On bound entanglement assisted distillation

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Abstract

We investigate asymptotic distillation of entanglement in the presence of an unlimited amount of bound entanglement for bi-partite systems. We show that the distillability is still bounded by the relative entropy of entanglement. This offers a strong support to the fact that bound entanglement does not improve distillation of entanglement.

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Recent years have witnessed an explosion of interest in understanding and quantifying the amount of entanglement in a given state of bi-partite and multi-partite quantum systems (for a review see [1]). Entanglement of bi-partite systems in a pure state is well understood asymptotically; it can be quantified by a single measure - von Neumann’s reduced entropy [2], [3], [4]. In the finite (non-asymptotic) case [5], however, there are still many surprises. For example, one of these surprises is the existence of catalytic processes recently found by Jonathan and Plenio [6]. Multi-partite systems are, on the other hand, not well understood

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at all; even the bi-partite mixed states display a number of counter-intuitive properties such as the existence of bound entanglement [7]. Also, it is not known with what maximal efficiency can entanglement be distilled from a general bi-partite mixed state. Present work lies in the domain of asymptotic distillation of bi-partite quantum systems and investigates the usefulness of bound entanglement in this process. It is known that there are three different types of entanglement for bi-partite systems [7] and they are: 1. free entangled states (FE), i.e. states from which pure singlets can be distilled by local operations aided with classical communication (LOCC); 2. bound entangled states (BE), which are inseparable, but cannot be distilled by LOCC and 3. separable states, which contain no entanglement and consequently cannot be distilled either. There are several bounds for the yield of distillation (discussed below), the number which is sometimes also called the entanglement of distillation [3]. In this paper we consider the following problem (first posed by the family Horodecki in [7]). Suppose that we have an unlimited supply of bound entanglement in addition to LOCC. Can this help in distilling more singlets then we could have without the presence of bound entanglement? We provide an upper bound to the bound-entanglement assisted distillation which strongly suggests that the answer is no. If, in addition, the entanglement of formation for BE states was strictly non-zero (which would be true if the entanglement of formation was additive [3], but otherwise is not known), this would then make them more than useless: they would require a certain amount of entanglement to be created by local means even asymptotically, but would not be able to enhance distillation in any way.

A distillation procedure involves two parties, Alice and Bob, who share a certain number of bi-partite systems in some quantum state. Their task is to convert this into as large a number of singlets (or any other maximally entangled state of two qubits) as possible, by acting only locally, i.e. on their own systems, and communicating classically results of their actions to each other. This is a very important protocol since we know that maximally entangled pairs achieve a higher fidelity of quantum information transmission and processing in general. Quantum relative entropy (defined below) is a quantity that is non-increasing
under a general quantum evolution [8] and in particular under LOCC. It can therefore help
us to produce an upper bound on the amount of distillable entanglement [9]. We will show
that the number of singlets (or any other maximally entangled two qubit states) that can be
distilled from \( n \) copies of \( \sigma \) is bounded from the above by
\[
\min_{\omega \in \mathcal{B}} S(\sigma^\otimes n || \omega),
\]
where \( \omega \) is of the same dimensionality as \( \sigma^\otimes n \), \( S(\sigma || \omega) := tr \{ \sigma \log \sigma - \sigma \log \omega \} \) is the quantum relative entropy
between \( \sigma \) and \( \omega \) and \( \mathcal{B} \) is the set of BE and separable (disentangled) states together. In
other words, we will show that the entanglement of distillation
\[
D(\sigma) \leq \frac{1}{n} \min_{\omega \in \mathcal{B}} S(\sigma^\otimes n || \omega).
\]
This then provides an upper bound on distillation, which is achievable in distillation of pure
states [2] and Bell-diagonal states with only two non-zero eigenvalues [3]. A similar bound
and this general method of providing a bound was first proposed by Vedral et. al. [4] and
Vedral and Plenio [9], who performed the minimisation over separable states. This was
then recently improved by Rains [10], [11] by introducing a minimisation over BE states
in addition to separable states and enlarging the set of operations that Alice and Bob can
perform. Our main result will then be to show that this bound cannot be improved by
adding any amount of bound entanglement to aid the distillation.

