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Abstract

The eigenvalue problem of an electron in the plane in the presence of a repulsive scatterer is studied. The electron is subject to a weak in-plane electric field and a magnetic field perpendicular to the plane. The associated magnetic length is much larger than the range of the scatterer. In this parameter region it is natural to follow Prange and treat the scatterer basically as a repulsive δ -function. However, the finite range of the scatterer is essential in that it provides the cutoffs necessary to make the problem mathematically well posed. We demonstrate that a true δ -function is unable to trap an electron in a finite electric field, no matter how small. At high Landau levels we find semi-quantitative agreement with recent classical results on electron trapping. With sharp cutoffs one bound state per Landau level is found for sufficiently weak electric fields. As the strength of the electric field is increased, the role of the bound state is taken over by a metastable wave packet which remains close to the scatterer for an exceedingly long time. This wave packet is explicitly constructed. With smooth cutoffs, all bound states become submerged in the continuum, and only long-lived wavepackets remain.

1 Introduction

The motion of an electron in a static potential in two dimensions, under the influence of a strong perpendicular magnetic field, is basic to the understanding of the integer quantum Hall effect. The fundamental length scale for such phenomena is the magnetic length $l_B = \sqrt{\hbar/eB}$, where e is the magnitude of the elementary

charge and $\mathcal{B}$ the magnetic field. As is well known, when the potential landscape is sufficiently smooth that the variations are slow on the scale of the magnetic length, the electron states form ribbons of width l_B along equipotential lines in the landscape[1]. Except close to saddle points, where tunneling can become important, this qualitative picture is essentially the same in quantum and classical physics. Such a ribbon state encircling the top of a local potential maximum can be said to represent an electron trapped by a repulsive potential.

It is natural to ask what happens in the opposite extreme, where the linear dimension a of a localized scatterer is much smaller than the magnetic length, $a \ll l_B$. This problem was addressed by Prange[2, 1] shortly after the discovery of the quantum Hall effect. In that classic paper he studied electron states in 2 dimensions, in a perpendicular magnetic and an in-plane electric field, and in the presence of (in particular) a single repulsive delta-like scatterer. Two of his main conclusions were: (i) there is a bound state; (ii) the total current remains unchanged, since the current lost due to the bound state is redistributed to the extended ones. This latter fact helped explain the robustness of the quantum Hall effect.

Prange's model, but with an *attractive* δ -function scatterer, has been reexamined by Perez and Coutinho[3], and by Cavalcanti and de Carvalho[4], in zero electric field. They show that an attractive two-dimensional δ -function is "too strong", in the sense that the energies of the bound states tend to minus infinity. However, by renormalizing the strength of the δ -function (logarithmically weakening it), they show that the energies of the bound states can be made finite. No statements are made on *repulsive* scatterers.

In this paper we again return to Prange's problem, but focus exclusively on repulsive scatterers. Recently, the corresponding question, "Can a local repulsive potential trap an electron?" was asked in a classical setting[5]. The answer was found to be a guarded "yes", in the sense that rigorous mathematical results combined with complicated numerical phase portraits demonstrates that a finite phase space corresponding to trapping is found in the parameter region where, roughly speaking, $v_d T_c < a$. Here $v_d = \mathcal{E}/\mathcal{B}$ is the drift velocity in crossed electric and magnetic fields, $T_c = 2\pi/\omega_c$ is the cyclotron period, and a the linear dimension of the scatterer. The simple physics behind this condition is that the classical electron, in order to be trapped, must return to the scatterer for repeated collisions, and not drift away from it.

From the point of view of these classical results, certain aspects of Prange's findings begin to look mysterious. How is it possible that the range of the scatterer plays no role in his argument? Or, alternatively: Is it really true that a bona fide repulsive δ -function, with zero range, can trap the electron in quantum mechanics, whereas a finite range is essential in classical mechanics? Why should cutoffs due to the boundaries be an key ingredient in the trapping argument, as put forward by Prange? It is a fact that boundaries by themselves can trap electrons, both classically and in quantum mechanics, but that is not really the issue here. In the classical trapping story referred to above, boundaries play no role whatsoever.

In this paper we clarify these questions. Basically, we follow Prange's proce-

dures, but demonstrate that the cutoff due to the range a of the scatterer, *not* the boundaries, should be used in order to get a consistent picture of trapping. Keeping the range a of the repulsive potential strictly finite is necessary for an understanding of the relation between the classical and the quantum results, even if this limits the possibilities for constructing results in closed form. For clarity we, initially, take the system size L in the x -direction to be finite. When, subsequently, the limit $L \rightarrow \infty$ is taken, the intrinsically quantum effect of tunneling alters the problem in a subtle manner. Take, as an analogous example, the hydrogen atom in its ground state, subject to a weak electric field. Very far from the nucleus there are free states with the same energy as the ground state. As a consequence, there is a finite, but extremely small, probability that the electron tunnels through the barrier and escapes. The electric field turns all the bound states into (extremely sharp) resonances or, depending on the point of view, into long-lived metastable states. Usually the continuum states are, wisely, ignored when the deformation of the hydrogen atom due to the electric field is calculated. In our present problem one should apply similar wisdom. Most of our discussion is based on replacing a smooth cutoff function resulting from the finite range of the scatterer by a sharp cutoff. By this trick we (somewhat artificially) separate “truly” bound states from long-lived metastable ones.

In order to make clear where we differ from Prange, the problem is restated in Section 2, and the equations basic to the arguments are rederived. In Section 3 we discuss the various cutoffs of relevance to the problem. As a preparation for later sections, Section 4 is devoted to the simple case of (essentially) zero electric field, and we calculate explicitly the non-trivial form of the lowest eigenfunction. We then proceed to discuss, in Section 5, the properties of the eigenvalues on the basis of sharp cutoffs and apply that to the calculation of the current in Section 6. We find that with sharp cutoffs there is a correspondingly sharp distinction between bound and extended states. The problems associated with passing to the limit of large systems are highlighted in Section 7, and metastable and extremely long-lived localized wave packets are explicitly constructed. The results obtained are finally summarized and discussed in Section 8.

When this paper was being prepared for publication, parallel work by Gyger and Martin[6] extending that of Ref.[4] to small but finite electric fields was brought to our attention. Similarities and differences with their work are also briefly commented upon in Sec.8.

2 The problem

In our geometry we orient the magnetic field $\mathcal{B}$ along the z -axis and the electric field $\mathcal{E}$ along the y -axis. We measure lengths in units of the magnetic length l_B , and energies in units of $\hbar\omega_c$, where $\omega_c = e\mathcal{B}/m$ is the cyclotron frequency (with m the effective mass). The dimensionless electric field ϵ is taken to be

$$\epsilon = e\mathcal{E}l_B/\hbar\omega_c. \quad (1)$$

The usual connotation that ϵ is a small quantity also applies here since we focus on weak electric fields, $\epsilon \ll 1$. We work in the Landau gauge in which the vector potential $\mathbf{A} = (-y\mathcal{B}, 0)$ has a component along the x -axis only. In our dimensionless units the Schrödinger equation then reads

$$\mathcal{H}\Psi_\alpha(\mathbf{r}) = E_\alpha\Psi_\alpha(\mathbf{r}), \quad (2)$$

with $\mathcal{H}$ given by

$$\mathcal{H} = -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{y}{i} \frac{\partial}{\partial x} + \frac{y^2}{2} + \epsilon y + V(x, y) = \mathcal{H}^0 + V. \quad (3)$$

$V(x, y)$ is the potential of the scattering object, which we take as purely repulsive. We assume that $V(x, y)$ is of microscopic dimensions, with a dimensionless linear size of the order $a/l_B \ll 1$. With a of atomic dimensions, this inequality is satisfied even for quite high magnetic fields.

Thus $V(x, y)$ is constrained to a very narrow region around the origin in our scaled coordinates. This makes it advantageous to expand the wave function $\Psi_\alpha(\mathbf{r})$ in the complete basis $\Phi_{n,k}(\mathbf{r})$ of the unperturbed problem. These functions are given by the following expression:

$$\Phi_{n,k}(\mathbf{r}) = \frac{e^{ikx}}{L^{\frac{1}{2}}} e^{-y_k^2/2} \frac{H_n(y_k)}{(\sqrt{\pi} 2^n n!)^{\frac{1}{2}}}, \quad (4)$$

where y_k is given by

$$y_k = y - k + \epsilon \quad (5)$$

and L is the size of the system in the x -direction, with periodic boundary conditions in that direction. Consequently, the values of k run through the set

$$k = \frac{2\pi}{L}(0, \pm 1, \dots). \quad (6)$$

As one sees from (5), k is not only the wavenumber in the x -direction but also the offset of the wave function in the y -direction. Finally, with n the Landau level index ($n = 0, 1, \dots$), H_n is the corresponding Hermite polynomial. The associated unperturbed energy levels are,

$$E_{n,k} = n + \frac{1}{2} + \epsilon k - \frac{\epsilon^2}{2}. \quad (7)$$

Expand the eigenfunction in the complete set (4),

$$\Psi_\alpha(\mathbf{r}) = \sum_{n,k} c_{n,k}^\alpha \Phi_{n,k}(\mathbf{r}), \quad (8)$$

and insert the expansion into the terms of (3), except for the one involving the potential of the scatterer. One then obtains a formal expression for $c_{n,k}^\alpha$ by projecting out the (n, k) -term

$$c_{n,k}^\alpha = \frac{\langle n, k | \mathcal{V} | \alpha \rangle}{E_\alpha - E_{n,k}}, \quad (9)$$

where the matrix element is given by

$$\langle n, k | \mathcal{V} | \alpha \rangle = \int d\mathbf{r} \Phi_{n,k}^*(\mathbf{r}) V(\mathbf{r}) \Psi_\alpha(\mathbf{r}). \quad (10)$$

Eq.(9) is, of course, not yet an explicit solution, since the matrix elements still contain the unknown eigenfunction Ψ_α .

