Profiles of dark haloes: evolution, scatter, and environment

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ABSTRACT
We study dark-matter halo density profiles in a high-resolution N-body simulation of a ΛCDM cosmology. Our statistical sample contains ~5000 haloes in the range \(10^{11} - 10^{14} h^{-1} M_\odot\) and the resolution allows a study of subhaloes inside host haloes. The profiles are parameterized by an NFW form with two parameters, an inner radius \(r_s\) and a virial radius \(R_{\text{vir}}\), and we define the halo concentration \(c_{\text{vir}} \equiv R_{\text{vir}}/r_s\). First, we find that, for a given halo mass, the redshift dependence of the median concentration is \(c_{\text{vir}} \propto (1 + z)^{-1}\). This corresponds to \(r_s(z) \sim \text{constant}\), and is contrary to earlier suspicions that \(c_{\text{vir}}\) does not vary much with redshift. The implications are that high-redshift galaxies are predicted to be more extended and dimmer than expected before. Second, we find that the scatter in halo profiles is large, with a \(1\sigma \Delta (\log c_{\text{vir}}) = 0.18\) at a given mass, corresponding to a scatter in maximum rotation velocities of \(\Delta V_{\text{max}}/V_{\text{max}} = 0.12\). We discuss implications for modelling the Tully-Fisher relation, which has a smaller reported intrinsic scatter. Third, subhaloes and haloes in dense environments tend to be more concentrated than isolated haloes, and show a larger scatter. These results suggest that \(c_{\text{vir}}\) is an essential parameter for the theory of galaxy modelling, and we briefly discuss implications for the universality of the Tully-Fisher relation, the formation of low surface brightness galaxies, and the origin of the Hubble sequence. We present an improved analytic treatment of halo formation that fits the measured relations between halo parameters and their redshift dependence, and can thus serve semi-analytic studies of galaxy formation.

Key words: cosmology — dark matter — galaxies: formation — galaxies: structure

1 INTRODUCTION
In the “standard” picture of galaxy formation, dark-matter (DM) haloes provide the framework for the formation of luminous galaxies (e.g., ?; ?; ?). The DM haloes are assumed to form hierarchically bottom-up via gravitational amplification of initial density fluctuations. The haloes carry with them gas, which eventually cools and contracts to form luminous disc galaxies at the halo centres. The halo profile has a direct dynamical role in determining the observable rotation curve of the disc. It also affects gas cooling and infall and therefore the structural properties of the resultant disc, such as size, luminosity and surface brightness. In order to model properly the dissipative stages of galaxy formation and obtain meaningful predictions for observable quantities (such as the Tully-Fisher relation), it is important to perform detailed dynamical studies of the evolution of halo structure, and to obtain statistical characteristics based on a fair sample of the simulated halo population.

Most naturally, the density profiles of haloes are expected to be a two-parameter family. This is because, assuming that the formation of haloes can be approximated by spherical collapse, each proto-halo perturbation can be characterized by two quantities, e.g., mass and radius (or density contrast) at some fiducial cosmological time. In the approximation of spherical collapse, these parameters specify the full evolution of each halo, including the epoch at which it collapses and its virial radius. A successful two-parameter functional form for the halo profiles has been proposed by Navarro, Frenk, & White (?; ?, ?, hereafter NFW95, NFW96, and NFW97):

\[
\rho_{\text{NFW}}(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2},
\]

where \(r_s\) is a characteristic “inner” radius, and \(\rho_s\) a corresponding inner density. As we show in § 2, one of the inner parameters can be replaced by a “virial” parameter, either the virial radius \(R_{\text{vir}}\), mass \(M_{\text{vir}}\), or velocity \(V_{\text{vir}}\). A very useful alternative is the concentration parameter \(c_{\text{vir}}\), which relates the inner and virial parameters. NFW found
that this functional form provides a good fit to haloes over a large range of masses, and for several different cosmological scenarios. It has been tested for the Einstein-deSitter model with a standard CDM power spectrum of initial fluctuations (SCDM), a flat cosmological model with Ω_m = 0.3, Ω_Λ = 0.7 and a corresponding CDM power spectrum (ACDM), and several models with power-law power spectra (confirmed by ?, and ?, hereafter KKK97).

NFW then noticed that, for a given cosmology, their haloes show a strong correlation between the model’s two parameters, e.g., an increase in ρ_s for decreasing M_c. A natural reason for the fact that low-mass haloes tend to show higher densities is that they typically collapsed earlier, when the universe was denser. To model this trend, NFW proposed a toy model (outlined in Appendix A) which assumes that ρ_s is a constant multiple of the universal density ρ_0(z_c) at a collapse redshift z_c, and that the collapsing mass at z_c is a constant fraction f of the total halo mass that has just collapsed and virialized at z = 0. The general trend of the relation between the two profile parameters at z = 0 is reproduced well for a proper choice of values for the constants k and f, with different values for the different cosmological models.

Since the halo profiles are expected to be a two-parameter family, it is important to study the scatter about this mean relation between the two halo parameters. This scatter could provide the second parameter which is necessary in order to explain the observed variations in galaxy properties, such as bulge-to-disc ratio, size and surface brightness. It may be argued that the scatter in halo spin parameter may also contribute to these variations, but it cannot account for all of them because, e.g., it does not properly correlate with the environment. The scatter in halo profiles should also have direct implications for understanding the surprisingly tight scatter observed in the Tully-Fisher (TF) relation for disc galaxies. NFW found a small scatter among their simulated haloes, which could have provided a convenient explanation for the small TF scatter, but other theoretical studies predict a larger scatter (?), and the generality of the NFW result is limited by the small number of haloes simulated per cosmology (∼ 20) and by their selective choice of haloes near virial equilibrium. We therefore here set to study in detail the scatter in a large, “fair” sample of simulated haloes.

The accumulating data of galaxies at high redshifts provide a great incentive for studying the properties of the halo population as a function of redshift. NFW97 tried to extend their toy model in order to predict this redshift dependence by assuming that k and f are both constant in time. In order to actually study in detail the redshift dependence of halo profiles, we use our large statistical sample of simulated haloes. Our results below, which differ from the NFW97 toy-halo profiles, we use our large statistical sample of simulated haloes. The outer, virial radius R_vir, of a halo of virial mass M_vir, is defined as the radius within which the mean density ρ_vir is a constant fraction of the mean density at the virial radius ρ_vir:

\[ M_{\text{vir}} \equiv \frac{4\pi}{3} \Delta_{\text{vir}} \rho_s R_{\text{vir}}^3. \]  

The associated virial velocity is defined by\[ V_{\text{vir}}^2 \equiv GM_{\text{vir}}/R_{\text{vir}}. \] The one-to-one relations between the three virial parameters are fully determined by the background cosmology (independent of the inner halo structure), so only one of them at a time can serve in the pair of independent parameters characterizing the profile. The virial overdensity \[ \Delta_{\text{vir}} \] is provided by the dissipationless spherical top-hat collapse model (?); it is a function of the cosmological model, and it may vary with time. For the Einstein-deSitter cosmology, the familiar value is \[ \Delta_{\text{vir}} \approx 200 \] at all times. For the family of flat cosmologies (Ω_m + Ω_Λ = 1), the value of \[ \Delta_{\text{vir}} \] can

- The sampling is “fair”, in the sense that haloes are found in any environment, field and clustered, and irrespective of the dynamical stage of the halo after virialization.
- The resolution is high, allowing a distinction between “distinct” haloes and “subhaloes”, and a study of environmental trends.
- The time evolution and scatter about the one-parameter family are studied in detail.

In § 2 we discuss further the parametric functional form used for the halo profiles. In § 3 we present the revised toy model for predicting the mean relation between the halo profile parameters and its redshift dependence. In § 4 we describe our N-body simulations and our method of halo finding and classification. In § 5 we present our results for haloes at z = 0; we compare the mean result to our model prediction, and quantify the intrinsic scatter. In § 6 we discuss implications for observable rotation curves and the TF relation. In § 7 we investigate the redshift dependence of halo properties, and the toy-model fits. Finally, in § 8, we summarize our results and discuss further implications.

### 2 PROFILE CHARACTERISTICS

We choose to fit the density profiles of all haloes at all redshifts with the NFW two-parameter functional form (Eq. 1). This is a convenient way to parameterize the profiles, without implying that it necessarily provides the best possible fit. Indeed, a similar analysis could be carried out using alternative functional forms (see an example below). In this section, we discuss the various parameters that are associated with the halo density profile, and the relations between them.

The inner radius, r_s, is where the effective logarithmic slope of the profile is −2, a characteristic radius which we term r_s. For much smaller radii, ρ_NFW ∝ r_s, and for much larger radii, ρ_NFW ∝ r_s. The inner density parameter of the NFW profile is related to the NFW density at r_s by ρ_s = 4ρ_{NFW}(r_s), and equals the local density at about half r_v: \[ \rho_s = \rho_{NFW}(r = 0.466 r_s). \]

The outer, virial radius R_vir, of a halo of virial mass M_vir, is defined as the radius within which the mean density is \[ \Delta_{\text{vir}} \] times the mean universal density \[ \rho_s \] at that redshift: \[ M_{\text{vir}} \equiv \frac{4\pi}{3} \Delta_{\text{vir}} \rho_s R_{\text{vir}}^3. \]

The associated virial velocity is defined by\[ V_{\text{vir}}^2 \equiv GM_{\text{vir}}/R_{\text{vir}}. \] The one-to-one relations between the three virial parameters are fully determined by the background cosmology (independent of the inner halo structure), so only one of them at a time can serve in the pair of independent parameters characterizing the profile. The virial overdensity \[ \Delta_{\text{vir}} \] is provided by the dissipationless spherical top-hat collapse model (?); it is a function of the cosmological model, and it may vary with time. For the Einstein-deSitter cosmology, the familiar value is \[ \Delta_{\text{vir}} \approx 200 \] at all times. For the family of flat cosmologies (Ω_m + Ω_Λ = 1), the value of \[ \Delta_{\text{vir}} \] can

\[ M_{\text{vir}} \approx 10^{11} h^{-1} M_{\odot} \left( \Omega_{\text{dm}} \Delta_{\text{vir}}(z)/200 \right)[R_{\text{vir}}(1+z)/75 h^{-1} \text{kpc}]^3, \]

\[ V_{\text{vir}} \approx 75 \text{km/s}\left[R_{\text{vir}}(75 h^{-1} \text{kpc})/(\Omega_0 \Delta_{\text{vir}}(z)/200)^{1/3}(1+z)^{3/2}\right]. \]
ΛCDM relative to SCDM mainly reflect a difference in virial radii when taking into account the model dependence of ∆ concentrations of haloes in different cosmologies. For example, our definition reduces confusion associated with comparing

be approximated by \( \Delta_{\text{vir}} \approx (18\pi^2 + 82x - 30x^2)/\Omega(z) \)

where \( x \equiv \Omega(z) - 1 \), and \( \Omega(z) \) is the ratio of mean matter density to critical density at redshift \( z \). For example, in the ΛCDM cosmological model that serves as the basis for our analysis in this paper (\( \Omega_m = 0.3 \)), the value at \( z = 0 \) is \( \Delta_{\text{vir}} \approx 340 \).

