Relativistic quantum mechanics
of a Dirac oscillator

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Abstract

The Dirac oscillator is an exactly soluble model recently introduced in the context of many particle models in relativistic quantum mechanics. The model has been also considered as an interaction term for modelling quark confinement in quantum chromodynamics. These considerations should be enough for demonstrating that the Dirac oscillator can be an excellent example in relativistic quantum mechanics. In this paper we offer a solution to the problem and discuss some of its properties. We also discuss a physical picture for the Dirac oscillator’s non-standard interaction, showing how it arises on describing the behaviour of a neutral particle carrying an anomalous magnetic moment and moving inside an uniformly charged sphere.

Resumen

El oscilador de Dirac es un modelo exactamente resoluble que ha sido introducido recientemente en el contexto de la mecánica cuántica relativista de muchos cuerpos. El problema ha sido también explorado como posible fuente de un término de interacción para modelar confinamiento en cromodinámica cuántica. Estas consideraciones establecen sin lugar a dudas que el oscilador de Dirac puede servir como ejemplo interesante en mecánica cuántica relativista. Este artículo ofrece una solución al problema y la discusión de algunas de sus propiedades. También discutimos una imagen física que se ha introducido para el término de interacción del oscilador, mostrando como surge de considerar el comportamiento de una partícula neutra con momento magnético anómalo que se mueve dentro de una esfera cargada uniformemente.

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1. Introduction

Interesting and exactly soluble problems in relativistic quantum mechanics (RQM) are scarce. Disregarding the one dimensional examples — of which the book of Greiner (1991) is probably the best source — the typical problems for which the Dirac equation is exactly soluble are (Schweber 1964) the hydrogen atom, a particle in an homogeneous magnetic field, a particle in the field of an electromagnetic plane wave, and the Morse oscillator. There seem to be no new additions to the traditional textbook set in the last few years. Though in some situations it would be desirable to discuss examples in RQM (not necessarily exactly soluble) more related with contemporary applications, most of these are too technical and this usually forbids the discussion.

Our purpose in this paper is to discuss a problem in RQM, the so-called Dirac oscillator which, despite its various uses, has an straightforward exact solution. Though variants of the model were discussed some years ago (see, for example, Itô et al. 1967, Katriel and Adam 1969), the model was recently rediscovered in the context of relativistic many body theories (Moshinsky and Szczepaniak 1989). The Dirac oscillator has been also studied in connection with supersymmetric RQM, with quark confinement models in quantum chromodynamics (QCD) and with relativistic conformally invariant problems (Moreno and Zentella 1989, Benítez et al. 1990, Martínez-y-Romero et al. 1990, 1991, Martínez-y-Romero and Salas-Brito 1991). On the other hand, as we hope to make clear in this work, the problem has interesting features which can make it a helpful example in RQM. Besides the Dirac oscillator is closely related to the non-relativistic harmonic oscillator, as we exhibit in section 2.

Our paper is organized as follows. In section 2 we introduce the Dirac oscillator and discuss its principal properties. In section 3 we offer a physical interpr-
tation for the potential of the Dirac oscillator, and cast its equation of motion in a manifestly covariant form. A complete solution for the problem is given in section 4. We also show that the radial part of the components of its spinor eigenfunctions have the same form as the radial eigenfunctions of a 3-D non-relativistic harmonic oscillator. Conclusions and comments are given in section 5.

2. The Dirac oscillator

2.1. Relativistic quantum mechanics

Before introducing the Dirac oscillator, let us first briefly recall the fundamentals of Dirac’s RQM (Bjorken and Drell 1964, Greiner 1991). The basic equation in the theory is the free particle Dirac equation (for conciseness we use units such that $\hbar = c = 1$):

$$H_{\text{free}} \psi = (\alpha \cdot p + m \beta) \psi = i \frac{\partial \psi}{\partial t}$$

where $H_{\text{free}}$ is the free-particle Dirac Hamiltonian, $p = -i \nabla$ is the momentum, the $4 \times 4$ Dirac matrices are

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the $\sigma$ are $2 \times 2$ Pauli matrices and the 1’s and 0’s in $\beta$ stand, respectively, for $2 \times 2$ unit and zero matrices. If we introduce the gamma matrices $\gamma^0 = \beta$, $\gamma = \beta \alpha$, then, on multiplying by $\beta$, the Dirac equation can be put in the manifestly Lorentz covariant form

$$(i \gamma^\mu \partial_\mu - m) \psi = 0.$$  

(3)