At this point Prange takes the scattering potential to be a δ -function which allows him to write

$$\langle n, k | \mathcal{V} | \alpha \rangle = \lambda \Phi_{n,k}^*(\mathbf{0}) \Psi_\alpha(\mathbf{0}), \quad (11)$$

where λ is the δ -function strength of $V(\mathbf{r})$

$$\lambda = \int d\mathbf{r} V(\mathbf{r}). \quad (12)$$

On the basis of (11) the explicit solution is easily constructed. Insertion into (9) yields

$$c_{n,k}^\alpha = \frac{\lambda \Phi_{n,k}^*(\mathbf{0}) \Psi_\alpha(\mathbf{0})}{E_\alpha - E_{n,k}}, \quad (13)$$

where the value of $\Psi_\alpha(\mathbf{0})$ follows from normalization

$$1 = \sum_{n,k} |c_{n,k}^\alpha|^2 = \lambda^2 |\Psi_\alpha(\mathbf{0})|^2 \sum_{n,k} \frac{|\Phi_{n,k}(\mathbf{0})|^2}{(E_\alpha - E_{n,k})^2}. \quad (14)$$

Finally, the eigenvalue equation is found by insertion of the expression (13) into the definition of $\Psi_\alpha(\mathbf{0})$, as given by (8). The result is

$$\frac{1}{\lambda} = \sum_{n,k} \frac{|\Phi_{n,k}(\mathbf{0})|^2}{E_\alpha - E_{n,k}}. \quad (15)$$

All has now formally been determined: The eigenvalues E_α are the roots of (15), $\Psi_\alpha(\mathbf{0})$ follows from (14) and the wave function from (8) through its expansion coefficients as given by (13). The trouble with this formal solution is that when summations over n and k are extended to infinity, divergences appear. In other words, cutoffs are needed.

3 Cutoffs

With a passing reference to cutoffs due to the finite core of the scatterers, Prange in his discussion[2] of this problem relies on the finite size of the system for regularizing the sums. In our opinion this is misleading, and for two reasons. First, the problem of a localized repulsive potential as a possible trap in crossed electric and magnetic fields seems to be a well-posed problem, even on the infinite plane. In this idealization of a physics problem, boundaries should play no role. Second, it is well known that a wall can trap electrons in crossed $\mathcal{E}$ and $\mathcal{B}$ fields,

even without a scatterer¹. If walls are made to play a crucial role in a discussion of trapping, the associated physics due to the wall itself must be taken seriously. The machinery developed in Sec.2 for treating scatterers of short range is eminently unsuited for that task.

The cutoff of fundamental importance in this context is the cutoff in n . That this is so is already apparent from the special case of zero electric field. Set $\epsilon = 0$ in (15). The sum over k reduces, in the limit $L \rightarrow \infty$, to the normalization integral, and the eigenvalue equation (15) becomes

$$\frac{2\pi}{\lambda} = \sum_n \frac{1}{E_\alpha - (n + 1/2)}. \quad (16)$$

Without a cutoff this sum diverges logarithmically. We conclude, as did Prange[2], that this divergence is intrinsic to a true δ -function on the infinite plane. For a mathematical treatment with an attractive potential, see [6, 3, 4].

Rather than appeal to boundary effects, we shall base our arguments on the fact that cutoffs appear naturally as consequences of the finite size of the scatterer. From (10) it is clear that the matrix element reduces to the form (11) only when the function $\Phi_{n,k}^*(\mathbf{r})$ varies sufficiently slowly over the spatial extent of the scatterer. The eigenfunctions (4) show oscillatory behavior, which for sufficiently high frequency will make the value of the corresponding matrix element tend to zero. In the x -direction the eigenfunctions are governed by e^{ikx} . In the y -directions the oscillations are those of the Hermite polynomials. Although we have a scatterer with circular symmetry in mind, this asymmetry makes it conceptually advantageous to distinguish the extent of the scatterer in the two directions, and write $a \rightarrow \mathbf{a} = (a_x, a_y)$.

The Hermite polynomials oscillate with a period of order $1/\sqrt{2n+1}$. As a result, we find a cutoff at $n = N$, with

$$2n + 1 \leq 2N + 1 \equiv \left(\frac{l_B}{a_y}\right)^2. \quad (17)$$

This cutoff is essential for all results of this paper. Clearly, the cutoff provided by the oscillations of $H_n(y_k)$ in the y -directions is a smooth one. However, what matters in our analysis is that the cutoff makes sums over n convergent by effectively restricting them to a region below N . This allows us to replace the smooth cutoff by a sharp one at $n = N$. On the other hand, since the matrix elements with $n \sim N$ depend on the detailed properties of the scatterer, we can only meaningfully discuss Landau levels with n well below $n = N$. As a consequence, we shall restrict ourselves to $n < N/2$.

The finite range a_x in the x -direction implies the following cutoff in k ,

$$|k| < K \equiv \frac{l_B}{a_x}. \quad (18)$$

¹As an example, take a channel in the x -direction, confined in the y -direction by a parabolic potential $(1/2)m\omega_0^2 y^2$. With $\mathcal{E}$ - and $\mathcal{B}$ -fields in the usual directions, it is easy to show that in every Landau subband a zero current state exists, for which (in unscaled variables) $k = -(e^2/\hbar m\omega_0^2)\mathcal{B}\mathcal{E}$. This state describes electrons trapped at the lower wall.

This cutoff is not needed for convergence of the sums or integrals over k and does, therefore, not have the fundamental significance of the cutoff in n . The integrands always include a Gaussian factor which, together with the Hermite polynomials, effectively limits the integration to the interval $|k| < \sqrt{2n+1}$. This range is, moreover, smaller than that given by the cutoff (18), except for the largest allowed values of $n = \mathcal{O}(N)$, when the two are essentially the same.

In spite of its less fundamental nature, we shall use the k -cutoff extensively in the arguments below. The reason for this is that it helps disentangle the bound states from the current carrying ones. Strictly speaking, this distinction is only sharp if the cutoff at $|k| = K$ makes the integrands vanish beyond $|k| = K$. For convenience (in order to allow partial integrations without surface terms), we assume that the integrands go smoothly to zero “at” $|k| = K$.

The introduction of a sharp k -cutoff in the above sense is an artificial trick introduced to clarify the structure of the problem at hand. We return to the qualitatively significant, if quantitatively small, changes introduced by finite tails in the integrands for large $|k|$ in the last part of this paper.

4 Eigenvalues and eigenfunctions in vanishing electric field

Before embarking on a discussion of the the full eigenvalue problem (15), we first focus on the degenerate case of $\epsilon \rightarrow 0$, where the eigenvalue equation reduces to the simple form (16). In zero electric field all states are in a sense bound by the magnetic field alone. Here we shall concentrate on those states that are bound to the radially symmetric perturbing potential (we again put $a_x = a_y = a$). For these states the angular momentum with respect to the origin is conserved, and the natural gauge in which to discuss them is the circular one $\mathbf{A} = (-y\mathcal{B}/2, x\mathcal{B}/2)$, rather than the Landau gauge generally used in this paper. The problem is simplified by the observation that the states with non-zero angular momentum have vanishing wave functions at the origin and, as a consequence, they are not perturbed by the presence of the scatterer². These states, therefore, are degenerate states with unperturbed eigenvalues. It is the states with zero angular momentum that are significantly affected by the scatterer and that will be considered in the following.

The unperturbed eigenvalues are $E_n = n + 1/2$, and (16) shows that each of the perturbed eigenvalues is located between two consecutive unperturbed ones. It is, therefore, convenient to write $E_\alpha = n_\alpha + 1/2 + \xi_\alpha$, with $0 < \xi_\alpha < 1$. Eq.(16) then reads,

$$\frac{2\pi}{\lambda_\alpha} = \frac{1}{\xi_\alpha}, \quad (19)$$

²With a scatterer of finite strength and radius this statement is not strictly true, but for Landau levels well below cutoff it closely approximates the real situation.

where the effective strength λ_α is given by

$$\frac{2\pi}{\lambda_\alpha} = \frac{2\pi}{\lambda} - \sum_{n \neq n_\alpha}^N \frac{1}{n_\alpha + \xi_\alpha - n}, \quad (20)$$

and where we have taken the cutoff at $n = N \gg 1$ to be sharp.

