Leptogenesis with Heavy Majorana Neutrinos Revisited

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Abstract

The mass term for Majorana neutrinos explicitly violates lepton number. Several authors have used this fact to create a lepton asymmetry in the universe by considering CP violating effects in the one loop self-energy correction for the decaying heavy Majorana neutrino. We compare and comment on the different approaches used to calculate the lepton asymmetry including those using an effective Hamiltonian and resummed propagators. We also recalculate the asymmetry in the small mass difference limit.

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After Sakharov’s seminal paper on the conditions for obtaining a matter asymmetric universe several authors have attempted to realise these conditions in the context of different models[1]. One approach is to create a lepton asymmetric universe by the asymmetric decay of heavy scalars or Majorana neutrinos and to convert the lepton asymmetry into a baryon asymmetry by B+L violating sphaleron processes. In the original papers on leptogenesis CP violation was realised by the interference between the tree level diagram and the one loop vertex correction[2]. Later it was realised that there was an additional contribution from the one loop self-energy correction to the external heavy scalar or neutrino[3, 4].

The self-energy contribution has been calculated differently by different authors. Refs.[4, 5] obtain the asymmetry by considering the interference of the tree level diagram for decay and the diagram containing the one loop self-energy correction. However this approach breaks down in the degenerate heavy particle mass limit. In refs.[6, 7, 8, 9, 10] the authors obtain the one loop effective Hamiltonian, or more precisely, the one loop effective mass matrix for the heavy species. Because particles in the self-energy loop go on shell the effective mass matrix is non-Hermitian. This, coupled with CP violation, results in different evolution of the scalars and their antiparticles, or of the right-handed and left-handed neutrinos, and ultimately gives rise to an asymmetry when the heavy particles decay. In refs.[7, 10] the authors find an enhancement in the degenerate scalar or neutrino mass limit[11]. The authors of refs.[6, 7, 8, 9] and [10] differ in their use of the effective Hamiltonian. In this paper we compare and comment on both approaches and on the approach in refs.[12, 13] where the authors use resummed propagators to evaluate cross sections and decay rates. We shall discuss below models involving decays of heavy Majorana neutrinos. However our comments also apply to models with heavy scalars. In refs.[8, 10] the authors consider models with charged scalar fields; to facilitate discussion we have modified their definitions to suit the Majorana fields \( N_a \).

In section 1 we discuss some subtleties associated with a non-Hermitian Hamiltonian, particularly with respect to the normalisation of eigenstates of the Hamiltonian. In section 2 we compare the approaches of refs.[7] and [10] both of whom use the effective Hamiltonian. In section 3 we discuss some field theoretic concerns regarding the definition of the decay amplitude for eigenkets of a non-Hermitian Hamiltonian and the existence of a Fock space for mixed fields. In section 4 we compare the matrices used to diagonalise the effective Hamiltonian or projections of the resummed propagator in refs.[7, 9, 12]. We also comment on the approach of ref.[13]. We re-evaluate the asymmetry in the small mass difference limit.
In section 5 we summarize our results.

1 Diagonalisation of the effective mass matrix

Let us start with the lagrangian

\[ L_{int} = \sum_i M_i [ (\bar{N}_R^i) c N_R^i + (\bar{N}_L^i) c (N_R^i) c ] \]

\[ + \sum_{\alpha,i} h_{\alpha i}^* \bar{N}_R^i \phi^\dagger \ell_L^\alpha + \sum_{\alpha,i} h_{\alpha i} \ell_L^\alpha \phi N_R^i \]

\[ + \sum_{\alpha,i} h_{\alpha i}^* (\ell_L^\alpha)^c \phi^\dagger (N_R^i)^c + \sum_{\alpha,i} h_{\alpha i} (N_R^i)^c \phi (\ell_L^\alpha)^c \]

(1)

as in ref.[7]. \( N_R^c = (N_i^c R)^c \) and \( N^c = C \bar{N}^T \). \( \ell_L^i \) are light leptons and \( \phi \) is a scalar field. For concreteness the \( \ell_L^i \) can be the left-handed lepton doublet of the Standard Model, \( \phi \) can be the Standard Model Higgs doublet. We suppress the SU(2) indices carried by the lepton and the Higgs fields which are contracted by the \( \epsilon_{ab} \) tensor. \( \alpha \) and \( i \) are generation indices and we assume one heavy Majorana field per generation of light leptons. For simplicity we shall work with two generations. The lagrangian may also be rewritten in terms of \( \eta_i = N_i^R + N_i^C \).

In what follows we shall replace \( N_i^C \) by \( N_i^L \).

