoff for the zeroth order in the gauge coupling, since
it has nothing to do with gauge fields. While, for higher orders, the dimensional regularization is used. The result is

\[
Z[A]_0 = \exp \left[ -\int d^4x \left\{ m^2 \frac{g_2}{2} + 18\pi^2 \left( \Lambda^4 \left( 1 - m^2 \Lambda^2 \right) e^{-m^2\Lambda^2} + m^4 \epsilon R (m^2/\Lambda^2) \right) 
+ 13 \left( 1\epsilon + \ln \mu^2 m^2 \right) F_{\mu\nu} F_{\mu\nu} - 145 m^4 \left( F_{\mu\nu} F_{\mu\nu} \right)^2 \right] 
+ 2315 m^8 \left[ 2 \left( F_{\mu\nu} F_{\mu\nu} \right)^3 - 13 \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right)^2 F_{\mu\nu} F_{\mu\nu} \right] + O(F^8) \right] \right], \tag{3.8}
\]

where

\[
E_1(z) = \int_1^\infty dte^{-zt},
\]
\[
1\epsilon = 1\epsilon - \gamma + \ln 2\pi, \tag{3.9}
\]
and \(\mu\) is a renormalization scale. [Note that \(E_1(z) > 0\) for any real \(z(> 0)\).] Recall that the gauge action is written as \((-1/4e^2_{\text{bare}})F_{\mu\nu} F_{\mu\nu}\) to define a renormalized charge such that

\[
1\epsilon R (\mu) = Z_3 \epsilon R (\mu) + 112\pi^2 1\epsilon, \tag{3.11}
\]

where the first term of the right-hand side is the bare part and \(Z_3\) is the wave function renormalization constant.

Now we turn our attention to the functional integration of the gauge field strength. The total partition function is given by combining Eq. (3.8) with the result in the preceding section:

\[
Z[m] = \exp \left[ -\int d^4x V_0(m) \right] \int D\beta \alpha D\rho [\det |\nabla|]^{-2} \times \exp \left[ -18\pi^2 \int d^4x \left\{ \left( 4\pi^2 \epsilon R (\mu) + 13 \ln \mu^2 m^2 \right) \nabla_{\mu} \nabla_{\mu} + \text{higher orders} \right\} \right], \tag{3.12}
\]

where

\[
V_0(m) = m^2 g^2 + 18\pi^2 \left( \Lambda^4 \left( 1 - m^2 \Lambda^2 \right) e^{-m^2\Lambda^2} + m^4 \epsilon R (m^2/\Lambda^2) \right) \tag{3.13}
\]

If we adopt the dimensional regularization from the zeroth order we have a different critical coupling due to the lack of a quadratic term of the renormalization scale \(\mu\) in the effective action. Compare two expressions of the zeroth order effective action, Eq. (3.13) by the proper-time cutoff after expanding in terms of \(m^2/\Lambda^2\),

\[
V_0(m) = m^2 g^2 + m^4 16\pi^2 \left[ \Lambda^4 m^4 - 2\Lambda^2 m^2 - \gamma + 32 + \ln \Lambda^2 m^2 \right],
\]

and one by the dimensional regularization,

\[
V_0(m) = m^2 g^2 + m^4 16\pi^2 \left[ 1\epsilon + \ln 2\pi - \gamma + 32 + \ln \mu^2 m^2 \right].
\]
“higher orders” are those of Eq. (3.8) with $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, and $\tilde{F}_{\mu\nu}$ are given by Eqs. (2.42) and (2.43). All variables in Eq. (3.12) are $\tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}$ and $\tilde{F}_{\mu\nu} \bar{F}_{\mu\nu}$, so we can change the integration variables from $\beta_{\alpha}$ and $\rho$ to $S$, $T$, and $U$:

\begin{align}
  \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} &= 2(S^2 + T^2 + U^2), \\
  \tilde{F}_{\mu\nu} \bar{F}_{\mu\nu} &= 2(S^2 - T^2),
\end{align}

yielding

\[ D\beta_{\alpha} D\rho [\text{Det} |\nabla|]^{-2} = DSDT DU \text{Det}[-\Box_{\mathrm{g}}]^{-1}, \]  

from Eq. (2.44). As for $\tilde{F}_{\mu\nu} \bar{F}_{\mu\nu}$, though it cannot be expressed by the total divergence any more, its expectation value in the space-time integral would vanish,

\[ \int d^4 x_E \langle \tilde{F}_{\mu\nu} \bar{F}_{\mu\nu} \rangle = \int d^4 x_E (2(S^2 - T^2)) = 0, \]  

due to a symmetry $S \leftrightarrow T$ of the effective action (3.12); the dependence on $S$ and $T$ appears only in the forms $S^2 + T^2$ and $(S^2 - T^2)^2$.

