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A RELATIVISTIC QUARK MODEL FOR MESONS AND THE
"WEISSKOPF-VAN ROYEN PARADOX"

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A B S T R A C T

Non-relativistic internal motion of the quark-antiquark system does not necessarily imply a non-relativistic spinor structure. A model with a relativistic spinor structure successfully describes meson annihilation processes; large symmetry breaking according to an essentially arbitrary prescription is required to obtain these results using the usual model.

We assume that heavy quarks exist ($M > 5 \text{ GeV}$) as fundamental constituents of hadrons. Mesons must occur as poles in quark-antiquark ($Q-\bar{Q}$) scattering and we can therefore use the Bethe-Salpeter (BS) equation in the $Q-\bar{Q}$ channel to discuss them (this does not imply a belief that the properties of mesons are dominated by a simple $Q-\bar{Q}$ state).

The graphical notation is taken from the accompanying paper ¹⁾. We denote the BS wave function ($\begin{smallmatrix} \rightarrow P+q \\ \leftarrow P-q \end{smallmatrix}$) by $\chi(P, q)$. $\chi_{\alpha\beta}$ transforms like $u_{\alpha} \bar{u}_{\beta}$ for a $Q-\bar{Q}$ state and we will write it as a matrix. For a pseudoscalar state the most general form of χ is:

$$\chi = \chi_5 (A + B \not{q} \not{P} + C \not{P} + D (\not{q} \not{P} - \not{P} \not{q})). \quad (1)$$

For a state with charge parity $+$ (e.g. the π^0 meson) A , B , C and D are functions of q^2 , P^2 and $(q \cdot P)^2$. In the rest frame we define:

$$\chi(\vec{P}=0) = \begin{pmatrix} \chi_{+-} & \chi_{++} \\ \chi_{--} & \chi_{-+} \end{pmatrix}. \quad (2)$$

For a weakly bound non-relativistic state of particle and anti-particle of mass M :

$$\chi_{+-} \sim \chi_{-+} \sim \frac{\vec{q} \cdot \vec{q}}{M} \chi_{++},$$

$$\chi_{--} \sim \frac{\vec{q}^2}{M^2} \chi_{++}.$$

In the non-relativistic quark model it is, in effect, assumed that such relations are also true for tightly bound systems so that χ_{+-} , χ_{-+} and χ_{--} can be neglected when $\vec{q}^2 \ll M^2$.

NECESSARY CONDITIONS FOR THE USUAL NON-RELATIVISTIC MODEL

Explicitly, for a $J^P = 0^-$ state:

$$\chi_{++} = A - \mu(Bq_0^2 + C),$$

$$\chi_{--} = A + \mu(Bq_0^2 + C),$$

where $2\mu = \text{bound state mass } (M_B)$. We take the average quark mass $=1$, so that $\mu \ll 1$ for the states of interest. For a $J^{PC} = 0^{-+}$ state, A, B and C are functions of q^2 , $(q_0\mu)^2$ and μ^2 *); assuming that they are sufficiently smooth functions so that they can be expanded in μ^2 for fixed q^2 and q_0^2 we find either

$$\chi_{++} = \chi_{--} + O(\mu)$$

or

$$\chi_{++} = -\chi_{--} + O(\mu).$$

The usual quark model requires $\chi_{--} \ll \chi_{++}$ in the range of variables where the wave function is appreciable. The "smoothness" assumption is therefore incompatible with the usual model for vanishingly small μ .