An associated useful characteristic is the concentration parameter, \( c_{\text{vir}} \), defined as the ratio between the virial and inner radii,

\[
c_{\text{vir}} \equiv \frac{R_{\text{vir}}}{r_s}. \tag{3}
\]

Note that our definition of \( c_{\text{vir}} \) differs slightly from that originally used by NFW, \( c_{\text{NFW}} \equiv R_200/r_s \), where \( R_{200} \) is the radius corresponding to a density of 200 times the critical density, independent of the actual cosmological model.

A third relation between the parameters of the NFW profile is

\[
M_{\text{vir}} = 4\pi \rho_s^3 A(c_{\text{vir}}), \quad A(c_{\text{vir}}) \equiv \ln(1 + c_{\text{vir}}) - \frac{c_{\text{vir}}}{1 + c_{\text{vir}}}. \tag{4}
\]

The three relations (Eqs. 2, 3 and 4) allow the usage of any pair out of the parameters defined so far (excluding degenerate pairs consisting only of virial quantities) as the two independent parameters that fully characterize the profile.

Since the more observable quantities have to do with rotation curves, it is worth translating the density profile into a circular velocity curve for the halo,

\[
V_c^2(r) \equiv \frac{GM(r)}{r} = \frac{V_{\text{max}}^2}{r} \frac{c_{\text{vir}}}{A(c_{\text{vir}})} A(x), \tag{5}
\]

where \( x \equiv r/r_s \). The maximum velocity occurs at a radius \( r_{\text{max}} \approx 2.16 r_s \) and is given by

\[
\frac{V_{\text{max}}^2}{V_{\text{vir}}^2} \approx 0.216 \frac{c_{\text{vir}}}{A(c_{\text{vir}})}.
\]

For typical \( c_{\text{vir}} \) values in the range \( 5 - 30 \), \( V_{\text{max}} \) varies in the range \((1 - 1.6)V_{\text{vir}}\). Figure 1 shows the ratio \( V_{\text{max}}/V_{\text{vir}} \) as a function of \( c_{\text{vir}} \). Note that for haloes of the same mass, a larger \( c_{\text{vir}} \) goes with a larger \( V_{\text{max}} \). Because of the relationship between \( r_{\text{max}} \) and \( r_s \), haloes with \( c_{\text{vir}} \leq 10 \) have velocity curves which continue to rise gradually out to an appreciable fraction of their virial radii, while those with \( c_{\text{vir}} > 10 \) rise more steeply, and possibly represent galaxies in which \( r_{\text{max}} \) is identifiable observationally (though the effects of baryonic contraction should be taken into account before the observed and simulated values of \( r_{\text{max}} \) can be compared).

We stress again that the specific choice of the NFW functional form does not limit the generality of our analysis in a severe way. This is largely due to the association of the specific \( r_s \) with the more general \( r_{\text{max}} \). When the NFW function is fitted to a generic halo whose profile even vaguely resembles a similar shape, the fitting procedure is likely to return an \( r_s \) value that is close to the effective \( r_{\text{max}} \) of that halo. The concentration parameter can then be interpreted as a general structure parameter not necessarily restricted to the specific NFW function. In particular, any spread in \( c_{\text{vir}} \) can be attributed to a real scatter in a “physical” inner radius such as \( r_{\text{max}} \) rather than to inaccuracies in the assumed universal profile.

The interpretation of \( r_s \) as \( r_{\text{max}} \) allows one to map the NFW parameters to appropriate parameters of other functional forms. For many purposes, such as determining \( V_{\text{max}} \) or modelling gas cooling and galaxy formation, the NFW form is sufficiently accurate. However, a comparison with alternative profiles with different core behaviors may be important when much smaller radii are concerned \((r \lesssim 0.02 R_{\text{vir}})\), where there are indications of deviations from an extrapolation of the NFW shape (\( ? \), hereafter KKBP; \( ? \), ?).

A specific example of an alternative profile functional form is the Burkert profile (\( ? \)):

\[
\rho_b(r) = \frac{\rho_s}{[1 + (r/r_b)^2]^{1 + r/r_b}}, \tag{7}
\]

This profile resembles the NFW profile for \( r \gtrsim 0.02 R_{\text{vir}} \), is similar to the best-fit profile of KKBP at all radii, and seems to do well in fitting haloes from very high resolution N-body simulations (\( ? \)). The Burkert profile has a log slope of \(-2 \) at \( r_{\text{max}} \approx 1.52 r_b \), so one can relate the scale radii of NFW and Burkert by \( r_b \approx r_s/1.52 \), and then relate the concentration parameters by \( R_{\text{vir}}/r_b \approx 1.52 c_{\text{vir}} \).

Note also that the relationship between \( c_{\text{vir}} \) and \( r_{\text{max}} \) is robust, regardless of profile shape. For example, with the Burkert profile we have an implied velocity maximum at \( r_{\text{max}} \approx 3.25 r_b \). If one assumes instead the relation as gleaned from an NFW fit, with \( r_{\text{max}} \approx 2.16 r_s \approx 3.28 r_b \), there is good agreement between the values of \( r_{\text{max}} \).

3 A REVISED TOY MODEL

In any investigation based on computer simulations it is useful to have a simple toy model that helps interpret the nu-
merical results and allows an easy application of the conclusions in subsequent analytic or semi-analytic investigations. As mentioned in the Introduction and outlined in Appendix A, NFW96 and NFW97 proposed such a model, with 2 free parameters, which successfully recovers the mean \( c_{\text{vir}} \) and \( M_{\text{vir}} \) relation at \( z = 0 \) for the several different cosmological models simulated by them. Our simulations (see §5 below) indeed confirm the success of this model at \( z = 0 \). However, we find (§7 below) that the NFW97 model does not reproduce properly the redshift dependence of the halo profiles as seen in the simulation; it significantly over-predicts the concentration of haloes at early times, \( z \gtrsim 1 \). We therefore propose a revised toy model that is shown below to recover properly the full behavior of the mean \( c_{\text{vir}} \) and its redshift dependence. We present the model in this preparatory section, so that we can refer to it when we describe and interpret the results from the simulations in the following sections.

3.1 The model

We seek a model for the typical halo concentration, denoted in this section by \( c_{\text{vir},M,(a)} \), for a given mass \( M_{\text{vir}} \) and epoch \( a = (1 + z)^{-1} \). Following the general spirit of the NFW97 model, we assign to each halo an epoch of collapse, \( a_c \). Unlike their formulation, which utilizes the extended Press-Schechter approximation, we define \( a_c \) in a simpler way as the epoch at which the typical collapsing mass, \( M_s(a) \), equals a fixed fraction \( F \) of the halo mass at epoch \( a \),

\[
M_s(a) = FM_{\text{vir}}. \quad (8)
\]

The typical collapsing mass at an epoch \( a \) is defined by \( \sigma[M_s(a)] = 1.686 \), where \( \sigma(M) \) is the linear rms density fluctuation on the comoving scale encompassing a mass \( M \), and 1.686 is the linear equivalent of the density at collapse according to the familiar spherical collapse model. The typical collapsing mass is therefore a known function of the linear power spectrum of fluctuations and the linear growth rate \( D(a) \) for the cosmology in hand. For some purposes we find it convenient to measure the halo mass in units of the typical halo mass at the same epoch,

\[
\mu \equiv M_{\text{vir}}(a)/M_s(a). \quad (9)
\]

The second relation of the model arises by associating the mean density of the universe \( \rho_u \) at \( a_c \), with a characteristic density of the halo at \( a \). NFW97 used the inner density parameter \( \rho_a \), which is related to \( M_{\text{vir}} \) and \( r_\Delta \) via the specific shape of the NFW profile. Instead, we define a more general characteristic density \( \rho_s \) by combining inner and virial quantities:

\[
M_{\text{vir}} \equiv \frac{4\pi}{3} r_\Delta^3 \rho_s. \quad (10)
\]

The NFW profile implies \( \rho_s = 3\rho_u A(c_{\text{vir}}) \). The association of the halo density \( \rho_s \) with the universal density at collapse is made via a second free parameter, \( K \):

\[
\rho_s = K^3 \Delta \rho_u(a_c) = K^3 \Delta \rho_u(a)(\frac{a}{a_c})^3. \quad (11)
\]

The parameter \( K \) represents contraction of the inner halo beyond that required for top-hat dissipationless halo virialization, and it is assumed to be the same for all haloes. Using Eqs. 2, 10, and 11, we obtain a simple expression for \( c_{\text{vir}} \) in terms of \( a_c \) as our second model relation:

\[
c_{\text{vir}}(\mu, a) = K \frac{a_c}{a}. \quad (12)
\]

The model is fully determined by Eqs. 8 and 12 given the values of the two parameters \( F \) and \( K \). We find below (§5 and §7) that by adjusting \( F \) and \( K \) we are able to reproduce the full behavior of \( c_{\text{vir}}(\mu, a) \) as measured in our simulations. The small differences in the definitions of \( F \) and \( K \) compared to the analogous parameters of NFW97, \( f \) and \( k \), make a big difference in the success of the model.

Note in Eq. 8 that, for any cosmology, \( a_c \) is uniquely determined by \( M_{\text{vir}} \), independent of \( a \). This implies via Eq. 12 that, for a fixed halo mass,

\[
c_{\text{vir}}(a) \propto a. \quad (13)
\]

Our model thus predicts that for haloes of the same mass the concentration is proportional to \( (1 + z)^{-1} \). This is different from the NFW prediction in which the concentration is a much weaker function of redshift.

In order to gain a basic understanding of the important elements of this model, we discuss its predictions in the context of three cosmological models of increasing complexity: (1) a self-similar model of Einstein-deSitter cosmology and a power-law power spectrum of fluctuations, (2) standard CDM, in which the universe is still Einstein-deSitter but the power spectrum deviates from a power law, and (3) the relevant cosmology of the current investigation, \( \Lambda \)CDM, with \( \Omega_m \neq 1 \) and a non-power-law spectrum.

