Optimal ensemble length of mixed separable states

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Abstract

The optimal (pure state) ensemble length of a separable state, A, is the minimum number of (pure) product states needed in convex combination to construct A. We study the set of all separable states with optimal (pure state) ensemble length equal to \( k \) or fewer. Lower bounds on \( k \) are found below which these sets have measure 0 in the set of separable states. In the bipartite case and the multiparticle case where one of the particles has significantly more quantum numbers than the rest, the lower bounds are sharp. A consequence of our results is that for all two particle systems, except possibly those with a qubit and those with a nine dimensional Hilbert space, and for all systems with more than two particles the optimal pure state ensemble length for a randomly picked separable state is with probability 1 greater than the state's rank. In bipartite systems with each particle having the same Hilbert space, with probability 1 it is greater than \( 1/4 \) the rank raised to the 3/2 power and in a system with \( p \) qubits with probability 1 it is greater than \( 2^p/(1 + 2p) \), which is almost the maximal rank squared.

1 Introduction

One of the important mathematical problems in quantum information theory is the characterization of separable states. In the case of pure separable states, much progress has been made. For instance, if one considers a quantum system of \( p \) particles with state space \( H = \bigotimes_{j=1}^{p} \mathbb{C}^{n_j} \), then the pure states are rays in \( H \). Mathematically, this is the complex projective space \( \mathbb{CP}(N-1) \), which is a real manifold of dimension \( 2N - 2 \), where \( N = n_1 \cdots n_p \). The separable pure states are product pure states and so correspond to a submanifold isomorphic to the Cartesian product, \( \mathbb{CP}(n_1 - 1) \times \cdots \times \mathbb{CP}(n_p - 1) \), which has real dimension \( \sum_{j=1}^{p} (2n_j - 2) \). Thus the set of separable pure states is a measure 0, closed,
non-dense subset of the set of pure states. In particular if one randomly picks a pure state in $H$, the probability it is entangled (i.e. not separable) is one. Moreover, every entangled state has an open set of entangled states around it.

The situation for separable mixed states is quite different. To see why, first recall that mixed states are described in terms of density matrices. These are $N \times N$, complex, positive semi-definite, Hermitian matrices with trace equal to 1. If $N = n_1 \cdots n_p$, then the separable density matrices are those which are convex combinations of product matrices, where by product matrix we mean one of the form $A = A_1 \otimes \cdots \otimes A_p$. Unlike the pure state case, the set of separable density matrices, $\Sigma (n_1, \ldots, n_p)$, is not of measure 0 in the set of all density matrices, $DM(N)$ - it is not negligible. In fact the vector space of $N \times N$ Hermitian matrices has bases which consist solely of product density matrices. This means $\Sigma (n_1, \ldots, n_p)$ contains an open subset of $DM(N)$, since the convex hull of a vector space basis contains a set which is open in the hyperplane that contains the basis elements. In the case of $\Sigma (n_1, \ldots, n_p)$, that hyperplane is the set of matrices with trace equal to 1. Thus $\Sigma (n_1, \ldots, n_p)$ is a compact, convex subset of $DM(N)$, which is the closure of its non-empty interior. The interior, moreover, contains an element which is in some sense the center of $DM(N)$, the totally mixed state ([1] [2] [3]).

One might think that $\Sigma (n_1, \ldots, n_p)$ would thus be easy to characterize. After all, such is the case for common convex, compact sets with non-empty interiors such as balls and polytopes. But $\Sigma (n_1, \ldots, n_p)$ is not simple at all. For instance, unlike balls and polytopes, there is no easy way to determine the minimum number of product states needed in convex combination to construct a given separable mixed state.