Since smoothness is not established, it is instructive to consider a model. We take the simplest one-body propagator

$$S(p) = \frac{i}{(2\pi)^4(\not{p} - M)},$$

and a potential of the form:

*) χ is only defined at the bound state pole, i.e., for fixed $P^2 = \mu^2$. Here we have in mind varying the potential or quark masses so that μ^2 changes.

$$-(2\pi)^4 \hat{V} \chi = V_1 \chi + V_2 \chi_S \chi_S + V_3 \chi_P \chi_P.$$

With a common quark mass $M=1$ and putting $p^2+1 \simeq 1$, the four coupled parts of the BS equation for a $J^{PC}=0^{-+}$ state become

$$A(1-q^2) + 2p^2(Bq_0^2 + C) - 4p^2 \vec{q}^2 D = \int (V_1 + V_2 - 4V_3) A d^4k. \quad (3)$$

$$2A + Bq_0^2(1-q^2) + C(1-q^2) = \int (V_1 - V_2 + 2V_3) \left(\frac{q \cdot k \cdot k_0}{q_0} B + C \right) d^4k \quad (4)$$

$$2A + Bq_0^2(1-q^2) + C(1+q^2 - 2q_0^2) - 4\vec{q}^2 D = \int (V_1 - V_2 + 2V_3) (Bk_0^2 + C) d^4k \quad (5)$$

$$A + C + (1+q^2) D = \int (V_1 + V_2 - 2V_3) 4 \frac{\vec{q} \cdot \vec{k}}{\vec{q}^2} D d^4k. \quad (6)$$

In the usual model it is assumed that $\vec{q}^2 \ll 1$, $\chi_{--} \ll \chi_{++}$ and $\chi_{+-} \ll \chi_{++}$ so that $A \simeq -p(Bq_0^2 + C)$, $B \ll A$ and $D \ll A$. In this case Eq. (3) becomes:

$$A(1-q^2) \simeq \int (V_1 + V_2 - 4V_3) A d^4k, \quad (7)$$

and Eqs. (4) and (5) become

$$A(1-q^2) \simeq \int (V_1 - V_2 + 2V_3) A d^4k, \quad (8)$$

where we have used the condition $A \simeq -p(Bq_0^2 + C)$. Equation (6) gives D in terms of A . Equations (7) and (8) are incompatible (and the usual model cannot occur) unless $V_2 \simeq V_3 \simeq 0$ or $V_2 \simeq 3V_3$ (in these cases, however, the equations for χ_{++} and χ_{--} approximately decouple^{2),3)} and $\chi_{--} \ll \chi_{++}$ does not necessarily occur).

To make more definite statements a soluble model must be considered. Using a factorizing potential we find:

1. as $p \rightarrow 0$, always $X_{++} \rightarrow \pm X_{--}$;
2. if the potential is not a Dirac scalar and the terms in it do not conspire, then $X_{++} = \pm X_{--} + O(p)$ for $p \ll 1$. The coefficient of p is non-zero and of order X_{++} ;
3. if the potential is a Dirac scalar (or conspiratorial) then two regions can be distinguished:
 - a) $\langle \vec{q}^2 \rangle^2 \gg p^2$, in which case again, $X_{++} = \pm X_{--} + O(p)$.
 - b) $\langle \vec{q}^2 \rangle^2 \ll p^2$, in which case $X_{--} \ll X_{++}$ or $X_{++} \ll X_{--}$.

We conclude that the usual non-relativistic quark model requires a Dirac scalar potential ^{*)} as well as $\vec{q}^2 \ll 1$. In other cases the wave function has a highly relativistic structure.

RELATIVISTIC MODELS

We shall consider two models with relativistic wave functions which are suggested by the considerations above. As usual, we assume that the quarks are approximately degenerate and the potentials approximately SU(3) invariant ^{**)} . Hence mesons will occur in multiplets with approximately

*) We prefer an SU(3) scalar, pseudoscalar $Q-\bar{Q}$ interactions, which only allows annihilation graphs in the potential which binds the $J^P = 0^-$ singlet state. This lifts the singlet/octet degeneracy for the pseudoscalar mesons but not for other mesons (for which we predict a mixing angle given by $\tan \theta \simeq 1/\sqrt{2}$). This is supported by a quark model analysis of symmetry breaking for the established multiplets ⁴⁾.