3.2 Example 1: the self-similar case

As an illustrative example, assume a fully self-similar case: Einstein-deSitter cosmology, \( \Omega_m = 1 \), for which the growth rate is \( D(a) \propto a \), and a power-law power spectrum of fluctuations, \( P(k) \propto k^n \), for which \( \sigma(M) \propto M^{-\alpha} \) with \( \alpha = (n + 3)/6 \). In this case,

\[
M_s(a) \propto a^{1/\alpha}. \quad (14)
\]

Together with Eqs. 8 and 9, we have \( a_c/a_0 = (\mu F)^{\alpha} \). Then, using Eq. 12, we obtain

\[
c_{\text{vir}}(\mu, a) = K(F\mu)^{-\alpha}. \quad (15)
\]

Note that in the self-similar case the two parameters \( F \) and \( K \) can be replaced by one parameter, \( K F^{-\alpha} \). Equivalently, we may vary only \( K \) and adopt the natural value \( F = 1 \), namely, apply the model to the collapse of the whole halo. This is a special feature of our revised model, not valid in the original NFW model.

The slope of the function \( c_{\text{vir}}(\mu) \) at \( a \) is completely determined by the power index \( \alpha \) (i.e., \( n \)):

\[
c_{\text{vir}} \propto \mu^{-\alpha}. \quad (\text{This simple mass dependence can be checked against the results of the simulations of NFW97 in their Figure 6, which presents } c_{\text{vir}}(\mu) \text{ at } z = 0 \text{ for Einstein-deSitter cosmology with four different power-law spectra: } n = -1.5, -1, -0.5, \text{ and } 0 \text{ (} \alpha = 1/4, 1/3, 5/11 \text{ and } 1/2 \text{). In each case, our model predicts the simulated slope quite well, even slightly better than the NFW model. As power-law } M^{-\alpha} \text{ becomes steeper (} n \text{ larger), the difference in collapse epochs for haloes of a given mass difference becomes larger, which is reflected in a steeper } c_{\text{vir}}(\mu) \text{ relation.}})
\]
The collapse factor $K$ may be determined from the simulations at the present epoch by matching the value of $c_{\text{vir}}$ at any desired $\mu$. We find for a typical simulated halo ($\mu = 1$) at $a = 1$ that $c_{\text{vir}} \approx 10$, implying that the additional collapse factor for the whole halo ($F = 1$) must be $K \sim 10$.

Clearly, for the fully self-similar case one would expect any dimensionless properties of $M$, haloes to be invariant in time. This is easily verified by setting $\mu = 1$ in Eq. 15: $c_{\text{vir}}(1,a) = K$ (for $F = 1$). In this case, the concentration is also fixed for any other fixed value of $\mu$. Accordingly, the concentrations are different for haloes of the same mass that are addressed at different redshifts. For the Einstein-deSitter cosmology, $\Omega = 1$ is well approximated using Eq. 16.

The relation $c_{\text{vir}} \propto a$ (Eq. 13) implies (via Eq. 3) that the value of $r_s$ is the same (in physical coordinates) at all redshifts. Again, this is different from the NFW97 model prediction.

### 3.3 Example 2: SCDM

As a second example, consider the standard CDM cosmology (SCDM: $\Omega_m = 1$, $h = 0.5$, $\sigma_8 = 0.7$). Although the time evolution here is still self-similar, the power spectrum is not a power law: $\sigma(M)$ has a characteristic bend near $\mu \sim 1$ today [which corresponds to a mass $M_\nu(a = 1) \approx 2.5 \times 10^{13}h^{-1}M_\odot$]. The local slope varies from $\alpha \approx 0$ for $\mu \ll 1$ to $\alpha = 2/3$ for $\mu \gg 1$.

The model solution for $c_{\text{vir}}(\mu)$ does not have a closed form in this case, but it is easy to obtain a useful approximation. The slope of the $c_{\text{vir}}(\mu)$ relation at a specific $\mu$ is determined by the effective slope of the power spectrum on a scale corresponding to the mass $\mu F M_\nu$ (not $\mu M_\nu$), since this is the mass scale used to characterize the halo collapse time in Eqs. 8 and 9. We now obtain an approximate relation similar to Eq. 15, but with the effective local $\alpha$ replacing the constant $\alpha$:

$$c_{\text{vir}} \simeq K(\mu F)^{-\alpha(\mu F)}. \quad (16)$$

Unlike the power-law example, the value chosen for the constant $F$ does play a role in the slope of the $c_{\text{vir}}(\mu)$ relation at a given $\mu$. In order to determine the best values of $F$ and $K$, we match the model predictions of $c_{\text{vir}}(\mu)$ to the results of the $N$-body simulations of the SCDM model at the present epoch. Using $F = 0.01$ and $K = 4.3$ in Eqs. 8 and 12, we are able to reproduce quite well the SCDM results of NFW97 (their Figure 6) over the range $\mu \approx 0.01 - 100$. The relation about $\mu \sim 1$ is well approximated using Eq. 16, where $\alpha(\mu F = 0.01) \approx 0.15$.

But now, using no extra parameters, we are able to reproduce the time dependence of this relation, which we test using the $c_{\text{vir}}$ values as determined from our small box ($7.5h^{-1}\text{Mpc}$) SCDM simulation. In our simulation, we find that the best-fit parameters at $a = 1$ are $F = 0.01$ and $K = 3.8$ (rather than the $K = 4.3$ we found for the NFW97 simulations). For a fixed $M_{\text{vir}}$, we find that the model-predicted scaling of $c_{\text{vir}}(a) \propto a$ indeed describes very well the time evolution of the halo population.

In our toy model, the values of $F$ and $K$ are assumed to be constants as a function of both $a$ and $\mu$. Such a behavior is naturally expected in the fully self-similar case, in which no special time or scale is present in the problem. However, the success of this toy model in the SCDM case, which is not scale invariant, is somewhat surprising. The reason for this success is linked to the small value of $F$, which pushes the relevant mass scales of the problem to values much below that of the bend in the power spectrum (near $\mu \sim 1$), where the spectrum is approaching a power-law behavior. For small values of $\mu$ ($\mu \ll 1$), the slope of $c_{\text{vir}}(\mu)$ is almost independent of the actual value of $F$ as long as the latter is smaller than 0.05. The specific preferred value of $F = 0.01$ arises from the need to match the model $c_{\text{vir}}(\mu)$ with the simulations in the range $\mu \geq 1$.

### 3.4 Example 3: ΛCDM

Our third example concerns a currently popular ΛCDM cosmological model ($\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$, $\sigma_8 = 1.0$, where $M_\nu \approx 2 \times 10^{13}h^{-1}M_\odot$ at $z = 0$). In this case, self similarity is violated due to the non-power-law spectrum as before, and also by the time-dependent fluctuation growth rate associated with the low $\Omega_m$. Using similar reasoning to our discussion in the previous example, one may expect our toy model with constant $F$ and $K$ to break down, but, again, this worry is alleviated as long as $F$ is small. As in the SCDM case, this pushes the relevant mass scales to the power-law regime away from the bend in the power spectrum. In addition, a small value of $F$ demands that the collapse epoch is early, when the mean density was near the critical value, $\Omega_m(a_c) \approx 1$, and therefore the fluctuation growth rate was close to that in the self-similar cosmology, $D(a) \propto a$. The deviation from the self-similar growth rate introduces in Eq. 16 a multiplicative correction factor, given by the growth of fluctuations between the epochs $a_c$ and $a$ in the given cosmology compared to the Einstein-deSitter case. In the case of ΛCDM and $a_c \ll 1$ this factor at $a = 1$ is about 1.25.

As we show in § 5 and § 7 below, using $F = 0.01$ and $K = 4.0$ in the model equations 8 and 12, we are able to reproduce the full behavior of the median $c_{\text{vir}}(\mu, a)$ in our ΛCDM simulations. At $a = 1$, we have $c_{\text{vir}}(\mu, a) \propto \mu^{-\alpha(F\mu)}$, where $\alpha \approx 0.13$ for $\mu \sim 1$. For haloes of fixed mass ($\mu M_\nu = \text{const}$), we have $c_{\text{vir}}(a) \propto a$ as before.

We will show below that by setting the value of $K$ to 2.6 and 6.0 we are able to artificially parameterize the $-1\sigma$ and $+1\sigma$ scatter respectively in the value of $c_{\mu}$ for the simulated population of haloes. A similar range of $K$ values accounts for the scatter for all masses and at all cosmological epochs.

\footnote{The normalization of $c_{\text{vir}}(\mu, 1)$ in our simulation is slightly lower than in the NFW simulations, by 15%. This may be due to the small box size; we find a similar effect when we compare results from our large-box ($60h^{-1}\text{Mpc}$) and small-box ($7.5h^{-1}\text{Mpc}$) ΛCDM simulations. The presence of large waves induces larger environments of high background density in which the halo concentrations tend to be higher (see § 5).}
4 SIMULATING HALOE S

4.1 The numerical simulations

Only recently have large cosmological N-body simulations reached the stage where detailed structural properties of many dark-matter haloes can be resolved simultaneously. One of the most successful methods for high force resolution and mass resolution is the Adaptive Refinement Tree (ART) code (KKK97). The method makes use of successive refinements of the grid and time step in high density environments. The simulations based on the ART code provide, for the first time, a compilation of a statistical sample of well-resolved DM haloes, as well as substructure of haloes within haloes. In previous simulations, haloes were picked “by hand” using certain selection criteria from a low-resolution cosmological simulations, to be re-simulated with high resolution. This selection induces a certain bias into the sample.

We have used the ART code to simulate the evolution of DM in a low-density flat ΛCDM model for which Ωm = 0.3, Ωλ = 0.7, h = 0.7, and σ8 = 1.0 at z = 0. The simulation followed the trajectories of 256^3 cold dark matter particles within a cubic, periodic box of comoving size 60 h^{-1}Mpc from redshift z = 40 to the present. We have used a 512^3 uniform grid, and up to six refinement levels in the regions of highest density, implying a dynamic range of 32,768. The formal resolution of the simulation is thus \( f_{res} = 1.8 \times 10^5 \text{h}^{-1} \text{kpc} \), and the mass per particle is \( m_p = 1.1 \times 10^9 \text{h}^{-1} M_\odot \). In the present paper, we analyze 12 saved time steps from z = 5 to 0. We have also used two simulations in smaller boxes to check for resolution and cosmology dependence. One of these is a 30 h^{-1}Mpc box simulation of the same ΛCDM cosmology, with 256^3 particles, \( m_p = 1.4 \times 10^6 \text{h}^{-1} M_\odot \), and \( f_{res} = 0.9 \text{h}^{-1} \text{kpc} \). The other, in a 7.5 h^{-1}Mpc box, is of the SCDM cosmology (\( \Omega_m = 1, h = 0.5 \), and \( \sigma_8 = 0.7 \) at \( z = 0 \)), and it consists of 128^3 particles, \( f_{res} = 0.5 \text{h}^{-1} \text{kpc} \), and \( m_p = 5.5 \times 10^7 \text{h}^{-1} M_\odot \). Tests of the ART code for numerical effects on halo density profiles are discussed in KKB.