Let the bare λ be of $\mathcal{O}(1)$. First consider bound states close to the lowest Landau levels, i.e., with $n_\alpha = \mathcal{O}(1)$. Neglect of ξ_α in the expression (20) for $1/\lambda_\alpha$ yields, for large N , $1/\lambda_\alpha \sim \ln N$. From (19) then follows that $\xi_\alpha \sim 1/\ln N$, demonstrating the consistency in the neglect of ξ_α in (20) for the lowest Landau levels. Note, however, that this formal argument can be somewhat misleading. Even for N as large as $N = 10\,000$, the “small” quantity is $1/\ln N = 0.11$, still a sizeable fraction of unity, the Landau level separation.

The upward shift in energy, ξ_α , which is of $\mathcal{O}(1/\ln N)$ for the lowest Landau levels, slowly grows with the Landau index n_α . This is readily seen from (20): Increase of n_α by unity removes one positive term contributing to $1/\lambda_\alpha$ and adds a negative one. When $n \sim N/2$ the positive and negative parts of the sum in (20) essentially cancel, ξ_α becomes of $\mathcal{O}(1)$ and can no longer be neglected in (20). For $n_\alpha = N/2$ the eigenvalue equation (19) becomes

$$0 = \frac{1}{\xi_\alpha} - \frac{1}{1 - \xi_\alpha} + \sum_{n=1}^{N/2} \frac{1}{n + \xi_\alpha} - \left(\frac{2\pi}{\lambda} + \sum_{n=2}^{N/2} \frac{1}{n - \xi_\alpha} \right). \quad (21)$$

The near symmetry of this equation shows that the corresponding solution ξ_α is close to $1/2$. The energy shift ξ_α continues to grow with growing Landau level index, gradually approaching unity. However, the details of this approach depends on the precise nature of the the cutoff around $n = N$, which we are not prepared to specify³. On general grounds one would expect correspondence between quantum and classical results for “large” n . As we shall see in Sec.8, this expectation is borne out also in the present problem. In order that the details of the cutoff do not interfere with the arguments, “large” n should here be taken to mean $n \sim N/2$.

After this discussion of the spectrum, we now summarize the properties of the corresponding bound state eigenfunctions in zero electric field. In Appendix A we give the details, using the symmetric gauge mentioned above. In the standard Landau gauge, the wave functions for states with zero angular momentum can be written

$$\Psi_\alpha(r) = \frac{e^{ixy/2 - r^2/4}}{\sqrt{2\pi}} f_\alpha(r). \quad (22)$$

The factor $\exp(ixy/2)$ is absent in the symmetric gauge. The function $f_\alpha(r)$ is given by

$$f_\alpha(r) = \frac{1}{A} \sum_{n=0}^N \frac{L_n(r^2/2)}{\xi_\alpha + n_\alpha - n}, \quad (23)$$

³The precise nature of the cutoff is irrelevant for the properties of the low-lying states. In discussing those states we were, consequently, justified in using a sharp cutoff at $n = N$.

where the normalization factor A is given in Appendix A. Furthermore, $L_n(r^2/2)$ is the Laguerre polynomial[7], normalized such that $L_n(0) = 1$. For the low lying states, the term with $n = n_\alpha$ dominates the sum, but it is not justified to neglect the others (as Prange does), since the sum is slowly convergent. We have plotted the function $f_\alpha(r)$ for the lowest Landau level in Fig.1, using a sharp cutoff at $N = 10\,000$. One observes a dip in the wave function towards the origin, as

Figure 1: The correction function $f_\alpha(r)$ for the lowest Landau level wave function.

a result of the repulsive influence of the scatterer. This decrease of probability density near the origin is compensated by an increase for larger r . Note that the function is cutoff dependent: As $N \rightarrow \infty$, $f_\alpha \rightarrow 1$ everywhere except at the origin, where it vanishes. This follows from the properties of the Laguerre polynomials. However, the convergence to this limiting form is extremely slow, governed by $1/\ln N$.

5 Eigenvalues in non-zero electric field

5.1 Preliminaries

In this section and the following one, we discuss the eigenvalue problem under the simplifying assumption that the cutoffs in n and k are sharp. In particular, we assume that the k -integrands go smoothly to zero “at” $|k| = K = l_B/a_x$ and remain strictly zero beyond that value. We return to a discussion of this point in later sections.

First we rewrite (15) in a form suitable for our purposes. For the summation over k we use the grid of points

$$k_j = \frac{2\pi}{L}j, \quad (24)$$

where the j 's are integers. We write the perturbed energy E_α in terms of the corresponding band index and a variable k_α or, rather,

$$x_\alpha = \epsilon(k_\alpha - \epsilon), \quad (25)$$

such that

$$E_\alpha = n_\alpha + 1/2 + \epsilon k_\alpha - \epsilon^2/2 = n_\alpha + 1/2 + x_\alpha + \epsilon^2/2. \quad (26)$$

Here k_α is specified with respect to the grid points as

$$k_\alpha = k_l + \frac{2\pi}{L}\delta_l, \quad (27)$$

where k_l is the grid point to the left next to k_α and thus, $0 < \delta_l < 1$. Furthermore, we introduce the abbreviation

$$f_n(q) = \frac{1}{\sqrt{\pi}2^n n!} e^{-q^2} H_n^2(q), \quad (28)$$

which turns the eigenvalue equation into, with $q_j = k_j - \epsilon$,

$$\frac{1}{\lambda} = \frac{1}{L} \sum_{n,j} \frac{f_n(k_j - \epsilon)}{\epsilon(k_\alpha - k_j) + n_\alpha - n} = \frac{1}{L} \sum_{n,j} \frac{f_n(q_j)}{x_\alpha - \epsilon q_j + n_\alpha - n}. \quad (29)$$

Now introduce an effective λ , in analogy with (20),

$$\frac{1}{\lambda_\alpha(x_\alpha)} = \frac{1}{\lambda} - \sum_{n \neq n_\alpha}^N \frac{1}{L} \sum_j \frac{f_n(q_j)}{x_\alpha - \epsilon q_j + n_\alpha - n}. \quad (30)$$

By the same arguments as in Sec.4, we can assume that $\lambda_\alpha(x_\alpha)$, for the lowest Landau levels, depends on x_α only very weakly. Furthermore, define the function $A_n(p)$ by

$$A_n(p) = \frac{1}{L} \sum_j \frac{f_n(q_j)}{p - q_j}. \quad (31)$$

In terms of (30) and (31), the eigenvalue equation (29) reads,

$$\frac{\epsilon}{\lambda_\alpha} = A_{n_\alpha}(k_\alpha - \epsilon). \quad (32)$$

It is on this simple form of the eigenvalue equation, with the very weak k_α -dependence of λ_α neglected, that we shall base our subsequent discussion.

The distinction between bound states and current carrying ones turns out to be quite subtle. In that connection a smooth version of the function $A_n(p)$ in (31) will prove useful. With reference to Appendix B it is defined as

$$B_n(p) = \int \frac{dq}{2\pi} \mathcal{P} \frac{f_n(q)}{p - q}. \quad (33)$$

In Fig.2 $B_0(p)$ is shown as a function of p for $p > 0$. From the definition (33) it immediately follows that $B_n(p)$ is antisymmetric in p . For large L the functions

Figure 2: The function $2\pi B_0(p)$ for $p > 0$. $B_n(-p) = -B_n(p)$.

$B_n(p)$ and $A_n(p)$ coincide when the argument p lies outside the range of summations in (31), i.e., outside the band. Inside the band, the smooth continuation $B_n(p)$ is no longer a faithful representation of $A_n(p)$. In fact, a rapidly varying function must be added, as discussed in Appendix B. That function is essential for the discussion of states *inside* the band, as we shall see below.

We have now assembled the tools necessary to discuss bound versus current carrying states when the k -cutoff is considered as sharp.

5.2 Bound states

The discussion of the eigenvalue problem (29), or (32), simplifies considerably if the summation over k_j is cut off in such a way that the Landau bands do not overlap. With a “sharp” cutoff at $|k| = K$, overlap is prevented when

$$\epsilon K < \frac{1}{2}. \quad (34)$$

In this subsection we shall exploit the corresponding simplifications.

With non-overlapping Landau bands, the eigenvalue equation (29) retains the property exploited in Sec.4 that each of the perturbed eigenvalues is located between two consecutive unperturbed ones. The *unperturbed* spectrum consists of Landau levels split up by the electric field to become quasi-continuous Landau bands. Within these bands the level splitting is of $\mathcal{O}(\epsilon/L)$. In Fig.3 we present a schematic graphical solution of (29), with the repulsive potential present. The

Figure 3: The graphical determination of the perturbed energies

level splitting inside the Landau bands clearly remains infinitesimally small (of the order ϵ/L). These perturbed states form the quasi-continuum of extended states. The single state *between* two Landau bands is split off by a finite amount and is a localized, or bound, state. In the next section we shall confirm this picture by calculating the current associated with the different eigenstates.