**) Since the relative positions of the nearby baryon-antibaryon thresholds deviate badly from SU(3) symmetry, this is a strong assumption. See Ref. 3) for more discussion.

degenerate binding energies and approximately identical BS wave functions (apart from normalization). The normalization chosen is such that amplitudes constructed from χ are invariant Feynman amplitudes ^{*) 1)}; in comparing χ^π with χ^K etc. below, we have taken the leading terms in the normalization condition for small p^2 and assumed that the Dirac scalar parts of V (the generalizations of the V_i above to arbitrary potentials) can be expanded in terms of p^2 .

Model 1

$$\chi_{++} = \chi_{--} + O(M_B) ; \chi_{++}^\pi \simeq \chi_{++}^K ; \frac{\chi_{++}^\pi - \chi_{--}^\pi}{M_\pi} \simeq \frac{\chi_{++}^K - \chi_{--}^K}{M_K} \text{ etc.}$$

(We assume the order of magnitude relation $A \sim B \sim C \sim D$ suggested by setting $p^2 = 0$ in Eqs. (3)-(6).)

Model 2

$$\chi_{++} = -\chi_{--} + O(M_B) ; \chi_{++}^\pi \simeq \chi_{++}^K ; \chi_{++}^\pi - \chi_{--}^\pi \simeq \chi_{++}^K - \chi_{--}^K \text{ etc.}$$

(We assume $A \sim p^2 B \sim p^2 C \sim p^2 D$, as suggested by putting $A \sim p^2 B$ and then letting $p^2 \rightarrow 0$ in Eqs. (3)-(6).)

TEST OF THE RELATIVISTIC MODEL FOR PSEUDOSCALAR MESONS

We assume the conventional current $\times$ current theory of weak interactions with a quark current:

$$J_\mu(x) = \bar{\psi}_p(x) \gamma_\mu (1 + R \gamma_5) (\psi_n(x) \cos \theta + \psi_\lambda(x) \sin \theta), \quad (9)$$

^{*)} This point is essential below when we discuss the effect of the $K-\pi$ mass difference on branching ratios.

where ψ_p , ψ_n and ψ_λ are the quark fields, θ is the Cabibbo angle and R is a constant (the non-relativistic quark model for mesons gives $R = 3/5 \times G_A/G_V \approx 1/\sqrt{2}$). Hence

$$\begin{aligned} \langle 0 | J_p(0) | \pi^+ \rangle &= \langle 0 | T(\bar{\psi}_p(0) \gamma_\mu \gamma_5 R \cos \theta \psi_n(0)) | \pi^+ \rangle \\ &= R \cos \theta \int \text{Trace } \chi^\pi \gamma_\mu \gamma_5 d^4 q \end{aligned} \quad (10)$$

to all orders of the strong interactions [graphically Eq. (10) represents

]. Conventionally the currents are parameterized:

$$\langle 0 | J_p(0) | \pi^+ \rangle = f_\pi \cos \theta_A K_\pi^\pi,$$

$$\langle 0 | J_p(0) | K^+ \rangle = f_K \sin \theta_A K_\pi^K.$$

Assuming $\theta \approx \theta_A$ ^{*)}, we find:

$$\frac{f_\pi}{f_K} = \frac{\int (\bar{B}_q^\pi + C^\pi) d^4 q}{\int (\bar{B}_q^K + C^K) d^4 q} = \frac{M_K}{M_\pi} \frac{\int (\chi_{+}^\pi - \chi_{-}^\pi) d^4 q}{\int (\chi_{+}^K - \chi_{-}^K) d^4 q}$$

Therefore Model 1 gives $f_\pi \approx f_K$.

Model 2 gives $M_\pi f_\pi \approx M_K f_K$.

Experimentally $f_\pi \approx f_K$ (taking θ from other decays). Model 1 explains this naturally. Model 2 requires symmetry breaking of the order M_π/M_K to obtain $f_\pi \approx f_K$. In the non-relativistic quark model, symmetry breaking of the order $\sqrt{M_\pi/M_K}$ is required in the value of the wave function at the origin ($\vec{v}=0$) ^{5), 6)} ^{**)} . Such large symmetry breaking is difficult to reconcile with the idea of $SU(3)$ symmetry [broken by terms of order $(M_K - M_\pi)/M$] which is the basis of the quark model; this is the "Weisskopf-Van Royen paradox". The relativistic model 1 gives $f_\pi \approx f_K$ without large symmetry breaking.