Note that $Z[m]$ (3.12) is a trivial product of integrals at each space-time point. We evaluate these integrals with the help of the WKB approximation. For a technical reason we discard $O(F^8)$ terms in Eq. (3.8). (This enables us to obtain the stationary point analytically. See below.) Discretizing space-time with (lattice spacing)$^4 = 32\pi^2/\Lambda^4$ and neglecting the irrelevant factor $\text{Det}[-\Box_{\mathrm{g}}]^{-1}$, we obtain

\begin{align}
  Z[x] &= \prod_{\text{site}} 8(32\pi)^{32} \int_{[0,\infty)^3} ds \, dt \, du \\
  &\times \exp \left[ - \left\{ v_0(x) + \left( 16\pi^2 e_\alpha^2(\mu) + 43 \ln \mu^2 x \Lambda^2 \right)(s^2 + t^2 + u^2) \\
  - 145 x^2 \left[ 4(s^2 + t^2 + u^2)^2 - 7(s^2 - t^2)^2 \right] \\
  + 2315 x^4 \left[ 8(s^2 + t^2 + u^2)^3 - 13(s^2 + t^2 + u^2)(s^2 - t^2)^2 \right] \right\} \right], \tag{3.18}
\end{align}

where the first coefficients are normalization factors to cancel the Gaussian integration; dimensionless parameters, $x = m^2/\Lambda^2$, $s = S/\Lambda^2$, etc. have been introduced; and

\begin{equation}
  v_0(x) \equiv 32\pi^2 V_0(m)\Lambda^4 = 16\pi^2 x g^2 \Lambda^2 + 2(1 - x)e^{-x} + 2x^2 E_1(x). \tag{3.19}
\end{equation}

Introducing a polar coordinate $(r, \theta, \varphi)$ with $r^2 = s^2 + t^2 + u^2$ and $r^2 \sin^2 \theta \cos 2\varphi = s^2 - t^2$ and integrating with respect to $\theta$ and $\varphi$, we find

\begin{equation}
  Z[x] = \prod_{\text{site}} 4\pi (32\pi)^{32} \int_0^\infty dr \exp \left[ - \left\{ v_0(x) + v_\rho(r; x) \right\} \right], \tag{3.20}
\end{equation}

10
where
\[ v_F(r; x) = A(x)r^2 - B(x)r^4 + C(x)r^6 - \ln r^2, \]  
(3.21)
\[ A(x) = 16\pi^2 e_{n}^2 (\mu) + 43 \ln \mu^2 x \Lambda^2 = 16\pi^2 e_{n}^2 (\Lambda) - 43 \ln x, \]  
(3.22)
\[ B(x) = 32675 x^2, \]  
(3.23)
\[ C(x) = 1364725 x^4, \]  
(3.24)
and \( O(r^8) \) terms in the exponent have been neglected.