An electromagnetic interaction is usually introduced in the free particle Dirac equation (1) (or (3)) using the standard minimal-coupling prescription $p \rightarrow p$
$-eA$, and $H \rightarrow H - e\phi$, or, in a Lorentz covariant fashion, $p^\mu \rightarrow p^\mu - eA^\mu$, where $A$ is the vector potential, $\phi$ the scalar electromagnetic potential and $A^\mu = (\phi, A)$. This prescription suffices for writing down the Dirac equation for most but not for all the problems, some are not amenable to this procedure. For example, for describing the electromagnetic interaction of a neutron, the Dirac equation must be written as (Bjorken and Drell 1964)

\[(i\gamma^\mu \partial_\mu + \frac{\kappa e}{4m_n}\sigma_{\mu\nu}F^{\mu\nu} - m_n)\phi = 0,\]

where $F^{\mu\nu}$ is the electromagnetic field tensor, $e$ is the magnitude of the electron charge, $\kappa$ is the anomalous magnetic moment and $m_n$ is the mass of the neutron. As we discuss in the next section, the coupling term used for the Dirac oscillator can be interpreted as an anomalous magnetic interaction. Let us recall that anomalous interactions are simply a phenomenological way of describing the effect of residual electromagnetic interactions between electromagnetic fields and globally neutral composite particles with charged constituents.

2.2. The Dirac oscillator Hamiltonian

As a simplified model for complex interactions the harmonic oscillator has many uses in non-relativistic quantum mechanics, that it can be exactly solved is thus particularly useful. This fact is intimately related with its Hamiltonian being quadratic in both the momenta and the spatial coordinates. With these considerations in mind and trying to obtain an analogous model in RQM, the Dirac equation has been explored in the search of a potential that can make it linear in both the momenta and the spatial coordinates (Moshinsky and Szczepaniak 1989); the resulting problem has been termed the Dirac oscillator. To obtain the non-trivial equation of motion for this oscillator, linear in both $r$ and $p$, we have to perform the following non-minimal substitution in the free particle Hamiltonian
\[ H_{\text{free}} \]

\[ p \rightarrow p - im\omega \beta r, \quad (5) \]

where \( m \) is the mass of the particle described and \( \omega \) is the oscillator frequency. The Dirac equation for the system is then

\[ i \frac{\partial \psi}{\partial t} = H\psi = (\alpha \cdot (p - im\omega r \beta) + m\beta)\psi. \quad (6) \]

At this point, we may wonder why the interaction is introduced as we have done and not by simply turning on a linearly growing potential in \( H_{\text{free}} \)? There are two reasons for this. First, the pure linear term can always be gauged away—to see this, introduce a linearly growing potential to \( H_{\text{free}} \) and take \( \Lambda = r^2/2 \) as a gauge-function for recovering the free particle equation. Second, as we show in section 3, the inclusion of the \( \beta \) matrix is crucial for obtaining a Lorentz covariant interaction. We also remark that although the \( p - im\omega r \beta \) term is obviously not hermitian, the complete Hamiltonian remains Hermitian due to the presence of the \( \alpha \) matrix. The prescription (5) also guarantees the C, P, and T invariance properties of the Dirac oscillator (Moreno and Zentella 1989). In particular note that the parity invariance of \( H \), i.e. the purely spatial invariance under \( r' = -r \), is fairly obvious from equation (6) if you remember that in Dirac’s RQM the parity operator is

\[ P = e^{i\varphi} \gamma^0, \quad (7) \]

where \( e^{i\varphi} \) is an unobservable phase factor conventionally chosen as one of the four values \( \pm 1 \) or \( \pm i \) (Greiner 1991).

2.3. Properties of the Dirac oscillator
We turn now to establishing the relationship between the Dirac oscillator and the harmonic oscillator. Let us first take the square of the Hamiltonian of the Dirac oscillator, in this way, and after a few straightforward manipulations, we get

\[ H^2 = \alpha \cdot (p - i\omega r\beta) + m\beta)^2 = p^2 + m^2\omega^2 r^2 + (4S \cdot L - 3)m\omega\beta, \]  

(8)

where

\[ S = \frac{1}{2}\sigma \]  

(9)
is the spin and

\[ L = r \times p \]  

(10)
is the orbital angular momentum of the oscillating particle described.