Since the right handed neutrinos \( N_i^R \) decay to leptons while the left-handed neutrinos \( N_i^L \) decay to antileptons, the lepton number violation is in the mass term for the heavy neutrinos. Note that the last two lines in the lagrangian above are equivalent as \( \chi_1 \chi_2 = \bar{\chi}_2 \bar{\chi}_1 \). One loop effects give rise to mixing between heavy neutrino flavours as shown in fig. 1. The one loop effective mass term due to the diagrams in fig. 1 is given by

\[
\begin{pmatrix}
0 & 0 & M_1 + H_{11} & H_{12} \\
0 & 0 & H_{12} & M_2 + H_{22} \\
M_1 + H_{11} & \bar{H}_{12} & 0 & 0 \\
\bar{H}_{12} & M_2 + H_{22} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
N_{1L} \\
N_{2L} \\
N_{1R} \\
N_{2R}
\end{pmatrix}
\]

(2)

Ignoring the masses of the light leptons and of the scalar field, the expressions for \( H_{ij} \) and \( \bar{H}_{ij} \) are given in ref.[7] as

\[
H_{ij} = H_{ji} = \left[ M_i \sum_{\alpha} h_{\alpha i}^* h_{\alpha j} + M_j \sum_{\alpha} h_{\alpha i} h_{\alpha j}^* \right] \left( g_{\alpha ij}^{\text{dis}} - \frac{i}{2} g_{\alpha ij}^{\text{abs}} \right)
\]

(3)

\[
\bar{H}_{ij} = \bar{H}_{ji} = \left[ M_i \sum_{\alpha} h_{\alpha i} h_{\alpha j}^* + M_j \sum_{\alpha} h_{\alpha i}^* h_{\alpha j} \right] \left( g_{\alpha ij}^{\text{dis}} - \frac{i}{2} g_{\alpha ij}^{\text{abs}} \right)
\]

(4)

3
Figure 1: One loop contributions to the mass matrix
and

\[ H_{ii}^{(1)} = \tilde{H}_{ii}^{(1)} = \begin{bmatrix} 2M_i \sum_{\alpha} h_{\alpha i} h_{\alpha i}^* \end{bmatrix} \left( g_{\alpha ii}^{\text{dis}} - \frac{i}{2} g_{\alpha ii}^{\text{abs}} \right) \tag{5} \]

where \( g_{\alpha ij}^{\text{abs}} = \frac{1}{16\pi} \).

The effective mass matrix \( M_{\text{eff}} = M_0 + H \) in eq. (2) can be written as \( M = -\frac{i}{2} \Gamma \) where the dispersive part of \( H \) (i.e. terms proportional to \( g_{\alpha ij}^{\text{dis}} \)) is included in \( M \) and the absorptive part (proportional to \( g_{\alpha ij}^{\text{abs}} \)) is included in \( \Gamma \). The absorptive part reflects the fact that the particles in the loop have gone on shell. Both \( M \) and \( \Gamma \) are Hermitian but not \( M_{\text{eff}} \). This is analogous to the formulation used to obtain an effective mass matrix for neutral kaons[14]. The one loop corrections in \( M \) can be absorbed away by renormalisation. (The off-diagonal elements of \( M \) are renormalised by non-diagonal elements of the wave function counterterms[15].) However, there are no absorptive counterterms[15], nor are they needed for the absorptive parts are finite. It will be the mixing via the absorptive terms that will be important for creating the lepton asymmetry.

Since the effective mass matrix is non-Hermitian it can not be diagonalised by a unitary matrix, i.e. \( VM_{\text{eff}}V^{-1} = M_D \), where \( M_D \) is diagonal, but \( V^{-1} \neq V^\dagger \). The columns of \( V^{-1} \) are the eigenkets of \( M_{\text{eff}} \) and the rows of \( V \) are the suitably normalised eigenbras of \( M_{\text{eff}} \), i.e., \( x M_{\text{eff}} = \lambda x \). (The subtleties of normalisation are discussed below.) If \( M_{\text{eff}} \) were Hermitian the eigenbras of \( M_{\text{eff}} \) would be the Hermitian conjugate of the eigenkets and \( V^{-1} \) would equal \( V^\dagger \). This is not true in our case, though the bra-eigenvalues and ket-eigenvalues are the same.

Therefore we rewrite the mass term of eq. (2) as

\[ \bar{\tilde{N}} M_{\text{eff}} N = \bar{\tilde{\psi}} M_D \psi = \bar{\tilde{\psi}} M' \psi \tag{6} \]

where \( \psi = V N \) and \( \bar{\tilde{\psi}} = \bar{\tilde{N}}V^{-1} \), and only \( M_D \) is diagonal above. Besides the choice mentioned above one can diagonalise \( M_{\text{eff}} \) by a bi-unitary transformation, i.e, \( U^\dagger M_{\text{eff}} V = M_D \), where \( U \) and \( V \) are unitary. However in this case the kinetic energy term will not be diagonal in the new \( (\tilde{\psi}, \psi) \) basis.