Let us first consider the lowest correction, up to the \( O(r^2) \) term. The \( r \)-integration is analytically performed to give
\[ Z[x] = \prod_{\text{site}} 2^{15/2} \pi^3 \exp[-v(x)] \]  
(3.25)
where
\[ v(x) = v_0(x) + 32 \ln A(x) = 4xG + 2(1 - x)e^{-x} + 2x^2 E_1(x) + 32 \ln [4\pi \alpha(\Lambda) - 43 \ln x] \]  
(3.26)
with \( G = g^2 \Lambda^2/4\pi^2 \) and \( \alpha(\Lambda) = \epsilon^2_n(\Lambda)/4\pi \). The argument of the logarithm becomes negative at the Landau pole \( x = \exp[3\pi/\alpha(\Lambda)] \), but we do not care such a heavy fermion and thus assume that the argument is always positive. The stationary point \( x^* \) defined by
\[ 14 \frac{\partial v(x)}{\partial x} \bigg|_{x=x^*} = 1G - e^{-x^*} + x^* E_1(x^*) - 3\alpha(\Lambda)8(3\pi - \alpha(\Lambda) \ln x^*)x^* = 0 \]  
(3.27)
should satisfy the stability condition
\[ 14 \frac{\partial^2 v(x)}{\partial x^2} \bigg|_{x=x^*} = E_1(x^*) + 9\pi \alpha(\Lambda) - 3\alpha(\Lambda)^2(\ln x^* + 1)8(3\pi - \alpha(\Lambda) \ln x^*)^2 x^* > 0, \]  
(3.28)
for which \( x^* < e^{-1} \approx 0.368 \) is a sufficient condition. To solve Eq. (3.27) we set a condition:
\[ 1 \gg -x \ln x \gg (1 - \gamma)x \gg x^2, \]  
(3.29)
which is fulfilled if \( x < 1. \times 10^{-2} \). (In the actual situation \( m \sim 1 \) MeV and we should take \( \Lambda > 1 \) TeV, a lower bound of compositeness from experiments[16], to give \( x < 10^{-12} \).) Under this condition Eq. (3.27) becomes
\[ 1G - 1 - x \ln x - 3\alpha(\Lambda)8(3\pi - \alpha(\Lambda) \ln x)x = 0, \]  
(3.30)
where we have used the expansion of \( E_1(x) \) for \( x \ll 1 \),
\[ E_1(x) = -\gamma - \ln x + O(x). \]  
(3.31)
Figure 1: Squared dynamical mass of fermion $m^2/\Lambda^2$ shown as a function of the four-fermion coupling constant $g^2\Lambda^2/4\pi^2 = G$ for several fixed gauge coupling constants $\alpha(\Lambda) = e^2_r(\Lambda)/4\pi$, obtained from Eq. (3.27).

For $\alpha(\Lambda) = 0$ there exists a nonvanishing solution only if $G \geq 1$; therefore the critical coupling $G_c$ is 1. The solution for $G \simeq 1$ is

$$x^* \simeq -G - 1\ln[G - 1].$$

(3.32)

For $\alpha(\Lambda) > 0$ the solution is obtained in two separate regions: (i) $x \ll \exp[-3\pi\alpha(\Lambda)]$;

$$x^* \simeq -3G8\ln(3G8),$$

(3.33)

which is independent of $\alpha(\Lambda)$. (ii) $\exp[-3\pi\alpha(\Lambda)] \ll x < \alpha(\Lambda)/4$;

$$x^* \simeq \alpha(\Lambda)G8\pi(1 - G).$$

(3.34)

Thus there always exists a nonvanishing solution for a given $G$, that is, $G_c = 0$.

[Solutions at several $\alpha(\Lambda)$'s are depicted in Fig. 1.] The actual situation, $\alpha(\Lambda) \simeq 1/137$, $x \simeq 10^{-12}$, and $\Lambda \simeq 1$ TeV, lies in the case (ii) ($\exp[-3\pi\alpha(\Lambda)] \simeq 1.7 \times 10^{-561}$) and from Eq. (3.34)

$$\pi G = g^2\Lambda^24\pi \simeq 8\pi^2x\alpha(\Lambda) \simeq 1.1 \times 10^{-8};$$

(3.35)

g$^2$ is highly suppressed even at this scale ($\Lambda \simeq 1$ TeV).