We can see now that, as equation (8) shows, \( H^2 \) becomes essentially a Klein-Gordon Hamiltonian with harmonic oscillator interaction plus a spin-orbit coupling term. In this sense, we may say that the Dirac oscillator is something like the “square root” of a linear harmonic oscillator. The squared Hamiltonian (8) can be used to obtain in a simple way the energy eigenvalues of the Dirac oscillator, as we show in section 4. We must remark that the squared Hamiltonian (8) is composed only of operators commuting with the \( \beta \)-matrix —these operators are called even. This property of \( H^2 \) implies that a closed form of the Foldy-Wouthuysen (FW) transformation can be found for the problem. This useful point is discussed in Martínez-y-Romero et al. (1990).

If we define the total angular momentum of the Dirac oscillator in the usual way as \( J = L + \sigma/2 \), it is easy to show that the system conserves the total angular
momentum. To this end, let us show first that neither \( \mathbf{L} \) nor \( \sigma \) are separately conserved:

\[
[\mathbf{L}, H] = i(\alpha \times \mathbf{p}) - m \omega (r \times \alpha) \beta, \tag{11}
\]

and

\[
[\sigma/2, H] = -i(\alpha \times \mathbf{p}) + m \omega (r \times \alpha) \beta, \tag{12}
\]

where \([A, B] \equiv AB - BA\). But, obviously, \( \mathbf{J} = \mathbf{L} + \sigma/2 \) does commute with \( H \) and, therefore, the total angular momentum \( \mathbf{J} \) is conserved by a Dirac oscillator

\[
[\mathbf{J}, H] = 0. \tag{13}
\]

3. Physical origin and Lorentz covariance of the interaction term.

To obtain a physical model for the interaction term in the Hamiltonian of the Dirac oscillator we follow here the ideas of Moreno and Zentella (1989), see also Benítez et al. (1990). Let us begin with a simple electrostatic problem. Consider an uniformly charged dielectric sphere of radius \( R \), the electric field produced inside the sphere varies as \( \mathbf{E} = -\lambda \mathbf{r} \) (\( \lambda \) is a constant), whereas the magnetic field vanishes everywhere \( \mathbf{B} = 0 \). We can always consider a very large sphere to safely disregard any edge effects. We are going to show how a particle with an anomalous magnetic moment coupled to the electromagnetic field produced by the sphere leads to the Hamiltonian of the Dirac oscillator. In the process we also exhibit the Lorentz covariant properties of the latter.

In the proper frame of the sphere (let us call this the laboratory frame), we may calculate the electromagnetic potential \( A^\mu \) associated with its static electro-
magnetic field. Good expressions for this are given by \( \mathbf{A} = 0, \phi = \lambda tr \), that is

\[
\hat{\mathbf{A}}_\mu = \lambda(0, tr).
\] (14)

To put this expression in a Lorentz covariant form, we take advantage of the gauge invariance of the electromagnetic field. To this end we select the gauge function

\[
\Lambda = -\frac{\lambda}{4}(tr^2 - \frac{t^3}{3}),
\] (15)

to obtain the new potential

\[
A^\mu_{lab} = \hat{\mathbf{A}}_\mu_{lab} - \partial^\mu \Lambda = \frac{\lambda}{4}(t^2 + r^2, 2tr).
\] (16)

Introducing the unit four-vector \( U^\mu_{lab} = (1, 0) \) —which may be interpreted as the four-velocity of a point particle attached to the origin of the laboratory frame (Núñez-Yépez et al. 1989)— we can rewrite the 4-potential (16) in the form

\[
A^\mu = \frac{\lambda}{4}[2(U \cdot x)x^\mu - x^2U^\mu].
\] (17)

This expression is manifestly Lorentz covariant, the electromagnetic field tensor produced by the sphere can now be calculated as

\[
F_{\mu\nu} = \lambda(U^\mu x^\nu - U^\nu x^\mu).
\] (18)