If one can expand the \( \psi, \tilde{\psi} \) and \( N \) fields in terms of annihilation and creation operators (later we comment on this assumption), then \( \bar{\tilde{\psi}}_c = |N_a\rangle V^{-1} \), where \( |\tilde{\psi}_c\rangle \) represents a particle created by the action of the creation operator in \( \tilde{\psi}_c(x) = \bar{\tilde{N}}_a(x)V^{-1} \) on the vacuum and \( |N_a\rangle \) represents a particle created by the creation operator in \( \bar{\tilde{N}}_a(x) \) acting on the vacuum. Note
that while $\sum_a |N_a\rangle\langle N_a| = 1$, $\sum_a |\psi_a\rangle\langle \psi_a| \neq 1$ where $\langle \psi_a| = |\psi_a\rangle^\dagger$. Instead $\sum_a |\tilde{\psi}_a\rangle\langle \psi_a| = 1$ and $\langle \psi_a|\tilde{\psi}_a\rangle = 1$ where $\langle \psi_a|$ represents a particle created by the action of the destruction operator in $\bar{\psi}(x)$ acting to the left on the vacuum\(^2\).

## 2 Obtaining the lepton asymmetry

Refs.[6, 7, 8, 9] and [10] differ in their choice of the initial state relevant for the calculation of the lepton asymmetry. Let $\epsilon$ be a measure of CP violation due to the self-energy correction. In refs.[6, 7, 8, 9] the authors define

\[
\epsilon_{\psi_a} = \left[ \frac{\Gamma_{\psi_a \rightarrow l} - \Gamma_{\tilde{\psi}_a \rightarrow l^*}}{\Gamma_{\psi_a \rightarrow l} + \Gamma_{\tilde{\psi}_a \rightarrow l^*}} \right] \quad (7)
\]

while the definition of the asymmetry parameter in ref.[10] is

\[
\epsilon_{N_a} = \left[ \frac{\Gamma_{N_a \rightarrow l} - \Gamma_{N_a \rightarrow l^*}}{\Gamma_{N_a \rightarrow l} + \Gamma_{N_a \rightarrow l^*}} \right] \quad (8)
\]

where, as stated earlier, the definitions of the asymmetry parameter in refs.[8, 10] have been modified for a scenario involving Majorana neutrinos. Below we compare the approaches of refs.[7] and [10].

Before we proceed let us reiterate here the mechanism of the creation of the lepton asymmetry. Starting with two relativistic neutrino species the second (heavier) species becomes non-relativistic at $t_{nr2}$. After $t_{nr2}$ the physical states are the eigenstates of an effective Hamiltonian for one relativistic neutrino species with a Lorentz suppressed mass and one non-relativistic neutrino species. They can be represented by $N_1$ and $\psi_2'$ fields, where $\psi_2'$ is a linear combination of the $N_a$ (but is not $\psi_2$). We assume the heavier $\psi_2'$ decay while the $N_1$ neutrinos are in equilibrium. At time $t_{eq}$ the lighter neutrinos $N_1$ go out of equilibrium and later at $t_{nr}$ they become non-relativistic, i.e., we assume $K = M_1/T_{eq} \ll 1$, where $T_{eq}$ is the temperature at $t_{eq}$. Thus at $t_{nr}$ one will have a universe with only $N_1$ neutrinos with equal numbers of right-handed and left-handed neutrinos, $N_{1R}$ and $N_{1L}$. After $t_{nr}$ the effective

\(^1\)In ref.[9] the authors discuss the fact that because of the non-Hermitian nature of the effective mass matrix $\psi \neq \psi^c$. They state that $\psi$ still represents a Majorana field and hence we sum only over particle states. In models involving charged scalars one certainly needs to consider both particle and antiparticle states. In that case the sum over all states will include particle and antiparticle states.

\(^2\)The normalisation of $\psi$ states has also been discussed elsewhere, as for example, in ref.[16] and references thereof.
Hamiltonian is as in eq. (2) and the $\psi_a$ are the eigenstates of the effective Hamiltonian. The state of the universe with equal numbers of $N_{1R}$ and $N_{1L}$ can now be described as a linear combination of the $\psi_a$. More precisely, a state $|N_{1R}\rangle$ (or $|N_{1L}\rangle$) at $t_{nr}$ can be rewritten in terms of $|\tilde{\psi}_a\rangle$. As in the standard neutrino flavour oscillation scenario, the right- and left-handed content of an originally $|N_{1R}\rangle$ or $|N_{1L}\rangle$ state will now start to oscillate. One can see from eq. (2) that the amplitude for $|N_{1R}\rangle \rightarrow |N_{2L}\rangle$ is proportional to $H_{12}$ while the amplitude for $|N_{1L}\rangle \rightarrow |N_{2R}\rangle$ is proportional to $\tilde{H}_{12}$. (The amplitudes for $|N_{1R,L}\rangle \rightarrow |N_{1L,R}\rangle$ are equal and proportional to $M_1 + H_{11}$.) Because of CP violation $H_{12} \neq \tilde{H}_{12}$. Therefore, though one starts with an equal number of $N_{1R}$ and $N_{1L}$, more $N_{1L}$ may transform into $N_{2R}$ than $N_{1R}$ into $N_{2L}$ and one ends up with an universe with an unequal content of $N_{1R}$ plus $N_{2R}$ and $N_{1L}$ plus $N_{2L}$. When the neutrinos subsequently decay one obtains a lepton number asymmetry since right-handed neutrinos decay to leptons and left-handed neutrinos decay to anti-leptons. (We ignore here asymmetry due to the vertex correction.)

