SINGLE-BEAM LIFETIME

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Abstract
An overview of the most relevant single beam effects, which cause a reduction in beam lifetime in storage rings, is given. The considerations, however, are limited to perturbations which have a random time character. After defining the aperture limitations which are responsible for the reduction in beam intensity, the lifetime due to quantum fluctuation and radiation damping for electrons (or positrons) is derived, followed by a treatment of the intensity reduction due to statistical fluctuations without damping. Furthermore the effects of beam gas scattering, the effects of Coulomb scattering inside a bunch (i.e. Touschek scattering and multiple intrabeam scattering) and the effects of resonance crossing on the lifetime are summarized.

1. INTRODUCTION

To give a general definition, the lifetime of a single beam is the time a beam can be kept, usually in a storage ring. Consequently all effects which reduce the beam intensity should be taken in account. The considerations given here, however, will be limited to the reduction of beam intensity caused by small perturbations which have a random time dependence. In this sense, lifetime is usually defined as the time the beam intensity decays to a certain fraction of its initial value (half or 1/e).

Such effects are for instance:

i) Effect of noise on betatron oscillation and synchrotron oscillation;
ii) Scattering by residual gas (elastic, inelastic);
iii) Quantum fluctuation of radiation;
iv) Scattering between particles inside a bunch (IBS, Touschek scattering);
v) Multiple traversal of resonances;

1.1 Aperture Limitations

The lifetime given by all these effects is related to the final aperture available for the particle motion. Without this limitation the average intensity, i.e. average stored current would remain constant (usually used to define the lifetime). Only the current densities would be changed and the bunch would become longer and wider and might not be useful anymore for experiments (the luminosity for colliders would be decreased and the brilliance for radiation sources would be reduced).

Aperture limitation, does not necessarily mean a physical limitation, but also the limitation in the transverse plane due to the dynamic aperture, or the limitation in the longitudinal plane due to the RF-bucket size or the dynamic aperture for off-momentum particles (i.e. particles with large synchrotron oscillation amplitudes).

1.1.1 Transverse Limitation

In the linear treatment of betatron oscillation the particle is lost, if its amplitude exceeds the aperture of the vacuum chamber.

Normally, non-linear magnetic fields present in the accelerator (as for instance sextupoles which are introduced to compensate chromatic effects) cause a limitation of the
maximum betatron amplitude, described by the dynamic aperture. Also in this case the particle is lost at the physical aperture, but non-linear effects blow up the betatron motion and limit the "stable" initial amplitudes to values far below the physical aperture.

Figure 1 shows the relation between initial amplitude and maximum final amplitude due to the non-linear effects. The initial amplitude for which the final one approaches infinity is usually called the dynamic aperture.

![Dynamic aperture graph](image)

**Fig. 1** Dynamic aperture reductions due to non-linearities. The increased betatron amplitude by non-linearities is drawn as a function of the initial amplitude.

1.1.2 Longitudinal Limitation

In the longitudinal plane the particle is lost either at the RF-acceptance limit or the momentum acceptance of the dynamic aperture.

**RF acceptance**
This is the limit of self focusing for the longitudinal motion, which is described by the separatrix in longitudinal phase space (see figure 2).

![Longitudinal phase space separatrix](image)

**Fig. 2** Aperture limitation in the longitudinal plane due to the self-focusing of the RF. Potential and phase space separatrix are shown.
with:

\[ H_s = \frac{eU}{\pi \hbar \alpha_c E} \left[ \cos \Phi_0 + \left( \phi_0 - \frac{\pi}{2} \right) \sin \phi \right] \]  

(1)

and the meaning:
- \( U \): RF voltage,
- \( \Phi_0 \): synchronous phase,
- \( h \): harmonic number,
- \( E \): total particle energy,
- \( \alpha_c \): momentum compaction factor.

**Momentum acceptance of dynamic aperture**

For particles with momentum deviation, the dynamic aperture can be strongly reduced as compared to the zero momentum case. Figure 3 shows the dynamic aperture reduction in ELETTRA with increasing momentum deviation, i.e. synchrotron oscillation amplitude.

![Graph showing dynamic aperture reduction](image)

**Fig. 3** Reduction in off-momentum aperture for ELETTRA.

### 1.2 Particle Distribution in a Storage Ring in the Presence of Small Statistically Independent Perturbations

If we want to evaluate the fraction of particles lost by a limitation in aperture, we have to know the distribution function, i.e. what part of the distribution is cut off by the aperture limitation.

![Distribution function with aperture limitation](image)

**Fig. 4** Sketch of particle loss of a distribution function due to an aperture limitation.
The particles from the shadowed area in figure 4 are lost by the aperture limitation. For an instantaneous limitation one has an initial intensity drop, but then the intensity remains constant, if now diffusion process exists to refill the shadowed area.

The effect of statistically independent perturbations can be described by the Fokker-Planck equation [1,2]:

\[
\frac{\partial w(x,t)}{\partial t} = -\frac{\partial}{\partial x} [ \langle x \rangle w(x,t) ] + \frac{\partial^2}{\partial x^2} \left[ \left( \frac{\langle x^2 \rangle}{2} \right) w(x,t) \right]
\]  

(2)

where \( w \) is the distribution function, \( \langle x \rangle \) is the mean increase per unit time and \( \langle x^2 \rangle \) is the mean square of the increase per unit time. Physically the Fokker-Planck equation is a continuity equation:

\[
\frac{\partial w}{\partial t} = \text{div} \ I = -\frac{\partial}{\partial x} \left[ \langle x \rangle w - \frac{\partial}{\partial x} \left( \frac{\langle x^2 \rangle}{2} \right) w \right]
\]

(3)

with \( I \) the current along the x-axis. The first term represents a systematic component corresponding to the mean displacement of the velocity (i.e. damping or excitation) and the second term describes a diffusion component proportional to the gradient of the concentration.

2. LIFETIME DUE TO QUANTUM FLUCTUATION AND RADIATION DAMPING

Lifetime effects due to quantum fluctuations and radiation damping are essentially only important for electrons [3], since the radiation effects are small for higher particle masses and therefore usually insignificant for protons. Although protons in the SSC, at 20 TeV reveal a decent radiation damping effect, very welcome there to combat beam growth due to intrabeam scattering.

2.1 Transverse Plane

The Fokker-Planck equation for the distribution of the emittance becomes [4]:

\[
\frac{\partial w(\epsilon,t)}{\partial t} = -\frac{\partial}{\partial \epsilon} \left[ \langle \delta \epsilon \rangle w(\epsilon,t) \right] + \frac{\partial}{\partial \epsilon} \left[ \left( \frac{\langle \delta^2 \epsilon \rangle}{2} \right) w(\epsilon,t) \right]
\]

(4)

where \( \langle \delta \epsilon \rangle \) is the mean change of the emittance per unit time and \( \langle \delta^2 \epsilon \rangle \) is the mean square of the change per unit time. The equation can be solved, after these quantities have been derived. There are two effects contributing to these parameters. First the statistical nature of the quantum emission which increases the beam size (excitation), and then the restoring of the energy in the accelerating cavities, which acts to decrease the beam dimensions (damping).

