Neutrino mixing in Seesaw model

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We study the neutrino mixing matrix (MNS matrix) in seesaw model. Assuming large mass hierarchy for heavy right-handed Majorana mass, we show that MNS matrix is determined by a unitary matrix which transforms neutrino yukawa term into a triangular form. Large mixing may occur even if yukawa matrix for charged lepton and that for neutrino is simultaneously diagonalized by biunitary transformation. It naturally may satisfy the experimental lower bound of the mixing from the atmospheric neutrino oscillation.

§1. Introduction

After the observation of the neutrino oscillations, the mixing matrix of lepton sector (MNS matrix) has been discussed in many literatures. Among the various proposal of explanation of the light neutrino masses, seesaw mechanism is an attractive scenario. In this paper, we study MNS matrix in the framework of the seesaw model. Specifically, we assume that the gauge group is $SU(2) \times U(1)$. Since there are three light neutrinos, each light neutrino may have its partner, a gauge singlet neutrino with large Majorana mass. Moreover, the neutrino oscillation experiments suggest that there is mass hierarchy. In the context of the seesaw model, the mass hierarchy may be originated from the yukawa term and/or Majorana mass term. To account for the hierarchy and the mixings, there are two extreme cases. The first case is that the hierarchy comes from the Majorana mass term. The other case is that the heavy right-handed neutrinos have a degenerate mass and the hierarchy comes from the yukawa term. In this paper, we are interested in the former case. If the mass hierarchy of neutrinos comes from the hierarchy of the Majorana mass term, the diagonalization of the seesaw matrix goes through in a very different way compared to that of charged fermions. This is what we will discuss in this paper. The seesaw mass matrix is $2N_f \times 2N_f$ matrix ($N_f = 3$ in our case) and has a simple structure,

$$M_\nu = \begin{pmatrix} 0 & y_\nu \frac{v}{\sqrt{2}} \\ y_\nu^T \frac{v}{\sqrt{2}} & m_N \end{pmatrix}. \tag{1}$$

Its $N_f \times N_f$ submatrix corresponding to the Majorana mass term of light neutrinos is zero and the submatrix corresponding to the Majorana mass term of heavy neutrinos can be real diagonal matrix. The origin of the flavor mixing is Dirac type yukawa

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term denoted by $y_{\nu} \sqrt{2}$. Because the seesaw matrix is a symmetric matrix, it can be
diagonalized by a unitary matrix $V$ as $V^T M_{\nu} V$. Despite of the simple structure,
it is not possible to diagonalize the matrix analytically because we have to treat
$6 \times 6$ matrix. We perform the approximate diagonalization of the mass matrix
and obtain the parametrization of $V$. This tells us the mixing matrix of neutrino
sector. Combining it with the mixing matrix of the charged lepton sector, we may
determine the flavor mixing in lepton sector, namely, MNS matrix. The method we
employ is very similar to the diagonalization of the seesaw matrix for quark mass in
the context of a left-right model. In the model, isosinglet quarks with large mass
hierarchy play the similar role to the right-handed heavy neutrino in the present case.
The approximate diagonalization procedure of the seesaw type matrix is developed
in Ref. 8) In this paper, we extend the method to the seesaw model for the neutrino
mass with the same gauge group and higgs as those of the standard model.

The paper is organized as follows. In section 2, by showing the procedure of
the diagonalization, we explain how the MNS matrix comes out. In section 3, we study
the special case that the yukawa matrices for charged leptons and for neutrinos are
simultaneously diagonalized through unitary transformation. If this were the case
for the two yukawa matrices of up and down type of quarks, the KM matrix would
be trivial. However, in the seesaw model for neutrino mass, the large mixing may
still occur. We discuss on the phenomenological implications of our analysis. Finally
our conclusions are presented in section 4.