The formalism of ref.[10] starts with a universe with equal numbers of $N_{1R}$ and $N_{1L}$, or, with states $|N_{1R}\rangle$ and $|N_{1L}\rangle$ with equal probability of existing at $t_{nr}$. The asymmetry obtained reflects the oscillations in the right-left $N$ content that takes place prior to decay. On the other hand, the final asymmetry is obtained in ref.[7] by considering only $|\tilde{\psi}_1\rangle$. One may argue that typically one does consider the asymmetry generated by only the lightest neutrino. But what that entails in this scenario is that the heavier $\psi'_2$ neutrino decays while the interactions of the lighter $N_1$ are still in equilibrium, leading to the erasure of any asymmetry generated in the decay of $\psi'_2$. Subsequently the lightest neutrino $N_1$ goes out of equilibrium, becomes non-relativistic and is re-expressed as a linear combination of both $|\tilde{\psi}_a\rangle$ states. Thus one has to consider the lightest $N_a$ species and not the lightest $\psi_a$ species as the initial state at $t_{nr}$. Nevertheless, since $|\tilde{\psi}_1\rangle$ is an asymmetric mixture of $|N_{aR}\rangle$ and $|N_{aL}\rangle$ its decay gives a lepton asymmetry. Ref.[16] also argues against starting with the $\psi_a$ states as these are “propagation” states which the authors claim are not physical. However our observations are based on a cosmological argument rather than a field theoretic argument. (We shall also state the field theoretic argument later.) The importance of the initial state is also discussed in ref.[10]. Our arguments above do not apply to scenarios where $K \gg 1$, i.e., when the neutrinos are non-relativistic when they go out of equilibrium. In such cases, from a cosmological standpoint, $\epsilon_{\psi_a}$ will be the relevant asymmetry parameter.

In the limit of large mass splittings, i.e., $|M_1 - M_2| \gg |H_{ij}|$, the asymmetry obtained
in ref.[10] is similar to that obtained in ref.[7]. We find that the approximations needed to obtain this result, i.e., ignoring terms proportional to $W_1^*W_2$, $W_2^*W_1$ and $|W_2|^2$ in eq. (17) of ref.[10] and assuming a small mixing angle is equivalent to ignoring the asymmetry from the interference between $\psi_1$ and $\psi_2$ and from $\psi_2$ decay and to setting $N_1 \approx \psi_1$. However in the small mass splitting case the asymmetry obtained in refs.[7, 10] are not similar, as we show later.

Our arguments imply that one has a mixed state (as opposed to a pure state) of $|N_1R\rangle$ and $|N_1L\rangle$ with equal probabilities at $t_{eq}$. Thereafter one evolves the $|N_{1R,L}\rangle$ states independently. Since the mass term is not significant when $N_1$ is relativistic there will be no significant mixing till $t_{nr}$. After $t_{nr}$ both $|N_{1R,L}\rangle$ can be rewritten as pure states of $|\tilde{\psi}_1\rangle$ and $|\tilde{\psi}_2\rangle$ and hence one gets interference between $|\tilde{\psi}_1\rangle$ and $|\tilde{\psi}_2\rangle$ states which is necessary for left-right oscillations. However if one argues that one instead has a mixed state of $|\tilde{\psi}_a\rangle$ at $t_{nr}$ then one would obtain $\epsilon_{\psi_a}$ as in ref.[7]. In that case the initial number densities of the $\psi_a$ particles that enter into the solution of the Botzmann equation would be weighted by the probabilities of obtaining the $\psi_a$ at $t_{nr}$, given by $|\langle \psi_a|N_1\rangle|^2$. One would then include the contributions of both $\psi_a$ to the total asymmetry since the asymmetry generated by the decay of $\psi_2$ will not be erased as $\psi_1$ interactions are out of equilibrium at $t_{nr}$.

3 Some field theoretic concerns

We now raise more fundamental issues with both the approaches using the effective Hamiltonian. The asymmetry parameter is defined in refs.[6, 7, 8, 9] using decay rates for rotated fields $\psi_a$ with complex masses. It is shown in ref.[17] (in the context of kaon physics) that the lack of orthonormality of the $|\tilde{\psi}_a\rangle$ states raises ambiguities in the normalisation of states making it difficult to interpret $\langle f|\tilde{\psi}_a\rangle$ as the probability amplitude for $\psi_a \rightarrow f$. If one normalises $|\tilde{\psi}_a\rangle$ as $|\tilde{\psi}_a\rangle/\sqrt{\langle \tilde{\psi}_a|\tilde{\psi}_a\rangle}$ then the state is scale dependent. This is because $|\tilde{\psi}_a\rangle \rightarrow c|\tilde{\psi}_a\rangle$ implies $\langle \psi_a | \rightarrow |\tilde{\psi}_a|c^{-1}$ since $|\tilde{\psi}\rangle = |N\rangle V^{-1}$ and $\langle \tilde{\psi} | = V\langle N\rangle$. On the other hand, if one normalises the $\tilde{\psi}$ states as $|\tilde{\psi}_a\rangle/\sqrt{\langle \tilde{\psi}_a|\tilde{\psi}_a\rangle}$ then $\sum_f |\langle f|\tilde{\psi}_a\rangle|^2/|\langle \tilde{\psi}_a|\tilde{\psi}_a\rangle| \neq 1$. This can be better understood by considering the quantum mechanical example of kaon mixing with $|\tilde{\psi}_a\rangle$ corresponding to $|K_L\rangle$ (or $|K_S\rangle$) and $|f\rangle$ corresponding to $\langle K_{L,S}\rangle$ and $\langle \tilde{K}_{L,S}\rangle$. A similar argument can be extended to field theoretic scenarios. If $\psi_a$ are intermediate particles, i.e., if one includes a production amplitude for the $\psi_a$, the above concerns do not apply [17]. If one does include a production amplitude for the $\psi_a$ in $\epsilon_{\psi_a}$ it will cancel out in the numerator.
and denominator leaving behind the asymmetry parameter as defined in refs. [6, 7, 8, 9].