What is the relativistic quantum behaviour of a neutral quantum particle moving inside our dielectric sphere? In the laboratory frame only the electric field is non-vanishing and, at first sight, it would seem that a neutral particle could not interact with this field. But, as we already pointed out in section 2, this is not the
case if the particle carries an anomalous magnetic moment. When we substitute our expression (18) for the electromagnetic field in the interaction term appearing in equation (5), we get
\[ \frac{\kappa e}{4m} \sigma_{\mu\nu} F^{\mu\nu} = \frac{\kappa e}{2m} \lambda (i\vec{\alpha} \cdot \vec{r}), \] (19)
to obtain an expression for the Hamiltonian in the Dirac equation, we only need to multiply equation (19) by the \( \beta \) matrix (Greiner 1991). What we get in this way is an interaction term which grows linearly with \( r \). Hence, the Hamiltonian of this problem can also be obtained, as we did in section 2 for the Dirac oscillator, by means of a non-minimal coupling prescription:
\[ p \rightarrow p - \frac{i}{2m} e\kappa \lambda r \beta. \] (20)
We have to choose
\[ \lambda = \frac{2m^2\omega}{e\kappa}, \] (21)
for reproducing the Dirac oscillator equation (6). That is, for obtaining a Dirac oscillator the charge density of the sphere, and hence the electric field, must be taken as proportional to the Dirac oscillator frequency \( \omega \). We have thus exhibited that the Hamiltonian of the Dirac oscillator may be regarded as describing a neutral particle (a neutron for example) with anomalous magnetic moment and interacting with a static radially growing electric field.

4. Eigensolutions for the problem.

We now proceed to calculate the complete solution for the Dirac oscillator problem. We find that each component of the radial spinorial eigenfunctions have the same form as the radial wavefunctions of the 3-D non-relativistic oscillator.
Hence, the name oscillator applied to the problem may also be justified at this level.

As we have seen in section 2, the total angular momentum $J$ commutes with $H$ and, therefore, the angular momentum is conserved in our system. From the rules of sum for angular momentum, we know that $j = l \pm 1/2$, where $j$ is the total angular momentum and $l$ is the orbital angular momentum quantum number, respectively. We also know that in our system parity is a good quantum number, so we use it to classify the energy eigenfunctions. As the parity of the spinorial eigenfunctions of radially symmetric problems in Dirac’s RQM is of the form $(-1)^l$ (Bjorken and Drell 1964), it is useful to define

$$\epsilon = \begin{cases} 
+1 & \text{if parity is } (-)^{j+1/2}, \\
-1 & \text{if parity is } (-)^{j-1/2}.
\end{cases}$$ (22)

in both cases, we have $l = j + \epsilon/2$.

Since the Dirac oscillator conserves angular momentum, it is convenient to split the energy eigenfunction into a radial part and an angular part according to ($\pm$ corresponds to the value of $\epsilon$)

$$\psi(r, t) = \frac{1}{r} \left( \begin{array}{c} F(r) Y_{j ml}^{\pm}(\theta, \phi) \\
G(r) Y_{j ml'}^{\mp}(\theta, \phi) \end{array} \right) \exp(-iEt),$$ (23)

where $Y_{j ml}^{\pm}$ and $Y_{j ml'}^{\mp}$ are the spinor spherical harmonics (Greiner 1991)

$$Y_{j ml}^{\pm}(\theta, \phi) = \begin{pmatrix} 
\sqrt{\frac{l+m+\frac{3}{2}}{2l+1}} Y_{l,m-\frac{1}{2}}(\theta, \phi) \\
\pm \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} Y_{l,m+\frac{1}{2}}(\theta, \phi) 
\end{pmatrix},$$ (24)

and the $Y_{l,n}$ are standard spherical harmonics. Notice that due to the fact that the parity operator $P$ (eq. 7) is proportional to $\beta(= \gamma^0)$, $Y_1$ has to be of opposite parity to $Y_1'$; the only way to accomplish this is by taking $l' = j - \epsilon/2$.  

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To obtain the eigenfunctions and the energy spectrum of the Dirac oscillator we use $H^2$ (equation (8)). From that equation, we obtain that the differential equations governing the behaviour of both components of the wave function, are the sum of a harmonic oscillator equation plus a spin-orbit coupling term (this is another way to see that the Hamiltonian of the Dirac oscillator conserves angular momentum). Notice that for $l = j + \epsilon/2$ we should have

$$(j + 1/2)(j + 1/2 + \epsilon) = l(l + 1),$$  \hspace{1cm} (25a)$$

whereas, for $l' = j - \epsilon/2$, we should have

$$(j + 1/2)(j + 1/2 - \epsilon) = l'(l' + 1).$$  \hspace{1cm} (25b)$$

In this way, we have established that the square of the angular momentum satisfies (cf. the expression for $\beta$ in equation (2))

$$L^2 = (j + 1/2)(j + 1/2 + \epsilon \beta).$$  \hspace{1cm} (26)$$

From equation (26), we can get the contribution to the energy spectrum of the spin-orbit coupling term. Using the relation