If a radiation quantum is emitted, the energy of the particle is reduced, which changes the corresponding closed orbit (particles with different energies have different closed orbits, as described by the dispersion). If a quantum with energy \( \delta E \) is emitted, the betatron oscillation coordinates are changed according to:

\[
x \rightarrow x + D \frac{\delta E}{E}
\]

\[
x' \rightarrow x' + D' \frac{\delta E}{E}
\]

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substituting these changes in the expression for the emittance:

\[ \varepsilon \beta = \left( x + D \frac{\delta E}{E} \right)^2 + \left[ ( x + D \frac{\delta E}{E} ) \alpha + ( x' + D' \frac{\delta E}{E} ) \beta \right]^2 \]  

(5)

we find after some manipulations:

\[ <\delta e> = -2 a_\beta \varepsilon + M \]
\[ <\delta e^2> = 2 \varepsilon M \]

with "a_\beta" the damping decrement, which is made up by the excitation term due to quantum emission and the damping term created by the recompensation of the energy in the accelerating cavities. \( M = H <\delta E^2/E > \), with the Courant Snyder invariant \( H = D^2 \gamma + 2DD' \alpha + D' ^2 \beta \).

After substitution of these quantities the Fokker-Planck equation becomes:

\[ \frac{\partial w(\varepsilon, t)}{\partial t} = \frac{\partial}{\partial \varepsilon} \left[ 2 \varepsilon a_\beta w(\varepsilon, t) + \frac{\varepsilon M}{\partial \varepsilon} \frac{\partial w(\varepsilon, t)}{\partial \varepsilon} \right] \]

(6)

for the stationary distribution the time derivative vanishes and we get the solution:

\[ w_\varepsilon(\varepsilon) = \frac{2a_\beta}{M} e^{-\frac{2a_\beta}{M} \varepsilon} \]

(7)

Replacing the emittance \( \varepsilon \) by the transverse coordinate \( x \), we find the well-known gaussian distribution for the transverse beam dimensions due to betatron oscillations:

\[ w_x(x) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} \]

(8)

Since the distribution extends to infinity, the vacuum chamber walls will cause a loss of particles. Assuming a maximum value of the emittance \( \varepsilon_m \) due to aperture limitations, the solution of the Fokker-Planck equation must now vanish at this boundary:

\[ w_\varepsilon(\varepsilon_m) = 0 \]

In the case of a limitation of the emittance by \( \varepsilon_m \), the time dependence of the distribution function can be separated as:

\[ w(\varepsilon, t) = e^{-\frac{t}{\tau}} \bar{w}(\varepsilon) \]

(9)

\[ \frac{\bar{w}(\varepsilon)}{\tau} = \frac{\partial}{\partial \varepsilon} \left[ 2 \varepsilon a_\beta \bar{w}(\varepsilon) + \frac{\varepsilon M}{\partial \varepsilon} \frac{\partial \bar{w}(\varepsilon)}{\partial \varepsilon} \right] \]

(10)

After performing the integration \( \int ... \, d\varepsilon \), we find:
\[-\frac{1}{\tau} = \varepsilon_m \, M \frac{\partial \overline{w}(\varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon_m}\]

(11)

if we substitute here as an approximation the solution without limitation, we get for the lifetime:

\[\tau = \frac{1}{2 \, a \beta \, r_{\beta}} \, e^{r_{\beta}} \quad \text{with} \quad r_{\beta} = \frac{2a \beta}{M} \, \varepsilon_m = \frac{1}{2} \left( \frac{x_{\beta}}{\sigma_{\beta}} \right)^2\]

The equation can also be solved more accurately by using a series expansion and solving [5]:

\[\overline{w}'' \varepsilon M + \overline{w} (2a + M) + \left( 2a + \frac{1}{\tau} \right) \overline{w} = 0\]

(12)

2.2 Longitudinal Plane

The same equation derived for the longitudinal plane (assuming that the limitation in momentum is given by the RF-bucket size) leads to:

\[\tau = \frac{1}{2 \, a_s \, r_s} \, e^{r_s} \quad \text{with} \quad r_s = \frac{1}{2} \left( \frac{\Delta E}{\sigma_E} \right)^2\]

where \(\Delta E\) is the energy acceptance of the bucket provided by the RF-system and \(\sigma_E\) the standard deviation of the energy distribution.

3. LIFETIME FOR STATISTICAL FLUCTUATIONS WITHOUT DAMPING

If the statistical perturbation is independent of the variable the mean value does not contain a damping term and the Fokker-Planck equation (6) reduces to a pure diffusion equation. Such random perturbations are for instance caused by fluctuations of the magnetic field or scattering on residual gas particles:

\[\frac{\partial w}{\partial t} = M \, \frac{\partial}{\partial \varepsilon} \left( \varepsilon \, \frac{\partial w}{\partial \varepsilon} \right)\]

(13)

we assume that the motion is limited by a maximum value of the emittance \(\varepsilon_m\). A change in variables of the form:

\[\tau = \left( \frac{M}{\varepsilon_m} \right) t\]

and

\[\theta = \frac{\varepsilon}{\varepsilon_m}\]

simplifies the equation to:

\[\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial \theta} \left( \frac{\partial w}{\partial \theta} \right)\]

(14)

With the initial condition and boundary condition respectively:

\[w(\theta,0) = w_0(\theta)\]
\[w(1,\tau) = 0\]

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the differential equation has solutions of the form [1,6]:

$$w(\theta, \tau) = \sum_n C_n J_0(\lambda_n \sqrt{\theta}) e^{-\frac{\lambda_n^2 \tau}{4}}$$  

(15)

with the coefficients:

$$C_n = \frac{1}{J_1(\lambda_n)^2} \int_0^1 w_0(\theta) J_0(\lambda_n \sqrt{\theta}) d\theta$$  

(16)

where $\lambda_n$ is the n'th zero of the Bessel function $J_0(\theta)$.