§2. MNS matrix in Seesaw model

We start with the mass terms for lepton sector. Without loss of generality, the
charged lepton matrix is a real diagonal matrix. Furthermore, Majorana mass matrix
$m_N$ is set to be a real diagonal matrix, $m_{Nij} = m_{Ni}\delta_{ij}$. Writing $M_{\nu}$ in Eq. (1)
explicitly, the mass terms are,

$$\mathcal{L}_{\text{lepton}} = \bar{\nu}_L y_{\nu} \frac{v}{\sqrt{2}} \nu_R + \bar{\nu}_L \bar{N}_R y_{\nu} \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & y_{\nu} \frac{v}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{\nu} \frac{v}{\sqrt{2}} & m_{N1} \\ m_{N2} \\ m_{N3} \end{pmatrix} \left( \begin{pmatrix} \nu_L \\ \bar{N}_R \end{pmatrix} \right) + \text{h.c.}$$

We focus on the case that the Majorana mass term is much bigger than Dirac mass
term, $m_N \gg y_{\nu} \frac{v}{\sqrt{2}}$. We are interested in the case that the diagonal Majorana masses
have hierarchy, $m_{N_i} \gg m_{N_j}$ ($i < j$) and the rank of yukawa matrix $y_{\nu}$ is $N_f$.

The approximate diagonalization procedure of the seesaw type matrix goes as
follows. $M_{\nu}$ is diagonalized by a $2N_f \times 2N_f$ unitary matrix as $V^T M_{\nu} V = M_{\nu}^{\text{diagonal}}$. 

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Let us parametrize the unitary matrix as follows:\textsuperscript{9)

\[ V = \begin{pmatrix} S & K \\ T & A \end{pmatrix}, \]

where \( S, K, T, A \) are \( N_f \times N_f \) submatrices. The unitarity relation is written in terms of the submatrices,

\[ SS^\dagger + KK^\dagger = 1, \]
\[ TT^\dagger + AA^\dagger = 1, \]
\[ ST^\dagger + KA^\dagger = 0. \]

Then \( M_\nu \text{diagonal} \) is given by,

\[ M_\nu \text{diagonal} = V^T M_\nu V =
\begin{pmatrix}
T^T y_\nu^T \frac{v}{\sqrt{2}} S + S^T y_\nu \frac{v}{\sqrt{2}} T + T^T m_N T & T^T y_\nu^T \frac{v}{\sqrt{2}} K + S^T y_\nu \frac{v}{\sqrt{2}} A + T^T m_N A \\
(T^T y_\nu^T \frac{v}{\sqrt{2}} K + S^T y_\nu \frac{v}{\sqrt{2}} A + T^T m_N A)^T & A^T y_\nu^T \frac{v}{\sqrt{2}} K + K^T y_\nu \frac{v}{\sqrt{2}} A + A^T m_N A
\end{pmatrix}. \]

We can see easily that the submatrices \( T^T y_\nu^T \frac{v}{\sqrt{2}} S + S^T y_\nu \frac{v}{\sqrt{2}} T + T^T m_N T \) and \( A^T y_\nu^T \frac{v}{\sqrt{2}} K + K^T y_\nu \frac{v}{\sqrt{2}} A + A^T m_N A \) must be diagonal and the off-diagonal submatrix becomes zero,

\[ T^T y_\nu^T \frac{v}{\sqrt{2}} K + S^T y_\nu \frac{v}{\sqrt{2}} A + T^T m_N A = 0. \]

In the leading order of \( 1/m_N \), the submatrices are given by,

\[ A = 1, \quad SS^\dagger = 1, \quad T = -\frac{v}{\sqrt{2}m_N} y_\nu^T S, \quad K = y_\nu^* \frac{v}{\sqrt{2}m_N}, \]

and the seesaw mass matrix is written as,

\[ V^T M_\nu V = \begin{pmatrix}
-S^T y_\nu \frac{v^2}{2m_N} y_\nu^T S & 0 \\
0 & m_N + \mathcal{O}(1/m_N)
\end{pmatrix}. \]