If $A \in \Sigma (n_1, \ldots, n_p)$, we say its optimal ensemble length is the minimum number of product states needed in convex combination to construct $A$. When we require all the product states to be pure, we call the minimum number needed the optimal pure state ensemble length. This latter quantity was studied for two particle systems with $H = C^n \otimes C^n$ by Uhlmann [4] and by DiVincenzo, Terhal and Thapliyal [5] among others. Uhlmann showed the optimal pure state ensemble length is at least equal to the rank of the density matrix and no greater than its square. DiVincenzo, Terhal and Thapliyal took up the question of whether one actually needed more than the rank. This is an important question, for the spectral theorem assures that every density matrix can be expressed as the convex combination of pure states, the number equalling the rank of the matrix. They found examples of states with optimal pure state ensemble length greater than their rank. We shall see for systems with three or more particles and for systems of two particles other than possibly those modelled on $C^2 \otimes C^n$ or $C^3 \otimes C^3$ that almost every separable state has an optimal pure state ensemble length greater than its rank.

In this paper we examine the size of the set of all separable mixed states which have optimal ensemble length of $k$ or fewer and the set of all which have optimal pure state ensemble length of $k$ or fewer. The first set will be denoted by $\Sigma^k (n_1, \ldots, n_p)$ and the second by $\Sigma_{\text{pure}}^k (n_1, \ldots, n_p)$. We completely determine
the $k$ for which $\Sigma^k(n_1, \ldots, n_p)$ has measure 0 in $\Sigma(n_1, \ldots, n_p)$ in both the bipartite case and the case in which one of the particles has substantially more quantum numbers than all the rest - for instance a molecule and photons. This result is the content of theorem 1. In theorem 2 (respectively theorem 3) a lower bound on $k$ for which $\Sigma^k(n_1, \ldots, n_p)$ (respectively $\Sigma^k_{\text{pure}}(n_1, \ldots, n_p)$) has measure 0 in $\Sigma(n_1, \ldots, n_p)$ is given. Moreover, in theorem 2 an upper bound on $k$ for which $\Sigma(n_1, \ldots, n_p)$ has positive measure and contains an open subset is also given. In order to put the main theorems in context, I should mention that a classical theorem of Caratheodory assures one never needs more than $N^2$ pure product states to construct a separable state. Thus $\Sigma(n_1, \ldots, n_p) = \Sigma^k(n_1, \ldots, n_p) = \Sigma^k_{\text{pure}}(n_1, \ldots, n_p)$ for $k = N^2$. However, it is not the case one always needs this many. For instance Sanpera, Tarrach and Vidral [8] have shown in the 2-qubit case one needs no more than four pure product states.

Our main results are the following:

**Theorem 1** Let $N = n_1 \cdots n_p$ with $n_1 \leq n_2 \leq \cdots \leq n_p$ and $n_1 \cdots n_{p-1} \leq n_p$. Then $\Sigma^k(n_1, \ldots, n_p)$ has the following properties: a) It is a connected, compact subset of $\Sigma(n_1, \ldots, n_p)$. In particular if it is not all of $\Sigma(n_1, \ldots, n_p)$, then it is not dense and its complement in $\Sigma(n_1, \ldots, n_p)$ is an open subset. b) If $k < n_1^2 \cdots n_{p-1}^2$, then $\Sigma^k(n_1, \ldots, n_p)$ has measure 0 in $\Sigma(n_1, \ldots, n_p)$. c) If $k \geq n_1^2 \cdots n_{p-1}^2$, then $\Sigma^k(n_1, \ldots, n_p)$ has positive measure in $\Sigma(n_1, \ldots, n_p)$ and in fact contains an open subset.

**Theorem 2** The set $\Sigma^k(n_1, \ldots, n_p)$ has the following properties: a) It is a connected, compact subset of $\Sigma(n_1, \ldots, n_p)$. In particular if it is not all of $\Sigma(n_1, \ldots, n_p)$, then it is not dense and its complement in $\Sigma(n_1, \ldots, n_p)$ is an open subset. b) If $k < (n_1^2 \cdots n_p^2) / (1 + p + \sum_{j=1}^p n_j^2)$, then $\Sigma^k(n_1, \ldots, n_p)$ has measure 0 in $\Sigma(n_1, \ldots, n_p)$. c) If $n_1 \leq n_2 \leq \cdots \leq n_p$ and $k \geq n_1^2 \cdots n_{p-1}^2$, then $\Sigma^k(n_1, \ldots, n_p)$ has positive measure in $\Sigma(n_1, \ldots, n_p)$ and in fact contains an open subset.