It is bound states outside the band that we now focus on. To be specific, we consider the state with energy close to the lowest Landau level. For this state the eigenvalue equation (32) for the determination of k_0 can, by the correspondence between (31) and (33), be written as

$$\frac{\epsilon}{\lambda_0} = A_0(k_0 - \epsilon) = B_0(k_0 - \epsilon), \quad (35)$$

with $\lambda_0(x)$ given by (30). By similar arguments as those used in Sec.4, the weak dependence on $x = \epsilon(k_0 - \epsilon)$ in (30) can be neglected and one realizes that, for small ϵ , $1/\lambda_0$ is of $\mathcal{O}(\ln N)$. In spite of the fact that $1/\lambda_0 = \mathcal{O}(\ln N)$, the left hand side of (35), for small ϵ , becomes arbitrarily small. Thus, k_0 becomes large, and to leading order in small ϵ , we can replace $B_0(k_0 - \epsilon)$ by its asymptotic form for large arguments⁴, $B_0(p) \simeq 1/(2\pi p)$. From this the simple relation $(\lambda_0/2\pi) = x = \epsilon(k_0 - \epsilon)$ follows. In Appendix C we have shown that this argument can be developed into a perturbative expansion that gives the energy eigenvalue, as represented by k_0 , in powers of ϵ , as follows

$$k_0 = x(\epsilon)/\epsilon = x_0/\epsilon + (x_2 + 1)\epsilon^2 + \dots, \quad (36)$$

where x_j are constants⁵. In particular, x_0 is the solution of the equation

$$\frac{2\pi}{\lambda} = \sum_{n=0}^N \frac{1}{x_0 - n}, \quad (37)$$

which shows that $x_0 = \mathcal{O}(1/\ln N)$. The value of $x_2 + 1$ is given in Appendix C.

In order that the solution found is outside the band cut off at $|k| = K$, the lowest order version of (36) shows that

$$\epsilon K < x_0. \quad (38)$$

Since x , and thus x_0 is of $\mathcal{O}(1/\ln N)$, this condition is more stringent than that of non-overlapping bands, (34).

Similar arguments as those presented here for the bound state associated with the lowest Landau level hold for higher ones. The main changes are: (i) The function $B_n(p)$ will not start its decay already at $p \sim 1$ but, because of the influence of the Hermite polynomials, only decay from $p \sim \sqrt{2n+1}$ on. (ii) The value of $1/\lambda_\alpha$ will slowly decrease with n , from $\mathcal{O}(\ln N)$ when $n \sim 1$ to $\mathcal{O}(1)$ when $n \sim N/2$. From this follows that when $n \sim N/2$, the condition for finding a bound state outside the band reverts to (34).

⁴From Fig.2 it is clear that already from $p = 2$ on, $B_0(p)$ is well approximated by its asymptotic form.

⁵In order to keep the notation simple, we have not made explicit that the x of (36) refers to the zeroth Landau level. Had we done so the x_0 of (36) would read x_{00} etc. Instead we note that the x_0 used in the text depends weakly on the Landau level index.

5.3 States inside the band

For states inside the band the discussion of eigenvalues must, in principle, be based on the sum $A_n(p)$. For simplicity, we again focus on the lowest Landau band. Furthermore, we again lump the sum over $n \neq 0$ into an effective λ_0 . For large L we then appeal to the results of Appendix B and write the eigenvalue problem (32) as

$$\frac{\epsilon}{\lambda_0} = A_0(k_\alpha - \epsilon) = B_0(k_\alpha - \epsilon) + \frac{1}{2}f_0(k_\alpha - \epsilon) \cot(\pi\delta_l), \quad (39)$$

where $(2\pi/L)\delta_l$ ($0 \leq \delta_l < 1$) is the shift from the unperturbed value k_l as defined in (27). When L is large we can regard $B_0(k_\alpha - \epsilon)$ and $f_0(k_\alpha - \epsilon)$ as functions of a continuous argument $(k - \epsilon)$. The (eigenvalue) equation (39) then becomes an equation for the determination of δ_l . For sufficiently large L then, $\delta_l \rightarrow \delta(k)$ can be treated as a function of a continuous variable. Thus, we can rewrite (39), from which $\delta(k)$ is to be determined, as

$$\frac{1}{2}f_0(k - \epsilon) \cot[\pi\delta(k)] = \frac{\epsilon}{\lambda_0} - B_0(k - \epsilon). \quad (40)$$

The shift $\delta(k)$ amounts to a tiny shift of the unperturbed eigenvalues, and as such it may seem totally irrelevant. However, in the next two sections we shall demonstrate that the behavior of $\delta(k)$ is essential for the determination of the current associated with eigenstates in the band, or with linear combinations of such states.

The qualitative behavior of $\delta(k)$ is easily read off from (40). For all $k < \epsilon$ the function B_0 is negative, with magnitude decaying as $|k|^{-1}$ when $-k \gg 1$. Thus, to the left of the middle of the band, the right hand side (40) is positive and, at least, of $\mathcal{O}(\epsilon \ln N)$. The exponential prefactor will generally dominate all other factors so that, with $k - \epsilon \rightarrow k$ for small ϵ ,

$$\delta(k) \sim e^{-k^2} \quad ; \quad -K < k < -1. \quad (41)$$

Now let k increase. For $k > \epsilon$ the function B_0 becomes positive and (for not too large ϵ) the right hand side of (40) will subsequently change sign. At the corresponding value of k , $\delta = 1/2$. When k is increased further, $\delta(k)$ will rapidly approach unity. One finds,

$$1 - \delta(k) \sim e^{-k^2} \quad ; \quad 1 < k < k_0. \quad (42)$$

Beyond $k - \epsilon \sim 1$, $B_0(k - \epsilon)$ starts to decrease (see Fig.2) and, eventually, the right hand side of (40) changes sign again. Comparison with (35) shows that this happens at the value $k = k_0$, previously assumed to be larger than the cutoff K and thus characterizing a bound state. However, as the electric field increases, k_0 decreases since, for reasonably small fields, (36) shows that $k_0 \simeq x_0/\epsilon$, with x_0 independent of the field. Eventually, therefore, $k_0 < K$, and k_0 then refers to a state *inside* the band, even though k_0 may still be large compared to unity. The

exponentially small prefactor on the left hand side of (40) then causes $\delta(k)$ to sharply decrease from $(1 - \mathcal{O}(e^{-k_0^2}))$ to $\mathcal{O}(e^{-k_0^2})$ as k is increased through k_0 . As can be seen from (40), this sharp decrease of $\delta(k)$ is completed within a range of k -values $\Delta k = \mathcal{O}(e^{-k_0^2})$. From $k = k_0 + \Delta k$ on, $\delta(k)$ remains exponentially close to zero. The qualitative behavior of $\delta(k)$ is sketched in Fig.4.

Figure 4: Sketch of $\delta(k)$ (a) near the middle of the band, and (b) close to $k = k_0$.

5.4 Summary

In summary, the picture with a sharp cutoff at $|k| = K$ is this: As long as $\epsilon K < 1/2$, the different Landau bands do not overlap. For very small electric fields, the solution $k_0 = x_0/\epsilon$ of (35) is outside the (lowest) band and corresponds to a bound state with energy $E_0 + x_0$, with x_0 a constant when effects quadratic in ϵ are neglected. Whereas the eigenvalue of the bound state is essentially unaffected by the electric field, the upper edge of delocalized states moves linearly with the field, $E_0 + \epsilon K$. As a consequence, the bound state will, eventually, become submerged in the continuum and lose its status as bound.

6 The Current

We calculate, in this section, the current carried by an eigenstate, since the *total* current in the x -direction discriminates between bound and extended states. Details will be relegated to Appendix D. In our scaling the unit for the (particle) current is $\hbar/l_B m$ and the dimensionless current is given by the expression

$$\langle \alpha | j_x | \alpha \rangle = \int d\mathbf{r} \left\{ \frac{1}{2i} \left[\Psi_\alpha^*(\mathbf{r}) \frac{\partial \Psi_\alpha(\mathbf{r})}{\partial x} - \frac{\partial \Psi_\alpha^*(\mathbf{r})}{\partial x} \Psi_\alpha(\mathbf{r}) \right] - y |\Psi_\alpha(\mathbf{r})|^2 \right\} \quad (43)$$

For bound states this expression should vanish whereas for continuum states it will be finite.

Before we embark on the calculations, we note that the total current summed over all the levels must be the same for the perturbed as for the unperturbed states,

$$\sum_\alpha \langle \alpha | j_x | \alpha \rangle = \sum_{n,k} \langle n, k | j_x | n, k \rangle. \quad (44)$$

This is the consequence of the invariance of the trace under the canonical transformation from the set $\Phi_{n,k}$ to the set Ψ_α . Thus the current missing in a bound states must show up as contributions to the current in the extended states. As mentioned in the Introduction, this fact (as pointed out by Prange[2]), has some bearing on the robustness of the integer quantum Hall effect.

Using the definition (5) of the variable y_k we may write

$$\langle \alpha | j_x | \alpha \rangle = \epsilon - \sum_{n,k} \sum_{n',k'} c_{n,k}^{\alpha*} c_{n',k'}^\alpha \langle n, k | y_k | n', k' \rangle \quad (45)$$

In order to evaluate the current, we first remark that the matrix elements of y_k in the unperturbed basis follow from the properties of the Hermite polynomials as

$$\langle n, k | y_k | n', k' \rangle = \delta_{k,k'} \left\{ \left(\frac{n}{2} \right)^{1/2} \delta_{n,n'+1} + \left(\frac{n+1}{2} \right)^{1/2} \delta_{n,n'-1} \right\} \quad (46)$$

Eq.(46) shows that one cannot restrict the discussion of the current to a single Landau level. This would automatically give the unperturbed value ϵ for each state, as y_k has matrix elements between states belonging to different Landau levels only.