In ref. [10] the asymmetry parameter is defined in terms of decay rates of the mixed \( N_a \) fields. It has been pointed out in ref. [18] that in general there does not exist a Fock space for mixed fields. If one assumes that the annihilation and creation operators of the \( \psi_a \) and \( \bar{\psi}_a \) fields respectively obey canonical anticommutation relations then the annihilation and creation operators of the \( N_a \) fields satisfy, for example, [18]

\[
\{ A_i(k, h, t), A^\dagger_j(k', h', t) \} = \delta^3(k - k')\delta_{hh'} \sum_a V^{-1}_{ia} V_{aj} \frac{E'_a - h|k|}{2E'_a} \tag{9}
\]

where \( h, h' = \pm 1 \) represent helicity, \( E'_a = (|k|^2 + m_a^2 - im_a \Gamma_a)^{\frac{1}{2}} \), the sum is over the mass eigenstates, \( A_i \) and \( A^\dagger_j \) include spinor elements as in ref. [18] and we have used \( N = V^{-1}\psi \) and \( \bar{N} = \bar{\psi}V \). These anticommutation relations are non-diagonal in the \( i, j \) indices and are hence not canonical. Therefore if \( \bar{N} = \bar{\psi}V, |N\rangle = |\bar{\psi}\rangle V \) is not a well defined statement in general. Furthermore, the two cases in which one can define an approximate Fock space for the \( N \) fields, namely, the extreme relativistic limit and almost degenerate masses with real \( V \) [18] do not apply to our problem.

In the limit that all the neutrino species are non-relativistic, as is the case for \( t \geq t_{nr} \), the above commutation relations are diagonal in the \( i, j \) indices. However, as pointed out in ref. [18], the annihilation and creation operators of the \( N \) fields are no longer time-independent in this case. But if one starts with an \( N_a \) state as the initial state at \( t_{nr} \) and evolves it as a linear combination of \( \psi_a \) states then this may not be a problem since one requires that the relation \( |N\rangle = |\bar{\psi}\rangle V \) hold at only one instant, namely, at \( t_{nr} \). Note that in ref. [10] the states \( |\bar{\psi}\rangle \) and \( |N\rangle \) are in co-ordinate space. The creation and annihilation operators for the mixed fields in co-ordinate space, namely, \( \bar{N}(x) \) and \( N(x) \), do satisfy diagonal anticommutation relations.

## 4 Final comments

It was pointed out by Veltman that it is inappropriate to have unstable particles as asymptotic in or out states [20]. To avoid this problem different effective approaches have been adopted to obtain the one loop decay amplitude. In refs. [6, 7, 8, 9, 10] the authors use an one loop effective Hamiltonian. In ref. [12] the decay amplitude is obtained by splitting a two particle scattering amplitude with the unstable particle as the intermediate particle. In
ref.[13] one obtains the decay amplitude from an effective LSZ formulation. We discuss here the differences and similarities in the above approaches to obtain the asymmetry.\textsuperscript{3}

Broadly speaking, one can further divide the different approaches into those that use unrotated fields $N_a$ to define the initial state, such as refs.[10, 13] and those that work with rotated fields to define the initial state, such as refs.[6, 7, 8, 9, 12]\textsuperscript{4}. Though refs.[7, 9, 12] deal with rotated fields their approaches are somewhat different. Below we first compare the rotating matrices for all these approaches in the small and large mass difference limits. We also comment on ref.[13]. We finally re-evaluate the asymmetry in the small mass difference limit.

In ref.[7] the authors obtain the eigenvalues and eigenvectors of the effective mass matrix in eq. (2). In ref.[9] the authors obtain the eigenvalues and eigenkets of the two off-diagonal blocks of $M_{eff}$. In general, the eigenvalues of the two off-diagonal blocks of the matrix do not equal each other or the eigenvalues of the full matrix and the eigenkets of the off-diagonal blocks describe particles different from those related to the eigenkets of the full matrix.