**Theorem 3** The set $\Sigma^k_{\text{pure}}(n_1, \ldots, n_p)$ has the following properties: a) It is a connected, compact subset of $\Sigma(n_1, \ldots, n_p)$. In particular if it is not all of $\Sigma(n_1, \ldots, n_p)$, then it is not dense and its complement in $\Sigma(n_1, \ldots, n_p)$ is an open subset. b) If $k < (n_1^2 \cdots n_p^2) / \left(1 - 2p + \sum_{j=1}^p 2n_j\right)$, then $\Sigma^k_{\text{pure}}(n_1, \ldots, n_p)$ has measure 0 in $\Sigma(n_1, \ldots, n_p)$.

The proofs of these theorems will be presented in the next section. First though let us look at a few consequences.

In the bipartite case considered by Uhlmann and by DiVincenzo et.al., $H = \mathbb{C}^n \otimes \mathbb{C}^n$. By theorem 1 there is an open set of separable mixed states with optimal ensemble length of $n^2$ or fewer. Note $n^2 = R_{\text{max}}$, the maximal rank of density matrices in this case. By theorem 3, however, the set of separable mixed states with optimal pure state ensemble length equal to $n^3 / (4 - 3/n)$ or fewer is of measure 0. Thus one must almost always use more than $(R_{\text{max}})^{3/2} / 4$ pure
product states to construct a mixed separable state in the bipartite case. This is quite a disparity. However, it is not indicative of all situations.

For instance, consider a system consisting of \( p \) qubits. Caratheodory’s theorem assures that every separable state can be decomposed into a convex combination of \( 2^{2p} \) pure product states or fewer. From our theorems 2 and 3, it is seen that for large values of \( p \) one must almost always use close to that number, whether one uses pure product states or general ones. In particular, \( \Sigma_k(2, \ldots, 2) \) has measure 0 for \( k < \frac{2^{2p}}{1 + 3 \log(2^{2p})} \) and \( \Sigma_{\text{pure}}(2, \ldots, 2) \) has measure 0 for \( k < \frac{2^{2p}}{1 + 2 \log(2^{2p})} \). In the terms of the maximal rank, these inequalities are \( k < R_{\text{max}}^2/(1 + 3 \log(R_{\text{max}})) \) and \( k < R_{\text{max}}^2/(1 + 2 \log(R_{\text{max}})) \). On the other hand, theorem 2 implies \( \Sigma_k(2, \ldots, 2) \) has positive measure and a non-empty interior if \( k \geq \frac{2^{2p}}{2} - 2 \). This is not sharp. For instance, when \( p = 3 \) one gets an open set with \( k = 13 \).

Turning to the general multiparticle system, we note that the maximum rank of a density matrix on \( H = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p} \) is \( n_1 \cdots n_p \). When this is less than \( n_1^2 \cdots n_p^2 / \left( 1 - 2p + \sum_{j=1}^p 2n_j \right) \), we can conclude from theorem 3 that the optimal pure state ensemble length of a separable state is almost always greater than the rank of the state. In particular, one must almost always use entangled pure states in the spectral (i.e. eigenvalue) decomposition of separable states. This occurs for all systems with three or more particles and for systems with two particles except possibly those with \( H = \mathbb{C}^2 \otimes \mathbb{C}^n \) or \( H = \mathbb{C}^3 \otimes \mathbb{C}^3 \). That there are exceptions was shown by Sanpera et.al. in [8]. As mentioned before, in that paper they showed every separable state on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) can be written as the convex combination of four or fewer pure product states. It would be interesting to see if the other \( \mathbb{C}^2 \otimes \mathbb{C}^n \) and \( \mathbb{C}^3 \otimes \mathbb{C}^3 \) are also exceptions.