Now we include the higher order terms. We evaluate the $r$-integration with the WKB approximation;

$$\int_0^{\infty} dr \exp[-v_p(r; x)] \simeq (2\pi)^{12} \exp[-\{v_p(r_0(x); x) + 12\ln[v''_p(r_0(x); x)]\}],$$

(3.36)
where the superscript \( ' \) denotes the \( r \)-differentiation and \( r_0(x) \) is a solution of the stationary point equation
\[
v'_p (r; x) = 2r \left[ A(x)r^2 - 2B(x)r^4 + 3C(x)r^6 - 1 \right] = 0,
\]
which is the cubic equation of \( r^2 \). There exists only one real-positive solution for \( r^2 
\]
\[
\begin{align*}
 r_0^2(x) &= 56153x^2 + 32(175102)^{x^4} \left\{ [P(x) + Q(x)]^{13} - [P(x) - Q(x)]^{13} \right\},
\end{align*}
\]
where
\[
\begin{align*}
 P(x) &= \sqrt{1 + 80281647403225x^2} - 112153x^2A(x) - 313670227x^4A(x)^2 + 175102x^4A(x)^4, \\
 Q(x) &= 1 + 40140847403225x^2 - 56153x^2A(x),
\end{align*}
\]
since \( C(x) > 0 \) from (3.24) and \( 9A(x)C(x) - 4B(x)^2 > 0 \) turns out to be
\[
\alpha(\Lambda) < 103275\pi896 \simeq 3.62 \times 10^2.
\]
Positivity of \( v''_p(r_0(x); x) \)
\[
v''_p(r; x) = 2 \left[ 15C(x)r^4 - 6B(x)r^2 + A(x) + 1r^2 \right] > 0
\]
is also guaranteed, since the sufficient condition \( 9B(x)^2 - 15A(x)C(x) < 0 \) is fulfilled. [This leads to a similar condition as Eq. (3.41).]

The final form of the partition function is
\[
Z[x] = \prod_{\text{site}} 2^{10\pi^3} \exp[-v(x)],
\]
with
\[
\begin{align*}
 v(x) &= v_0(x) + v_p(r_0(x); x) + 12 \ln \left[ v''_p(r_0(x); x) \right] \\
 &= v_0(x) + 13 + 23A(x)r_0(x)^2 - 13B(x)r_0(x)^4 \\
 &+ 12 \ln \left[ 12C(x)\left\{ 3 - 2A(x)r_0(x)^2 + 2B(x)r_0(x)^4 \right\} - A(x)r_0(x)^2 + 2Br_0(x)^4 \right],
\end{align*}
\]
where Eq. (3.37) has been used. Differentiating this with respect to \( x \) gives us the gap equation, whose numerical solution is shown in Fig. 2. In comparison with Fig. 1, there is a large deviation in the region where both \( x = m^2/\Lambda^2 \) and \( G = g^2/4\pi^2 \) are small, and the mass is increased by higher order corrections.

For sufficiently small \( x \), \( xA(x) \ll 1 \) or \( x \ll \alpha(\Lambda)/4\pi \), Eq. (3.44) can be expanded:
\[
\begin{align*}
 v(x) &= v_0(x) + 13 - 12 \ln 525x^4544 - 83(7255)^{2/3}x^{2/3} \\
 &+ 253(7255)^{1/3}x^{4/3}(3\pi\alpha(\Lambda) - 1792172125 - \ln x) \\
 &+ 4481377x^2(3\pi\alpha(\Lambda) - 22420655 - \ln x) + O(x^{8/3}),
\end{align*}
\]
yielding the gap equation
\[
14\partial v(x)\partial x = 1G - e^{-x} + xE_1(x) - 12x - (7255)^{2/3}49x^{1/3} \\
+ 259(7255)^{1/3}x^{1/3}(3\pi\alpha(\Lambda) - 523543688500 - \ln x) \\
+ 2241377x(3\pi\alpha(\Lambda) - 2110341310 - \ln x) + O(x^{5/3}).
\]
Figure 2: Squared dynamical mass of fermion $m^2/\Lambda^2$ shown as a function of the four-fermion coupling constant $g^2\Lambda^2/4\pi^2 = G$ for several fixed gauge coupling constants $\alpha(\Lambda) = e_r^2(\Lambda)/4\pi$, obtained from Eq. (3.44).