One can show that the eigenvalues of the two off-diagonal blocks are the same as that of the larger matrix, and the eigenvectors of the larger matrix are trivial combinations of corresponding eigenvectors of the smaller blocks when the two off-diagonal blocks are the same. In the limit of a small mass difference $|M_1 - M_2| \ll |H_{12}|$, $H_{12} \approx \tilde{H}_{12}$ and we find that the diagonalising matrices are equivalent and one gets the same asymmetry. For example, the matrix relating the right handed field projections $N_{aR}$ and $\psi_{aR}$ is given by

$$
\begin{pmatrix}
1 & -\frac{M'_1 - M'_2}{2H_{12}} \\
1 - \frac{M'_1 - M'_2}{2H_{12}} & 1
\end{pmatrix}
$$

(10)

for both approaches. The above matrix is obtained from $V^{-1}$ of Section 1 and we have not shown the normalisation. $M'_i = M_i + H_{ii}$. However in the limit of large mass difference $|M_1 + H_{11} - M_2 - H_{22}| \gg |H_{12}|$ the relation between $N_{aR}$ and $\psi_{aR}$ is given by

$$
\begin{pmatrix}
1 & -\frac{M'_1 H_{12} + M'_2 H_{12}}{M'_1 - M'_2} \\
\frac{M'_1 H_{12} + M'_2 H_{12}}{M'_1 - M'_2} & 1
\end{pmatrix}
$$

(11)

\textsuperscript{3}In section 7.3 of ref.[22] a further approach to obtain the decay rate for unstable particles using the optical theorem is given.

\textsuperscript{4}It is pointed out in section 5 of ref.[21] that in some cases diagonalising the effective mass matrix to obtain the $\psi_a$ fields may not be possible.
and
\[
\begin{pmatrix}
1 & -\frac{H_{12}}{M_1' - M_2'} \\
\frac{H_{12}}{M_1' - M_2'} & 1
\end{pmatrix}
\]  \tag{12}

for refs.[7] and [9] respectively. One can see that they are not equivalent. We find that working with the 2x2 matrices in this limit gives an asymmetry that is approximately 1/2 of the asymmetry obtained using the 4x4 approach. This can be traced to the absence of the contribution of the term proportional to $M_1'\tilde{H}_{12}$. Once again, if $h_{ai}$ are real $H_{12} = \tilde{H}_{12}$ and the diagonalising matrices of the two approaches become identical.

In ref.[8] the authors also consider the eigenvalues of the two blocks of a 4x4 effective mass matrix. However our above arguments do not apply to this case as the two blocks are along the diagonal and are transpose of each other, necessitating that the eigenvalues of the two 2x2 blocks are the eigenvalues of the larger matrix and that the eigenvalues of each block are the same, and the eigenvectors of the larger matrix are a trivial combination of the eigenvectors of the blocks.

Ref.[7] diagonalises the mass matrix and obtains the relationship between $|\tilde{\psi}_a\rangle$ and $|N_a\rangle$ states given by $|\tilde{\psi}\rangle = |N\rangle V^{-1}$. On the other hand, ref.[12] diagonalises projections of the resummed propagator, $S_{RR} = \langle N_R\tilde{N}_L\rangle$ and $S_{LL} = \langle N_L\tilde{N}_R\rangle$. Using this to factorise the amplitude for $l\phi^* \rightarrow \psi \rightarrow l^*\phi$ and $l^*\phi \rightarrow \psi \rightarrow l\phi^*$ they obtain the amplitudes for $\tilde{\psi} \rightarrow l\phi^*, l^*\phi$ in the large mass difference limit\(^5\). In the limit of large mass difference, $|M_1' - M_2'| \gg |H_{12}|$, the matrix relating $N_{aR}$ and $\psi_{aR}$ fields for ref.[12] is given by
\[
\begin{pmatrix}
1 & -\frac{2M_2\tilde{H}_{12}}{2M_1\tilde{H}_{12}} \\
\frac{2M_1\tilde{H}_{12}}{2M_1\tilde{H}_{12}} & 1
\end{pmatrix}
\]  \tag{13}

Though the matrix above differs from that for ref.[7] the ultimate asymmetry is the same as that of ref.[7] because the factor of 2 in the off-diagonal entries above compensates for the presence of the terms proportional to $H_{12}$ in eq. (11) \(^6\). We have also obtained the matrix relating $N_{aL,R}$ and $\psi_{aL,R}$ fields in the limit of small mass difference, $|M_1' - M_2'| \ll |H_{12}|$, by diagonalising the projections of the resummed propagator as in ref.[12]. We obtain the same matrix as refs.[7, 9], as given in eq. (10)\(^7\).

\(^5\)In section 9 of ref.[21] a similar factorisation of two particle scattering amplitudes was done (in a different context) to obtain the amplitude for decays of unrotated fields.

\(^6\)Note that the lagrangian of ref.[7] as given in eq. (1) is twice that of ref.[12].