Before turning to the proofs, two things need to be mentioned about measurability. First of all in this paper, ”almost always” is used in the strict mathematical sense of meaning ”except on a set of measure 0”. Secondly, there is a controversy over the proper measure to use for the set of density matrices. That does not apply to the results presented here since any two measures which are absolutely continuous with respect to each other have the same sets of measure 0. Since the hyperplane of Hermitian matrices with trace equal to 1 is a real \( N^2 - 1 \) dimensional vector space and \( \Sigma(n_1, \ldots, n_p) \) is a compact, convex subset of it with non-empty interior, we shall use \( N^2 - 1 \) dimensional Lebesgue measure for both.

2 Proofs

Suppose \( M \) and \( N \) are two finite dimensional \( C^\infty \) manifolds and \( f \) is a \( C^\infty \) function from \( M \) to \( N \). A point \( m \in M \) is a critical point for \( f \) if \( df_m : TM_m \to TN_{f(m)} \) is not onto. In words: \( m \) is a critical point for \( f \) if the differential of \( f \) at \( m \), which is a linear transformation from the tangent space of \( M \) at \( m \), \( TM_m \), to the tangent space of \( N \) at \( f(m) \), \( TN_{f(m)} \), is not onto. A point \( n \in N \) is a critical value for \( f \) if it is the image of a critical point. A classical theorem in differential topology due to Sard[6] states that the set of critical values in \( N \)
is of measure 0. This will be the key to our proofs. We shall apply it, along with the rank theorem [7], to the length $k$ mixing function which we shall define shortly.

For $w$ an integer, let $\text{Herm}(w)$ denote the set of $w \times w$ complex Hermitian matrices. $\text{Herm}(w)$ is a real vector space of dimension $w^2$. The subset of positive semi-definite matrices in $\text{Herm}(w)$ form a closed, convex cone with non-empty interior. For $r$ a real number take $\tau_r(w)$ to be the subset of $\text{Herm}(w)$ consisting of those matrices with trace equal to $r$. Each $\tau_r(w)$ is a $w^2 - 1$ dimensional hyperplane in $\text{Herm}(w)$. They are all parallel to $\tau_0(w)$, which is a vector space. The intersection of $\tau_1(w)$ with the cone of positive semi-definite matrices in $\text{Herm}(w)$ is the set of density matrices, $\mathcal{D}M(w)$. As mentioned before, it is a compact, convex set with non-empty interior in $\tau_1(w)$. Also, note that the tangent space of $\tau_1(w)$ at $Q$ is $\tau_0(w)$, since the hyperplanes are parallel.

Let $N = n_1 \cdots n_p$. The length $k$ mixing function

$$\mu_k : \mathbb{R}^{k-1} \times (\tau_1(n_1) \times \cdots \times \tau_1(n_p))^k \to \tau_1(N)$$

is defined for $Q = (\lambda_1, \ldots, \lambda_{k-1}, A_{11}, \ldots, A_{kp})$ by

$$\mu_k(Q) = \lambda_1 A_1 \otimes \cdots \otimes A_p + \left( - \sum_{j=1}^{k-1} \lambda_j \right) A_1 \otimes \cdots \otimes A_p$$

When $\mu_k$ is restricted to $\Lambda_k \times (\mathcal{D}M(n_1) \times \cdots \times \mathcal{D}M(n_p))^k$, where $\Lambda_k = \{ (\lambda_1, \ldots, \lambda_{k-1}) : \lambda_j \geq 0 \text{ and } \sum_{j=1}^{k-1} \lambda_j \leq 1 \}$, it yields elements in $\mathcal{D}M(N)$. Moreover, it does so by forming convex combinations of product states. Since $\mu_k$ is an algebraic function, it is infinitely differentiable and so the criteria for Sard’s theorem are satisfied. The differential of $\mu_k$ at the point $Q$ applied to the tangent vector $V = (r_1, \ldots, r_{k-1}, H_{11}, \ldots, H_{kp})$ is given by:

$$d\mu_k(Q) V =$$

$$\sum_{j=1}^{k-1} \lambda_j \left[ H_{j1} \otimes A_{j2} \otimes \cdots \otimes A_{jp} + A_{j1} \otimes H_{j2} \otimes A_{j3} \otimes \cdots \otimes A_{jp} + \cdots + A_{j1} \otimes \cdots \otimes A_{j-1} \otimes H_{jp} \right]$$

$$+ \left( 1 - \sum_{j=1}^{k-1} \lambda_j \right) \left[ H_{k1} \otimes A_{k2} \otimes \cdots \otimes A_{kp} + \cdots + A_{k1} \otimes \cdots \otimes A_{kp-1} \otimes H_{kp} \right]$$

$$+ \sum_{j=1}^{k-1} r_j A_{j1} \otimes \cdots \otimes A_{jp} - \sum_{j=1}^{k-1} r_j \left( A_{j1} \otimes \cdots \otimes A_{kp} \right)$$

5
We need to determine when \( d\mu_k \) is never onto. To this end observe that 
\( \tau_0(N) \), the tangent space at each point of \( \tau_1(N) \), equals

\[
\tau_0(n_1) \otimes \text{Herm}(n_2) \otimes \cdots \otimes \text{Herm}(n_p) + \\
\text{Herm}(n_1) \otimes \tau_0(n_2) \otimes \cdots \otimes \text{Herm}(n_p) + \\
\cdots + \text{Herm}(n_1) \otimes \cdots \otimes \text{Herm}(n_{p-1}) \otimes \tau_0(n_p).
\]  

(2)

(Note, this is sum, not direct sum. There is a great deal of overlap in the terms. In particular, do not add dimensions.)

Let us first prove part \( b \) of theorem \( 1 \) for the bipartite case. Thus \( N = n_1 n_2 \), \( n_1 \leq n_2 \), and \( k < n_2^2 \). We need to show \( d\mu_k(Q) \) is not onto for any \( Q = (\lambda_1, \ldots, \lambda_{k-1}, A_{11}, A_{12}, \ldots, A_{k1}, A_{k2}) \). To begin, notice that \( k < n_2^2 \) means that neither \( \{A_{j1}\} \) spans \( \text{Herm}(n_1) \) nor \( \{A_{j2}\} \) spans \( \text{Herm}(n_2) \). Hence if either the projections of the \( A_{j1} \) onto \( \tau_0(n_1) \) do not span \( \tau_0(n_1) \) or the projections of the \( A_{j2} \) onto \( \tau_0(n_2) \) do not span \( \tau_0(n_2) \), then \( d\mu_k(Q) \) cannot be onto. Indeed, without loss of generality suppose the projections of the \( A_{j2} \) onto \( \tau_0(n_2) \) do not span. Then there is a \( C \in \tau_0(n_2) \) which is orthogonal to the span of those projections. Since \( \{A_{j1}\} \) does not span \( \text{Herm}(n_1) \) there is a \( B \in \text{Herm}(n_1) \) which is orthogonal to the span of \( \{A_{j1}\} \). The product \( B \otimes C \) is then both in \( \tau_0(n_1 n_2) \) and orthogonal to every term in equation (1) and so \( d\mu_k(Q) \) is not onto.

Since \( \dim \tau_0(n_2) = n_2^2 - 1 \), the situation just considered occurs if any of the following hold \( k < n_2^2 - 1 \), \( n_1 < n_2 \), any of the \( \lambda_j \) are 0, or the \( \lambda_j \) add to 1. Therefore, to finish this part of the proof let us assume \( n_1 = n_2 = n \), \( k = n^2 - 1 \), none of the \( \lambda_j \) are 0 and the \( \lambda_j \) do not add to 1.