4.2 Finding and fitting haloes and subhaloes

In this investigation, we sample all types of DM haloes independent of their environment. In particular, we identify both the standard kind of “distinct” haloes, of the type identified using common friends-of-friends algorithms and considered in Press-Schechter approximations, and also “subhaloes”, whose centres are located within the virial radius of a larger “host” halo. Our halo finding/classifying algorithm, which is based on the Bound Density Maxima technique (?), has been specifically designed to simultaneously identify distinct haloes and subhaloes (Appendix B).

We fit every identified DM halo using the NFW profile (Eq. 1). Before fitting, we check the halo radial density profile to see if it has a significant upturn, \( dp(r)/dr > 0 \), and if so, we declare this point to be the truncation radius \( R_t \). Our measured \( R_t \) values are comparable or somewhat smaller than the expected tidal radii. For haloes with no significant upturn in density, we fit the NFW density profile out to \( R_{vir} \), while for haloes with a truncation radius, we fit the profile out to \( R_t \), and extrapolate the NFW function in order to assign virtual values of \( R_{vir} \) and \( M_{vir} \).

The profile fitting is performed as follows. After identifying a centre for the halo, we count particles in logarithmically spaced radial bins, and assign corresponding Poisson errors based on the count in each bin. We then fit an NFW profile (by \( \chi^2 \) minimization) to the counts in bins using the bin errors in the covariance matrix, and obtain best-fit values for the two free parameters \( R_{vir} \) and \( r_s \) (or equivalently \( M_{vir} \) and \( c_{vir} \), etc.) along with the corresponding errors in these parameters. We then remove unbound particles from each halo, as described in Appendix B, and iterate the process of determining \( R_t \) and fitting a profile until the halo contains only bound particles.

We present results for haloes with masses in the range \( 1.5 \times 10^{13} \text{–} 10^{14} h^{-1} M_\odot \); the smallest haloes thus contain \( \sim 150 \) particles. A profile fit to a halo of only a few hundred particles may carry large errors. We therefore make a rigorous attempt to estimate the errors and take them into account in every step of the process. Poor fits due to small number statistics are marked by large errors that are incorporated in the analysis and the results we present.

The profile fit of haloes in crowded regions clearly involves ambiguities in the mass assignment to the subhaloes and the host. Our fitting procedure provides a well-defined recipe for mass assignment based on the value of \( M_{vir} \) even when the fit is actually performed inside an \( R_t \) that is smaller than the \( R_{vir} \) obtained by extrapolation. The concentration parameter is defined in the same way for all haloes, \( c_{vir} = R_{vir}/r_s \). Because in most cases of subhaloes the extrapolation procedure adds much less mass than the mass that actually lies between \( R_t \) and \( R_{vir} \), the double counting is not severe; most of the mass associated with the upturn in the profile is assigned to a different subhalo or to the host. On the other hand, a small subhalo does not cause a significant upturn in the profile of its host halo, and its mass is therefore also included in the mass assigned to the host. This partial double counting introduces some uncertainty to any recipe for assigning a luminous galaxy mass to a halo of a given mass.

The outcome of the halo finder/classifier is a statistical halo catalog that includes all the bound virialized systems in the simulation above the minimum mass threshold. We include distinct haloes and subhaloes, but not subhaloes of a second generation, i.e., those whose hosts are themselves subhaloes of a larger host. The output for each halo includes the list of its bound particles, the location of its centre, the NFW profile parameters (e.g., \( c_{vir} \) and \( M_{vir} \)), the corresponding errors (\( \sigma_c \) and \( \sigma_M \)), and the truncation radius, if relevant.
5 HALO PROFILES TODAY

We start by studying the halo profiles at the current epoch in the simulation. First, we study the median $c_{\text{vir}}$ at $z = 0$ for distinct haloes. The Poisson errors reflect the number of haloes within each mass bin. In order to account for the fit errors, we generated 500 Monte Carlo realizations in which the measured $c_{\text{vir}}$ and $M_{\text{vir}}$ of each halo were perturbed using random Gaussian deviates with standard deviations equal to $\sigma_{c_{\text{vir}}}$ and $\sigma_{M_{\text{vir}}}$ respectively. Median values of $c_{\text{vir}}$ were then determined using the Monte Carlo ensemble. The lowest-mass and highest-mass bin have $\sim 2000$ and $20$ haloes respectively (we avoid using bins with less than $10$ haloes.), and the Poisson errors grow with mass accordingly.

The median $c_{\text{vir}}$ decreases with growing mass, in qualitative agreement with the toy models, and therefore consistent with the assertion that small mass haloes are more concentrated because they typically collapse earlier than haloes of larger masses. The NFW97 model outlined in Appendix A has been slightly adjusted to yield $c_{\text{vir}}$ rather than $c_{\text{NFW}}$. Its predicted slope is in reasonable agreement with that derived from the simulations, but there is some indication that the slope is too shallow. Using our revised toy model outlined in § 3, with $F = 0.01$ and $K = 4.0$, we reproduce the median relation even better, as shown by the central thin solid line. Near $\mu \sim 1$ ($M_{\text{vir}} \sim M_* \sim 2 \times 10^{13} h^{-1} M_\odot$ at $z = 0$), the model prediction is

$$c_{\text{vir}}(\mu, z = 0) \simeq 1.25 K (\mu F)^{-\alpha(\mu F)} \approx 9\mu^{-0.13}. \quad (17)$$

Indeed, the slope of the $c_{\text{vir}}(\mu)$ curve is closely related to the varying slope of the mass power spectrum, which influences the relative difference in collapse epochs for typical objects on different mass scales. The factor of $1.25$, as explained in § 3, is a measure of the deviation from the Einstein-deSitter self-similar linear fluctuation growth rate $D(\alpha) \propto \alpha$ between some high redshift and $z = 0$ (where the corresponding collapse epoch, for a given mass, is earlier in the $\Lambda$CDM case).

5.2 Subhaloes and environmental dependence

If the median $c_{\text{vir}}(M_{\text{vir}})$ indeed reflects different formation epochs, one might expect the $c_{\text{vir}}$ of haloes of a given mass to vary with the density of the environment, since haloes in dense regions typically collapse earlier. In particular, the concentration should tend to be larger for subhaloes compared to distinct haloes. Another effect that may lead to higher $c_{\text{vir}}$ values in subhaloes is the expected steepening of their outer profile due to tidal stripping. Since stripping is likely to be more effective for small mass haloes, this process may lead to a stronger mass dependence in subhaloes. A third effect is that haloes that are embedded in a high-density environment are likely to experience extreme collapse histories and frequent merger events which may affect their final concentrations.

Figure 3 shows the relation $c_{\text{vir}}(M_{\text{vir}})$ at $z = 0$ for subhaloes. We see that subhaloes on galactic scales ($M_{\text{vir}} \sim 10^{12} h^{-1} M_\odot$) are indeed more concentrated than distinct

\begin{figure}
\centering
\includegraphics[width=\textwidth]{halo_profiles}
\caption{Concentration versus mass for distinct haloes at $z = 0$. The thick solid curve is the median at a given $M_{\text{vir}}$. The error bars represent Poisson errors of the mean due to the sampling of a finite number of haloes per mass bin. The outer dot-dashed curves represent only the true, intrinsic scatter in $c_{\text{vir}}$, after eliminating both the Poisson scatter and the scatter due to errors in the individual profile fits due, for example, to the finite number of particles per halo. The central and outer thin solid curves are the predictions for the median and 68% values by the toy model outlined in the text, for $F = 0.01$ and three different values of $K$. The thin dot-dashed line shows the prediction of the NFW97 model for $F = 0.01$ and $K = 4.0$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{subhalo_profiles}
\caption{Concentration versus mass for subhaloes at $z = 0$. The curves and errors are the same as in Figure 2.}
\end{figure}
haloes of the same mass. The dependence on mass seems to be stronger for subhaloes than for distinct haloes, with $c_{\text{vir}} \propto \mu^{-0.3}$ (compared to $\mu^{-0.13}$), though the large errors in the case of subhaloes make this trend only marginal.

We address directly the dependence of concentration on background density in Figure 4, which shows $c_{\text{vir}}$ as a function of local density for all haloes (both distinct and subhaloes) in the mass range $M_{\text{vir}} = 0.5 - 1.0 \times 10^{12} h^{-1} M_{\odot}$. The local background density is defined for the dark matter within spheres of radius $1 h^{-1} \text{Mpc}$ in units of the average density of the universe in the simulation, $\rho_u = 8.3 \times 10^{-3} h^2 M_{\odot} / \text{Mpc}^3$. We see that haloes in more dense environments indeed tend to be more concentrated than their more isolated counterparts. Note that this trend is in fact stronger than the dependence of $c_{\text{vir}}$ on mass.

We find a similar trend when the local density is determined within spheres of radius $1.5 h^{-1} \text{Mpc}$, but the trend becomes weaker when spheres of radius $0.5 h^{-1} \text{Mpc}$ are used. Similarly, we find that the trend holds for haloes of mass $\lesssim 5 \times 10^{12} h^{-1} M_{\odot}$. For larger masses, the trend seems to become less pronounced, but this is quite inconclusive because we have only a few massive haloes ($\lesssim 15$) with local densities $\gtrsim 100 \rho_u$. A similar trend has been seen for distinct haloes alone, but it is difficult to make a definitive assessment in this case because there are only a few distinct haloes with local densities $\gtrsim 100 \rho_u$.

The results at low and high densities are comparable to those for distinct haloes and subhaloes respectively, consistent with a significant correlation between being a subhalo (or a distinct halo) and residing in a low-density (or a high-density) environment.

Our toy model is not sophisticated enough to account for the dependence of concentration on environment, and for the exact relation for subhaloes. However, in the framework of our toy model (§ 3), we can artificially parameterize these trends by varying the collapse parameter $K$ as a function of local density. We are left with the qualitative speculations mentioned above for the interpretation of the trends with environment seen in the simulations.