\(^7\)In general $S_{RR}$ and $S_{LL}$ do not have the same poles making it non-trivial to define the particle mass. We do not address such field theoretic subtleties. We only mention that in the limit that $|M_1' - M_2'| \ll |H_{12}|$ $S_{RR}$ and $S_{LL}$ do have the same pole.
In ref.[13] the author obtains the resummed propagator as in ref.[12] but does not diagonalise it and obtains the amplitudes for $N_1 \rightarrow l\phi^*, l^*\phi$ by by truncating the Green’s function in momentum space with $(S_{11})^{-1} = \hat{\rho} - M_1 + \hat{\Sigma}_{11} - \hat{\Sigma}_{12}\hat{\rho} - M_2 + \hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}$, where the self-energy corrections to the propagator $\hat{\Sigma}_{ij} = i\Sigma_{abs}^{ij}$ are defined in eq. (4.2) of ref.[13]. In the LSZ formalism the truncating operator defines the corresponding incoming or outgoing particle. This implies that the incoming neutrino has a mass that is the (complex) pole of the component $S_{11}$ and not of the full resummed propagator $S$, i.e., one is including only one loop diagonal corrections to define the neutrino mass and excluding off-diagonal corrections. Excluding mixing in the incoming state allows one to use the classical Boltzmann equation as discussed later. The derivation of the LSZ formula in momentum space used in ref.[13] involves integrations over time in intermediate steps, whose range of integration must be modified from $(-\infty, +\infty)$ to $(0, +\infty)$ if the neutrino fields have a complex mass. However the decay amplitude obtained in ref.[13] can be embedded in an amplitude involving only stable particles in asymptotic states (analogous to the approach in section 9 of ref.[21]), if one equates $S_{11} = 1/[(\hat{\rho} - M_1 + \hat{\Sigma}_{11} - \hat{\Sigma}_{12}\hat{\rho} - M_2 + \hat{\Sigma}_{22})^{-1}\hat{\Sigma}_{21}]$ with $u_1\bar{u}_1/\ldots$ and again $u_1$ satisfies only a diagonal Dirac equation. (\ldots" is approximately equal to $iM_1\Gamma_1$ where $\Gamma_1$ includes width effects close to $N_1$ production.)

We reiterate here the nature of the incoming particle in various approaches. In ref.[5] the incoming particle is an unmixed particle corresponding to the original lagrangian of the theory, i.e., $N_{1L,R}$ of eq. (1). In ref.[10] the incoming particle is a particle corresponding to the mixed field of the one loop effective theory, i.e, $N_{1L,R}$ of eq. (2). In ref.[6, 7, 8, 9] the incoming particle is a particle corresponding to the rotated field of the one loop effective theory, i.e., $\psi_1$ of eq. (6) or its equivalent. In ref.[12] the incoming particle corresponds to the rotated field obtained by diagonalising projections of the resummed propagator. The incoming particle has a one loop effective mass. In ref.[13] the incoming particle is unrotated but has a one loop (complex) effective mass corresponding to the pole of the $S_{11}$ component of the resummed propagator. In the limit that mixing is small compared to the mass difference, i.e, $|H_{12}| \ll |M_1' - M_2'|$ the amplitudes for decay in refs.[5, 12, 13] are similar though the spinor $u$ in the amplitude satisfies $u\bar{u} = \hat{\rho} + m$ with a different $m$ for each approach. As implied earlier the amplitude for ref.[7] differs in form from the others but refs.[5, 7, 12, 13] give approximately the same asymmetry in this limit.

 Integrating over a semi-infinite time interval has been taken into account in ref.[10] (and, for example, in ref.[23] in a different context).
In the opposite limit that \(|M'_1 - M'_2| \ll |H_{12}|\) the rotating matrices of refs.[7, 9, 12] are identical and give the same asymmetry. It was first shown in ref.[7] that there is an
enhancement in the asymmetry when \(|M_1 - M_2| \leq |H_{12}|\) and \(|H_{11} - H_{22}| \ll |H_{12} + \tilde{H}_{12}|\). We have reevaluated the asymmetry in this limit and obtained

\[
\epsilon_{\psi_1} = -\frac{Im(h^*_{\alpha_1}h_{\alpha_2}h^*_{\beta_1}h_{\beta_2})}{8\pi} \frac{\eta}{\eta^2[2|h_{\beta_2}|^2 + Re(h^*_{\gamma_1}h_{\gamma_2})] + 4(\eta^{\alpha\beta})^2 Re^2(h^*_{\alpha_1}h_{\alpha_2})|h_{\delta_1} + h_{\delta_2}|^2}
\]  

(14)

where \(\eta = (M_2 - M_1)/M_1\) and the sum over \(\alpha, \beta, \gamma, \delta\) and \(\epsilon\) is implied\(^9\). The expression for the asymmetry obtained in ref.[13] in the limit \(|M'_1 - M'_2| \ll M'_1\) is

\[
\epsilon_{N_1} = \frac{Im(h^*_{\alpha_1}h_{\alpha_2}h^*_{\beta_1}h_{\beta_2})}{8\pi|h_{\delta_1}|^2} \frac{r_N}{r_N^2 + 4A_{22}^2}
\]