Suppose \( A_{j1} = E_j + 1nI \) and \( A_{j2} = F_j + 1nI \) where \( \{E_j\} \) and \( \{F_j\} \) are bases for \( \tau_0(n) \). In order to establish \( d\mu_k(Q) \) is not onto, we only need to show it does not send a basis of \( \mathbb{R}^{k-1} \times (\tau_0(n) \times \tau_0(n))^{k} \) onto a basis of \( \tau_0(n^2) \). The elements of \( \mathbb{R}^{k-1} \times (\tau_0(n) \times \tau_0(n))^{k} \) are of the form \( V = (r_1, \ldots, r_{k-1}, H_{11}, H_{12}, \ldots, H_{k1}, H_{k2}) \). By successively picking one \( r_j \) to be 1 and all the other entries in \( V \) to be 0 and then picking all \( r_j \) to be 0 and successively picking \( H_{ji} \) to be one of the \( E_s \) or \( F_i \) depending upon whether \( i = 1 \) or 2, we obtain a basis for \( \mathbb{R}^{k-1} \times (\tau_0(n) \times \tau_0(n))^{k} \). Applying \( d\mu_k(Q) \) to this basis, we obtain the set

\[
\left\{ \begin{array}{c}
E_s \otimes F_t + E_s \otimes 1nI, E_s \otimes F_t + 1nI \otimes F_t, \\
E_s \otimes F_t + 1nI \otimes F_t + E_s \otimes 1nI - E_k \otimes F_k - 1nI \otimes F_k - E_k \otimes 1nI
\end{array} \right\}
\]

where \( s \) and \( t \) range independently from 1 to \( n^2 - 1 \). Subtracting the first group of these elements from the second and third groups and adding the first group with \( s = t = k \) to the third, we get the set

\[
\{E_s \otimes F_t + E_s \otimes 1nI, 1nI \otimes F_t - E_s \otimes 1nI, 1nI \otimes F_t - 1nI \otimes F_k\}
\]

(4)
Subtracting the last group from the second and adding the result to the first group, we obtain

\[ \{ E_s \otimes F_t + 1n_l \otimes F_k, 1n_l \otimes F_k - E_s \otimes 1n_l, 1n_l \otimes F_k - 1n_l \otimes F_k \} \quad (5) \]

Since \( \dim \tau_0 (n) = n^2 - 1 \), there are \( (n^2 - 1) (n^2 - 1) \) elements in the first group of this last set. There are \( n^2 - 1 \) elements in the second group and there are \( n^2 - 2 \) in the third group. Thus all told there are \( n^2 - 2 \) elements in the set. But \( \dim \tau_0 (n^2) = n^4 - 1 \) and so the set cannot form a basis, which means \( d\mu_k (Q) \) is never onto if \( k < n^2 \).

Hence if \( k < n^2 \), then every point in \( \mathbb{R}^{k-1} \times (\tau_1 (n_1) \times \tau_1 (n_2))^k \) is a critical point for \( \mu_k \). It follows from Sard’s theorem that the image of \( \mu_k \) is of measure 0 in \( \tau_k (N) \). The bipartite case of part b of theorem 1 is then a result of the facts that \( \Sigma^k (n_1, n_2) \) is in the image of \( \mu_k \) and any measure 0 subset of \( \tau_k (N) \) has measure 0 in \( \Sigma (n_1, n_2) \).

To finish the proof of part b of theorem 1, we only need to note that \( \Sigma^k (n_1, \ldots, n_p) \subset \Sigma^k \left( \prod_{j=1}^{p-1} n_j, n_p \right) \) and use what we have just proved for the bipartite case.

Let us now prove part c of theorems 1 and 2. We shall use the rank theorem [7] which states that if \( d\mu_k \) is onto at a point \( Q = (\lambda_1, \ldots, \lambda_{k-1}, A_{11}, \ldots, A_{kp}) \), then \( \mu_k \) maps some open ball centered at \( Q \) onto an open set containing \( \mu_k (Q) \). Thus we need to find a \( Q \) in the interior of \( \Lambda_k \times (\mathcal{D}M (n_1) \times \cdots \times \mathcal{D}M (n_p))^k \) at which \( d\mu_k (Q) \) is onto.