We discuss possible implications of the environment dependence in § 8.

### 5.3 Scatter in the concentration parameter

One of the most interesting of our results at $z = 0$ is the spread in $c_{\text{vir}}$ values for fixed $M_{\text{vir}}$. A significant scatter about the median relations may have intriguing observational implications. Before we report on our results, we briefly describe our methods of ascertaining the intrinsic scatter.

There are two sources of scatter on top of the intrinsic spread in halo concentrations. First, Poisson noise due to the sampling of a finite number of haloes in each mass bin adds a significant spread, especially in the case of large-mass haloes. Small-mass haloes, on the other hand, are plentiful, but the relatively small number of particles in each halo introduces a significant error in the measured value of the halo profile parameters $M_{\text{vir}}$ and $c_{\text{vir}}$. This is the second source of additional scatter. The Poisson error due to the finite number of haloes is relatively straightforward to correct, but correcting the error in the profile parameters requires a more involved procedure.

We account for this error in the profile parameters using the errors obtained in the profile fits. Within each mass bin, we have performed $\sim 500$ Monte Carlo realizations in an attempt to undo the effect of the measurement errors as follows. Every measured $c_{\text{vir}}$ value has been perturbed by a one-sided random Gaussian deviate, positive or negative depending on whether the measured $c_{\text{vir}}$ is smaller or larger than the median respectively. The standard deviation of each Gaussian deviate was set to be the error in the value of $c_{\text{vir}}$ as estimated in the profile fit of that specific halo. The smaller scatter obtained in this set of Monte Carlo realizations provides an estimate for the spread excluding the fit errors. We then subtract in quadrature the Poisson error due to the finite number of haloes to obtain our estimate for the intrinsic scatter in $c_{\text{vir}}$.

We have checked our technique of measuring the intrinsic spread using an artificial ensemble of 1000 (spherical) haloes with a variety of numbers of particles and a known intrinsic distribution of $c_{\text{vir}}$. The technique reproduced the median concentration and true spread to within 5% when the particle number was varied from 100 to $10^5$, the range of interest for our simulated haloes.

We have also checked our procedure for measuring the intrinsic scatter using a simulation of higher resolution in a smaller box of side $30 h^{-1} \text{Mpc}$, in which there are on average 8 times as many particles in a halo of a given mass. This simulation was stopped at $z = 1.7$, so we pursued this test only at that epoch and earlier. Because the simulation of higher resolution contains only few haloes near the high mass end, we limit the comparison to the mass range $(3-10) \times 10^{11} M_{\odot}$. The values obtained by our procedure from the haloes simulated with higher resolution, both for the median and the intrinsic spread in $c_{\text{vir}}$, agree to within $\sim 5\%$ with the corresponding values obtained with lower resolu-
across the entire mass range studied. In addition, the spread in the mass range we consider. The simulated distributions (thick lines) include the 2,000 distinct haloes and 200 subhaloes within this mass range. Log-normal distributions with the same median and standard deviation as the measured distributions are shown (thin lines). Subhaloes are, on average, more concentrated than distinct haloes and they show a larger spread.

This level of agreement is consistent with the Poisson error of the high-resolution simulation, which is of order 10% due to the resolution of the simulation. This level of agreement is consistent with the Poisson error of the high-resolution simulation, which is of order 10%.

Now that the method has been discussed, we turn the attention back to the relation between \( \Delta(\log c_{\text{vir}}) \) and \( M_{\text{vir}} \) for distinct haloes, Figure 2. The measured 68% scatter is shown, as well as the “pushed in” corrected scatter which marks our (under-) estimated intrinsic scatter. As can be seen by noting the Poisson error bars, the correction at the small-mass end is almost entirely due to the measurement errors of the profile parameters, which are dominated by the small number of particles per halo.

We see that the intrinsic spread is large; it is comparable to the systematic change in the median value of \( c_{\text{vir}} \) across the entire mass range studied. In addition, the spread is roughly constant as a function of mass, with a 1σ deviation of \( \Delta(\log c_{\text{vir}}) \sim 0.18 \). We discuss possible observational implications of this scatter in the next section.

The spread in \( c_{\text{vir}} \) values as a function of \( M_{\text{vir}} \) for subhaloes is shown in Figure 3. Note that the scatter is larger for the subhalo population than for their distinct counterparts of the same mass, with a 1σ variation of \( \Delta(\log c_{\text{vir}}) \sim 0.24 \). This is clearly seen in Figure 5, where the probability distributions of concentrations for distinct haloes and subhaloes are compared (for \( M_{\text{vir}} = 0.5 - 1.0 \times 10^{12} h^{-1} M_{\odot} \)). It is possible that the larger scatter evaluated for subhaloes is a result of their more complicated formation histories, including for example more interactions and stripping. We point out that we have found no significant trend with the number of co-subhaloes within the same virialized host. Such a trend might have been expected if interaction among co-subhaloes plays an important role in determining the profile shape.

The 68% intrinsic spread in \( c_{\text{vir}} \) as a function of the local density (for \( 0.5 - 1.0 \times 10^{12} h^{-1} M_{\odot} \) haloes) is given in Figure 4. We can use the obtained distribution of halo concentrations as a function of local density to probe questions associated with the origin of LSB galaxies and the observed morphology density relation (§ 8).

In Figure 5 we show the distribution of concentration values for distinct haloes and subhaloes, along with lognormal functions with the same median and standard deviation. The log-normal forms describe the observed distributions reasonably well. Such a result has also been reported by Jing (1998). Our scatter for distinct haloes is similar to that reported by Jing, but it is difficult to make a quantitative comparison because the two investigations focus on different aspects of the problem.

In the context of the toy model presented in § 3, one can parameterize the spread in \( c_{\text{vir}} \) as spread in collapse epochs and/or collapse histories, via the parameters \( a_c \) and \( K \) respectively. Using Eq. 12, we find that the evaluated spread in \( c_{\text{vir}} \) can be matched by a spread of \( \Delta(\log K(a_0/a_c)) \sim \Delta(\log c_{\text{vir}}) \sim 0.18 \) in the toy model. If we absorb all the scatter in the collapse parameter \( K \), we find that the model matches the 50 ± 34% (comprising 68%) percentiles of the \( c_{\text{vir}} \) distribution with \( K = 6.0 \) and 2.6 respectively (for \( F = 0.01 \)). Note that the scatter in \( K \) is not symmetric about the median (of \( K = 4.0 \)); it rather reflects the log-normal nature of the \( c_{\text{vir}} \) distribution. The model predictions for the above values of \( K \) are shown as thin solid lines in Figure 2; they match the simulated scatter fairly well.

We show in § 7 that this parameterization also reproduces the simulated scatter as a function of \( z \). This is just a useful parameterization of the scatter using the toy model. A more detailed modelling of the spread with deeper physical insight is beyond the scope of the present paper.

### 6 Rotation curves and Tully-Fisher

The simulated distributions of \( c_{\text{vir}} \) values as a function of mass and environment have several observational implications. Here, we discuss only preliminary predictions involving rotation curves and the Tully-Fisher relation based on very crude assumptions about associating disc properties to those of the simulated dark-matter haloes. A more detailed study requires realistic modelling of physical processes involving gas and stars.

In order to illustrate what the spread in \( c_{\text{vir}} \) values may imply observationally, Figure 6 shows example NFW rotation curves for \( 3 \times 10^{11} h^{-1} M_{\odot} \) (distinct) haloes using the median, \( \pm 1 \sigma \), and \( \pm 2 \sigma \) values of \( c_{\text{vir}} \). These are raw rotation curves of the dark-matter haloes before they are affected by the infall of baryons, but they may still serve as a crude approximation for the final rotation curves. One can see that the rotation curves span a significant range of shapes and the corresponding spread in \( V_{\text{max}} \) values is substantial. The median rotation curve for \( 3 \times 10^{11} h^{-1} M_{\odot} \) subhaloes (not shown) is similar to the upper 1σ curve shown in Figure 6.
A clue for the expected TF relation of discs may be provided by the measured relation between the halo parameters \(M_{\text{vir}}\) and \(V_{\text{max}}\). The latter is derived from \(c_{\text{vir}}\) using Eq. 6. The \(M_{\text{vir}}-V_{\text{max}}\) relation is shown in Figure 7, separately for distinct haloes and subhaloes. The median distinct-halo relation is well approximated by the linear relation

\[
\log\left(\frac{M_{\text{vir}}}{h^{-1} M_\odot}\right) = \alpha \log\left(\frac{V_{\text{max}}}{\text{km/s}}\right) + \beta
\]  

with \(\alpha \simeq 3.4 \pm 0.05\) and \(\beta \simeq 4.3 \pm 0.2\), where the Poisson errors in each mass bin have been propagated to obtain the quoted error on each fit value. Note that the slope is steeper than that expected from the standard scaling of the virial parameters: \(M_{\text{vir}} \propto V_{\text{vir}}^3\). This is a direct result of the correlation between mass and concentration (\(M_{\text{vir}} \propto V_{\text{vir}}^{-0.13}\)). We may in fact derive the expected \(\alpha\) using the effective power law from Eq.6: \(V_{\text{max}}/V_{\text{vir}} \propto c_{\text{vir}}^{27}\), which implies \(M_{\text{vir}} \propto V_{\text{max}}^3 (V_{\text{vir}}/V_{\text{max}})^3 \propto V_{\text{max}}^{3+0.91} \propto V_{\text{max}}^{3.4}\). We point out that the linear relation provides a good fit, showing no obvious need for non-linear corrections in the TF relation.

The lower panel in Figure 7 shows \(M_{\text{vir}}\) vs. \(V_{\text{max}}\) for subhaloes. This relation is also well fit by the linear relation, Eq. 18, but now with \(\alpha \simeq 3.9 \pm 0.25\) and \(\beta \simeq 2.6 \pm 0.75\). The subhalo relation has a steeper slope compared to distinct haloes, and \(M_{\text{vir}} \simeq 10^{12} h^{-1} M_\odot\) subhaloes typically have a \(\sim 12\%\) higher \(V_{\text{max}}\). This difference between the slope and zero-point of distinct haloes and subhaloes may have implications for the use of cluster or group galaxies to calibrate the Tully-Fisher relation in the field.