(15)

where \(r_N = (M_1^2 - M_2^2)/(M_1M_2) \approx -2\eta\) and \(A_{ij} = \sum_\alpha h_{\alpha i}h^*_{\alpha j}/(16\pi)\). In obtaining the asymmetry above, \(\hat{p}\) was substituted by \(M_1\). To be consistent with the truncation of the Greens’s function \(\hat{p}\) should be replaced by the pole of \(S_{11}\). It is not trivial to solve for the pole. Using a naive substitution of \(\hat{p}\) by \(M_1 - \hat{\Sigma}_{11} + \hat{\Sigma}_{12}[M_1 - M_2 + \hat{\Sigma}_{22}]^{-1}\hat{\Sigma}_{21}\), where \(\hat{\Sigma}_{ij}\) are evaluated at \(\hat{p} = M_1\) and ignoring terms of order \(h_{\alpha i}^4\) at the amplitude level, as in ref.[13], one obtains\(^{10}\)

\[
\epsilon'_{N_1} = -\frac{Im(h^*_{\alpha_1}h_{\alpha_2}h^*_{\beta_1}h_{\beta_2})}{8\pi|h_{\delta_1}|^2} \frac{r_N(1 - a)}{r_N^2(1 - a)^2 + 4(A_{22} - A_{11} + aA_{22})^2},
\]

(16)

where \(a = |A_{12}|^2M_1^2/[(M_1 - M_2)^2 + A_{22}^2M_1^2]\). In the case of hierarchical Yukawa couplings the above result reduces to the asymmetry in eq. (15). In the limit that \(A_{11} \sim A_{22}\), or \(|h_{\alpha_1}| \sim |h_{\alpha_2}|\), and \(\eta \sim A_{22}\), which is comparable to the limit in which the enhancement was seen in ref.[7], \(a \sim 1/2\). Then the asymmetry reduces to

\[
\epsilon'_{N_1} = -\frac{Im(h^*_{\alpha_1}h_{\alpha_2}h^*_{\beta_1}h_{\beta_2})}{8\pi|h_{\delta_1}|^2} \frac{2r_N}{r_N^2 + 4A_{22}^2}.
\]

(17)

One can see that the asymmetry in eq. (14) is different from that in eq. (17). This is not surprising as eq. (14) represents the asymmetry in the decay of a \(\psi\) particle while eq. (17) reflects the asymmetry in the decay of a particle represented by an unrotated field with a complex mass. In the limit of large mixing these need not be equivalent.

\(^{9}\)This result differs from that of ref.[7] but we have been informed by Dr. U. Sarkar that there is a correction to the calculation of \(\epsilon_{\psi_1}\) in ref.[7] in this limit.

\(^{10}\)Finite temperature effects, which we have not included, can induce changes in particle masses which can be relevant [24].
We have also obtained the asymmetry in the small mass difference limit for the model involving sneutrino decays in ref. [10]. In the limit that $|M_1^2 - M_2^2| \ll M_1^2$ and $|\Gamma_{11} - \Gamma_{22}| \ll |\Gamma_{12}|$ (in the notation of ref. [10]), we find

$$\epsilon_{N_1} = \frac{\text{Im}(h_{11}^* h_{22} h_{31}^* h_{32})}{\pi (|h_{31}|^2 + |h_{32}|^2)} \frac{r_N}{r_N^2 + 16[(A_{22} - A_{11})^2 + 4 \text{Im}^2 A_{12}]}$$

(18)

where $A_{ij}$ and $r_N$ are defined as above. While comparing with the asymmetry obtained in other works one must keep in mind that eq. (18) is the asymmetry obtained from sneutrino decays and that the model in ref. [10] has twice as many decay channels for the sneutrinos as compared to the neutrinos in other works.

We would also like to add a note of caution when one uses the effective Hamiltonian approach and the Boltzmann equation to obtain the lepton asymmetry. The Boltzmann equation which is used to calculate the number densities of various species assumes that the nature of the species does not change over time. The l.h.s. of the Boltzmann equation is the time derivative of the number density of a species and one has to be careful as to exactly what one means by $n_N(t)$ or $n_\psi(t)$ at times when $N_a$ or $\psi_a$ may not be the physical states. This comment applies to scenarios where one starts with $N$ states at $t_{nr}$ which evolve subsequently as $\psi$ states, as well as to ref. [19] where one obtains the asymmetry by evolving the $\psi$ number density from when the neutrinos are relativistic.

It has been argued in ref. [24] that decoherence effects due to interactions with the thermal universe justify excluding mixing effects in the incoming neutrino state. Excluding mixing in the incoming state allows one to work with a classical (diagonal) Boltzmann equation.

5 Summary

In conclusion, we have argued that in leptogenesis scenarios where the heavy neutrinos or scalars are relativistic when they go out of equilibrium one should calculate the asymmetry in the decay of the lightest unrotated species. We have pointed out certain issues related to the definition of the decay amplitude for eigenkets of the non-Hermitian effective Hamiltonian and the existence of a Fock space for mixed fields. As different authors have used various approaches to rotate the mixed fields we have compared the rotating matrices of refs. [7, 9, 12] and discussed their similarities and differences. Finally we have re-evaluated the asymmetry in the small mass difference limit using different approaches.
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