We know there are bases of \( \text{Herm}(n_i) \) which consist of elements in the interior of \( \mathcal{D}M (n_i) \). We also know \( 1n_p I \) is in the interior of \( \mathcal{D}M (n_i) \). Therefore, since \( k \geq n^2_1 \cdots n^2_{p-1} = \dim \text{Herm} (n_1 \cdots n_{p-1}) \), we can pick the \( A_{ji} \) for \( j = 1, \ldots, k \), \( i = 1, \ldots, p - 1 \) to be in the interior of \( \mathcal{D}M (n_i) \) and such that \( \{ A_{ji} \otimes \cdots \otimes A_{jp-1} \} \) spans \( \text{Herm} (n_1 \cdots n_{p-1}) \). Choosing them so and also choosing all \( \lambda_j = 1k \) and all \( A_{jp} = 1n_p I \), we obtain a \( Q \) which satisfies our needs. To see this, note an element of \( \mathbb{R}^{k-1} \times (\tau_0 (n_1) \times \cdots \times \tau_0 (n_p))^k \) is of the form \( V = (r_1, \ldots, r_{k-1}, H_{11}, \ldots, H_{1p}, \ldots, H_{kp}) \). Let \( \Gamma_i \) be the set of all \( V \) for which the only non-zero component is an \( H_{ji} \). Since \( \{ A_{1j} \otimes \cdots \otimes A_{jp-1} \} \) spans \( \text{Herm} (n_1 \cdots n_{p-1}) \) and \( A_{ip} = 1n_p I \), we have that \( d\mu_k (Q) \) maps \( \Gamma_i \) onto \( \tau_0 (n_1 \cdots n_p) \otimes 1n_p I \) for \( i < p \) and \( \Gamma_p \) onto \( \text{Herm} (n_1 \cdots n_{p-1}) \otimes \tau_0 (n_p) \). These two sets span \( \tau_k (N) \) and so part c of theorems 1 and 2 is proved.

To finish the proofs of these two theorems we first note that if \( k \) satisfies the condition in part b of theorem 2, then the dimension of the domain of \( \mu_k \) is less than the dimension of its target. In such a case it is impossible for \( d\mu_k (Q) \) to ever be onto because the dimension of its domain is too small. And so the result follows again from Sard’s theorem. As for part a of theorems 1 and 2 it is a consequence of the fact \( \Sigma_k (n_1, \ldots, n_p) \) is the image of the connected, compact set \( \Lambda_k \times (\mathcal{D}M (n_1) \times \cdots \times \mathcal{D}M (n_p))^k \) under the continuous map \( \mu_k \).

Finally, as for theorem 3, let us recall that the set of pure states in \( \mathcal{D}M (q) \) is isomorphic to the complex projective space \( \mathbb{C}P (q - 1) \), which has real dimension \( 2q - 2 \). Hence we need to consider the composition of the embedding.
\( \iota : \mathbb{R}^{k-1} \times (\mathbb{C}P(n_1 - 1) \times \cdots \times \mathbb{C}P(n_p - 1))^k \rightarrow \mathbb{R}^{k-1} \times (\tau_1(n_1) \times \cdots \times \tau_1(n_p))^k \)

with \( \mu_k \). Part a of theorem 3 is a result of the fact \( \Sigma^k_{pure}(N) \) is the image of the connected, compact set \( \Lambda_k \times (\mathbb{C}P(n_1 - 1) \times \cdots \times \mathbb{C}P(n_p - 1))^k \) under the continuous map \( \mu_k \circ \iota \). As for part b, it is a simple consequence of Sard’s theorem and the observation that \( \mathbb{R}^{k-1} \times (\mathbb{C}P(n_1 - 1) \times \cdots \times \mathbb{C}P(n_p - 1))^k \) has dimension \( k \left( 1 - 2p + \sum_{j=1}^{k} 2n_j \right) - 1 \) while \( \tau_1(N) \) has dimension \( n_1^2 \cdots n_p^2 - 1 \). Since \( d\mu_k \circ \iota \) is a linear transformation, it cannot be onto if the dimension of the domain is strictly less than the dimension of its image, which is the case here if \( k < \left( n_1^2 \cdots n_p^2 \right) / \left( 1 - 2p + \sum_{j=1}^{k} 2n_j \right) \).

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