Now insert the expansion (8) for Ψ_α , use (46), (13) and (14) to find the current in the form,

$$\langle \alpha | j_x | \alpha \rangle = \epsilon - \frac{N_\alpha}{D_\alpha} \quad (47)$$

The expression for the denominator D_α follows immediately from the normalization requirement (14) as

$$D_\alpha = \frac{1}{L} \sum_{n,j} \frac{f_n(k_j - \epsilon)}{(\epsilon k_\alpha - n - \epsilon k_j)^2}, \quad (48)$$

with f_n given by (28). The expression for N_α has been derived in Appendix D. It reads

$$N_\alpha = -\frac{1}{L} \sum_{n,j} \frac{\dot{f}_n(k_j - \epsilon)}{\epsilon k_\alpha - n - \epsilon k_j} \quad (49)$$

with

$$\dot{f}_n(q) = \frac{df_n(q)}{dq}. \quad (50)$$

For the bound states, k_α is outside the band cut off at $|k| = K$. As a result, none of the denominators in D_α and N_α vanish for the bound states, and the transition from sums to integrals is straightforward. Partial integration of (49) then gives

$$N_\alpha = \epsilon D_\alpha \quad \Rightarrow \quad \langle \alpha | j_x | \alpha \rangle = 0. \quad (51)$$

Thus, the name “bound state” is justified.

For states inside the band, the situation is clearly different. We must keep careful track of the shift in k -position of the various eigenstates, as given by δ_l of (27). In fact, we shall now show that N_α/D_α and thus, the current, can be expressed in terms of δ_l 's. We start by writing the eigenvalue equation (29), with $n_\alpha = 0$, in the form

$$\frac{1}{\lambda} = \frac{1}{L} \sum_{n,j} \frac{f_n(k_j - \epsilon)}{\epsilon[k_l - k_j + (2\pi\delta_l/L)] - n}. \quad (52)$$

A relation between two consecutive δ_l 's can be established by subtraction of the corresponding equation with $l \rightarrow l+1$ from (52). The difference in the two denominators beyond the δ 's is eliminated by shifting the summation variable $k_j \rightarrow k_{j+1}$ and noting that $(k_{l+1} - k_{j+1}) = (k_l - k_j)$. The corresponding shift in the numerator is handled by expansion,

$$f_n(k_{j+1} - \epsilon) = f_n(k_j - \epsilon) + \frac{2\pi}{L} \dot{f}_n(k_j - \epsilon) + \dots \quad (53)$$

In this manner one arrives at

$$\begin{aligned} & \sum_{n,j} \frac{\dot{f}_n(k_j - \epsilon)}{\epsilon[k_l - k_j + (2\pi\delta_{l+1}/L)] - n} \\ &= \sum_{n,j} \frac{\epsilon(\delta_{l+1} - \delta_l) f_n(k_j - \epsilon)}{\{\epsilon[k_l - k_j + (2\pi\delta_l/L)] - n\} \{\epsilon[k_l - k_j + (2\pi\delta_{l+1}/L)] - n\}}. \end{aligned} \quad (54)$$

The difference $\delta_{l+1} - \delta_l$ is always, for large L , of $\mathcal{O}(1/L)$. To leading order we can, therefore, replace δ_{l+1} by δ_l in the denominators in (54). Reference to (48) and (49) then gives the desired relation

$$\frac{N_\alpha}{D_\alpha} = -\epsilon(\delta_{l+1} - \delta_l). \quad (55)$$

From this a remarkable formula for the current immediately follows

$$\langle \alpha | j_x | \alpha \rangle = \epsilon(1 + \delta_{l+1} - \delta_l) \rightarrow \epsilon \left(1 + \frac{2\pi}{L} \frac{d\delta(k_\alpha)}{dk_\alpha} \right). \quad (56)$$

One could suspect that this relation is no more than the standard result for the group velocity,

$$v_{gr} = \frac{\partial E(k)}{\partial k}, \quad (57)$$

with reference to the “dispersion relation” (26) with (27). This suspicion may well be justified. However, a dispersion relation presupposes the existence of a well defined wave number in the x -direction, i.e., of translational invariance, continuous or discrete. Without a scatterer the system is clearly invariant for *any* translation in the x -direction. With a scatterer the continuous invariance is broken, and our association of a “wave number” k_α with the perturbed states was *not* based on translational invariance. Also, the crucial term in (56) is of $\mathcal{O}(1/L)$, and vanishes in the limit when k becomes a continuous variable, as needed in (57). In order to remove such doubts on the validity of (57), we have given the detailed independent proof of (56) in this section.

The redistribution of currents due to the existence of a bound state is easily understood on the basis of (56). On the one hand the second term on the right is of $\mathcal{O}(1/L)$ with respect to the first. I.e., in the limit $L \rightarrow \infty$ every single state carries the current ϵ . However, as $L \rightarrow \infty$, the number of current carrying states inside the given band diverges with L . Summing the $\mathcal{O}(1/L)$ term over all the states in the band, one finds the additional contribution $\epsilon[\delta(\infty) - \delta(-\infty)] = \epsilon$, precisely equal to the current lost by the bound state when the scatterer is introduced.

7 Metastable states

7.1 The qualitative picture

The results of the previous section leave a puzzle. Assume, as we have so far done throughout, that there is a “sharp” cutoff at a fixed value $|k| = K$. In the summary of Sec.5, we pointed out that when the band edge $E_0 + \epsilon K$ reaches the discrete eigenvalue $E_0 + x_0$ of the bound state, that state becomes submerged in the continuum, and is no longer bound. Eq.(56) shows that, in the limit $L \rightarrow \infty$, all extended states carry the same current, ϵ . In other words, the arguments so far seem to predict an abrupt change in the current from 0 to ϵ for the state with energy $E_0 + x_0$ as, with increasing electric field, the band edge hits the bound state. Physically, one would expect the current to change smoothly. How can this be reconciled with the results obtained?

An important hint is provided by Fig.4. The rapid decrease of $\delta(k)$ around k_0 from exponentially close to unity to exponentially close to zero happens over a range of k -values $\Delta k \sim e^{-k_0^2}$. For $1/L > e^{-k_0^2}$ this sudden drop is picked up by a single eigenstate with exponentially small current. This is not really a bound state but, rather, an eigenstate masquerading as a narrow resonance. With increasing electric field, i.e., decreasing k_0 , this resonant eigenstate provides an interpolation between bound states proper and standard extended states.

However, as $L \rightarrow \infty$, the separate identity of the resonant eigenstate dissolves, and the loss of current is now shared by a large number of states within Δk around

k_0 . This *linear combination* of eigenstates, rather than the single eigenstate, exhibits the resonance phenomenon in the sense that it carries an exponentially small current. Since a finite energy width, of $\mathcal{O}(\epsilon\Delta k) = \mathcal{O}(\epsilon e^{-k_0^2})$ is now needed, the linear combination is no longer stationary, it represents a *metastable* wave packet that eventually leaks away from the scatterer. By the same token, the exponentially small current refers to the *initial* stage of this development: Eventually, the current grows to the standard value of ϵ , after the wave packet has left the scatterer for good.

7.2 Calculations

We now substantiate the above picture by explicitly constructing the desired wave packet $|\Gamma\rangle = \sum_{\alpha} \gamma_{\alpha} |\alpha\rangle$ from the normalized eigenstates $|\alpha\rangle$. The wave packet itself is normalized to 1, i.e., $\sum_{\alpha} |\gamma_{\alpha}|^2 = 1$. It is essential for the subsequent arguments that the superposition is over an energy band sufficiently narrow that the inequality

$$\epsilon(k_{\alpha} - k_{\beta}) \ll 1, \quad (58)$$

holds for all pairs of states contributing significantly to the packet. Under this condition it is shown in Appendix D that interference effects are unimportant and that the time dependent current associated with the wave packet is

$$\begin{aligned} \langle \Gamma | j_x(t) | \Gamma \rangle &= \epsilon - \frac{1}{2} \sum_{\alpha, \beta} \frac{\gamma_{\beta}^*(t) \gamma_{\alpha}(t)}{\sqrt{D_{\beta} D_{\alpha}}} (N_{\alpha} + N_{\beta}) \\ &= \epsilon \left\{ 1 + \text{Re} \left[S^*(t) \frac{2\pi}{L} \sum_{\alpha} \gamma_{\alpha}(t) \sqrt{D_{\alpha}} \frac{d\delta(k_{\alpha})}{dk_{\alpha}} \right] \right\}, \end{aligned} \quad (59)$$

with $S(t) = \sum_{\alpha} \gamma_{\alpha}(t) / \sqrt{D_{\alpha}}$ and D_{α} given by (48). Note that (59) reduces to (56) when $\gamma_{\beta} = \delta_{\alpha, \beta}$.