At this stage, the unitary matrix is parametrized as,

\[ V = \begin{pmatrix}
1 & 0 \\
-\frac{v}{\sqrt{2}m_N} y_\nu^T & 1
\end{pmatrix} \begin{pmatrix}
S & 0 \\
0 & 1
\end{pmatrix}. \]

Further S must be chosen so that the \(-S^T y_\nu \frac{v^2}{2m_N} y_\nu^T S\) is diagonal. For any matrix \( y \) of rank \( N_f \), we can find a unitary matrix which transforms \( y \) into a triangular form.\textsuperscript{8) We choose such matrix as \( ST \),

\[ ST y_\nu = \begin{pmatrix}
y_1 & 0 & 0 \\
y_2 & y_2 & 0 \\
y_3 & y_3 & y_3
\end{pmatrix} \equiv y_\Delta. \]
where the diagonal elements are real and the off-diagonal elements are complex in general. By choosing $S$ in this way, the seesaw mass matrix is approximately diagonalized. This is shown by writing submatrix $-S^T y_\nu^\dagger m_N y_\nu T S$,

$$-S^T y_\nu^\dagger m_N y_\nu T S = \left( \begin{array}{ccc} y_{11}^2 m_{N1} & y_{12} y_{31} m_{N1} & y_{13} y_{31} m_{N1} \\ y_{12}^\dagger y_{21} m_{N1} & y_{22}^2 m_{N1} + y_{21}^2 m_{N2} & y_{23} y_{31} m_{N1} + y_{23} y_{32} m_{N2} \\ y_{13}^\dagger y_{32} m_{N1} & y_{23} y_{32} m_{N2} + y_{21}^2 m_{N3} & y_{33}^2 m_{N3} + y_{31}^2 m_{N1} + y_{32}^2 m_{N2} \end{array} \right).$$

(13)

Because $m_{N1} \gg m_{N2} \gg m_{N3}$, the off-diagonal elements are much smaller than the difference of the diagonal elements. Therefore the mass matrix is regarded as approximately diagonalized,

$$V^T M_{\nu} V \simeq \left( \begin{array}{ccc} \frac{y_{11}^2 v^2}{2m_{N1}} & \frac{y_{22}^2 v^2}{2m_{N2}} & \frac{y_{33}^2 v^2}{2m_{N3}} \\ 0 & m_{N1} & 0 \\ 0 & 0 & m_{N2} \end{array} \right),$$

(14)

and the unitary matrix $V$ is rewritten as,

$$V = \left( \begin{array}{cc} S & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ -\frac{v}{\sqrt{2m_N}} y_\Delta^T \end{array} \right).$$

(15)

The MNS matrix is $3 \times 6$ submatrix of $V$, namely $V_{MNS} = S^*(1, y_\Delta \frac{v}{\sqrt{2m_N}})$, and leptonic charged current is written as,

$$J_{L\mu} = (\bar{e}_L \bar{\mu}_L \tau_L)(S^* y_\Delta \frac{v}{\sqrt{2m_N}}) \gamma_\mu \left( \begin{array}{c} \nu_{L1} \\ \nu_{L2} \\ \nu_{L3} \\ \eta_{L1} \\ \eta_{L2} \\ \eta_{L3} \end{array} \right),$$

(16)

where $\nu$ and $n$ are mass eigenstates of neutrinos.