Before we present the main result we explain our notation. The symbol \( \psi^\otimes n \) means \( n \) copies
of the state \( \psi \). On the other hand, \( \omega_m \) means that \( \omega \) lives in the space of \( m \) bi-partite
systems, but is not necessarily of the product nature. The dimensionality of these systems
will always be clear from the problem we are considering. In this case the subscript will be
omitted when there is no chance of confusion. \( \Lambda \) will denote a LOCC. Greek symbols \( \omega, \beta \)
and \( \gamma \) will denote bound entangled states. \( \psi \) always denotes a maximally entangled state
of two qubits, and \( \sigma \) will be the state from which we wish to distill some entanglement and
could be free entangled, bound entangled or separable (this is a bi-partite system of any
dimensionality). \( D \) will stand for the (unassisted) entanglement of distillation and \( D^{BE} \) for
the bound-entanglement assisted entanglement of distillation, where we allow an unlimited
amount of bound entanglement.
Lemma 1. If $\psi$ is any maximally entangled state of two qubits, then $\min_{\omega \in B} S(\psi \otimes n || \omega) = n$. (The state $\omega$ lives in the Hilbert space of dimension $2^n \times 2^n$).

Proof: By definition of the relative entropy we have that $\min_{\omega \in B} S(\psi \otimes n || \omega) = \min_{\omega \in B} -\langle \psi \otimes n | \log \omega | \psi \otimes n \rangle$. But, the logarithmic function is concave so that

$$\min_{\omega \in B} -\langle \psi \otimes n | \log \omega | \psi \otimes n \rangle \geq \min_{\omega \in B} -\log \langle \psi \otimes n | \omega | \psi \otimes n \rangle$$

However, according to the recent result of the family Horodecki [12], since $\omega$ is a bound entangled state, then its fidelity with the maximally entangled state cannot be larger than the inverse of the half dimension of that state, so that $\langle \psi \otimes n | \omega | \psi \otimes n \rangle \leq 1/2^n$. Thus,

$$\min_{\omega \in B} S(\psi \otimes n || \omega) \geq -\log(1/2^n) = n$$

But we know that this minimum is achievable by the (in fact separable) state $\omega = \rho \otimes n$, where $\rho$ is obtained from $\psi$ by removing the off-diagonal elements in the Schmidt basis. This therefore completes our proof $\Box$.

This lemma is very important since it tells us that the relative entropy can be used to produce an upper bound on distillable entanglement. Namely, if we are starting with $n$ copies of state $\sigma$, and obtaining $m$ copies of $\psi$ by LOCC, then

$$D = m/n = \frac{1}{n} \min_{\omega \in B} S(\psi \otimes m || \omega) \leq \frac{1}{n} \min_{\omega \in B} S(\sigma \otimes n || \omega)$$

where the equality follows from lemma 1 and the inequality from the fact that the relative entropy is non-increasing under LOCC (strictly speaking, $D = \lim_{n \to \infty} \frac{m}{n}$ and, of course, $m$ is a function of $n$, $m = m(n)$ [10]. We will omit limits for two reasons: 1. we wish to keep the notation as simple as possible and 2. all our results apply in the finite case as well). The above relies on the fact that the local operations acting on a bound entangled state cannot produce free entanglement (this follows trivially from the definition of bound entanglement). Note also that this achieves a tighter bound than previously proposed by Vedral and Plenio [9], since $\min_{\omega \in B} S(\sigma \otimes n || \omega)/n \leq \min_{\omega \in B} S(\sigma || \omega) := E_R(\sigma)$, where $E_R(\sigma)$
is the relative entropy of entanglement of \( \sigma \). As we have noted, this is closely related to the bound recently considered by Rains [11].

Before we prove the main result we state the assumption under which it is valid. We assume that if we have two BE states \( \beta_1 \) and \( \beta_2 \), then \( \beta_1 \otimes \beta_2 \) is also a BE state. This is true for all the BE states which have a positive partial transposition [7], and only these have been known to exist so far [13,14]. However, it could happen that a BE state has a negative partial transposition. In this case, a direct product of two such states might result in a state with free entanglement (this, of course, cannot be proven since we do not know whether a negative partial transposition BE states exist in the first place!). Furthermore, if this happens than it is even questionable whether these should be called BE states to start with. Thus we can safely assume that a direct product of two BE states is also a BE state.

We are now ready to state the main result of this paper:

**Theorem 1.** Suppose that the distillation of \( \sigma^{\otimes n} \) is aided by an unlimited amount of bound entanglement \( \beta^{\otimes k} \) (where \( k \) can be any number). Then the distillation yield is bounded from the above by \( \frac{1}{n} \min_{\omega \in B} S(\sigma^{\otimes n}||\omega) \).