We point out, however, that if for subhaloes we replace \(M_{\text{vir}}\) by the mass \(M_1\) inside the truncation radius \(R_t\), the logarithmic slope becomes \(\alpha_1 = 3.6 \pm 0.2\), consistent with the slope obtained by Avila-Reese et al. (1999) for haloes within clusters using a similar mass assignment procedure. The reason for the slope change is that the ratio of \(M_1/M_{\text{vir}}\) is roughly 1 for low-mass, high-\(c_{\text{vir}}\) haloes, and becomes less than 1 for high-mass, low-\(c_{\text{vir}}\) haloes. This slope is similar to the slope we find for distinct haloes. It is not obvious a priori which of the halo masses is more relevant to the mass of the cooled gas that ends up as the luminous disc, and thus to the observed Tully-Fisher relation. Therefore, the worry about the universality of the slope of Tully-Fisher is not conclusive. However, the zero-point difference between the two types of haloes exists regardless of the mass choice, and is a robust result.

The scatter in the TF relation is an issue of great interest. We find for distinct haloes a 1σ scatter at fixed \(V_{\text{max}}\) of \(\Delta (\log M_{\text{vir}}) \simeq 0.17\), while the corresponding scatter for fixed \(M_{\text{vir}}\) is \(\Delta V_{\text{max}}/V_{\text{max}} \simeq 0.12\). This scatter is in rough agreement with the spread predicted by Eisenstein & Loeb (1996) for a similar cosmology using Monte Carlo realizations of halo formation histories based on the Press-Schechter approximation. The subhalo relation shows an even larger scatter, with \(\Delta V_{\text{max}}/V_{\text{max}} \simeq 0.16\) at a fixed \(M_{\text{vir}}\).

Observational estimates for the intrinsic scatter in (I-band) TF range from \(\sigma(V)/V \sim 0.09\) (?) to \(\sim 0.03\) (?). At best, the observed scatter leaves no room for any intrinsic variation in the mass-to-light ratio of galaxies, and may imply that gas contraction and other hydrodynamical processes must somehow act to decrease the scatter. A simple idea that may resolve this discrepancy is discussed in Appendix C and we briefly outline the argument here. For a fixed mass and spin, a more concentrated halo (higher \(V_{\text{max}}\)) will induce more gas contraction, and therefore produce a smaller, brighter disc. Such a correlation between the mass-to-light ratio of galaxies and the deviation of \(V_{\text{max}}\) from the median \(V_{\text{max}}\) at a given \(M_{\text{vir}}\) could reduce the scatter to that required to match observations. Detailed modelling, including the back-reaction of the halo during disc formation, is needed to test this hypothesis in detail.
7 REDSHIFT DEPENDENCE

As data accumulate at high redshift, it becomes increasingly important to study the predicted evolution of the population of halo profiles as a function of redshift.

Figure 8 shows the median $c_{\text{vir}}$ as a function of $M_{\text{vir}}$ for distinct haloes at several different redshifts. We see that for a fixed mass, the typical $c_{\text{vir}}$ value changes quite dramatically, while the shape of the mass dependence remains roughly constant. The thin solid lines show our toy model predictions. This two-parameter model, which has been normalized to match the slope and normalization of the relation at $z = 0$, does remarkably well at all redshifts. As predicted by the toy model in §3, the concentration of haloes of a fixed mass scales as $c_{\text{vir}} \propto (1 + z)^{-1}$. A similar behavior has been confirmed using the SCDM simulation in a smaller box ($7.5h^{-1}\text{Mpc}$, described in §4). The redshift dependence of the subhalo concentrations seems similar, but we don’t have sufficient statistics for conclusive results involving subhaloes at high redshifts.

As mentioned in the Introduction, the dramatic evolution in the concentration of haloes of a fixed mass is different from the prediction of the NFW97 analytic toy model (see Appendix A). This is illustrated in Figure 9, which shows the median $c_{\text{vir}}$ of the distinct halo population of $M_{\text{vir}} = (0.5 - 1.0) \times 10^{12}h^{-1}M_\odot$ as a function of redshift. The NFW prediction (for $0.8 \times 10^{12}h^{-1}M_\odot$ haloes) overestimates $c_{\text{vir}}$ by $\sim 50\%$ at $z = 1$, and the disagreement grows with redshift. Our revised toy model reproduces the simulated redshift trend very well. The scatter about the relation is remarkably constant as a function of $z$: $\Delta (\log c_{\text{vir}}) \sim 0.18$. Also shown is how the spread can be parameterized by varying $K$ in our toy model, as discussed in §5.

The redshift dependence of the inner radius, $r_s$, can be deduced from that of $c_{\text{vir}}$ by recalling that the virial radius of fixed-mass haloes also varies like $R_{\text{vir}} \propto \Delta_{\text{vir}}^{1/3}/(1 + z)$. This implies that, on average, the inner radius of haloes of a given mass remains roughly constant as a function of redshift (aside from the $z$ dependence of $\Delta_{\text{vir}}$). We see this explicitly in Figure 10, which shows the evolution of the median and $68\%$ scatter of $r_s$ as a function of $z$ for distinct haloes in the mass range $0.5 - 1.0 \times 10^{12}h^{-1}M_\odot$. The fact that the median $r_s$ value declines slightly near $z = 0$ is due to the $z$ dependence of $\Delta_{\text{vir}}$ in the ΛCDM model simulated. The robustness of the characteristic length scale, $r_s$, may provide an interesting clue for the understanding of the build-up of DM halo structure.

The strong decline in the concentration of haloes of a fixed mass as a function of redshift should have an interesting impact on galaxy-formation modelling at high redshift — e.g., aimed at understanding the nature of Lyman Break Galaxies (?, ?) and the evolution of the Tully-Fisher relation (?). Although, in general, haloes, and therefore galaxies, are expected to be smaller at high redshift (reflecting the higher universal density) and to have higher circular velocities ($V_{\text{vir}} \propto R_{\text{vir}}^{1/2}$), the observed $c_{\text{vir}}$ behavior will tend to counteract this tendency.

Insight into the expected TF evolution of discs may be gained from Figure 11, which shows the $M_{\text{vir}}$ versus $V_{\text{max}}$ relation for (distinct) haloes at several redshift steps. The evolution in the zero-point is indeed less dramatic than would be expected from the scaling of $V_{\text{vir}}$. In fact (not shown) there is almost no evolution in the zero-point between $z = 0 - 0.5$. The slope of the relation is roughly constant as a function of redshift ($\alpha = 3.4 \pm 0.1$) and the scatter is roughly constant; $\Delta V_{\text{max}}/V_{\text{max}} \simeq 0.12$.

Furthermore, because disc size is expected to be a decreasing function of halo concentration (§8), the decline of $c_{\text{vir}}$ with $z$ implies a relative increase in disc sizes at high redshift. This should result in lower than expected surface brightnesses at high $z$, both because of the extended size...
and the corresponding lower efficiency of star formation. To this one could add the fact that the supply of cold gas for disc formation at high redshifts may be limited (not extending all the way to \(R_{\text{vir}}\)) because the smaller inner densities will lessen collisional cooling. These results may hinder the association of quiescently star-forming objects with Lyman-break galaxies as discussed e.g., by Mo, Mao, & White (1999) (see ?; ? for an alternative physical model for Lyman-break galaxies). They further argue for the slow evolution of the Tully-Fisher relation.

8 CONCLUSIONS AND DISCUSSION

The main direct conclusions of this paper, based on analyzing a statistical sample of dark-matter haloes in a high-resolution simulation of the ΛCDM cosmology, are as follows:

- The redshift dependence of the halo profile parameters has been measured in the simulations, and reproduced by an improved toy model. For example, \(c_{\text{vir}} \propto (1 + z)^{-1}\) for haloes of the same mass, predicting that at high redshift they are less concentrated and with larger inner radii than previously expected. The corresponding prediction for rotation curves is lower values of \(V_{\text{max}}/V_{\text{vir}}\) at high \(z\).

- The correlation between any two halo profile parameters has a significant scatter. For example, in the \(c_{\text{vir}}-M_{\text{vir}}\) relation, the spread in \(c_{\text{vir}}\) is comparable to the systematic change in \(c_{\text{vir}}\) across three orders of magnitude in \(M_{\text{vir}}\). The 1σ spread for fixed \(M_{\text{vir}}\) is \(\Delta(\log c_{\text{vir}}) \simeq 0.18\), corresponding to \(\Delta V_{\text{max}}/V_{\text{max}} \simeq 0.12\) at a given \(M_{\text{vir}}\).

- There are indications for environmental trends in halo profiles. Haloes in dense environments, or subhaloes, are more concentrated than their isolated counterparts of the same virial mass, and they exhibit a larger scatter in \(c_{\text{vir}}\).

The main implications of the above findings can be summarized as follows:

- Disc galaxies at high redshifts are predicted to be more extended and of lower surface brightness than expected previously. The constant inner radius at fixed mass may be a dynamical clue for understanding the formation of halo structure.

- The scatter in the halo mass-velocity relation is significantly larger than in the observed TF relation, which suggests that the luminosity of a disc forming inside a halo of a given mass should correlate with the maximum rotation velocity. We pointed out a possible simple explanation for that.

- The environmental trends of halo profiles may caution against the universality of the TF relation. In addition, these trends, together with the observed scatter, may provide insight into the origin of the Hubble sequence. Below, we argue that haloes of low concentration will tend to host blue galaxies and haloes of high concentration, red galaxies or spheroids. We also point out that extremely low \(c_{\text{vir}}\) haloes plausibly host LSB galaxies.

We have proposed an alternative to the toy model originally proposed by NFW97. It reproduces the correlations between the two parameters of the halo profiles, e.g., \(c_{\text{vir}}\) and \(M_{\text{vir}}\), as well as the redshift dependence of these correlations. This model also offers a simple parameterization that reproduces the scatter about the median relation observed in our simulation. The modified toy model is a useful tool for semi-analytic modelling of galaxy formation. In particular, analyses of the type performed by ?, which made predictions for disc properties at \(z \sim 3\), based on the halo toy model of NFW97, should be reconsidered using our modified toy model.