Assuming as previously an initial emittance distribution which corresponds to a Gaussian distribution in betatron coordinates, we have:

$$w_0(\epsilon) = \frac{1}{2\epsilon_0} e^{-\frac{\epsilon}{2\epsilon_0}}$$  

(17)

or in the new coordinates:

$$w_0(\theta) d\theta = \frac{\epsilon_m}{2\epsilon_0} e^{-\frac{\epsilon_m}{2\epsilon_0}} \theta d\theta$$  

(18)

where $\epsilon_0$ is the emittance corresponding to one sigma of the transverse phase space dimensions. We then find for the coefficients:

$$C_n = \frac{\epsilon_m}{2\epsilon_0} \frac{1}{J_1(\lambda_n)^2} \int_0^1 e^{-\frac{\epsilon_m}{2\epsilon_0}} J_0(\lambda_n \sqrt{\theta}) d\theta$$  

(19)

Generally the integration must be done numerically. Only if $\epsilon_m$ is large, can the integration be performed and we get for the coefficients:

$$C_n = \frac{1}{J_1(\lambda_n)^2} e^{-\frac{\lambda_n^2 \epsilon_0}{2 \epsilon_m}}$$  

(20)

For the development of the beam intensity in time, we get (for large $\epsilon_m$):

$$N(\tau) = \int_0^1 w(\theta, \tau) d\theta$$

$$= 2 \sum_n \frac{1}{\lambda_n J_1(\lambda_n)} e^{-\frac{\lambda_n^2}{4} \left( \frac{\tau + \epsilon_0}{\epsilon_m} \right)}$$  

(21)

The lifetime is defined as:

$$T = -\frac{dN(\tau)}{d\tau}$$  

(22)

which reaches an asymptotic value given by:

$$T_\infty = \frac{\lambda_1^2 \epsilon_m}{4 M}$$  

(23)
Figure 5 shows the variation of the lifetime for two different parameters of \( \frac{\varepsilon_0}{\varepsilon_m} \).

![Graph showing the variation of lifetime with time for two different parameters.](image)

Fig. 5 Change in lifetime for random perturbations without damping, for two different aperture limitations. For both initial conditions, the lifetime is approaching the asymptotic lifetime [6].

For small values, i.e. larger aperture limits, the lifetime is long at the beginning and decreases slowly to the asymptotic value. If the aperture is small, the initial lifetime is very short and then approaches rapidly the asymptotic value.

4 LIFETIME DUE TO BEAM-GAS SCATTERING

Particles are lost by scattering at the residual gas molecules. This effect can easily be controlled by providing sufficient pumping to reach a low residual gas pressure. There are two principally different effects:

**Elastic Scattering:** where the stored particle is transversally deflected and increases its betatron oscillation amplitude. If the change is large enough, the particle is lost at either the physical aperture or the dynamic aperture.

**Inelastic Scattering:** besides the deflection, where the particle does not loose energy, there is also the possibility that a light quantum is emitted during the collision and the energy of the particle is changed or the particle transfers energy to the atom of the rest gas. In both cases the particle looses energy and consequently gets lost at the RF acceptance limit or the off-momentum dynamic aperture limit.

4.1 Elastic Scattering [7]

An incoming homogeneous particle current with constant velocity is scattered at a point-like Coulomb field given by the charge of the nucleus of the residual gas atom.

4.1.1 Rutherford Scattering [8]

We are interested in the cross section of the scattering effect which is easily derived from the graph in figure 6:
Fig. 6 Scattering of an electron on a nucleus (Rutherford scattering) and geometry of the scattering process.

Geometry: \( \tan \frac{\theta}{2} = \frac{a}{p} \) \hspace{1cm} (24)

All scattering curves are hyperbolas with the impact parameter "p" for half of the main axis and "a" the half-size of the secondary axis.

The quantity "a" can be easily defined by looking at a particle which moves on axis in the direction of the Coulomb center. It is reflected at a position in which the kinetic energy equals the potential energy:

\[ m c^2 = \frac{Ze^2}{2a} \rightarrow a = \frac{Ze^2}{2m_0c^2\gamma} \] \hspace{1cm} (25)

The incoming ring \( d\sigma \) of particles is scattered in the solid angle \( d\Omega \).

\[ \frac{d\sigma}{d\Omega} = 2\pi p \frac{dp}{d\theta} \Rightarrow d\sigma = 2\pi \left( \frac{a}{\tan \theta/2} \right) d \left( \frac{a}{\tan \theta/2} \right) = \frac{a^2}{2} \frac{d\Omega}{\sin^4 \theta/2} \] \hspace{1cm} (26)
In the small angle approximation, which is valid for all practical cases we are interested in, we get for the differential cross section:

\[
\frac{d\sigma}{d\Omega} = \left( \frac{Ze^2}{m_0c^2 \gamma} \right)^2 \frac{2}{\theta^4}
\]  

(27)

The cross section has to be corrected for two extreme cases:

For small angles, i.e. large deviations, \( \theta \) from the central axis, the screening effect of the electrons on the atom has to be taken into account, which causes a fall off of the potential more rapidly than \( \frac{1}{r} \).

At large angles the cross section is changed due to the finite size of the nucleus. Figure 7 shows these corrections on the cross section given by eq. (27).

![Graph showing the change of differential cross section]

Fig. 7 Change of the differential cross section for the effects of electron screening at small angles and finite nuclear size at large angles [8].

The cross section flattens off at small angles to a finite value at \( \theta = 0 \). The modified cross section has the form:

\[
\frac{d\sigma}{d\Omega} = \left( \frac{Ze^2 \cdot 2}{m_0c^2 \gamma} \right) \frac{1}{(\theta^2 + \theta_1^2)^2}
\]  

(28)

where \( \theta_1 \), the minimum angle due to electron shielding.

Integration between \( \theta_0 \), the minimum angle for which a loss occurs and \( \theta_{\text{max}} \) (which we don't need to specify, because the dominant losses happen at small angles) leads to:

\[
\sigma = \frac{2\pi Z^2 r_0^2}{\gamma^2} \left\{ \frac{1}{\theta_0^2 + \theta_1^2} - \frac{1}{\theta_{\text{max}}^2 + \theta_1^2} \right\}
\]  

(29)

\[
\theta_1 = \frac{Z^{1/3}}{192} \cdot \frac{1}{\gamma}
\]  

(30)

which is approximately:
\[
\sigma = \frac{2\pi Z^2 r_0^2}{\gamma^2} \cdot \frac{1}{\theta_0^2}
\]

The maximum betatron amplitude generated by a deflection \( \theta_0 \) is given by:

\[
A_1 = \theta_0 \sqrt{\beta_i} \sqrt{\beta}
\]

(32)

The particle is lost if the amplitude reaches the half aperture \( H \) or the dynamic limit, i.e.:

\[
A_1 \geq H
\]

(33)

averaged over all positions of the ring, we get for the cross section:

\[
\sigma = \frac{2\pi Z^2 r_0^2}{\gamma^2} \cdot \frac{\beta \hat{\beta}}{H^2}
\]

(34)

Accordingly the cross section becomes smaller for an optics with small beta functions and for increasing machine energies.

### 4.2 Inelastic Scattering [7,9]

We have to deal here with two effects:

i) Bremsstrahlung scattering, where the particle emits a photon (and the nucleus is left unexcited).

ii) Inelastic scattering from an electron of the atom in which the momentum transfer excites the atom.