We want to construct a wave packet which at $t = 0$ has an exponentially small current. The desired wave packet is one for which the coefficient in front of the derivative of $\delta(k)$ in (59) is unity. Thus we choose $\gamma_{\alpha}(0)$ to be real, and given by

$$\gamma_{\alpha}(0) = \frac{1}{S(0) \sqrt{D_{\alpha}}}. \quad (60)$$

This is consistent with the normalization requirement and gives $S^2(0) = \sum_{\alpha} 1/D_{\alpha}$. With $\gamma_{\alpha}(0)$ chosen according to (60) we now go to the limit $L \rightarrow \infty$ in (59) to get for the initial value of the current

$$\langle \Gamma | j_x(0) | \Gamma \rangle = \epsilon \left(1 + \int_{\kappa} dk \frac{d\delta}{dk} \right), \quad (61)$$

where the subscript κ is a reminder that the integral should go over a narrow band. If we now choose that narrow band to be centered at k_0 , where Fig.4 shows $\delta(k)$ to drop from exponentially close to unity to exponentially close to zero over a k -interval of $\mathcal{O}(e^{-k_0^2})$, our goal has been reached. Then eq.(61) shows

that, initially, the total current associated with the corresponding wave packet is exponentially small.

More explicitly, expansion of the right hand side of (40) around k_0 gives, to lowest order in $(k - k_0)$,

$$\cot[\pi\delta(k)] = \frac{2}{f_0(k_0 - \epsilon)} \cdot \frac{k - k_0}{2\pi(k_0 - \epsilon)^2} \simeq \frac{k - k_0}{\pi k_0^2 f_0(k_0)} \quad (62)$$

since k_0 is large and ϵ small. This can be rewritten as the Lorentzian

$$\sin^2[\pi\delta(k)] = \frac{w_0^2}{w_0^2 + (k - k_0)^2} \quad (63)$$

with exponentially small width, $w_0 = \pi k_0^2 f_0(k_0)$.

Similarly, the explicit form for the expansion coefficients γ_α of (60) can be found from the dominant part of D_α for large L . From (48) and (89) one has

$$D_\alpha = L \frac{f_0(k_\alpha - \epsilon)}{4\epsilon^2} \cdot \frac{1}{\sin^2[\pi\delta(k_\alpha)]} + \mathcal{O}(1). \quad (64)$$

Since the coefficients $\gamma_\alpha(t)$ develop time dependent phase factors, the current does not remain exponentially small. In order to calculate its time dependence, we need the explicit form (64) in (60). From (59) and (60) one finds,

$$\frac{S^*(t)}{S^*(0)} = \frac{\sum_\alpha e^{i\epsilon t(k_\alpha - k_0)} / D_\alpha}{\sum_\alpha 1 / D_\alpha} = \frac{1}{\pi} \int \frac{dk}{w_0} \frac{w_0^2 e^{i\epsilon t(k - k_0)}}{w_0^2 + (k - k_0)^2} = e^{-\epsilon w_0 t}. \quad (65)$$

The second time dependent factor in (59) gives, in fact, precisely the same integral. Putting this together, one finds the time dependence of the current as

$$[\epsilon - \langle \Gamma | j_x(t) | \Gamma \rangle] = [\epsilon - \langle \Gamma | j_x(0) | \Gamma \rangle] e^{-2\epsilon w_0 t}. \quad (66)$$

In other words, the lifetime τ of the metastable state is

$$\tau = \frac{1}{2\epsilon w_0} = \frac{\epsilon}{2\sqrt{\pi x_0^2}} e^{(x_0/\epsilon)^2}. \quad (67)$$

8 Discussion

In this paper we have analyzed the spectrum and corresponding eigenstates of an electron in a plane with a single repulsive scatterer of short range a , in a perpendicular magnetic and very weak in-plane electric field. The magnetic field is taken to be sufficiently weak that the radius of the scatterer is small on the scale set by the magnetic length, $a/l_B \ll 1$. In this range of fields (which covers essentially all attainable magnetic fields when a is of atomic dimensions), the problem can to a large extent be discussed as if the scatterer were a δ -function. However, to make the discussion mathematically and physically meaningful, cut-offs are needed. Appeal to the finite size of the system is unphysical in this

context. The only sound physical basis for cutoffs is the finite size of the scatterer. With an in-plane electric field, free cyclotron motion acquires a drift. The size a_y of the scatterer transverse to this drift provides the fundamental cutoff at $2n + 1 = 2N + 1 = (l_B/a_y)^2$ on the sum over Landau levels, essential for the convergence of such sums. The longitudinal size a_x does not have the same fundamental significance. It adds a cutoff to already convergent k -integrals, at $|k| = K = l_B/a_x$.

Starting with zero electric field we find isolated states bound to the repulsive scatterer, in addition to very degenerate Landau levels. The isolated states are those with zero angular momentum with respect to the center and, for the lower Landau levels, they are split off by an energy of $\mathcal{O}(1/\ln N)$. This is in contrast to the case of an attractive scatterer, for which the bound state energies diverge for a δ -function attractor of finite strength[3]. With a sharp k -cutoff, the bound states found here are the states that remain localized when the electric field is gradually increased from zero. The degenerate Landau levels, on the other hand, broaden into bands of states with delocalized wave functions.

The interplay between these bands and the localized states has been the main theme of the present paper. For weak electric fields, restricted roughly by the condition of non-overlapping bands (34), $\epsilon < 1/2K$, one bound state exists per Landau level, with eigenvalue above the corresponding band. As the electric field is increased the bound state energy is (to linear order in the field) unaffected, whereas the band edge moves upward, linearly in the field. As a result, the bound state will, eventually, become submerged in the band. This happens first at the lowest Landau level (subsequently at the lowest but one, etc.), namely when the requirement

$$\epsilon < \frac{x_0}{K} = \frac{c}{K \ln N}, \quad (68)$$

with c a constant of $\mathcal{O}(1)$, is violated. This requirement is, for large N , slightly more restrictive than the condition of non-overlapping bands, (34). In fact, it shows that a strict *repulsive* δ -function ($N \rightarrow \infty$) is *unable* to bind an electron, no matter how small is the (finite) electric field.

As one moves up through the Landau levels, the condition (68) is slowly weakened, and in the range $n \sim N/2$ the bound state remains separated from the band as long as (34) is fulfilled. It is for these high Landau levels one would expect that contact could be made with the classical criterion for trapped states to exist. Indeed, in terms of unscaled quantities (34) reads

$$\frac{e\mathcal{E}l_B}{\hbar\omega_c} \cdot \frac{l_B}{a} < \frac{1}{2} \Rightarrow v_d T_c < \pi a, \quad (69)$$

where $v_d = \mathcal{E}/B$ is the drift velocity in the crossed electromagnetic fields and T_c is the period of the cyclotron motion. Except for a numerical factor of $\mathcal{O}(1)$ this condition coincides with the *classical* condition for trapped states to exist[5] near spherical scatterers, namely, that the drift during one completed cyclotron period is less than the linear size of the scatterer. Neither the classical nor the quantum calculation is sufficiently precise to determine factors of $\mathcal{O}(1)$, and in this sense the agreement is perfect.

As the bound state becomes submerged into the band, it changes character and becomes metastable. More precisely, as shown explicitly in Sec.7, in the limit of an infinite system, a normalized linear combination $|\Gamma\rangle$ of eigenstates within a very narrow energy band can be constructed, such that the corresponding wave packet lives close to the scatterer for an exceedingly long time (67), of the order⁶

$$\tau \sim \epsilon \exp(x_0/\epsilon)^2. \quad (70)$$

The initial current associated with this metastable wave packet is of the order

$$\langle \Gamma | j_x | \Gamma \rangle = \mathcal{O}\{\exp(-k_0^2)\} = \mathcal{O}\{\exp[-(x_0/\epsilon)^2]\}. \quad (71)$$

This demonstrates that for small ϵ the distinction between bound states and metastable ones is more academic than practical. Extremely long lived metastable states are practically indistinguishable from bound states in the mathematically precise sense.

The relevance of this distinction becomes even more questionable when we recall that our discussion, so far, has been based on the assumption that the cutoff at $|k| = K$ is “sharp”, in the sense that the numerator in the k -integrals is strictly zero beyond the cutoff. This is the assumption that allows the sharp distinction that we made between states outside the band, which are bound, and states within the band, which are not. In reality, however, the cutoff is not sharp. To an already exponentially decaying numerator (the square of an unperturbed wave function), the finite size of the scatterer adds a decaying cutoff function as a factor. The details (possibly oscillatory) of this decaying function depends on the details of the repulsive potential, and are beyond the scope of the present work. All we can say at this point is that even those states that we have characterized as bound states in this paper are, generally, metastable ones, with a lifetime even somewhat longer⁷ than that (70) set by the exponential tail of the unperturbed wave functions. (True bound states could still exist, but then by accidental coincidence with zeros in the cutoff function.)

With the distinction between “bound states” and current carrying states blurred by the cutoff function being continuous rather than sharp, the condition (68) on the electric field for the lowest Landau level bound state to remain bound is no longer that relevant. As a more relevant question we propose the following: What is the range of electric fields for which the initial current associated with the lowest “bound state” wave packet is (order-of-magnitude-wise) less than a given small fraction ξ of the “freely” propagating states, with current ϵ ? Eq.(71) gives the (rough) answer,

$$\mathcal{O}\{\exp[-(x_0/\epsilon)^2]\} = \mathcal{O}\exp[-(c/\epsilon \ln N)^2]\} < \xi\epsilon. \quad (72)$$

⁶This result has been established for the metastable state associated with the lowest Landau level. For higher Landau levels, the asymptotics for large arguments of the Hermite polynomials in $f_n(q)$ of (28) add a factor of $(\epsilon/x_0)^{2n}$ to the result (70), with x_0 weakly dependent on the Landau level index, in agreement with the result found by Gyger and Martin[6].