The stability condition

\[ 14\partial^2v(x)\partial x^2 = E_1(x) + 12x^2 + (7255)^{2/3}427x^{4/3} + (7255)^{1/3}2527x^{2/3}(3\pi\alpha(\Lambda) - 2589043688500 - \ln x) + 2241377(3\pi\alpha(\Lambda) - 6241341310 - \ln x) + O(x^{2/3}) > 0, \]

is fulfilled for $x < 1$ and $\alpha(\Lambda) < 20655002589043\pi$. Therefore an $\alpha$-independent solution exists:

\[ x^* \simeq G2(1 - G) \]  

(3.48)

for $x$ obeying $x \ll \alpha(\Lambda)/4\pi$ and Eq. (3.29). In this case $g^2$ is also highly suppressed:

\[ \pi G = g^2\Lambda^24\pi \simeq 2x \simeq 2 \times 10^{-12}, \]

(3.49)

for $x \simeq 10^{-12}$, that is, $\Lambda \simeq 1$ TeV.

4 Discussion

We give a gauge invariant recipe for calculating the effective action in QED with the four-fermion interaction. We use perturbation and the local potential approximation to study the dynamical fermion mass. In order to include the higher orders of the gauge coupling constant we just expand the completely known function (3.5) into a
polynomial of \( F_+^2 + F_-^2 \equiv \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \) and \( F_+^2 F_-^2 \equiv \frac{1}{4} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2 \) and integrate with respect to the proper-time \( \tau \). The last task is to evaluate the triple integral of \( s, t, \) and \( u \). In this article we employ the WKB approximation for it, which, however, cannot be performed in an elementary manner when higher orders are included further. Meanwhile the ordinary perturbative treatment, considering the Gaussian part the kernel and expanding the exponential of higher parts than the third order, can always be ensured.

As for the local potential approximation its efficacy is still open. In order to go beyond the local potential approximation, we must perform the functional integration of \( S, T, \) and \( U \), instead of the ordinary integration like (3.18). For example, an \( O(\alpha(\Lambda)) \) quantity after integrating with respect to the gauge fields would be

\[
I = \int d^D x E \langle J_\mu A_\mu(x) J_\rho A_\rho(y) \rangle / \int d^D x E
\]

\[
= \int d^D x E \langle J_\mu \partial_\nu F_{\mu\nu} \square_E(x) J_\rho \partial_\sigma F_{\rho\sigma} \square_E(y) \rangle / \int d^D x E
\]

\[
= \int d^D p_E (2\pi)^D \Pi(p^2) \langle p^2 \delta_{\rho\sigma} - p_\mu p_\rho \rangle p_\mu p_\rho \langle p^2 \rangle \langle F_{\mu\nu}(p) F_{\rho\sigma}(-p) \rangle,
\]

where

\[
\langle F_{\mu\nu}(p) F_{\rho\sigma}(-p) \rangle = 1 \langle p^2 \rangle \langle p^2 \rangle \langle p^2 \rangle \langle F_{\mu\nu}(p) F_{\rho\sigma}(-p) \rangle.
\]

so that

\[
I = \int d^D p_E (2\pi)^D \Pi(p^2)(D - 1) = \int d^D p_E (2\pi)^D \Pi(p^2)(F_{\mu\nu}(p) F_{\mu\nu}(-p)) 2.
\]

The local potential approximation is to expand \( \Pi(p^2) \) in terms of \( p^2 \) and keep the lowest. Since the change of variables, Eq. (3.14), gives

\[
F_{\mu\nu}(p) F_{\mu\nu}(-p) = S(p) \left( \alpha(\Lambda) - \pi/3 \right)
\]

which claim that there is a nonzero \( G_c \) if \( \alpha(\Lambda) < \pi/3 \). It is, however, too early to conclude, since our present work is restricted only to the tree level of auxiliary fields and thus no higher orders of \( G \) are included. As is seen from the Figs. 1 and 2, the small negative correction, linear in \( G \), to dynamical mass could easily swallow the broken region. From a recent study[17], the one-loop inclusion of the auxiliary fields is promising. Thus a further study must be necessary for a definite conclusion.

In QED the functional integral in terms of the field strength can be constructed, since the configuration space of the gauge potential as well as the field strength is trivial
enough for the gauge to be completely fixed. It is challenging to generalize Eq. (2.33) to QCD where an obstacle for gauge fixing, the Gribov ambiguity[18], exists. In order to examine the dynamics of chiral symmetry breaking and color confinement in QCD, this must be done and the choice of convenient variables describing the low energy phenomena[19] must be necessary.

These directions of study are in progress.

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References