(There is also an elastic scattering at the electrons, but its effect is negligible).

#### 4.2.1 Bremsstrahlung scattering at the nucleus [10]

\[
\frac{d\sigma}{de}_A = \alpha \frac{Z^2 r_0^2}{\varepsilon} \left\{ \left[ \frac{4}{3} \left( 1 - \frac{\varepsilon}{E} \right) + \frac{\varepsilon^2}{E^2} \right] \left[ \varphi_1(\varepsilon_1) - \frac{4}{3} \ln Z \right] \\
+ \left[ \frac{2}{3} \left( 1 - \frac{\varepsilon}{E} \right) \right] \left[ \varphi_2(\varepsilon_1) - \varphi_1(\varepsilon_1) \right] \right\}
\]

(35)

with:

\[ \alpha = \frac{e^2}{\hbar c} = \frac{1}{137} \]

\( \varepsilon \) = photon energy,

\( E \) = particle energy,

\( \varphi_1, \varphi_2 \) = screening functions.

For high energy electrons the screening of the nucleus by the outer electrons must be taken into account. The argument of the screening functions \( \varphi_1, \varphi_2 \), is given by:

\[
\varepsilon_1 = 100 \frac{\varepsilon}{Z^{1/3}} \cdot \frac{m_0 c^2}{E(E-\varepsilon)}
\]

(36)
and can be approximated by $\varepsilon = 0$ for high energies, which correspond to the case of complete screening. In this case we get for the screening functions:

\[
\varphi_1(0) = \frac{20.836}{3}
\]

\[
\varphi_2(0) - \varphi_1(0) = \frac{2}{3}
\]

and for the differential cross section:

\[
\left( \frac{d\sigma}{d\varepsilon} \right)_A = \alpha \frac{4Z^2 r_0^2}{\varepsilon} \left[ \frac{4}{3} \left( 1 - \frac{\varepsilon}{E} \right) + \frac{\varepsilon^2}{E^2} \right] \left[ \frac{\varphi_1(0)}{4} - \frac{1}{3} \ln Z \right] + \left[ \frac{1}{9} \left( 1 - \frac{\varepsilon}{E} \right) \right] \]

(37)

4.2.2 Inelastic scattering off the electrons of the residual gas atom [11]

\[
\left( \frac{d\sigma}{d\varepsilon} \right)_B = \alpha \frac{Z r_0^2}{\varepsilon} \left[ \frac{4}{3} \left( 1 - \frac{\varepsilon}{E} \right) + \frac{\varepsilon^2}{E^2} \right] \left[ \psi_1(\varepsilon_2) - \psi_1(\varepsilon_1) \right] + \left[ \frac{2}{3} \left( 1 - \frac{\varepsilon}{E} \right) \right] \left[ \psi_2(\varepsilon_2) - \psi_1(\varepsilon_1) \right] \]

(38)

The functions $\psi_1$, $\psi_2$ have the same meaning as $\varphi_1$, $\varphi_2$ before.

For:

\[
\varepsilon_2 = 100 \frac{\varepsilon}{Z^{2/3}} \frac{m c^2}{E (E - \varepsilon)} = 0
\]

we find:

\[
\psi_1(0) = 28.34 \quad , \quad \psi_2(0) - \psi_1(0) = \frac{2}{3}
\]

(40)

and:

\[
\left( \frac{d\sigma}{d\varepsilon} \right)_B = \alpha \frac{4Z r_0^2}{\varepsilon} \left[ \frac{4}{3} \left( 1 - \frac{\varepsilon}{E} \right) + \frac{\varepsilon^2}{E^2} \right] \left[ \frac{\psi_1(0)}{4} - \frac{2}{3} \ln Z \right] + \left[ \frac{1}{9} \left( 1 - \frac{\varepsilon}{E} \right) \right] \]

(41)

The total differential cross section is given by the sum of both contributions. We can now derive the cross section of inelastic scattering in terms of $\varepsilon_m$ which is the lower boundary for which the particles start to be lost:

\[
\sigma = \int_{\varepsilon_m}^{E} \left( \frac{d\sigma}{d\varepsilon} \right)_E d\varepsilon = 4 \alpha r_0^2 \left\{ \int_{\varepsilon_m}^{E} \left[ \frac{4}{3\varepsilon} - \frac{4}{3\varepsilon} + \frac{\varepsilon}{E^2} \right] d\varepsilon \right. \\
\left. + \frac{1}{9} Z (Z + 1) \int \left[ \frac{1}{\varepsilon} - \frac{1}{E} \right] d\varepsilon \right\}
\]

(42)

if $\varepsilon_m \ll E$ we find:

\[
\sigma = 4\alpha r_0^2 \left\{ \int_{\varepsilon_m}^{E} \left[ \ln \frac{E}{\varepsilon_m} - \frac{5}{8} \right] + \frac{1}{9} Z (Z + 1) \int \left[ \ln \frac{E}{\varepsilon_m} - 1 \right] \right\}
\]

(43)
with:

\[ F = Z^2 \left[ \frac{\phi_1(0)}{4} - \frac{1}{3} \ln Z \right] + Z \left[ \frac{\psi_1(0)}{4} - \frac{2}{3} \ln Z \right] \]

\[ = Z^2 \ln \frac{e^{\varphi_1(0)}}{Z^{1/3}} + Z \ln \frac{e^{\psi_1(0)}}{Z^{2/3}} \]

\[ = Z^2 \ln \frac{183}{Z^{1/3}} + Z \ln \frac{1194}{Z^{2/3}} \quad (44) \]

For most of the practical cases the contribution from the electrons is negligible.

4.3 Beam-Gas Scattering Lifetime

The electrons lost after passing through a volume with residual gas atoms is given by:

\[ \frac{dn}{dt} = -n_0 \sigma N \, dx \]

which leads to the loss rate:

\[ \frac{1}{\tau} = -\frac{1}{n_0} \frac{dn}{dt} = c \sigma N \]

with \( n_0 \) the number of electrons and \( N \) the gas atoms per unit volume.

If desorption effects from the vacuum chamber walls are still relevant, the number of residual gas particles depends also on the number of electrons, \( n_0 \).