⁷Only the prefactor would, presumably, be affected. The lifetime would still be dominated by the inverse Gaussian.

Comparison with (68) shows that the factor K of (68) is absent here. I.e., (72) is a considerably milder condition on the field than (68). However, the factor $1/\ln N$ is still present, confirming that a true δ -function scatterer cannot trap an electron in an electric field, no matter how weak.

Three remarks on the region $n \sim N/2$ where contact can be made with the classical results: First note that, in this region, the ratio K does set the realistic scale for k_0 and, thus, for the current to remain small. In other words, the criterion (69) remains valid. Next, note that the details of the potential is less relevant in quantum than in classical mechanics. For instance, a δ -like scatterer with finite (a_x, a_y) , but without spherical symmetry would, in quantum mechanics, give results that were almost indistinguishable from those for a symmetric scatterer. In classical mechanics, however, asymmetry would cause major perturbations on the scattering. Third remark: In classical mechanics it is easy to realize[5] that only those electrons that initially are close to the scatterer can be trapped. Electrons coming from minus infinity will, after a few collisions, move on towards plus infinity. This situation is closely mimicked by our quantum results. Choosing the expansion coefficients $\gamma(k)$ real as in (60) we have put the initial wave packet close to the scatterer, where it remains for a very long time. *In principle*, we could start the wave packet far from the scatterer with expansion coefficients designed in such a way that they would develop into the same $\gamma(k)$ when the scatterer is reached. However, that would require extremely delicate fine-tuning of the initial phases. In physical practice, one does not have the necessary control over the initial phases. As a result, a typical wave packet started off far to the left of the scatterer with the correct energy distribution, would *not* end up with the required $\gamma(k)$ when reaching the scatterer, and would, consequently, not be trapped for long.

We finally comment on the relations between our work and the parallel one by Gyger and Martin[6]. The main part of their paper makes the important extension to small but finite electric fields of previous work by Cavalcanti and de Carvalho[4], using an elegant formalism developed for a renormalized δ -function scatterer. Gyger and Martin find that the electric field turns the bound states into narrow resonances, with associated lifetimes in complete agreement with our results. Technically, their work is restricted to attractive scatterers, but they conjecture that the resulting lifetimes also apply to repulsive ones. We confirm this. Our approaches are, indeed, complementary: In order to make semi-quantitative contact with the classical results of Ref.[5], it was essential for us that the scatterer be *repulsive*, and with a finite radius, even if that limits the possibilities for obtaining results in closed form. It is gratifying that the two approaches mutually support each other.

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A The lowest bound state

In this appendix we discuss the bound states of a delta-potential in a magnetic field. Thus the electric field is set equal to zero. Then it is advantageous to take the symmetric gauge $\mathbf{A} = \mathcal{B}(-y/2, x/2, 0)$ and to use polar coordinates r, ϕ . The unperturbed eigenfunctions are given by

$$\Phi_{n,l}(r, \phi) = e^{-r^2/4} r^{|l|} L_n^l(r^2/2) \frac{e^{il\phi}}{\sqrt{2\pi}} \quad (73)$$

where $L_n^l(x)$ is the associated Laguerre polynomial, with normalization such that $L_n^0(0) = L_n(0) = 1$. The corresponding unperturbed eigenvalues are

$$E_{n,l} = n + 1/2 + (|l| + l)/2 \quad (74)$$

Note that all the states with a negative l are degenerate with $l = 0$.

The computation of the level shifts due to the delta-potential proceeds along the same lines as in Sec.2. There is a simplifying feature due to the fact that only states with $l = 0$ overlap with the delta-potential. Thus the set of states with $l \neq 0$ and a fixed n are unperturbed and those with $l < 0$ remain degenerate. As we may restrict ourselves to the class of states with $l = 0$ we drop this index. Then the eigenvalue equation, which is the equivalent of (15), becomes

$$\frac{2\pi}{\lambda} = \sum_n \frac{1}{E_\alpha - E_n}, \quad (75)$$

since $L_n(0) = 1$. Likewise, the perturbed wavefunction in $r = 0$ is given by (see (14))

$$\frac{2\pi}{\lambda^2 |\Psi_\alpha(0)|^2} = A^2 = \sum_n \frac{1}{(E_\alpha - E_n)^2}. \quad (76)$$

Finally the expansion coefficients c_n^α are as in (13) expressed as

$$c_n^\alpha = \frac{\lambda \Psi_\alpha(0)}{E_\alpha - E_n}. \quad (77)$$

Putting these elements together we get for the wavefunction

$$\Psi_\alpha(r) = \frac{e^{-r^2/4}}{\sqrt{2\pi} A} \sum_{n=0}^N \frac{L_n(r^2/2)}{E_\alpha - E_n} = \frac{e^{-r^2/4}}{\sqrt{2\pi}} f_\alpha(r), \quad (78)$$

which defines the correction function $f_\alpha(r)$.

B Discrete sums versus integrals

The problem is the evaluation, for large L , of discrete sums of the type

$$F(p) = \frac{1}{L} \sum_j \frac{h(k_j)}{p - k_j}, \quad (79)$$

where k_j runs through the values

$$k_j = \frac{2\pi}{L}j \quad ; \quad j = 0, \pm 1, \dots, \quad (80)$$

and $h(k)$ is a smooth function. The essential element is to see how the grid of points k_j interfere with the zero of the denominator. We, therefore, specify the value of p with respect to the grid as

$$p = k + \frac{2\pi}{L}\delta, \quad (81)$$

where k is the largest the grid point smaller than p , and $0 \leq \delta < 1$.

We seek an asymptotic expression for $F(p)$ for large L . For that purpose we define a small zone of size Δq , symmetrically located around the point k , such that

$$\frac{2\pi}{L} \ll \Delta q \ll 1. \quad (82)$$

Outside the symmetric interval of size Δq the summation can safely be converted into an integral. For sufficiently small Δq it is accurately approximated by the principal value integral. Inside the interval the smooth function $h(q)$ may be replaced by the constant value $h(k)$. The summation inside the interval yields⁸

$$\sum_j \frac{1}{\delta - j} = \pi \cot(\pi\delta). \quad (83)$$

As it stands in (83), the sum is only conditionally convergent, but the symmetry prescribed by the problem removes all ambiguities, and the summation boundaries should tend to infinity in a symmetric way.

Combining the principal value integral and the sum, we get for (79) the desired expression,

$$F(p) = \int \frac{dq}{2\pi} \mathcal{P} \frac{h(q)}{p - q} + \frac{h(p)}{2} \cot(\pi\delta). \quad (84)$$

A corollary of this follows by taking the derivative of (79) with respect to p

$$G(p) = \frac{\partial F(p)}{\partial p} = -\frac{1}{L} \sum_j \frac{h(k_j)}{(p - k_j)^2}. \quad (85)$$

We carry out the differentiation by keeping the grid fixed and varying δ , i.e., $dp = (2\pi/L)d\delta$. Introduction of the infinitesimal dp is compensated by a shift in the integration variable in the principle value integral to give,

$$\int \frac{dq}{2\pi} \mathcal{P} \frac{h(q)}{p + dp - q} = \int \frac{dq'}{2\pi} \mathcal{P} \frac{h(q' + dp)}{p - q'}, \quad (86)$$

⁸The derivation of (83) involves the Fourier sum of the function $\exp(2\pi i\delta x)$ on the interval $0 \leq x < 1$. The Fourier coefficients of this function are basically the summand of (83). The periodic function, represented by the Fourier sum, has a discontinuity at integer values of x . At $x = 0$ the Fourier sum converges to the midway value: $[\exp(2\pi i\delta) + 1]/2$.

so that

$$\frac{\partial}{\partial p} \int \frac{dq}{2\pi} \mathcal{P} \frac{h(q)}{p-q} = \int \frac{dq}{2\pi} \mathcal{P} \frac{\dot{h}(q)}{p-q}, \quad (87)$$

with $\dot{h}(q) = dh(q)/dq$. Differentiation of (83) gives

$$\sum_j \frac{1}{(\delta-j)^2} = \frac{\pi^2}{\sin^2(\pi\delta)}. \quad (88)$$

Combining these results we find

$$G(p) = \int \frac{dq}{2\pi} \mathcal{P} \frac{\dot{h}(q)}{p-q} + \frac{\dot{h}(p)}{2} \cot(\pi\delta) - \frac{Lh(p)}{4} \frac{1}{\sin^2(\pi\delta)}. \quad (89)$$

Note that $G(k)$ diverges with L as it should, according to (85). The last, leading, term in (89) follows directly from (88) as the contribution of a narrow region Δq around p .