The large intrinsic scatter we find in the correlations between the halo profile parameters makes the haloes a two-parameter family, as expected, and should be taken into ac-
profiles of dark haloes

In order to gain a qualitative understanding of how \( c_{\text{vir}} \) may affect galactic disc formation, we may assume that the exponential disc forms by the adiabatic contraction of gas inside the dark-matter halo (\(?; \?, \?, \text{?}\)). The final disc size, \( r_d \), can be derived from the halo parameters \( c_{\text{vir}} \) and \( R_{\text{vir}} \), under the following further assumptions that (a) the disc forms from cold gas of mass \( \sim 0.03M_{\odot} \) which follows the original density profile of the halo out to \( R_{\text{vir}} \), and (b) the specific angular momentum of the gas is equal to that of the halo, which has an original spin parameter of \( \lambda = 0.035 \). Under these assumptions, we find the following fitting formula is good to within 1\% for \( 1 < c_{\text{vir}} < 50 \) (\(?\)):

\[
r_d \simeq 5.7 \, h^{-1}\text{kpc} \left( \frac{R_{\text{vir}}}{100 \, h^{-1}\text{kpc}} \right) [1 + (c_{\text{vir}}/3.73)^{0.71}]^{-1}.
\]

(A similar fitting formula, which allows more varied assumptions about the halo and disc make-up, is provided by \?). The general result is that \( r_d \), and thus the disc surface brightness, is a decreasing function of \( c_{\text{vir}} \).

The spread in exponential disc sizes implied from our 68\% spread in concentration values for an \( R_{\text{vir}} = 200 \, h^{-1}\text{kpc} \) halo at \( z = 0 \) (\( M_{\text{vir}} \sim 10^{12}h^{-1}M_{\odot} \), \( c_{\text{vir}} = 9.2 \rightarrow 21.3 \)) is \( r_d : 4.0 \rightarrow 2.6 \, h^{-1}\text{kpc} \). This is roughly the same spread in disc sizes resulting from a spin parameter variation of \( \lambda : 0.05 \rightarrow 0.03 \), which is approximately the intrinsic spread in \( c_{\text{vir}} \) inferred from N-body simulations (\(?; \?, \text{?}\)).

The two quantities, \( c_{\text{vir}} \) and \( \lambda \), are thus of comparable importance for determining observable properties of galaxies, and define a plane in parameter space for haloes of fixed mass. We suggest that the \( c_{\text{vir}}-\lambda \) plane can perhaps be linked to the observed variations of galaxies and help explain the Hubble sequence. For example, we argued above that haloes with very low concentrations would tend to lead to discs of low surface brightness, and with slowly rising rotation curves (i.e., \( V_{\text{max}}/V_{\text{circ}} \sim 1 \)). This argument will only apply, however, if \( \lambda \) is sufficiently large to prevent extensive gas infall. Similarly, haloes of high-\( c_{\text{vir}} \) and low-\( \lambda \) will likely be unable to produce dynamically stable discs (see, e.g., \( \text{Mo et al.} 1998 \)), and instead host spheroids. Other combinations of these parameters, may, perhaps, lead naturally to a range of galaxy morphologies.

This kind of mapping is further motivated by the understanding that high-\( c_{\text{vir}} \) haloes collapse earlier than low-\( c_{\text{vir}} \) haloes (as predicted by the toy model and explicitly demonstrated by NFW97 using simulated haloes). A natural association is then that high-\( c_{\text{vir}} \) haloes host old, red galaxies, and lower-\( c_{\text{vir}} \) haloes host young, blue galaxies. Furthermore, the environmental trend, that haloes in low-density environments tend to be less concentrated than haloes of the same mass in high-density environments, fits nicely into this picture. Indeed, LSB galaxies are observed to be more isolated than galaxies of higher surface brightness (\(?; \?, \text{?}\)), and spheroids tend to inhabit high-density environments (\(?; \?, \text{?}\)).

We have made in this paper only crude preliminary attempts to study the implications of our results concerning halo profiles. It would be desirable to incorporate our measured halo properties into semi-analytical modelling of gas processes and star formation in order to make detailed predictions for observable galaxy properties and their evolution.

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APPENDIX A: THE NFW97 MODEL

For completeness, we briefly review the NFW prescription for determining \( c_{\text{vir}}(M, \lambda) \). The goal of the NFW procedure is to provide the density profile of a halo of mass \( M_{\text{vir}} \) at epoch \( a_0 \) assuming that the profile is of the NFW form (1). The collapse redshift is now determined using the Press-Schechter approximation, which, given \( M_{\text{vir}} \) at epoch \( a_0 \), can be used to approximate the probability distribution for the epoch \( a \) when a halo trajectory was first more massive than some fraction \( f \) of \( M_{\text{vir}} \) (Lacey & Cole 93)

\[
P(> f M_{\text{vir}}, a|M_{\text{vir}}, a_0) = \frac{\delta_{\text{crit}}(a) - \delta_{\text{crit}}(1)}{\sqrt{2(\sigma^2_{\text{a}}(f M_{\text{vir}}) - \sigma^2_{\text{a}}(M_{\text{vir}}))}}.
\]

The quantities \( \sigma_0 \) and \( \delta_{\text{crit}} \) are defined in \( \S \) 3. One determines \( a_c \), by solving Eq. A1 for the most probable value of \( a \) by setting \( P(> f M, a|M, a_0) = 0.5 \). One now assumes that the central density of the halo is proportional to the density of the universe at \( a_c \), which implies:

\[
\rho_s = k \rho_c(a_0) \left( \frac{a_c}{a_0} \right)^3,
\]

where \( k \) is a numerical constant. Now, given \( M_{\text{vir}} \) and \( a_0 \), Eqs. A1 and A2 determine \( \rho_s \) and thus completely specify the density profile.

This procedure has two free parameters, \( f \) and \( k \), which may be adjusted in order to match the slope and normalization respectively of \( c_{\text{vir}}(M) \) at \( a = 1 \). NFW97 show that this two parameter model reproduces the \( a = 1 \) relations of simulated haloes in various cosmologies, including power law and open models. For the \( \Lambda \)CDM model we discuss, their favorite parameters are \( f = 0.01 \) and \( k = 3.4 \times 10^3 \), and these provide a reasonable reproduction of the median \( c_{\text{vir}}(M) \) relationship at \( a = 1 \) in our simulations.

Although useful in its ability to provide the correct \( z = 0 \) relation, this model fails to reproduce the observed \( z \) evolution of halo concentrations (\( \S \) 7). In \( \S \) 3, we present a revised toy model which, using the same number of free parameters, reproduces the full observed behavior of \( c_{\text{vir}}(M, z) \).

APPENDIX B: THE HALO FINDER

Most commonly used halo finders, which work either by the location of overdensities in a spatial window of fixed shape (usually spherical) or by friends-of-friends algorithms, do not
account for haloes within haloes. Since our projects specifically address the question of substructure, we have been obliged to devise a halo finder and classification algorithm suited for this purpose.

If one were only interested in distinct, virialized objects, haloes would be easily identified — there is little confusion as to where one halo ends and another begins because the physical extent of an object is determined by the virial overdensity criterion. However, we are interested in objects within the virial radius of large haloes, therefore certain ambiguities arise. How close must two density maxima be in order for them to represent a single object? How does one differentiate substructure from a collision in progress? How does one assign mass to haloes and subhaloes appropriately?

We have found a solution to these problems by assigning to each halo two length scales — an inner radius, \( r_s \), and an outer radius \( R_{v_{ir}} \). We do so by modelling the density profile of each halo using Eq. 1. The virial radius \( R_{v_{ir}} \) determines each halo mass and radial extent, and \( r_s \) determines when two density maxima/haloes should be combined into one. The details are described below.

Because the modelling process requires fitting a density profile to each halo, we attempt to find only haloes with more than \( N_p^{\text{min}} \) particles within their virial radii. If \( m_p \) is the mass of each particle, this means the minimal virial mass of haloes identified is \( M_p^{\text{min}} = N_p^{\text{min}} \times m_p \). Equivalently, using Eq. 2, we have a minimum virial radius \( R_{p_{ir}}^{\text{min}} \). The value of \( N_p^{\text{min}} \) is the first free parameter of this algorithm. We use \( N_p^{\text{min}} = 50 \).

Our density maxima finding routine is based on a Cloud-in-Cell (CIC) process (i) on the largest grid of the simulation \( \Delta L \), and rank the particles according to their local density as determined on this grid.

We then search for the possible halo centres, using two sets of smoothing spheres; one, with a small radius, \( r_{s1} \), in order to locate of tight, small clumps; and the other, with a larger radius, \( r_{s2} \), in order to locate the centre of haloes with diffuse cores. The smaller radius is \( r_{s1} = \alpha R_{\text{enc}} \), where \( R_{\text{enc}} \) is the highest force resolution in the simulation and \( \alpha \) is a free parameter of order unity. We use \( \alpha = 2 \). The second set of spheres have \( r_{s2} = \frac{R_{p_{ir}}^{\text{min}}}{4\alpha} \).

For each set of spheres, we take from the ranked list the particle with the highest local density and place a sphere about its location. A second sphere is placed about the next particle with the highest local density and place a sphere containing fewer than \( N_{p_{sp2}}^{\text{min}} \) particles. The minimum number of particles required for a kept sphere is determined separately for each radius.

For the \( r_{s1} \) spheres, we use the following conservative halo density profile:

\[
\rho(r) = \begin{cases} 
C/r_{s1}^{1.5} & r < r_{s1} \\
C/r_{s2}^{1.5} & r > r_{s2} 
\end{cases}
\]  

(\( \text{where } C \text{ is determined my fixing the minimum halo mass to be } M_p^{\text{min}} \)) in order to estimate the minimum core number of particles within \( r_{s1} \):

\[
N_{p_{s1}} = \frac{M_p^{\text{min}}}{\frac{1}{6}\left(\frac{R_{p_{ir}}^{\text{min}}}{r_{s1}}\right)^{1/2}}. \tag{B2}
\]

For the \( z = 0 \) timestep of the 60\(^{-1}\)Mpc simulation we analyze, \( N_{p_{s1}} = 3.9 \rightarrow 3. \) Spheres of size \( r_{s2} \) with fewer than \( N_{p_{sp2}}^{\text{min}} \) particles are discarded. Similarly, all of the \( r_{s2} \) spheres containing fewer than \( N_{p_{sp2}} = N_p^{\text{min}} \) particles are discarded.

The final list of candidate halo centres is made up of all of the (small) \( r_{s1} \) spheres, together with each of the \( r_{s2} \) spheres that do not contain an \( r_{s1} \) sphere.

(ii) For each sphere of radius \( r_{sp} = r_{s1} \) or \( r_{s2} \), whichever applies, we use the particle distribution to find the centre of mass and iterate until convergence. We repeat the procedure using a smaller radius, \( r = r_i \), where \( r_i = r_{sp}/2^i \). We continue this method until \( r_i = r_L \), where \( r_L \) is defined by the criterion \( r_L > f_{\text{enc}} > r_{L+1} \), or until reduction leads to an empty sphere.