With:

\[ N = N_0 + G \cdot n_0 \]

we find:

\[ \frac{dn}{dt} = -\sigma c \left[ n_0 N_0 + n_0^2 G \right] \]

Defining the lifetime as the time it takes for the initial particle intensity to be reduced by \( \frac{1}{e} \), we get:

\[ \frac{1}{\tau} = (\sigma c N_0) \ln \frac{N_0 + G n_0}{e N_0 + G n_0} \]

which reduces to the previous expansion if \( G = 0 \), i.e.:

\[ \frac{1}{\tau} = (\sigma c) N_0 \]

For \( N_0 = 0 \), the vacuum is dominated by the desorption effects, we get:

\[ \frac{1}{\tau} = \frac{(\sigma c) G n_0}{e - 1} \]

Since the residual gas is made up of different molecules each of which may contain more than one atom, we must replace \( N \) by the sum:

\[ N = \sum_{i,j} k_{ij} N_i \quad (46) \]

\( k_{ij} \) = number of atoms \((j)\) in the molecule \((n)\)
\( N_i \) = number of molecules of type \((i)\)
With \( N_i = \frac{p_i}{kT} \) we get for the lifetime (neglecting desorption effects):

\[
\frac{1}{\tau} = -\frac{1}{n_0} \frac{dn}{dt} = \frac{c \sigma}{kT} \sum_{i,j} k_{i,j} p_i
\]

(47)

where \( p_i \) is the partial pressure for gas species \( i \).

4.4 Lifetime Reduction Due to Ion Trapping [12-14]

So far we have only considered the normal pressure given by the residual gas in the vacuum chamber. But there are also effects which can locally enhance the gas pressure inside the volume of the circulating beam. This mechanism is called ion trapping and is only relevant to electron machines.

An ion created by a collision of the electron with the rest gas, gets affected by the electromagnetic force of the circulating bunches, which impose a transverse focusing on the ion, given by:

\[
\begin{align*}
\Delta x' &= a_x \cdot x \\
\Delta y' &= a_y \cdot y 
\end{align*}
\]

(48)

(For larger ion displacements \( x,y \) the expressions are not valid anymore and the force becomes highly non-linear.) The focusing parameter is given by:

\[
a_{x,y} = \frac{2 r_p c N_B}{A \sigma_{x,y} (\sigma_x + \sigma_y)}
\]

(49)

with \( A \) the atomic mass number, \( N_B \) the number of electrons per bunch and \( \sigma_{x,y} \) the rms beam sizes.

The principles for linear stability can now be applied for the small displacement approximation. For a beam with equally spaced bunches and identical particle population, one period of the force acting on the ions is given by the kick described above, followed by a drift equal to the bunch separation, i.e.:

\[
M = \begin{bmatrix} 1 & t_B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}
\]

(50)

with \( t_B \) the time separation between two bunches and "a" the focusing parameter. The motion is stable, i.e. ions are trapped, if:

\[-2 < T_r (M) = 2 - a t_B < 2\]

(51)

This limit is reached for a critical ion mass:

\[
(A_c)_{x,y} = \frac{r_p N_B s_B}{2 \sigma_{x,y} (\sigma_x + \sigma_y)}
\]

(52)

with \( s_B \) the bunch distance. All ion masses above the critical mass are trapped.

The ion accumulation continues until defocusing forces due to the ion space charge in the drift region are strong enough to establish an equilibrium. The drift region has to be replaced now by the defocusing transfer matrix which is given by:
\[ M_{\text{ion}} = \begin{bmatrix} \cosh(gt) & \frac{1}{g} \sinh(gt) \\ g \sinh(gt) & \cosh(gt) \end{bmatrix} \]  \hspace{1cm} (53)

with:

\[ g^2 = \frac{e}{Am_p} \frac{\partial E_y}{\partial y} = \frac{4\pi n_p}{A} c^2 \frac{d_i}{1 + \frac{\sigma_y}{\sigma_x}} \]  \hspace{1cm} (54)

where \( d_i \) is the ion density.

Applying again the stability criterion for one period, we get the condition for trapping.

\[ \frac{1}{A} [ A_c - k d_i ] < 1 \]  \hspace{1cm} (55)

with \( k = \frac{\pi r_p S R^2}{1 + \frac{\sigma_y}{\sigma_x}} \)

For zero initial density only masses \( A > A_c \) can accumulate. But if ions have already been accumulated, i.e. \( d_i \neq 0 \), the stability threshold decreases:

\[ A > A_c - k d_i \]  \hspace{1cm} (56)

The accumulation limit for a bunched beam is reached, if the focusing space charge force of the bunch corresponds to the integrated defocusing force of the already accumulated ions in the drift space between two bunches. This is the case, if the total number of trapped ions corresponds to the total number of electrons in the machine. The maximum ion density is then given by:

\[ d_i = \frac{A_c}{k} = \frac{N}{2\pi R} \cdot \frac{\beta}{2\pi \sigma_x \sigma_y} \]  \hspace{1cm} (57)

defining the neutralization factor as:

\[ \eta = \frac{d_i}{d_p} \]  \hspace{1cm} (58)

where \( d_p \) is the particle density in the bunch, the maximum value is given by:

\[ \eta_{\text{max}} = \frac{n_B l_B}{2\pi R} \]  \hspace{1cm} (59)

with \( n_B \), the number of bunches, \( l_B \) the bunch length and \( R \) the average machine radius.

For the local pressure increase, we then find:

\[ p = p_0 \left( 1 + \eta \frac{d_p}{d_m} \right) \]  \hspace{1cm} (60)

with the density of molecules in the residual gas (at \( T = 300 \text{ °K} \)):

\[ d_m = 3.2 \cdot 10^{13} p_0 [\text{nTorr}] \]  \hspace{1cm} (61)
Especially for small emittances the pressure increase could be large. On the other hand low emittance beams generate a strong over focusing for the ion, which leads to a higher critical mass. For a more complete picture of the pressure increase second ionization should be taken into account.

Another potentially harmful effect comes from a tune spread and tune shift generated by the linear ion space charge force, which is given by:

\[
\Delta Q_y = \frac{r_p}{\gamma} \int \frac{d_i \beta_y}{1 + \left( \frac{\sigma_y}{\sigma_x} \right)^2} \, ds
\]

(\(\Delta Q_y\) is larger then \(\Delta Q_x\) for flat beams).

5. LIFETIME DUE TO TOUSCHEK SCATTERING [15-17]

5.1 Cross Section

Particles inside a bunch perform transverse betatron oscillations around the closed orbit. Due to a scattering effect two particles can transform their transverse momenta into longitudinal momenta.

If the new longitudinal momenta of the two particles are outside the momentum acceptance, the particles are lost. This effect was first recognized by Bruno Touschek at the ADA storage ring of Frascati. The differential cross section for Coulomb scattering of two particles with equal but opposite momenta in the non-relativistic approximation is given by the Möller formula:

\[
\frac{d\sigma}{d\Omega} = \frac{4r_0^2}{(v/c)^2} \left[ \frac{4}{\sin^4 \Theta} - \frac{3}{\sin^2 \Theta} \right]
\]

where: \(r_0\) is the classical electron radius;  
\(v\) is the relative velocity in the centre of mass system (CMS);  
\(\Theta\) is the scattering angle.