C Bound State Eigenvalue

In this appendix we study in detail the eigenvalue equation (29) for $n_\alpha = 0$, and under the assumption that the eigenvalue represented by $x = \epsilon(k_0 - \epsilon)$ is outside the quasi-continuum. In other words, we study the eigenvalue equation for bound states.

$$\frac{2\pi}{\lambda} = \sum_n \frac{2\pi}{L} \sum_j \frac{f_n(q_j)}{x - n - \epsilon q_j} = \frac{2\pi}{L} \sum_j \frac{f_0(q_j)}{x - \epsilon q_j} + R, \quad (90)$$

where R is the contribution from the higher Landau levels. We can make it explicit by expanding the denominator

$$\frac{1}{x - n - \epsilon q} = \frac{1}{x - n} + \frac{\epsilon q}{(x - n)^2} + \frac{\epsilon^2 q^2}{(x - n)^3} - \dots \quad (91)$$

This serves a double purpose: We order the terms in powers of ϵ , and in the successive terms the sums over q_j can be converted into integrals over q that can be carried out by appeal to the properties of Hermite polynomials. We express the result in terms of the functions

$$F_j(x) = \sum_{n=1}^N \frac{1}{(x - n)^j} \quad (92)$$

Note that only $F_1(x)$ needs a cutoff in the summation over n , for $j > 1$ the sums are convergent without a cutoff. Straightforward calculations give

$$R = F_1(x) + \epsilon^2[-F_2(x) + (x + 1/2)F_3(x)] + \dots \quad (93)$$

Next we must evaluate the zeroth term in (90). Since the bound state is outside the cutoff we can, neglecting exponentially small terms, evaluate the corresponding integral in analogy with (93) as

$$\int dq \frac{f_0(q)}{x - \epsilon q} = \frac{1}{x} + \frac{\epsilon^2}{2x^3} + \dots \quad (94)$$

Combination of (93) and (94) leads to the equation for x

$$\frac{2\pi}{\lambda} = F_1(x) + \frac{1}{x} + \epsilon^2 \left[-F_2(x) + \left(x + \frac{1}{2}\right)F_3(x) + \frac{1}{2x^3} \right] + \dots \quad (95)$$

We can satisfy this equation by the expansion

$$x = x_0 + x_2\epsilon^2 + \dots \quad (96)$$

where the coefficients have to fulfill the equations

$$\frac{2\pi}{\lambda} = F_1(x_0) + \frac{1}{x_0} = \sum_{n=0}^N \frac{1}{x_0 - n}, \quad (97)$$

and

$$x_2 + 1 = \left[x_0 + \frac{1}{2} \right] \left[F_3(x_0) + \frac{1}{x_0^3} \right] \left[F_2(x_0) + \frac{1}{x_0^2} \right]^{-1}. \quad (98)$$

Insertion of this result finally yields for $k_0 = \epsilon + x/\epsilon$,

$$k_0 = x_0/\epsilon + (x_2 + 1)\epsilon \dots \quad (99)$$

The procedure used in this appendix produces a power series in ϵ^2 for the bound state eigenvalue. Although this series is an asymptotic one, it confirms that, for sufficiently small ϵ , we are justified in treating x and ϵk_0 as essentially ϵ -independent quantities.

For states *inside* the lowest Landau band, we can still use the same procedure in evaluating the remainder R . However, for the integral over the lowest band, the singularity in the denominator requires handling by the method of Appendix B.

D The current

In this appendix we give the details on the transformations for the expression for the current in a single eigenstate. Subsequently we study the current associated with a linear combination of a narrow (in energy) band of such states. For notational simplicity, all formulas will refer to states close to the lowest Landau band. Generalization to higher Landau bands is straightforward.

D.1 A single eigenstate

Using the expression for the current as given in (45)-(46) and (13)-(14) we can write the correction to the free current as a ratio of which the numerator and denominator are given as

$$N_\alpha = \sum_n \frac{1}{\sqrt{\pi} 2^n n!} \frac{1}{L} \sum_j e^{-q_j^2} \frac{H_n(-q_j) H_{n+1}(-q_j)}{(x - n - \epsilon q_j)(x - n - 1 - \epsilon q_j)} \quad (100)$$

and as

$$D_\alpha = \frac{1}{L} \sum_{n,j} \frac{f_n(q_j)}{(x - n - \epsilon q_j)^2} \quad (101)$$

Here $x = x_\alpha$ is defined in (25), q_j in (29) and $f_n(q)$ in (28). The denominator of (100) can be decomposed as

$$\frac{1}{(x - n - \epsilon q_j)(x - n - 1 - \epsilon q_j)} = \frac{1}{x - n - 1 - \epsilon q_j} - \frac{1}{x - n - \epsilon q_j} \quad (102)$$

In the first term we shift the summation variable from $n+1$ back to n and combine the two terms to get

$$N_\alpha = \sum_n \frac{1}{L\sqrt{\pi}2^n n!} \sum_j e^{-q_j^2} H_n(-q_j) \frac{2nH_{n-1}(-q_j) - H_{n+1}(-q_j)}{x - n - \epsilon q_j} \quad (103)$$

We then use the following property of the Hermite polynomials

$$\frac{d}{dq} e^{-q^2} H_n^2(q) = e^{-q^2} H_n(-q) [H_{n+1}(-q) - 2nH_{n-1}(-q)] \quad (104)$$

to write

$$N_\alpha = -\frac{1}{L} \sum_{n,j} \frac{\dot{f}_n(q_j)}{x - n - \epsilon q_j} \quad (105)$$

Note that for the bound states discussed in Sec.5, the value of k_α is outside the cutoff on the k -integration. As a result, no zero exists in the denominators of (101) and (105), and the summation over j can be directly converted into an integral. Partial integration of (105) and comparison with (101) then immediately leads to the relation $N_\alpha = \epsilon D_\alpha$, demonstrating that the current vanishes.

For eigenstates inside the band, we showed in Sec.6 that the corresponding current can be written as

$$\langle \alpha | j_x | \alpha \rangle = \epsilon - \frac{N_\alpha}{D_\alpha} = \epsilon(1 + \delta_{l+1} - \delta_l). \quad (106)$$

Since, for large L , $\delta_{l+1} - \delta_l = \mathcal{O}(1/L)$, all eigenstates in the band will individually, to an arbitrary good approximation, carry the current ϵ , equal to that of the unperturbed states.

D.2 A narrow band of states

In this subsection we derive a generalization of (56) that can be used to construct a special linear superposition of states $|\Gamma\rangle$ which, initially, carries an exponentially small current.

First write the normalized eigenstate in the form

$$|\alpha\rangle = \frac{1}{\sqrt{D_\alpha}} \sum_{n,k} \frac{\Phi_{n,k}^*(0)}{E_\alpha - E_{n,k}} |n, k\rangle, \quad (107)$$

where the normalization follows from (14) and the definition (48) of D_α . From these states we construct a normalized wave packet

$$|\Gamma\rangle = \sum_{\alpha} \gamma_{\alpha} |\alpha\rangle \quad (108)$$

with the normalization requirement $\sum_{\alpha} |\gamma_{\alpha}|^2 = 1$. Our ultimate aim is to construct this wave packet such that the current initially is exponentially small. Use the result (45)-(46) and repeat the steps in the previous subsection to get

$$\begin{aligned} \langle \Gamma | j_x | \Gamma \rangle &= \epsilon \sum_{\alpha, \beta} \frac{\gamma_{\beta}^* \gamma_{\alpha}}{\sqrt{D_{\beta} D_{\alpha}}} \sum_n \frac{1}{\sqrt{\pi} 2^n n!} \frac{1}{L} \sum_j e^{-q_j^2} H_n(-q_j) H_{n+1}(-q_j) \\ &\times \left\{ \left[\frac{1}{x_{\beta} - \epsilon q_j - n - 1} - \frac{1}{x_{\alpha} - \epsilon q_j - n} \right] \frac{1}{1 + \epsilon(k_{\alpha} - k_{\beta})} \right. \\ &\left. + \left[\frac{1}{x_{\alpha} - \epsilon q_j - n - 1} - \frac{1}{x_{\beta} - \epsilon q_j - n} \right] \frac{1}{1 - \epsilon(k_{\alpha} - k_{\beta})} \right\}. \end{aligned} \quad (109)$$

We now insist that all states contributing to the superposition lie within a narrow energy band corresponding to a $\Delta k = w_0$ such that $\epsilon w_0 \ll 1$. This allows us to replace the factors $[1 \pm \epsilon(k_{\alpha} - k_{\beta})]^{-1}$ in (109) by unity. The resulting simplifications give, after some algebra,

$$\langle \Gamma | j_x | \Gamma \rangle = \epsilon - \frac{1}{2} \sum_{\alpha, \beta} \frac{\gamma_{\beta}^* \gamma_{\alpha}}{\sqrt{D_{\beta} D_{\alpha}}} (N_{\alpha} + N_{\beta}). \quad (110)$$

Now introduce the sum $S(t) = \sum_{\alpha} \gamma_{\alpha}(t) / \sqrt{D_{\alpha}}$, and use (55), (56) to rewrite (110) in the form

$$\langle \Gamma | j_x(t) | \Gamma \rangle = \epsilon \left\{ 1 + \text{Re} \left[S^*(t) \frac{2\pi}{L} \sum_{\alpha} \gamma_{\alpha}(t) \sqrt{D_{\alpha}} \frac{d\delta(k_{\alpha})}{dk_{\alpha}} \right] \right\}. \quad (111)$$

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