(iii) We unify the spheres whose centres are within \( r_L \) of each other. The unification is performed by making a density weighted guess for a common centre of mass, and then iterating to find a centre of mass for the unified object by counting particles. The size of sphere used to determine the centre of mass is the smallest radius that will allow the new sphere to entirely contain both candidate halo spheres.

(iv) For each candidate halo centre we step out in radial shells of \( 1 \) h\(^{-1}\)kpc, counting enclosed particles, in order to find the outer radius of the halo: \( R_t = \min(R_{v_{ir}}, R_t) \). The radius \( R_{v_{ir}} \) is the virial radius, and \( R_t \) is a “truncation” radius, defined as the radius (\( < R_{v_{ir}} \)) in which a rise in (spherical) density is detected (\( d \log \rho/d \log r > 0 \)). This is our method for estimating when a different halo starts to overlap with the current halo and is important for haloes in crowded regions. We estimate the significance of a measured upturn using the Poisson noise associated with the number of particles in the radial bins considered. Only if the signal to noise of the upturn is larger than \( \sigma_{R_t} \) do we define a truncation radius. The value of \( \sigma_{R_t} \) is a free parameter. We use \( \sigma_{R_t} = 1.5 \). **

(v) Among the halo candidates for which we have found an \( R_{v_{ir}} \), we discard those with \( N_{v_{ir}} < N_p^{\text{min}} \), where \( N_{v_{ir}} \) is the number of particles within \( R_{v_{ir}} \). Among the halo candidates for which we have found a rise in spherical density, we discard those which contain less than \( N_{R_t^{\text{min}}} \) particles, where \( N_{R_t^{\text{min}}} = N_p^{\text{min}} \) if \( R_t > R_{v_{ir}}^{\text{min}} \), otherwise

\[
N_{R_t^{\text{min}}} = N_p^{\text{min}} \left( \frac{R_t}{R_{v_{ir}}^{\text{min}}} \right)^{3/2}. \tag{B3}
\]

The above constraint follows from an extrapolation of the minimum mass halo using an isothermal profile \( \rho(r) \propto 1/r^2 \).

(vi) We model the density profile of each halo using the NFW form (Eq 1) and determine the best fit \( r_s \)

** The choice was motivated by several tests using mock catalogues of haloes in clusters designed to determine how varying \( \sigma_{R_t} \) affects our ability to fit the density profiles of subhaloes. Although our results were not strongly dependent on this choice, we did obtain the best fits using \( \sigma_{R_t} = 1.5 \).
and $\rho_c$ values, which determine $R_{\text{vir}}$ and $M_{\text{vir}}$. The fitting procedure uses logarithmically spaced radial bins from $\max(2f_{\text{esc}}, 0.02 \times \min(R_{\text{vir}}, R_c))$ out to $R_c$. If any bins are empty we decrease the number of bins by one until all bins are full. If the number of bins is reduced below 3 we discard the halo as a local perturbation.

The fit takes into account the Poisson error in each bin due to the finite number of particles, and we obtain errors on the fit parameters ($\sigma_{\text{fit}}$ and $\sigma_{\nu}$) using the covariance matrix in the fit routine. The errors on the fit parameters can be translated easily into errors for $R_{\text{vir}}$ ($\sigma_{R_{\text{vir}}}$) and the estimated NFW mass of each halo, $M_{\text{vir}}$ ($\sigma_{M_{\text{vir}}}$).

(vii) We unify haloes which overlap in $R_c$. Our criterion is met if two (or more) halo centres have $R_c$ radii which overlap with each other while at the same time having velocities which allow them to be bound to the common system. If such a case occurs, then along with the individual halo NFW fits, we fit another NFW profile about the common centre of mass of the two combined haloes and decide whether the candidate-united-haloes are bound/unbound to the common NFW fit using the radial escape velocity determined using the common NFW profile (see below). If both haloes are bound we combine the two haloes into one, and keep the common fit for the characteristic parameters. If at least one is not bound, we do not combine the haloes.

An exception to this unifying criterion occurs if the fit errors on $R_c$ are large ($\sigma_{R_c}/R_c > 1$), we replace $R_c \rightarrow \min(R_c, R_c)$. In addition if the $R_c$ of a halo obeys $R_c > R_{\text{vir}}^{\text{min}}$ then we relax the strict combining of overlapping haloes. This case, which we refer to as the "cD" halo case, is discussed below (see (ix).)

(viii) For each halo, we remove all unbound particles before we obtain the final fits. We loop over all particles within the halo and declare a particle at a distance $r$ from the centre of a halo to be unbound if its velocity relative to the centre of mass velocity of the halo obeys $v > \sqrt{2\Phi_{\text{NFW}}(r)}$, where the radial potential for NFW density profile is given by

$$
\Phi_{\text{NFW}}(r) = -4\pi G \rho_c R_c^2 \left[ \log(1 + x) \right]/x.
$$

After removal, we construct a new density profile and find new NFW fit parameters. The procedure is repeated until the number of unbound particles becomes < 1% of the bound particles or until $M_{\text{vir}}^{\text{NFW}} < M_{\text{vir}}^{\text{min}}$ in which case the halo is discarded.

An exception to this removal scheme occurs if two haloes lie within the virial radius of each other and the ratio of their masses is at least 0.75. We define haloes in this situation to be a "partner" pair. For each halo in this situation, we take not only its potential into account, but also that of its partner.

(ix) An interesting case of subhalo structure, which would otherwise be excluded from our finding algorithms, is that of one or more density peaks close to the centre of a large halo. We shall refer to these inner density peaks as cD cases. If a halo, after its unbound particles have been removed, obeys the following criteria, it is a candidate for containing cD haloes: a) the NFW fit has a standard GoF < 0.1, b) the halo is a host of at least one subhalo, and c) the halo is "large", with $r_s > R_{\text{vir}}^{\min}$.

We identify the potential centres of cD haloes by searching the bound particle distribution within $r_s$ of each candidate cD halo using a CIC process on a fine grid ($r_{\text{grid}} = \alpha r_{\text{esc}}$). We discard all candidate density peaks with local densities less than the extrapolated minimum density (above the background density) within the core region of our smallest halo (see item (i)).

For density maxima located farther than $r_{\text{grid}}$ from the centre of the candidate host halo, we find $R_s$ and fit a NFW profile with iterative unbound particle removal. These are our cD haloes. cD haloes are discarded if their extrapolated virial mass is lower than our minimal mass halo.

Because we have a strict mass limit $M_{\text{vir}}^{\text{max}} = N_{\text{vir}}^{\text{min}} \times m_p$, we expect our halo finder to be somewhat incomplete just above $M_{\text{vir}}^{\text{min}}$. We have also checked our completeness in two ways. First, we used a separate BDM halo finder that does not attempt to fit profiles and does not demand the unification of haloes within a specified radius. It does, however, unify haloes that have equal velocities within 15% as long as they have centres within $\sim 150$ h$^{-1}$kpc (see §). This procedure allows a complete identification of DM haloes down to much lower particle numbers than our own. In order to check our results we have assigned to each halo in the catalogue produced by the other finder a typical $r_s$ given its mass, and checked the returned halo list for consistency against our catalogue from the same simulation box. For $N_{\text{particles}} \sim 150$, we estimate $\sim 80\%$ completeness and for $N_{\text{particles}} \sim 500$ we obtain $\geq 95\%$ completeness. A second, and almost identical completeness determination is obtained by carefully analyzing the roll-over in our observed mass function (§). We attribute our incompleteness for small masses to our fitting procedure, and errors associated with this process.

### APPENDIX C: REDUCING TF SCATTER

As discussed in §6, we find for distinct haloes a 1σ scatter at fixed $M_{\text{vir}}$, of $\Delta V_{\text{max}}/V_{\text{max}} \sim 0.12$, which is between 1.3 and 4 times the range of the reported intrinsic TF scatter. If these ΛCDM haloes are to host galaxies like those observed, this excessive scatter must be reduced. The translation of the halo virial mass into a disc luminosity, and of the original halo $V_{\text{max}}$ into a final observed disc velocity, should somehow decrease the scatter. Following is a qualitative analysis of how this can come about in a natural way.

The idea is that for a fixed halo mass and spin, a higher $V_{\text{max}}$ should induce further gas contraction into smaller radii, and therefore higher gas density, star-formation rate and luminosity. This can be shown in a little more detail, as follows.

The size of the exponential disc, $r_\text{d}$, that forms by a dissipative contraction of gas inside a given dark-matter halo can be estimated under the adiabatic baryonic-infall approximation. We showed in §8 (see Equation 19) that $r_\text{d}$ is a decreasing function of $c_\text{vir}$ for haloes of a fixed virial mass and spin, as long as the disc mass is a constant fraction of $M_{\text{vir}}$. For a typical case of a $V_{\text{vir}} = 200$ km/s halo, we demonstrated that, in the range encompassing 68% of $c_\text{vir}$ for such haloes ($c_\text{vir}$, 9.2 → 21.3), the corresponding spread in disc...
sizes is $R_d : 4.0 \rightarrow 2.6 \, h^{-1}\text{kpc}$. Across this range, an effective power-law approximation would therefore be $r_d \propto c_{\text{vir}}^{-0.5}$. If (a) the gas density in the disc scales like $\rho \propto r_d^{-2}$, (b) the star formation rate obeys a typical Schmidt law, $\dot{\rho} \propto \rho^{1.5}$, and (c) the luminosity scales like $L \propto \dot{\rho} r_d^2$, then the luminosity at a given mass depends on $c_{\text{vir}}$ as $L \propto r_d^{-1} \propto c_{\text{vir}}^{0.5}$.

Since $V_{\text{max}}$ (for a fixed $V_{\text{vir}}$) is also a monotonic function of $c_{\text{vir}}$ (Eq. 6; the effective power-law approximation across the 68% range is $V_{\text{max}} \propto c_{\text{vir}}^{0.27}$), we have obtained a positive correlation between $L/M_{\text{vir}}$ and the deviation of $V_{\text{max}}$ from the median $V_{\text{max}}$ at a given $M_{\text{vir}}$. Ignoring, for the moment, any difference between the $V_{\text{max}}$ of the original halo and that of the disc, the obtained correlation would correspond to a reduced scatter in $L$ by a factor larger than 2, as required. The effect of the spread in spins on the TF scatter is expected to be reduced for similar reasons, namely, because the luminosity and the maximum velocity are both expected to correlate with the spin in the same sense.

More detailed modelling, which takes into account how $V_{\text{max}}$ changes as the baryons fall in, will be needed to test this hypothesis in detail.