$$J^2 + L^2 + 2\mathbf{S} \cdot \mathbf{L} + \frac{3}{4},$$  \hspace{1cm} (27)$$

and combining equations (26) and (27), we get

$$\mathbf{S} \cdot \mathbf{L} = -\frac{1}{4} \epsilon (2j + 1) \beta - \frac{1}{2},$$  \hspace{1cm} (28)$$

which only means that the spin-orbit coupling makes a constant (i.e. independent of $\mathbf{r}$) contribution to the energy spectra. Besides, we know that for problems with spherical symmetry the square of the momentum is related to the square of the angular momentum by the relation (Greiner 1991)
\[ p^2 = -\frac{d^2}{dr^2} + \frac{L^2}{r^2}, \]  
(29)
valid for radial functions defined like in (23). From this equation and (26), we get for the square of the Hamiltonian of the Dirac oscillator

\[ E^2 - m^2 = -\frac{d^2}{dr^2} + \frac{(j + \frac{1}{2})(j + \frac{1}{2} + \epsilon\beta)}{r^2} + m^2\omega^2 r^2 + m\omega[\epsilon(2j + 1) - \beta]. \]  
(30)

This equation shows that the radial solutions of the Dirac oscillator must be the same as the solution for the non-relativistic 3-D harmonic oscillator (Benítez et al. 1990). This has to be so because, in (30), the term \( E^2 - m^2 \) equals the radial part of a non-relativistic harmonic oscillator equation plus the constant term \( m\omega[\epsilon(2j + 1) - \beta] \).

Let us now exhibit the eigenfunctions of the problem. For the so-called “big” radial component of the wave function \( F(r) \), the energy is positive (Greiner 1991) and the orbital angular momentum must be \( l = j + \epsilon/2 \). Using the analogy with the harmonic oscillator, \( F(r) \) must be of the form

\[ F_{n,l}(r) = A(\sqrt{m\omega}r)^{l+1} \exp(-m\omega r^2/2)_{1}F_{1}(-n, l + 3/2, m\omega r^2); \]  
(31a)

For the “small” component \( G(r) \), the energy is negative and the angular momentum is \( l' = j - \epsilon/2 \). Consequently, in this case the solution must be given by

\[ G_{n,l'}(r) = A(\sqrt{m\omega}r)^{l'+1} \exp(-m\omega r^2/2)_{1}F_{1}(-n, l' + 3/2, m\omega r^2). \]  
(31b)

In these equations, the \( {}_{1}F_{1}(a, b, c) \) are confluent hypergeometric functions (Davydov 1967, Gradshteyn and Ryzhik 1980), \( A \) is a normalization constant and \( n = 0, 1, 2, \ldots \) Employing the normalization
the constant $A$ may be easily evaluated, the result is

$$A = \left( \frac{m\omega}{\pi} \right)^{\frac{1}{4}} \left[ \frac{n!2^{n+l-\frac{\epsilon}{2}+\frac{3}{2}}}{(2n + 2L + 1 - 2\epsilon)!!} \right]^{\frac{1}{2}} \left[ (n + l + 1 - \frac{\epsilon}{2})^3 + (n + l - \frac{\epsilon}{2})^2 \right]^{-1/2}.$$

(33)

To obtain the energy spectrum we proceed as follows: we know that the spectrum of a 3-D harmonic oscillator is given by (Davydov 1967)

$$E_N = \omega \left( N + \frac{3}{2} \right), \quad N = 0, 1, 2 \ldots$$

(34)

where $N = 2n + l$ is the principal quantum number. The energy spectrum of the Dirac oscillator is thus given by the contribution of the harmonic oscillator plus constant terms

$$E^2 - m^2 = 2mE_N + m\omega[\epsilon(2j + 1) - \beta].$$

(35)

Thus the energy spectrum is given by (Benítez et al. 1990):

$$E = \{m\omega[2(N + 1) + \epsilon(2j + 1)] + m^2\}^{1/2},$$

(36a)

for the positive energy states, and

$$E = -\{m\omega[2(N + 2) + \epsilon(2j + 1)] + m^2\}^{1/2},$$

(36b)

for the negative energy states. It is to be noted that every state with quantum numbers $(N \pm s, j \pm s)$ for $\epsilon = -1$ have the same energy than a state with quantum numbers $(N \pm s, j \mp s)$ ($s$ an integer). This is a consequence of the dynamical symmetry of the problem (Quesne and Moshinsky 1990). The spectrum is shown
in figure 1 where this property and a characteristic supersymmetric pattern may be discerned (Benítez et al. 1990). It is important to point out here the difference between this spectrum and the bound state spectrum of, say, a Dirac hydrogen atom. In the latter all the bounded levels are always described by admixtures of the big and the small components of the wave function, it would not possible then to employ for solving it the explicit separation we used for solving the Dirac oscillator problem (Berrondo and McIntosh 1970, Benítez et al. 1990). In this sense, we may say that the Dirac oscillator is more akin to a free particle than to the Coulomb problem.