§3. Large mixing

In this section, we study the MNS matrix, in particular, mixing between charged leptons and light neutrinos. As we show in the previous section, it is determined by a unitary matrix $S$ which transforms Yukawa term of neutrino into a triangular form. Therefore, this is a different matrix from the one which we encounter for the biunitary transformation of charged lepton Yukawa terms. To clarify, we consider the case that charged lepton Yukawa term and that of the neutrino are diagonalized by a common unitary transformation on their left-handed chirality. It means that the Yukawa terms satisfies the following commutation relation,

$$[y_\nu y_\nu^\dagger, y_e y_e^\dagger] = 0.$$

(17)
If there were only Dirac mass term and were not Majorana mass term, the MNS matrix would be trivial under the relation. However, in the presence of Majorana mass term with large hierarchy, the mixing can still occur. To show this, it is convenient to work in the weak base that charged lepton yukawa term is diagonal. In this base, Eq.(17) means $y_\nu y_\nu^\dagger$ is also diagonal. Because of Eq.(12), $S^* y_\Delta y_\Delta^\dagger S^T$ must be diagonal. Therefore, given $y_\Delta$, we may determine $S$. Now we show that large mixing occurs without fine tuning of the matrix elements of $y_\Delta$. We illustrate this by considering two flavor case ($N_f = 2$) and apply to the explanation of the large mixing of the atmospheric neutrino. In two flavor case, $y_\Delta y_\Delta^\dagger$, $S$ and the approximately diagonalized light neutrino mass matrix $m_\nu$ are respectively expressed as,

$$y_\Delta y_\Delta^\dagger = \begin{pmatrix} y_2^2 & y_2 y_3^* \frac{y_2 y_3^*}{|y_3^2| + y_3^2} \\ y_2 y_3 & |y_3^2| + y_3^2 \end{pmatrix},$$

(18)

$$S = \begin{pmatrix} \cos \theta & -\sin \theta e^{i\phi} \\ \sin \theta e^{-i\phi} & \cos \theta \end{pmatrix},$$

(19)

with

$$\tan 2\theta = \frac{2|y_3^2|y_2}{-y_2^2 + |y_3^2|^2 + y_3^2}, \quad \phi = \arg(y_{33}),$$

(20)

and

$$m_\nu = -\left( \begin{pmatrix} \frac{m_{N2}^2}{m_{N2}^2} & \frac{y_2 y_3}{m_{N2}} \\ \frac{y_2 y_3}{m_{N2}} & \frac{m_{N3}^2}{m_{N2}} \end{pmatrix} \right) \frac{\nu^2}{2},$$

(21)

where $m_{N2} \gg m_{N3}$ is assumed. In order that Eq.(21) can be regarded as approximately diagonal matrix, the sufficient conditions on the elements of the yukawa term $y_\Delta$ are,

$$O(y_2) = O(y_3) \equiv y,$$

(22)

$$O\left( \frac{|y_{33}|}{y_2} \right) \leq 1.$$  

(23)

The first condition is that the hierarchy of the light neutrino mass comes from the hierarchy of the Majorana mass term, $m_{N2} \gg m_{N3}$. The second condition suppresses the further mixing from the form of Eq.(21). From Eq. (20), for $y_2^2 \sim |y_{33}|^2 + y_3^2$, the mixing angle is large even for tiny non-zero off-diagonal element $y_{33}$. For example, by setting $y_2 = y_3 \equiv y$, Eq. (20) is,

$$\tan 2\theta = \frac{2y}{|y_{33}|} \geq 2 \quad (y \geq |y_{33}|).$$

(24)

This leads to,

$$\sin^2 2\theta \geq 0.8.$$  

(25)
This can be compared with the result of atmospheric neutrino. The SuperKamiokande collaboration\(^3\) has reported the range of \(\sin^2 2\theta_{\text{atm}}\) as,
\[
\sin^2 2\theta_{\text{atm}} \geq 0.8.
\]

In Fig. 1, we show the \(\sin^2 2\theta\) as a function \(|\frac{y_{32}}{y_2}|\) by changing the ratio \(r = \frac{y_3}{y_2}\). For \(r = 1\), the maximal mixing occurs at infinitesimally small \(|y_{32}| = \epsilon\) and the mixing decreases as \(|y_{32}|\) increases. For \(r < 1\), the maximal mixing occurs at non-zero \(y_{32}\) because the diagonal elements are degenerate at \(|y_{32}| \neq 0\). For \(r > 1\), the mixing angle is suppressed because the diagonal elements cannot be degenerate. For \(r \neq 1\), the small mixing is also possible for \(|y_{32}| \ll |y_3 - y_2|\).