^{*)} With any other assumption the approximate equality of the values of $\theta_{\text{expt.}}$ determined in all other processes would be accidental.

^{**)} This result can, of course, be derived using our formalism. Taking the simplest one-particle propagators and an instantaneous potential, and neglecting $\partial V / \partial P_F$ in the normalization condition, it is possible to derive Weisskopf and Van Royen's value ⁶⁾ of $\psi^\pi(\vec{v}=0)$. This checks the consistency of the normalization.

It is fashionable (but not necessary) to assume that particle physics would not be radically changed if M_π were zero. In the quark model, $g_{\pi NN}$ depends linearly on χ^π ; taking the leading term in χ^π for small M_π , both Model 1 and 2 suggest that $g_{\pi NN}$ is approximately independent of M_π *). Hence Model 1 suggests:

$$g_{\pi NN} f_\pi \sim \text{const.},$$

while Model 2 (and also a model with $\chi_{--} \ll \chi_{++}$ and small symmetry breaking) suggests

$$\frac{g_{\pi NN} f_\pi}{M_\pi} \sim \text{const.}$$

The Goldberger-Treiman relation $[(g_{\pi NN} f_\pi)/(\sqrt{2}M_N) \simeq G_A/G_V]$ therefore suggests that particle physics is only stable in Model 1 as $M_\pi \rightarrow 0$.

We propose Model 1 for pseudoscalar mesons.

TEST OF RELATIVISTIC MODELS FOR VECTOR MESONS

A similar analysis can be applied to vector mesons ^{2), 3)}. Two simple relativistic models are suggested:

Model 1.

$$\vec{\chi}_{++} = \vec{\chi}_{--} + O(M_\pi); \quad \vec{\chi}_{++}^\rho \simeq \vec{\chi}_{++}^\rho; \quad \frac{\vec{\chi}_{+-}^\rho - \vec{\chi}_{-+}^\rho}{M_\rho} \simeq \frac{\vec{\chi}_{++}^\rho - \vec{\chi}_{--}^\rho}{M_\rho}$$

*) This is supported by the measured values of $g_{\pi NN}^2$, $g_{\pi NK}^2$ and $g_{\eta NK}^2$ ⁷⁾. Multiplying the SU(3) predictions by a factor R thus: $g_{\pi NK}^2 = R^2 g_{\pi NN}^2 (1+2f)^2/3$, $g_{\eta NK}^2 = R^2 g_{\pi NN}^2 (1-2f)^2$, the data give $0.87 < R < 1.12$. If $g_{\pi NN}$ depended strongly on M_π , R would depend on M_π/M_K and be far from one (a model with $\chi_{--} \ll \chi_{++}$ and small symmetry breaking in the wave function suggests $R = \sqrt{M_\pi/M_K} \simeq 0.53$).

Model 2

$$\bar{X}_{++} = -\bar{X}_{--} + O(M_0); \quad \bar{X}_{+-} = \bar{X}_{-+} \text{ etc.}$$

The models are tested by calculating the electric current:

$$\langle 0 | T_j(0) | \rho \rangle = \sum a_i e_i \int \text{Trace } X^j_{\rho} d^4 q$$

where a_i are the Clebsch-Gordan coefficients for the coupling $\rho \rightarrow Q - \bar{Q}$, with a quark of charge e_i . Conventionally the current is parameterized:

$$\langle 0 | T_j(0) | \rho \rangle = \frac{e M_0^2 g_{\rho} \phi^j}{2 \gamma \rho}$$

Hence Model 1 gives

$$\gamma_{\rho}^2 : \gamma_{\eta}^2 : \gamma_{\omega}^2 = \frac{M_{\rho}^2}{\cos^2 \theta} : \frac{3 M_{\eta}^2}{\cos^2 \theta} : \frac{3 M_{\omega}^2}{\sin^2 \theta}$$

where θ is the ω - η mixing angle. With $\tan \theta = 1/\sqrt{2}$ and $M_{\rho} \approx M_{\omega}$ we obtain 1:8:9 for the ratios and with the "physical" values of $\theta (=39.9^\circ)$ and of the masses we obtain 1:9.2:7.8.