Figure 8 shows the geometry of the scattering process.

![Diagram](image)

Fig. 8 Geometry of the process for Touschek scattering.
Following the original derivation by Bruck, we can introduce the dimensionless momentum:

\[ q = \frac{p_x}{m_0c} = \frac{1}{2} \left( \frac{v}{c} \right) \quad (64) \]

where \( p_x \) is the horizontal momentum of one particle and \( v \) the relative velocity in the CMS.

For the moment we consider a flat beam where the major collision effects happen in the horizontal plane. After elastic collision of the two particles in the CMS, the components of the particle momenta along the longitudinal direction are \( q(m_0c) \cos \chi \). Transferred back in the laboratory system, by

\[ p_x = \gamma \left[ p'_x + \frac{B E'}{c} \right] = \gamma P_x \]

Since we assumed a non-relativistic motion in the CMS, we can neglect the second term in the bracket and get:

\[ q(m_0c) \gamma \cos \chi \]

The particle is lost if the component along the longitudinal direction is larger than the momentum acceptance, i.e.:

\[ |q \cos \chi| > \frac{\Delta p_{RF}}{\gamma} \rightarrow \]

\[ |\cos \chi| > \frac{\Delta p_{RF}}{|q| \gamma} = \mu \quad (65) \]

where \( \Delta p_{RF} (m_0c) \) is the maximum momentum accepted by the RF.

Using the geometric relation from figure 9, i.e.:

\[ \cos \theta = \sin \chi \cos \phi \]

and

\[ d\Omega = \sin \chi \, d\chi \, d\phi \]

we get for the total cross section:

\[ \sigma = \frac{4 \pi r_0^2}{(v/c)^2} \int_{\arccos \mu}^{\pi} \sin \chi \, d\chi \int_{-\pi}^{\pi} d\phi \left[ \frac{4}{(1 - \sin^2 \chi \cos^2 \phi)^2} \right. \]

\[ - \left. \frac{3}{(1 - \sin^2 \chi \cos^2 \phi)} \right] \quad (66) \]

and finally after performing the integration:

\[ \sigma = \frac{8 \pi r_0^2}{(v/c)^4} \left[ \frac{1}{\mu^2} - 1 + \ln \mu \right] \quad (67) \]

(in the CMS)
5.2 Loss Rate and Lifetime

The differential loss rate of the particles, as sketched in figure 9, is given by:

\[
d\left(\frac{dN}{dt}\right) = (\sigma v)\rho dN
\]  

(68)

Since incoming and scattered particles belong to the same ensemble, i.e. \( dN = \rho dV \), we have:

\[
d\left(\frac{dN}{dt}\right) = (\sigma v)\rho^2 dV
\]  

(69)

For the total loss rate we get:

\[
\frac{dN}{dt} = \bar{\sigma v} \int p^2 dV \quad \text{(in the CMS)}
\]  

(70)

where we have assumed the approximation, that the product \( (\sigma v) \) is independent of the particle coordinates.

The integral can easily be performed in the Laboratory System (LS), where the distribution function is given by:

\[
\rho(x,y,s) = \frac{N}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_s} e^{-\left[ \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{s^2}{2\sigma_s^2} \right]}
\]  

(71)

assuming decoupled Gaussian distributions for all directions \((x,y,s)\).

Performing the integration we get:

\[
\int \rho^2 dV = \frac{N^2}{8\pi^{3/2}\sigma_x\sigma_y\sigma_s}
\]  

(72)

The average \( \bar{\sigma v} \) has to be done with regard to the change in relative momentum which leads to particle loss.
\[
\overline{\sigma V} = 2 \int_{\Delta p_{\text{RF}}/\gamma}^{x} g(q)(\sigma_{v})dq
\]  

where the factor 2 takes into account that two particles are lost in each scattering process.

The distribution function can be written as:

\[
g(q) = \frac{1}{\sqrt{2\pi} \sigma_{q}} \frac{e^{-\frac{q^2}{2\sigma_q^2}}}{\sigma_{q} = \frac{\gamma \sigma_{x}'}{\sqrt{2}}}
\]

and substituting relations (70), (72), (73) and the bunch volume \( V_{B} = (4\pi)^{3/2} \sigma_{x} \sigma_{y} \sigma_{s} \)

in \( \frac{1}{\tau} = \frac{1}{N} \frac{dN}{dt} \) we get:

\[
\frac{1}{\tau} = \frac{\sqrt{\pi} r_{0}^{2} c N_{B} (m_{0} c)^{3}}{\gamma^{2} V_{B} \sigma_{px}} \times
\]

\[
\int \frac{1}{q^{3}} \left[ \left( \frac{\gamma q}{\Delta p_{\text{RF}}} \right)^{2} - 1 - \ln \left( \frac{\gamma q}{\Delta p_{\text{RF}}} \right) \right] \frac{e^{-\frac{q^2}{\sigma_{px}^2}}}{dq}
\]

After introducing the quantities:

\[
u = \left( \frac{q}{\sigma_{px}} \right) \quad \text{and} \quad \varepsilon = \left( \frac{\Delta p_{\text{RF}}}{\gamma \sigma_{px}} \right)^{2}
\]

we get the final form of the Touschek growth rate for a flat beam:

\[
\frac{1}{\tau} = \frac{\sqrt{\pi} r_{0}^{2} c}{\gamma^{3}} \cdot \frac{N_{B}}{V_{B} \sigma_{x}'} \cdot \frac{1}{\left( \frac{\Delta p_{\text{RF}}}{p_{0}} \right)^{2}}
\]

\[
\left\{ \varepsilon \int_{\varepsilon}^{\infty} \frac{1}{u^{2}} \left[ \frac{u}{\varepsilon} \right] - \frac{1}{2} \ln \left( \frac{u}{\varepsilon} \right) - 1 \right\} e^{-u} du
\]

(74)

where the expression in the curly bracket is usually abbreviated by \( C(\varepsilon) \), which for \( \varepsilon < 1 \) can be approximated by:

\[
C(\varepsilon) = \ln \left( \frac{1}{1.78 \varepsilon} \right) - 1.5
\]

(75)

In a similar way we can derive the Touschek lifetime for a round beam [18]:

427
\[
\frac{1}{\tau} = \frac{2\pi r_0^2 c}{\gamma^4} \frac{N_B}{\gamma \sigma_x' \sigma_y'} \frac{D(\epsilon)}{\left( \frac{\Delta_{\text{PR}}}{p_0} \right)}
\]  

(76)

with:

\[
D(\epsilon) = \sqrt{\epsilon} \int_0^\infty \frac{1}{u^{3/2}} \left[ \frac{u}{\epsilon} - \frac{1}{2} \ln \left( \frac{u}{\epsilon} \right) - 1 \right] e^{-u} du
\]

(77)

6. INTRABEAM SCATTERING [19-21]

Intrabeam scattering is usually used as a synonym for multiple small-angle Coulomb (or also Touschek) scattering inside the beam. This effect has first been completely described by A. Piwinski for a weak focusing machine. Later on it has been extended to a strong focusing machine by Mtingwa and Bjorken.