As the above discussion was based in the square of the Hamiltonian of the Dirac oscillator and not in $H$ itself, one may wonder if the results presented are correct or not. We can show that in fact they are indeed correct. We only outline the principal steps. To begin with, recall that for systems with spherical symmetry the $\alpha \cdot p$ term of the Dirac equation —acting on functions defined like in (23)— can be written as

$$\alpha \cdot p = \alpha_r \left[ p_r - \frac{1}{r} \epsilon \left( j + \frac{1}{2} \right) \beta \right],$$  

(37)

where

$$\alpha_r \equiv \alpha \cdot r; \quad p_r \equiv -i \frac{\partial}{\partial r} r.$$

(38)

After substituting these expressions in the Hamiltonian of the Dirac oscillator we obtain, after some manipulations, the system of coupled differential equations

$$\left\{ \frac{d}{dr} + \frac{1}{r} \left[ \epsilon (j + \frac{1}{2}) + m \omega r^2 \right] \right\} G(r) = (E - m) F(r),$$

(39a)

and

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\[
\left\{ -\frac{d}{dr} + \frac{1}{r} \left[ \epsilon(j + \frac{1}{2}) + m\omega r^2 \right] \right\} F(r) = (E + m)G(r). \tag{39b}
\]

This system of equations can be solved using standard methods (Benítez et al. 1990), and the results are exactly the same as those given in Eqs. (27) and Eqs. (32).

5. Conclusions.

We have introduced and discussed the properties of a Dirac oscillator. We have shown that the Hamiltonian of the Dirac oscillator describes the interaction of a neutral particle with an electric field via an anomalous magnetic coupling and have found that the energy eigenfunctions are of the same form as those of the non-relativistic harmonic oscillator. The reason behind this is that the square of the Hamiltonian for the Dirac oscillator becomes essentially a harmonic oscillator plus constant terms. We have remarked that as the squared Hamiltonian of a Dirac oscillator is composed only of even operators then a closed form for the Foldy-Wouthuysen transformation can be found.

We have mentioned also that this system has some remarkable symmetry properties. First, as it conserves angular momentum it must be \(O(3)\) invariant, but in fact that it admits the larger dynamical symmetry algebra \(SO(3,1) \oplus SO(4)\), producing a more degenerate spectrum than expected from \(O(3)\) considerations only (Quesne and Moshinsky 1990). This can be seen in figure 1. The energy spectrum exhibits also a supersymmetric pattern (Benítez et al. 1990, Martínez-y-Romero and Salas-Brito 1991). This property and its relationship with the existence of an exact FW transformation is analysed in Martínez-y-Romero et al. (1990, 1991). The existence of a closed FW transformation implies that the positive and negative energy solutions never mix, independently of the intensity of
the interaction. This characteristic, shown by a class of systems besides this one (another example is a particle moving in an arbitrary, time independent, magnetic field), has been referred to as the stability of the Dirac sea (Martínez-y-Romero et al. 1990).

We hope to have made clear that the Dirac oscillator can be profitably used when modern examples are needed to illustrate the uses of the Dirac equation. It can be exactly solved with no more effort than a non-relativistic harmonic oscillator and many of its properties can be calculated and discussed without too much elaborate formalism, it can even be solved using purely algebraic techniques (Martínez-y-Romero and Salas-Brito 1991, De Lange 1991a,b). Furthermore, its usefulness for modeling confinement in a Lorentz covariant way can make its study more motivating for some people.

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Figure Caption

Figure 1.
The energy spectrum of a Dirac oscillator shown in two different ways. For convenience, the numbers given correspond to \((E^2 - m^2)/(m\omega)\) instead of \(E\).

Notice that states with quantum numbers \((N \pm s, j \pm s)\) for \(\epsilon = -1\) have the same energy than states with quantum numbers \((N \pm s, j \mp s)\) \((s\) an integer). This is a consequence of the dynamical symmetry of the problem.

There is also a typical supersymmetric pattern which is fairly clear in the squared spectra shown. The states with positive energy and \(\epsilon = +1\) have the same energy than the states with negative energy and \(\epsilon = -1\) excepting for the constant term \(2m\omega\).