In Model 2 the factors M_{ρ}^2 , etc., become M_{ρ}^4 , etc., giving 1:14:9 in the "ideal" case and 1:16.6:8.3 in the "physical" case.

Experimentally, the average of all experiments gives $1:5.85^{+3.1}_{-1.8}$: $9.0^{+3.9}_{-2.4}$ *) 8). We conclude that Model 2 is untenable. [The results of Model 1 are exactly the same as those obtained by Weisskopf and Van Royen using the non-relativistic quark model and a particular prescription for the (large) symmetry breaking ⁶⁾.] We propose Model 1 for the vector mesons.

*) The errors are large since one experiment disagrees with all others. Taking the Orsay colliding beam experiments ⁹⁾ alone gives

$$1:5.9^{+1.5}_{-1.3}:7.1^{+2.1}_{-1.6}.$$

SU(6) IN THE RELATIVISTIC MODEL

If $\vec{q}^2 \ll 1$ and the potential is approximately spin independent (i.e., contains $\vec{\gamma}$ only weakly in the rest frame) then all the terms in the BS equation are approximately scalars in the 2×2 spin subspaces ($++$, $--$, $+-$ and $-+$). Since the matrix structure of $J^P = 0^-$ and 3S_1 , $J^P = 1^-$ states only differs in these subspaces ${}^2, {}^3$, these states will be approximately degenerate with

$$\vec{X}_{++}^P \approx \vec{\sigma} X_{++}^\pi \text{ etc}$$

Hence SU(6) is possible (although we now assume $X_{++} \approx X_{--}$). It turns out ${}^2, {}^3$ that in Model 1:

$$\frac{\vec{X}_{++}^P - \vec{X}_{--}^P}{M_P} \approx \vec{\sigma} \frac{(X_{++}^\pi - X_{--}^\pi)}{M_\pi} , \text{ etc.}$$

and hence

$$\chi_P \approx \frac{R M_P}{\sqrt{2} f_\pi}$$

With $R \approx 1/\sqrt{2}$ [see Eq. (9) and the remarks following it] this equation becomes the famous Kawarabayashi-Suzuki ¹⁰⁾ relation which gives $\chi^2/4\pi = 0.61$. The average of all experiments ⁸⁾ gives $\chi^2/4\pi = 0.52^{+0.07}_{-0.06}$. (The more appealing value $R \approx 1$ gives $\chi^2/4\pi = 1.22$ which is incompatible with experiment.) (This result was also obtained by Weisskopf and Van Royen using their symmetry breaking prescription and the non-relativistic quark model ⁶⁾.)

FURTHER REMARKS

1. The BS equation can always be separated into equations which are odd or even under $P \rightarrow -P$ and hence only contain p^2 in the rest frame. Our assumption that the solution is analytic in p^2 therefore suggests quadratic mass formulae.
2. Unless further strong assumptions are made, the BS wave function can only be used to discuss meson annihilation processes. Taking only the simplest intermediate states (as is usual, but unjustifiable) the formalism can be used to discuss the possible effects of symmetry breaking in other processes ^{2),3)}.

CONCLUSIONS

We propose that the low lying mesons have the highly relativistic spinor structure $\chi_{++} \simeq \chi_{--}$; this model successfully describes meson annihilation processes. In the usual non-relativistic model our results can only be derived by assuming large symmetry breaking according to an arbitrary prescription ^{5),6)}.

A more primitive version of this work is contained in Ref. 2). More details and discussion of some points (particularly of the formalism) are contained in Ref. 3).

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R E F E R E N C E S

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