The derivation is only conceptually treated here, with the kinematics derived for the horizontal plane only.

6.1 Kinematics

The treatment given here closely follows that given in reference [20], but is for reasons of simplicity restricted to two dimensions only.

6.1.1 Coordinate systems

Figure 10 shows the momenta of two colliding particles, \( \hat{p}_1 \) and \( \hat{p}_2 \) in the laboratory system.

![Diagram of scattered particles](image)

Fig. 10 Momenta of scattered particles in the laboratory system.

The beam is moving in direction "s", with velocity "\( v_0 \)". The momenta in this coordinate system are given by:

\[
\hat{p}_i = p_i \left( \cos \left[ \arctg \left( x_i \right) \right], \sin \left[ \arctg \left( x_i \right) \right] \right)
\]

\[
= p_i \left( 1, x_i \right)
\]

(78)
where \( x_i' \) is the slope of the particle trajectory in the x-plane and the index \( i = 1,2 \) refers to the two particles.

In the \((u,v)\)-coordinate system the components for the momenta are:

\[
\begin{align*}
    p_{1u} &= p_1 \cos \alpha_1, \quad p_{2u} = p_2 \cos \alpha_2, \\
    p_{1v} &= -p_1 \sin \alpha_1, \quad p_{2v} = p_2 \sin \alpha_2
\end{align*}
\]

6.1.2 Transformation to the centre-of-mass system (CMS)

To treat the collision in a non-relativistic way, we now perform a transformation to the CMS (where the quantities are marked by primes). The relevant set of transformations is given by:

\[
\begin{align*}
    E_i' &= \gamma_u \left( E_i - \beta_u c p_{iu} \right) \\
    p_{iu}' &= \gamma_u \left( p_{iu} - \frac{\beta_u}{c} E_i \right) \\
    p_{iv}' &= p_{iv} \\
\end{align*}
\]

(79)

\( \beta_u \) and \( \gamma_u \) are then fixed by the conditions:

a) \[ p_{1u}' + p_{2u}' = 0 \]

with

\[ E_i = E_0 \beta_i p_i \]

we get:

\[
\beta_u = \frac{\beta_1 \gamma_1 \cos \alpha_1 + \beta_2 \gamma_2 \cos \alpha_2}{\gamma_1 + \gamma_2}
\]

b) \[ p_{1v}' + p_{2v}' = 0 \]

or

\[ \beta_1 \gamma_1 \sin \alpha_1 = \beta_2 \gamma_2 \sin \alpha_2. \]

6.1.3 Change of momenta in the CMS

In the CMS the two particles are deviated in their direction of motion by the angle \( \theta' \), as indicated in figure 11.

\[ \text{Fig. 11 Scattering process in the center-of-mass system.} \]
The components of the new momenta in the CMS can be derived by the rotation:

\[
\begin{bmatrix}
\dot{p}_{iu}' \\
\dot{p}_{iv}'
\end{bmatrix} = \begin{bmatrix}
\cos \theta' & \sin \theta' \\
-\sin \theta' & \cos \theta'
\end{bmatrix}\begin{bmatrix}
\dot{p}_{iu} \\
\dot{p}_{iv}
\end{bmatrix} = \begin{bmatrix}
p_{iu} \cos \theta' + p_{iv} \sin \theta' \\
-p_{iu} \sin \theta' + p_{iv} \cos \theta'
\end{bmatrix}
\]

(80)

6.1.4 Back transformation to the LS

The new momenta are now transformed back to the laboratory system:

\[
\begin{align*}
\tilde{p}_{iu}' &= \gamma \left( p_{iu} + \frac{\beta_{iu}}{c} E_i \right) \\
\tilde{p}_{iv}' &= \tilde{p}_{iv}
\end{align*}
\]

(81)

Since \( E_i' = E_0 \gamma_i \) and the motion has been considered to be non-relativistic in the CMS, we can approximate:

\[
\tilde{p}_{iu} = \gamma \tilde{p}_{iu}
\]

\((i = 1, 2)\)

(82)

We can now derive the change in momenta due to the scattering process:

\[
\tilde{\delta p}_i = \tilde{p}_i - \tilde{p}_i \quad \text{(in the u,v - system)}
\]

(83)

Expressed in components, we find:

\[
\begin{align*}
\delta p_{1u} &= p_1 \gamma \left[ \gamma \left( \cos \alpha_1 - \frac{\beta_{u1}}{\beta_1} \right) (\cos \theta' - 1) - \sin \alpha_1 \sin \theta' \right] \\
\delta p_{1v} &= -p_1 \left[ \gamma \left( \cos \alpha_1 - \frac{\beta_{u1}}{\beta_1} \right) \sin \theta' + \sin \alpha_1 (\cos \theta' - 1) \right] \\
\delta p_{2u} &= p_2 \gamma u \left[ \gamma \left( \cos \alpha_2 - \frac{\beta_{u2}}{\beta_2} \right) (\cos \theta' - 1) + \sin \alpha_2 \sin \theta' \right] \\
\delta p_{2v} &= -p_2 \left[ \gamma \left( \cos \alpha_2 - \frac{\beta_{u2}}{\beta_2} \right) \sin \theta' - \sin \alpha_2 (\cos \theta' - 1) \right]
\end{align*}
\]

(84)

6.2 Change of Equilibrium Beam Distribution

We now examine how the change in momenta influences the equilibrium beam parameters, such as emittance and momentum spread.

6.2.1 Emittance

The emittance of a particle with betatron coordinates \((x, x')\) at the position with optical parameters \((\beta_x, \alpha_x)\) is given by:

\[
\epsilon_x \beta_x = x^2 + (x \alpha_x + x' \beta_x)^2
\]

(85)

Due to the scattering process the coordinates \((x, x')\) are changed. The displacement is changed, because the corresponding closed orbit is modified, i.e.:
\[ \delta x = - \frac{\delta p}{p} D_x \]  

(86)

Correspondingly the derivative of the dispersion changes the slope of the betatron coordinate. But in addition there is a modification of the transverse momentum which also causes a change of the slope, i.e.:

\[ \delta x' = - \frac{\delta p}{p} D_x + \frac{\delta p_x}{p_0} \]  

(87)

For simplicity, we just consider the case \( \alpha_x = 0 \) and \( D_x' = 0 \). We then find for the change in emittance:

\[
\delta \epsilon_x \beta_x = \left[ 2x \delta x + \delta x^2 \right] + \beta_x^2 \left[ 2x' \delta x' + \delta x'^2 \right] \\
= \left[ -2x D_x \frac{\delta p}{p} + D_x^2 \left( \frac{\delta p}{p} \right)^2 \right] + \beta_x^2 \left[ 2x' \frac{\delta p_x}{p_0} + \left( \frac{\delta p_x}{p_0} \right)^2 \right]
\]  

(88)

6.2.2 Momentum spread

The invariant of the momentum spread for a bunched beam is given by:

\[ H = \eta^2 + \frac{\eta^2}{\Omega_s^2} \]  

(89)

with \( \eta \) the relative momentum deviation and \( \Omega_s \) the synchrotron frequency. If the total momentum is changed, we find for the variation of the invariant:

\[ \delta H = 2\eta \frac{\delta p}{p} + \left( \frac{\delta p}{p} \right)^2 \]  

(90)

6.3 Statistical Averaging of the Invariants

Only the method will be sketched here, how the final results have been achieved can be found in reference [20].