**Proof:** The proof follows from a chain of inequalities:

\[
D^{BE} = \frac{m}{n} = \frac{1}{n} \min_{\omega \in B} S(\psi^{\otimes m}||\omega_m) \\
\leq \frac{1}{n} S(\psi^{\otimes m}||tr_{k+n-m}A(\omega_n \otimes \omega_k)) \\
\leq \frac{1}{n} S(\psi^{\otimes m} \otimes \gamma_{k+n-m}||A(\omega_n \otimes \omega_k)) \\
\leq \frac{1}{n} S(\sigma^{\otimes n} \otimes \beta^{\otimes k}||\omega_n \otimes \omega_k) \\
= \frac{1}{n} S(\sigma^{\otimes n}||\omega_n) + \frac{1}{n} S(\beta^{\otimes k}||\omega_k)
\]

The second equality follows from lemma. The second inequality follows from the fact that partial tracing does not increase the relative entropy. The third inequality exists because LOCC (in fact, any quantum operations) cannot increase the quantum relative entropy. The last equality follows from the definition of the quantum relative entropy.
But now we arrange to have that $\omega_k = \beta \otimes k$, so that the second term reduces to zero. Also, we choose $\omega_n$ such that $\min_{\omega \in B} S(\sigma \otimes^n \| \omega) = S(\sigma \otimes^n \| \omega_n)$. Thus we have that

$$D^{BE} \leq \frac{1}{n} \min_{\omega \in B} S(\sigma \otimes^n \| \omega)$$

and this completes our proof $\square$.

**Corollary.** The bound entanglement assisted entanglement of distillation of the state $\sigma$ is bounded from the above by the relative entropy of entanglement of $\sigma$.

**Proof:** This follows immediately from the fact that $\min_{\omega \in B} S(\sigma \otimes^n \| \omega_n) \leq n \min_{\omega \in B} S(\sigma \| \omega_1)$. In particular, this limit is also smaller than or equal to the entanglement of formation of the state $\sigma$ since the entanglement of formation is always greater than or equal to the relative entropy of entanglement [9].

Note that the above result is more general in that it also applies if the assistance BE states are not all the same providing that they have a positive partial transposition. This is because of the special property of partial transposition,

$$(\beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_n)^T_2 = \beta_1^T_2 \otimes \beta_2^T_2 \otimes \ldots \otimes \beta_n^T_2$$

where betas are all bi-partite states and $T_2$ is a partial transposition over the second subsystem (the same is true if we partially transpose over the first subsystem). However, if the BE states can have a negative partial transposition, then the above might not in general apply as we discussed before. The reason is that by ”adding together” two of the BE states with negative partial transposition, $\beta_1 \otimes \beta_2$, we might produce a free entanglement state. So, if we work under the assumption that a direct product of two BE states is a BE state, then, the above result applies even when the distillation is aided by different BE states.

Can the above discussed bound be improved? Suppose that $\Lambda$ is the LOCC which distills $\sigma \otimes^n$ (for $n \rightarrow \infty$). Suppose also that we identify a set of states $\mathcal{G}$, such that $S(\psi \otimes^m \| \omega_m) \geq m$ for all $\omega_m \in \mathcal{G}$ and $\Lambda(\omega_m) \in \mathcal{G}$. Then

$$\lim_{n \rightarrow \infty} \frac{\min_{\omega_m \in \mathcal{G}} S(\sigma \otimes^n \| \omega_n)}{n}$$
is the upper bound on distillation. Of course, this might well be smaller than the previous bound which was taking minimum over BE states only. However, this kind of bound is not very useful since it is particularly tailored for $\Lambda$. Once $\Lambda$ is known to distill $\sigma^{\otimes n}$, then we can also apply it directly and see what the efficiency is. This implies that if we are to keep LOCC as general as possible, then the above bound can only be improved if we find a different distance measure, $D$, to quantum relative entropy with the following properties:

1. $D(\Lambda(\sigma)||\Lambda(\rho)) \leq D(\sigma||\rho)$ for all $\Lambda$ which are LOCC and all $\sigma$ and $\rho$; 
2. the minimum of $D$ over BE states for a maximally entangled pure state of two systems of dimension $n$ is $\log n$, just like in the case of the quantum relative entropy. The question of finding such a measure, if one should exist, remains an open problem.

Our result strongly suggests that bound entanglement cannot enhance distillation in any way. In fact, Alice and Bob need to invest some entanglement in order to create a BE state (although this might become arbitrarily small asymptotically), which than is not very useful in distillation as the above result suggests. The above, however, does not exactly prove this: it just shows that the upper bound on distillation without the presence of BE is the same as when we have an arbitrary number of BE systems present. So the above shows us that $D \leq D^{BE} \leq \min_{\omega \in B} S(\psi^{\otimes n}||\omega)/n$, but we could still have that $D < D^{BE}$. To strictly prove the equality we would have to show that the above upper bound to distillation can be achieved, i.e. that $D = \min_{\omega \in B} S(\psi^{\otimes n}||\omega)/n$. We leave this task for future research.

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