In Fig. 2, we constrain the ratios of the yukawa couplings by using the result of the SuperKamiokande, Eq.(26). The allowed range is between two curves in the parameter space, \((r, |\frac{y_{32}}{y_2}|)\). We can see that the experimental constraint can be easily satisfied for the range of the parameters in Eqs.(22), (23). The curves are obtained with the equation,
\[
\left(\frac{y_3}{y_2}\right)^2 + \left(\frac{|y_{32}|}{y_2} - \frac{1}{\tan 2\theta}\right)^2 = 1 + \frac{1}{\tan^2 2\theta},
\]
where \(\sin^2 2\theta\) takes the minimum and the maximum values of the experimental constraint, Eq.(26). Finally we comment on the mass scale \(m_{N3}\) of the lightest gauge singlet neutrino. According to the SuperKamiokande result, the mass squared difference of \(\mu\) and \(\tau\) neutrino is order of \(10^{-3}\)\,eV\(^2\). Assuming that this corresponds to the mass squared of the \(\tau\) neutrino and the yukawa coupling is the same order of the magnitude as that the gauge coupling,
\[
m_{\nu\tau} \simeq \frac{10^4}{m_{N3}}\,[\text{GeV}] \simeq \frac{10^{13}}{m_{N3}}\,[\text{eV}]
\]

Therefore, the mass of the lightest gauge singlet neutrino may be \(O(10^{15})\) \,[GeV].

§4. Conclusions and Discussion

We study MNS matrix in the seesaw model. If there is large mass hierarchy in the Majorana mass term, the flavor mixing in the neutrino sector is determined in a very different way compared with case without Majorana mass term. It is determined by a unitary matrix which transforms neutrino yukawa term into triangular matrix \(y_{\Delta}\). As a special case, we assume the yukawa terms of charged lepton and neutrino commute, (See Eq.(17)). Even this case, we show that the mixing still occurs and MNS matrix is determined by the unitary matrix which diagonalizes \(y_{\Delta}y_{\Delta}^\dagger\). We show that large mixing may occur within the natural choice of the range of parameters, i.e., for the universal diagonal elements and the small off-diagonal elements in \(y_{\Delta}\). About the origin of the commutation relation of the yukawa terms, it may come from the unification of the quark and lepton. Because KM matrix is close to unit matrix, this may suggest that up and down quarks yukawa terms may approximately satisfy the similar commutation relation. Such relation may also hold for lepton sector in
unified theories of quarks and leptons. Phenomenological study can be extended to three flavor case and may be applied to solar neutrino problem. We also note that A. Yu. Smirnov \(^{10}\) studied the mixing in the seesaw model. Because our approach differs from him and the transformation to the triangular form is essential in our approach, the relation between our work and his must be carefully examined. We address these issues for future study.

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**References**

Fig. 1. $\sin^2 2\theta$ as a function of $\frac{|y_{32}|}{y_2}$ for various $r = \frac{y_3}{y_2}$. Three curves correspond to $r = 1$ (solid line), $r = 1.2$ (dotted line), and $r = 0.8$ (dot-dashed line) respectively. The region between the two solid straight lines is allowed from the SuperKamiokande experiment on the atmospheric neutrino, $0.8 \leq \sin^2 2\theta_{\text{atm}} \leq 1$.

Fig. 2. The region in the parameter space ($\frac{y_3}{y_2}$, $\frac{|y_{32}|}{y_2}$) between the solid and dashed lines is allowed from the SuperKamiokande experiment on the atmospheric neutrino, $0.8 \leq \sin^2 2\theta_{\text{atm}} \leq 1$. 