First the invariants have to be averaged with respect to all scattering angles. Since the variations of the invariants are expressed by the changes in momenta, we have to find:

\[ \left\langle \delta p \right\rangle = \int \int \delta p \, d\sigma \]  

(91)

\[ \left\langle \delta p^2 \right\rangle = \int \int \delta p^2 \, d\sigma \]  

(92)

The lower limit of the first integral, \( \theta_m \) is given by a scattering angle, which corresponds to an impact parameter of half the beam size.

The differential of the cross section in the CMS is given by the non-relativistic Möller cross section, i.e.:

\[ d\sigma' = \frac{p_0^2}{16 \beta^4} \frac{\sin \theta' \, d\theta' \, d\phi'}{\sin^4 \frac{\theta}{2}} \]  

(93)

which has been approximated for small scattering angles.
In the second step the mean value of the invariants of one particle can be calculated by averaging over all possible coordinates of the second particle. To derive the mean values for all particles, one has to additionally average in a third step over all possible coordinates of the first particle. Combining the second and third steps, all invariants have to be integrated with the distribution function.

\[ P = \frac{1}{f_2^2(s) f_2^2(z) f_x(x_1) f(y_1) f_x'(x_1') f_y'(z_1')}. \]

The following relations hold between the particle coordinates:

\[
\begin{align*}
s_1 &= s_2 \\
z_1 &= z_2 \\
x_1 + \eta_1 D_x &= x_2 \eta_2 D_x
\end{align*}
\]

After having performed the averaging in the prescribed form, we can derive the invariant:

\[ \langle H \rangle \left( \frac{1}{\gamma^2} - \alpha_c \right) + \left( \frac{\epsilon_x}{\beta_x} + \frac{\epsilon_z}{\beta_z} \right) = \text{const.} \]  

(95)

There is a qualitatively different behaviour below and above transition. For \( \gamma < \gamma_T \), i.e. \( \frac{1}{\gamma^2} > \alpha_c = \frac{1}{\gamma_T^2} \), the motion is bounded.

An increase of one invariant can only happen at the expense of the other invariants. The particles behave like gas molecules in a closed box and an equilibrium of the distribution exists.

Above transition, i.e. for \( \frac{1}{\gamma^2} - \alpha_c < 0 \), the equilibrium quantities can increase beyond all limits and still fulfill the invariant condition.

7. LIFETIME DUE TO RESONANCE CROSSING [22]

Dangerous resonances are avoided by choosing an appropriate working point in the tune diagram. Due to various effects it might happen that the tune changes in time and resonances are crossed. One has to differ here between two effects depending on the speed of the tune variation.

7.1 Fast Crossing

For a sudden jump in tune within a time, which is small compared to the period a particle needs to complete its phase space cycle, the amplitude change is not very large. For the growth due to one rapid transversal of a one dimensional sum resonance we find:

\[ \int_{x^N}^{x} \frac{dx}{x^{N-1}} = \frac{\Delta e}{N} \frac{\pi}{\sqrt{\left| \Delta Q_t \right|}} \]  

(96)

where \( \Delta e \) is the resonance width, \( \Delta Q_t \) the tune change per turn and \( x \) the relative amplitude \( x_1/a_0 \). Depending on the order of the resonance, \( N \), we get for the growth: 

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\[
\begin{align*}
N = 1 & : x - 1 \\
N = 2 & : \ln x \\
N > 2 & : \frac{1}{N} \left[ 1 - \frac{1}{x^N} \right] \end{align*}
\]
\[
\Delta e = \frac{\Delta}{N} \pi \left[ \Delta Q_t \right]
\]
(97)

For several random crossings the average of the amplitude growth has to be multiplied with the square root of the crossings.

For a sinusoidal variation of the time, the resonance will get an infinite number of sidebands spaced by \( \frac{Q_s}{N} \), where \( Q_s \) is the synchrotron tune and \( N \) the order of the resonance. The lower order of these sidebands have usually to be avoided.

7.2 Slow Tune Variation

In the case of an adiabatic variation of the tune, the particles are trapped in the resonance island which is moving outwards and growing in phase space until the particle is lost at the aperture limit. Such an adiabatic variation of the tune can be generated for instance by a momentum diffusion (given by intrabeam scattering or beam gas scattering). If the diffusion constant is:

\[
D_p = \frac{d}{dt} \left( \frac{\Delta p}{p} \right)^2_{\text{rms}}
\]
(98)

we find a related diffusion of the tune (given for instance by the non-compensated terms of the chromaticity) with the diffusion constant:

\[
D_Q = (Q') D_p
\]
(99)

The fractional loss rate at time \( t \) is given by:

\[
\frac{1}{N} \frac{dN}{dt} = \frac{2P_T}{\Delta Q} \sqrt{\frac{D_Q}{\pi t}} \cdot e^{-\frac{8Q^2}{4D_Q t}}
\]
(100)

with \( P_T \) the trapping probability, which depends on the resonance strength, the particle amplitude and the rate of change.

8 LIFETIME MEASUREMENT

The measured particle loss rate in an accelerator is given by the sum over all contributions, i.e.:

\[
\frac{1}{\tau} = \sum_i \frac{1}{\tau_i}
\]
(101)

It is not always easy to derive from measurements, how the individual effects contribute to the total lifetime. The difference in dependence on various machine parameters (such as energy, gas pressure, beam size, aperture, etc.) must be utilized to distinguish between the various effects. This also requires a good knowledge of the vacuum pressure, which has to be derived from the monitored pressure. If vacuum gauges are placed near the vacuum pumps, the monitored average pressure could be substantially lower than the real one. A simulation can be performed to derive the average pressure, which again is difficult, since reliable information on the pumping speed and details on desorption effects are needed.
For the interpretation of lifetime measurements, the difference in dependence of the various lifetime effects on machine parameters such as energy, beam size, aperture, etc. has to be utilized.

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* * *

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